On recoverability of finite traces of square-summable sequences

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***Abstract*—The paper investigates recoverability of infinite sequences (discrete time signals) from incomplete observations. It is shown that under very mild restrictions on the location of the observed and missed data, recoverability of finite traces of signals can be achieved for wide classes of signals that everywhere dense in the space of square-summable signals.**

Keywords: discrete time signals, digital signals, sampling, signal recovery, spectrum degeneracy.

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I. Introduction

In general, possibility of recovery of a signal from a sample is usually associated with constraints for the classes of underlying signals such as restrictions on the spectrum support or signal sparsity. Recovery of signals from outside of these classes also can be considered in this setting; the corresponding recovery algorithms can be applied to the projections of the underlying processes on a recoverable class of signals. For example, in continuous time setting, band-limited functions can be recovered without error from a discrete sample taken with a sampling rate that is at least twice the maximum frequency present in the signal (the Nyquist critical rate). This defines the class of recoverable functions and the set of observations required for the recovery. It is known that, for continuous time signals with certain structure, the restrictions imposed by the Nyquist rate could be excessive for signal recovery; see e.g. [1, 14]. In particular, a sparse enough subsequence or a semi-infinite subsequence can be removed from an oversampling sequence [11, 17]. There is also a so-called Papoulis approach [16] allowing to reduce the sampling rate with additional measurements at sampling points.

For finite discrete time signals, some paradigm changing results were obtained in [2, 3, 4, 5, 6, 7] and consequent papers in the so-called *compressive sensing* setting (also known as compressed sensing, compressive sampling, or sparse sampling). This approach explores sparsity of signals, i.e. restrictions on the number of nonzero members of the underlying finite sequences.

Clearly, a process of a general type cannot be approximated by band-limited processes with a preselected band or by processes with a sparse spectrum with a preselected degree of sparsity. This leads to major limitations for data compression and recovery. For example, consider data compression via approximation of a continuous time function by samples of band-limited functions. A closer approximation would require wider spectrum band for these band-limited functions or more frequent sampling; respectively, a closer approximation leads to less efficient data compression.

In general, there is a difference between the problem of uniqueness of recovery and the problem of existence of a stable recovery algorithm. For continuous-time signals, this was investigated in the framework of the approach based on the so-called Landau's criterion [12, 13]; see [12, 13] and a recent literature review in [15]. As was emphasized in [13], the uniqueness results do not imply stable data recovery: any sampling below the Landau's critical rate cannot be stable. The Landau's rate mentioned here is a generalization of the critical Nyquist rate for the case of stable recovery, non-equidistant sampling and disconnected spectrum gaps.

The present paper considers infinite discrete time signals and shows that there exist wide classes of signals allowing robust and uniform recovery of their finite traces and such that they can approximate any signal of a general type (Theorem 1 below). This result represent a generalisation of results [8, 9, 10] obtained for some special sets of observed points and special types of spectrum degeneracy.

The paper is organized as following. Section II presents some definitions and preliminary results on predictability of sequences. Section III presents the main result. Section IV contains the proofs. Section V presents some discussion.

II. Definitions and background

Let $\mathbb{T} \triangleq \{z \in \mathbf{C} : \ |z| = 1\}$, and let $\mathbb{Z}$ be the set of all integers. Let $\mathbb{Z}^- = \{k \in \mathbb{Z} : \ k \le 0\}$, and let t $\mathbb{Z}^+ = \{k \in \mathbb{Z} : \ k > 0\}$.

We denote by ℓ_r the set of all sequences $x = \{x(t)\} \subset \mathbf{C}$, $t = 0, \pm 1, \pm 2, ...$, such that $\|x\|_{\ell_r} = \left(\sum_{t=-\infty}^{\infty} |x(t)|^r\right)^{1/r} < +\infty$ for $r \in [1, \infty)$ or $\|x\|_{\ell_\infty} = \sup_t |x(t)| < +\infty$ for $r = +\infty$.

