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An Analysis of Students' Abstraction and Use of Representations for Understanding the Operational Formal Definition of Function in Introductory Calculus

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DECLARATION

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

The research presented and reported in this thesis was conducted in accordance with the National Health and Medical Research Council National Statement on Ethical Conduct in Human Research (2007) – updated March 2014. The proposed research study received human research ethics approval from the Curtin University Human Research Ethics Committee (EC00262), Approval Number RDSE-55-15.

Signature : …………………………………

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ABSTRACT

The concept of function is very prominent in the Introductory Calculus course studied in the mathematics education undergraduate program in Indonesian universities. It is studied through the abstract formal theoretical approach characterized by precise mathematical definitions, upon which the theorems and propositions are logically deduced. The function concept became a source of cognitive problems for the students. In general, the student can construct or understand the formal definition of function. The formal definition of function was used to deductively verify a relation as a legitimate function. However, the students faced difficulties in providing the definition-based verification of a functional relation for their formal definition was not operational in nature. The operational definition could be formulated through the learning activity of constructing mathematical definitions. This study was conducted to investigate the students' construction of an understanding of the operational formal definition of function. To achieve this purpose, the research questions were twofold. First, how did students construct the operational formal definition of function? Second, how did students use mathematical representations in constructing the operational formal definition of function?

This study was designed as a multiple case study situated in the interpretivist paradigm and employing the explorative qualitative approach. It was implemented in the teaching interview setting which was an adaptation of the teaching experiment method. Ten students participating in this study were first year students undertaking an introductory calculus unit in the undergraduate mathematics education program in one large state university located in the eastern part of Indonesia. The calculus tasks were designed based on the learning path of the construction of the operational formal definition of function. The path was a learning trajectory developed on the basis of an *a priori* analysis which provided the structure of a sequence of the knowledge elements or concepts constituting the operational formal definition of function. The qualitative data were gathered from interviews, written documents, and observations. The students were interviewed while they were working in pairs to solve the calculus tasks by thinking aloud. The data were analyzed within the frameworks of the Abstraction in Context and the Three Worlds of Mathematics.

The research findings revealed that the students succeeded in formulating the operational formal definition of function through the theoretical abstraction process. Some students constructed their complete understanding of the whole concepts constituting the operational formal definition of function under investigation. The operational nature of the formulated definition was proved by using it to verify the functionality of a relation with a deductive argument. Abstraction in Context as a model of the theoretical abstraction was performed by the students in the abstraction processes. Recognizing, building-with, and constructing, as epistemic actions in the Abstraction in Context model, took place in a nested way during the process of constructing the operational formal definition of function. The learning trajectory for the operational formal definition of function was confirmed in the abstraction processes. It means that formulating the operational formal definition of function could be started from the very concepts of sets, namely, the association between sets' elements to the concepts of ordered pairs, the Cartesian product of two sets, relation, and the special properties of relation. Some students encountered problems in identifying and defining the special properties of relation. The difficulties were partly caused by their lack of an understanding of mathematical logic.

The mathematical representations functioned as the tools to perform the abstraction process. The research results showed that there were three kinds of representations dominantly used by the students in their solutions to the problems concerning the development of an understanding of the operational formal definition of function. The embodied representations were dominantly used in the students' responses to the concepts of association between sets' elements and ordered pairs. The connected formalizing-embodiment representations were used in the responses to the concept of the Cartesian product of twos sets and relation. The formal representations were used in the responses to the special properties of relation and the operational formal definition of function. Regarding the appropriateness, most of the responses commenced and finished with mathematical ideas expressed in appropriate representations. The responses categorized as formalizing-embodiment representations commenced and finished with correctly represented mathematical ideas. The students made some mistakes in representing mathematical ideas that were related to the use of symbols and notations and to the meaning and interpretation of mathematical logic.