For $x \in \ell_1$ or $x \in \ell_2$, we denote by $X = \mathcal{Z}x$ the Z-transform

$$X(z) = \sum_{t=-\infty}^{\infty} x(t) z^{-t}, \quad z \in \mathbf{C}.$$

Respectively, the inverse $x = \mathcal{Z}^{-1}X$ is defined as

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{i\omega}\right) e^{i\omega t} d\omega, \quad t = 0, \pm 1, \pm 2, ...$$

We have that $x \in \ell_2$ if and only if $\|X\left(e^{i\omega}\right)\|_{L_2(-\pi,\pi)} < +\infty$. In addition, $\|x\|_{\ell_\infty} \le \|X\left(e^{i\omega}\right)\|_{L_1(-\pi,\pi)}$.

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For a finite set S, we denote by $|S|$ the number of its elements.

For $\rho > 0$, we denote $B_\rho(\ell_2) = \{x \in \ell_2 : \ \|x\|_{\ell_2} \le \rho\}$.

The setting for the recovery problem

Let disjoint subsets $\mathcal{M}$ and $\mathcal{T}$ of $\mathbb{Z}$ be given, and let $\mathcal{V} \triangleq \mathbb{Z} \setminus (\mathcal{M} \cup \mathcal{T})$.

We are interested in the problem of recovery values $\{x(t)\}_{t\in\mathcal{T}}$ from observations $\{x(s)\}_{s\in\mathcal{M}}$ for $x \in \ell_2$, possibly, in the presence of a contaminating noise. We consider linear estimates only.

Definition 1: Let $\mathcal{X} \subset \ell_2$ be a set of signals. Consider a problem of recovery $\{x(t)\}_{t\in\mathcal{T}}$ from observations on $\mathcal{M}$ of noise contaminated sequences $x = \widetilde{x} + \xi$, where $\widetilde{x} \in \mathcal{X}$, and where $\xi \in \ell_2$ represents a noise. We say that $\mathcal{X}$ allows uniform and robust $(\mathcal{M}, \mathcal{T})$-recovery if there exists a sequence of mappings $h_n : \mathbb{Z} \times \mathbb{Z} \to \mathbf{R}$, $n = 1, 2, ...$, such that $\sup_{t\in\mathcal{T}} \|h_n(t, \cdot)\|_{\ell_2} < +\infty$ and that, for any $\varepsilon > 0$, and any finite set $I \subset \mathcal{T}$, there exists $\rho > 0$ and $N > 0$ such that

$$\sup_{t\in\mathcal{T}\cap I} |x(t) - \widetilde{x}_n(t)| \le \varepsilon \quad \forall x \in \mathcal{Y},\ \eta \in B_\rho(\ell_2), \tag{1}$$

where

$$\widetilde{x}_n(t) = \sum_{s\in\mathcal{M},\ |s|\le N} h_n(t, s)x(s). \tag{2}$$

Proposition 1: If a set $\mathcal{X}$ allows uniform and robust $(\mathcal{M}, \mathcal{T})$-recovery, then this set allows uniform and robust $(\bar{\mathcal{M}}, \bar{\mathcal{T}})$-recovery for any disjoint subsets $\bar{\mathcal{M}}$ and $\bar{\mathcal{T}}$ of $\mathbb{Z}$ such that $\mathcal{M} \subset \bar{\mathcal{M}}$ and $\bar{\mathcal{T}} \subset \mathcal{T}$.

Proof of Proposition 1. Let $\{h_n\}_{n=1}^{\infty}$ be such as required for $(\mathcal{M}, \mathcal{T})$-recoverability in in Definition 1. Then the conditions of Definition 1 hold for uniform and robust $(\bar{\mathcal{M}}, \bar{\mathcal{T}})$-recovery if one selects the corresponding functions $\bar{h}_n(t, s) = h_n(t, s)\mathbb{I}_{\{s\in\mathcal{M}\}}$. $\square$

III. The main results

Theorem 1: Assume that any of the following conditions holds:

(A) $|\mathcal{M} \cap \mathbb{Z}^-| = +\infty$ and $|\mathcal{T} \cap \mathbb{Z}^-| < +\infty$.
(B) $|\mathcal{M} \cap \mathbb{Z}^+| = +\infty$ and $|\mathcal{T} \cap \mathbb{Z}^+| < +\infty$.
(C) $|\mathcal{M} \cap \mathbb{Z}^-| = +\infty$ and $|\mathcal{M} \cap \mathbb{Z}^+| = +\infty$.