DEDICATION

To Ambo'ku, Emma'ku, Chana, Chaca, and Adyan

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INTRODUCTION

> "Human beings have survived because of their deeply ingrained habits of correcting one another, telling each other what they know, pointing out the moral, and supplying the answer."
>
> (Feiman-Nemser, 1983, p. 152)

1.1 Overview

This present study involves research on undergraduate mathematics education. It is an investigation of how first year university students in a mathematics education undergraduate program construct their understanding of the operational formal definition of function in an introductory calculus course. It seeks to explore the theoretical abstraction performed by the students in constructing their understanding of the definition of function, and the mathematical representations they use in constructing their understanding of the definition.

This introductory chapter begins with the background to the research which describes the current circumstance of introductory calculus as a course unit in an undergraduate program for mathematics education and the problem encountered by the students in learning the definition of function. The background also describes my experience as a teacher educator teaching an introductory calculus unit. After this, the following section provides the rationale for conducting the study. The next section formulates the research questions along with the overview of the research design implemented to seek the answers to the questions. This is followed by two further sections discussing the significance of the research and the role of the researcher respectively. The chapter concludes with the section overviewing the thesis organization.

1.2 Background to the Research

Introductory calculus is considered as a first university-level mathematics course. Traditionally, it is followed by intermediate calculus and advanced calculus. These courses are university mathematics subjects that dominate much of the landscape of undergraduate programs of mathematics education and mathematics (Bressoud, Johnston, et al., 2015). Introductory calculus is not only undertaken by the students of mathematics education or mathematics majors, but also by those of sciences and engineering majors. It is considered as a gateway unit required for most of the university students heading into sciences and engineering undergraduate programs. In the Indonesian university context, introductory calculus, sometimes named basic mathematics, is also a compulsory unit for students in other faculties such as fisheries, agriculture, and forestry (see, e.g., The Study Program of Agricultural Technology, 2015). Most of the students learning calculus express that they really want to comprehend the mathematical content in it, however, when studying calculus, they focus on the content they are able to do and on the concepts they consider important (Bressoud, 1994). In fact, there is a large number of students completing their calculus units but struggling to use their calculus knowledge (Bressoud, Johnston, et al., 2015).

Within the context of undergraduate programs of mathematics education and mathematics, the introductory calculus unit is one of several units where the students firstly engage with abstract theoretical mathematics in their transition to advanced formal mathematical thinking (Adiredja, 2014). By studying calculus, students are expected to develop their abstract mathematical thinking skills, and their knowledge of calculus will determine their success in advanced mathematics courses. The students' success in introductory calculus is simply a beginning phase because the following course units build upon the knowledge basis of calculus and necessitate language fluency through which calculus is represented (Bressoud, Mesa, & Rasmussen, 2015). For students in the undergraduate program for mathematics education, introductory calculus is a prerequisite fundamental unit covering many core mathematics concepts which are very important and need to be comprehensively mastered by those who are going to be school mathematics teachers. Calculus is the capstone in secondary school mathematics, therefore prospective school mathematics teachers must have the strong content knowledge and teaching knowledge of calculus (Ferrini-Mundy & Findell, 2001; Fothergill, 2011). Having a deep comprehension of the subject matter is a prerequisite for teachers' success in classroom teaching (U.S. Department of Education, Office of Postsecondary Education, & Office of Policy Planning and Innovation, 2002).

Calculus, generally defined as the study of rate and change, is a very important tool not only for mathematicians but also for people in many fields in which mathematical modeling and optimal solutions are part of their foci (Smart, 2013). However, we cannot start answering questions related to change if we do not know what the change is and how something changes (Rohde, Jain, Poddar, & Ghosh, 2012b). In this notion of change, the concept of function plays an essential role. Function is a prominent and important concept in introductory calculus. Calculus is a study of differentiating and integrating functions. A strong understanding of calculus has always been expected. Consequently, the mastery of calculus concepts and their prerequisite concepts, the concept of function among them, is absolutely important. The comprehension of this function is remarkably complex (Selden & Selden, 1992). The concept itself is unquestioningly central to much of the courses in the undergraduate mathematics program (Bagley, Rasmussen, & Zandieh, 2015). It is argued that the whole enterprise of mathematics could be unified by the concept of function (Viirman, 2014). "If it does nothing else, undergraduate mathematics should help students develop functions sense" (National Research Council, 1989, p. 51).