Then, for every bounded set $\mathcal{B} \subset \ell_2$ there exists a set of processes $\mathcal{B}_{\mathcal{M},\mathcal{T}}$ that allows robust and uniform $(\mathcal{M}, \mathcal{T})$-recovery and such that, for any $x \in \mathcal{B}$ and any $\varepsilon > 0$, there exists $\widehat{x} \in \mathcal{B}_{\mathcal{M},\mathcal{T}}$ such that $\|\widehat{x} - x\|_{\ell_2} \le \varepsilon$.

The conditions on the choice of the sets $\mathcal{M}$ and $\mathcal{T}$ imposed by Theorem 1 are very mild. For example, $\mathcal{M}$ can have arbitrarily located gaps, in particular, it can have periodic gaps as well as nonperiodic gaps.

Some examples where Theorem 1 holds

(i) If $\mathcal{M} = \{k,\ k \in \mathbb{Z}^-\}$ and $\mathcal{T} = \mathbb{Z}^+$, then condition (A) of Theorem 1 is satisfied. The corresponding recovery problem is a predicting problem.

(ii) If $\mathcal{M} = \{km,\ k \in \mathbb{Z}^-\}$ and $\mathcal{T} = \mathbb{Z}^+$, where $m \in \mathbb{Z}^+$ is given, then condition (A) of Theorem 1 is satisfied. The corresponding recovery problem is a predicting problem where only a periodic subsequence of past observations is available.

(iii) If $\mathcal{M} = \{k^d,\ k \in \mathbb{Z}^-\}$ and $\mathcal{T} = \mathbb{Z}^+$, where $m \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$ are given, then condition (A) of Theorem 1 is satisfied. The corresponding recovery problem is a predicting problem where only a non-periodic subsequence of past observations is available with possible large gaps between observations.

(iv) If $\mathcal{M} = \{k^d,\ k \in \mathbb{Z}^+\}$ and $\mathcal{T} = \{k \in Z,\ k < s\}$, where $d \in \mathbb{Z}^+$ and $s \in \mathbb{Z}$ are given then condition (B) of Theorem 1 is satisfied.

(v) If $\mathcal{M} = \{m|k|^d \mathrm{sign}\, k,\ k \in \mathbb{Z}\}$ and $\mathcal{T} = \mathbb{Z}$, where $m \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$, then condition (C) of Theorem 1 is satisfied. For the case where $d = 1$, this would correspond to the problem of recovery of a sequence from its m-periodic subsequence.

IV. Proof of Theorem 1

Let us prove first that Theorem 1 holds if condition (A) is satisfied. We refer it as Case (A). Let $\theta = -1 + \min_{t\in\mathcal{T}} t$. By condition (A) of the theorem, $\theta > -\infty$.

By Proposition 1, it suffices to consider the case where $\mathcal{M} \subset \{k \in \mathbb{Z} : \ t \le \theta\}$. Let us accept assume that this is the case.

Let the sequence $\{\tau(k)\}_{k\in\mathbb{Z}}$ be such that $t(k-1) < t(k)$ for all k,

$$\mathcal{M} = \{\tau(k)\}_{k=-\infty}^{\theta}, \quad \tau(k) = k \quad \text{for} \quad k > \theta. \tag{3}$$

Let us consider the processes $x \in \mathcal{B}$ and processes $y = y(x(\cdot))$ such that $y(k) = x(\tau(k))$ for all $k \in \mathbb{Z}$.

For $\delta > 0$, let $J(\delta) \triangleq \{\omega \in (-\pi, \pi] : \ |e^{i\omega} - 1| \le \delta\}$. Let $Y = \mathcal{Z}y$, and let $\widehat{Y}\left(e^{i\omega}\right) = \mathcal{Y}\left(e^{i\omega}\right)\mathbb{I}_{\{\omega\notin J(\delta)\}}$, $\widehat{y} = \mathcal{Z}^{-1}\widehat{Y}$. Let $\mathcal{B}^y_{\mathcal{M},\mathcal{T}}$ be the set of all corresponding processes $\widehat{y}$ for all $x \in \mathcal{B}$ and all $\delta > 0$.

Let $\mathcal{M}^y \triangleq \{k \in \mathbb{Z},\ k \le \theta\}$ and $\mathcal{T}^y \triangleq \{k \in \mathbb{Z},\ k > \theta\}$.

By Lemma 2 from [10], the set $\mathcal{B}^y_{\mathcal{M},\mathcal{T}}$ is uniformly $(\mathcal{M}^y, \mathcal{T}^y)$-recoverable in the sense of Definition 1 with some set of kernels $\{h^y_n\}_{n>0}$, such that the required estimate can be presented as

$$\widehat{y}_n(t) = \sum_{s\in\mathcal{M}} h_n(t - s)y(s).$$

Further, let $\widehat{x} \in \ell_2$ be defined such that

$$\begin{aligned} \widehat{x}(\tau(k)) &= \widehat{y}(k), && \text{if} \quad k \le \theta, \\ \widehat{x}(k) &= \widehat{y}(k), && \text{if} \quad k > \theta, \\ \widehat{x}(s) &= x(s) && \text{if} \quad s \le \theta,\ s \notin \mathcal{M}. \end{aligned}$$

It follows from the definitions and from the established recoverability of the set $\mathcal{B}^y_{\mathcal{M},\mathcal{T}}$ that the set $\mathcal{B}_{\mathcal{M},\mathcal{T}}$ allows uniform and robust $\mathcal{M}^y, \mathcal{T}^y$-recoverability in the sense of Definition 1 such that the required estimate can be presented as

$$\widehat{y}_n(t) = \sum_{s\in\mathcal{M}^y, |s|\le N} h_n(t-s)y(s)$$
$$= \sum_{s\in\mathcal{M},\ s\ge -N_1} h_n(t-\tau(s))x(\tau(s)).$$

Here $N_1>0$ is such that $N_1=-\tau(-N)$ It follows from the definitions that the set $\mathcal{B}_{\mathcal{M},\mathcal{T}}$ allows uniform and robust $\mathcal{M},\mathcal{T}$-recoverability in the sense of Definition 1 such that the required estimate can be presented as

$$\widehat{x}_n(t) = \sum_{s\in\mathcal{M},\ |s|\le N_1} h_n(t-\tau(s))x(s).$$

Similarly, we obtain that condition (B) is sufficient to ensure that the statement of the theorem holds. For this, we can just repeat the proof adjusted to the use of backward prediction. We refer it as Case (B).

Let us prove that condition (C) is sufficient to ensure that the statement of the theorem holds.

Let $\mathcal{M}_\pm \triangleq \mathcal{M}\cap\mathbb{Z}^\pm$ and $\mathcal{T}_\pm \triangleq \mathcal{T}\cap\mathbb{Z}^\pm$. Further, for $x\in\mathcal{B}$, let τ_+, y_+, $\widehat{x}_+$, and $\mathcal{B}^y_{\mathcal{M}_-,\mathcal{T}_+}$, be defined similarly to τ, y, $\widehat{x}$, and $\mathcal{B}_{\mathcal{M}_-,\mathcal{T}}$, respectively, defined for Case (A) with $\theta=0$. By the result obtained for the Case (A), it follows that the class $\mathcal{B}_{\mathcal{M}_+,\mathcal{T}_-}$ is uniformly $(\mathcal{M}_+,\mathcal{T}_-)$-recoverable, i.e., the conditions of Definition 1 hold, with the estimates

$$\widetilde{x}_{n,+}(t) = \sum_{s\in\mathcal{M}_-, s\ge -N} h_{n,+}(t-\tau_+(s))\widehat{x}_-(s), \quad t\in\mathcal{T}_+.$$

Here kernels $h^y_{n,+}$ are such as required in Definition 1.

Further, for $x\in\mathcal{B}$, let τ_-, y_- and $\widehat{x}_-$, and $\mathcal{B}_{\mathcal{M}_+,\mathcal{T}_-}$, be defined similarly to τ, y, $\widehat{x}$, and $\mathcal{B}_{\mathcal{M},\mathcal{T}}$, respectively, for Case (B) with $\theta=0$. By the result obtained for the case where condition (B) holds, it follows that the class $\mathcal{B}_{\mathcal{M}_+,\mathcal{T}_-}$ is uniformly $(\mathcal{M}_+,\mathcal{T}_-)$-recoverable, i.e., the conditions of Definition 1 hold, with the estimates

$$\widetilde{x}_{n,-}(t) = \sum_{s\in\mathcal{M}_+, s\le N} h_{n,-}(t-\tau_-(s))\widehat{x}_-(s), \quad t\in\mathcal{T}_-.$$

Here kernels $h^y_{n,-}$ are such as required in Definition 1.

Consider the process

$$\widetilde{x}_n(t) = \widetilde{x}_{n,+}(t)\mathbb{I}_{\{t\in\mathcal{M}_+\}} + \widetilde{x}_{n,-}(t)\mathbb{I}_{\{t\in\mathcal{M}_-\}}.$$

Clearly, an estimate (1) for a given ε holds for sufficiently large $\bar{n}$ and small ρ, since similar estimates hold for $\sup_{t\in\mathcal{M}_\pm}|\widetilde{x}_{n,\pm}(t)-x(t)|$.

Let us show that $\widehat{x}_n$ can be represented via (2) for with some choice of appropriate mappings $h_n:\mathcal{Z}\times\mathcal{Z}\to\mathbf{R}$. By the definitions, it follows that, for $t\in\mathcal{T}$,

$$\widetilde{x}_n(t) = \mathbb{I}_{\{t\in\mathcal{M}_-\}} \sum_{s\in\mathcal{M}_-, s\ge -N} h_{n,+}(t-\tau_+(s))\widehat{x}_-(s)$$
$$+\mathbb{I}_{\{t\in\mathcal{M}_+\}} \sum_{s\in\mathcal{M}_+, s\le N} h_{n,-}(t-\tau_-(s))\widehat{x}_-(s).$$

Hence

$$\widetilde{x}_n(t) = \sum_{s\in\mathcal{M}, |s|\le N} h_n(t,s)\widehat{x}(s),$$

where

$$\widetilde{h}_n(t,s)$$
$$= \mathbb{I}_{\{t\in\mathcal{M}_+\}} h_{n,+}(t-\tau_+(s)) + \mathbb{I}_{\{t\in\mathcal{M}_-\}} h_{n,-}(t-\tau_-(s)).$$

This gives representation (2). This completes the proof for the Case (C) as well as the proof of Theorem 1.

V. Discussion

The proofs above are using Lemma 2 from [10] on the robust and uniform predictability, which is rather technical. In fact, other similar results on extrapolation of semi-infinite sequences could be used instead. For example, it is known [11, 17] that a band-limited function is uniquely defined by a semi-infinite half of any periodic oversampling sequence; this could be used to prove a version of Theorem 1 where recoverability is not necessarily uniform and robust.

The choices of the sets $\mathcal{B}^y_{\mathcal{M},\mathcal{T}}$ in the proofs above could be different. In the proof above, these sets contain band-limited processes with spectrum gaps $J(\delta)$. Alternatively, they could be constructed from the processes featuring spectrum degeneracy at a single point only, as is allowed in Lemma 2 [10].

Furthermore, the predicting kernels used in the proof were selected to cover the most general case of the choice of $(\mathcal{M},\mathcal{T})$. For some important special cases, other kernels could be more effective with respect to numerical implementation. For example, the linear time-invariant recovering operators suggested in [9] would be preferable in the case where the set $\mathcal{T}$ is finite. Furthermore, the linear time-invariant recovering operators suggested in [8] would be preferable in the case where the set $\mathcal{T}$ is finite and where the underlying processes belong to ℓ_1. The linear recovery operators suggested in [10] would be preferable in the case where the set $\mathcal{M}$ represents a periodic subsequence of $\mathbb{Z}$.

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