In introductory calculus, the discussion of function covers its definition, its representations, its operations, and its properties. A representation of functions is a tangible configuration of signs, notations, diagrams, graphs, models, formulae, equations, pictures, or symbolic expressions that encode, express, or symbolize any aspects of the concept of functions. The operations on functions deal with the arithmetical operations, composition, and inverse operation. And, the properties of functions cover the injective, surjective, and bijective properties. The concept of function forms the basis of the study of limits and continuity. Selden and Selden (1992) argue that the importance of understanding the concept of function is broadly due to its role as a core concept and a basis for other advanced calculus concepts. The determinant role of function in the comprehension of concepts in introductory calculus justifies it as a threshold concept (Pettersson, Stadler, & Tambour, 2013; Scheja & Pettersson, 2010). Functions along with limits, continuity, derivatives, and integrals are threshold concepts in calculus (Pettersson, 2011). These concepts are initially troublesome for the students, but they are a portal to a new and previously unreachable view of the subject areas. Once the portal is passed, the understanding of these concepts will be transformed, irreversible, bounded, and integrative (Meyer & Land, 2006). Within the context of preparing prospective school mathematics teachers, introductory calculus should provide a beneficial perspective through a more explicit focus on how the general function formulation is applied in expressing and reasoning calculus concepts which are conceptually rich functions (Conference Board of the Mathematical Sciences, 2001). Research has identified the concept of function as one of the most important concepts of introductory calculus for the prospective mathematics teachers (Fothergill, 2011). However, it is not easy for the students to comprehensively master the concept of function.

In general, Thomas et al. (2015) claim that students' understanding of the concept of function is limited. The concept has become a source of cognitive problems for them. Working on functions as both a process and an object is already problematic for students. According to Asiala et al. (1996), to form a mathematical conception, an individual could start with transforming physical or mental objects to obtain other objects. Further, when this step-by-step action of transformation is interiorized, it turns to be a process, namely, a mental structure performing the same operation but entirely within the mind of the individual where he or she can do the transformation without having to perform each step explicitly (Asiala et al., 1996; Dubinsky, Weller, McDonald, & Brown, 2005). When an individual can imagine any element of the domain of a function being processed into another element of the codomain by any expressions, it means that he or she has a process conception of function. Furthermore, when the individual realizes the process as a totality and performs a transformation on it, then he or she has encapsulated the process to an object (Asiala et al., 1996; Dubinsky et al., 2005). With an object conception of function, the individual can perform transformations on function, such as defining a set of functions and applying arithmetical operations on a set of functions.

Students' understanding of function depends upon their conceptualization about what a function is. In this regard, the question is whether the students could construct or have an appropriate, correct definition of the concept of function. Studying functions based upon the formal perspective increases its abstract and general nature, and this makes the concept of function troublesome for the student (Artigue, Batanero, & Kent, 2007). Each student has a personal concept definition of function (Education

Committee of the EMS, 2014; Tall & Vinner, 1981) which defines what a function is in his or her individual mind. This personal definition is part of the student's concept image which is very illuminating as to his or her understanding of function (Jensen, 2009). The construction of conceptual understanding necessitates the personal definition being in accordance with the appropriate definition generally accepted within the community of mathematics. This is a real didactical challenge because, as argued by Ouvrier-Buffet (2006), the situation of students' attempting to comprehend a new mathematical concept is a far cry from the neat definition created by a professional mathematician.

The concept of function is no longer dealt with as a practical approach in introductory calculus. It is studied through an abstract formal theoretical approach. Introductory calculus in the undergraduate programs for mathematics education and mathematics is studied with formal theoretical approaches that are different from more practical approaches used in the same course for other undergraduate programs. As stated by Tall (1992), abstract formal mathematical thinking is characterized by precise mathematical definitions, upon which the theorems and propositions are logically deduced. The content comprises "so many definitions, so many theorems, so many applications, so much notation, so much 'abuse of language' (as the French call it), so much logical complexity, [and] so much abstraction" (Epp, 1987, p. 46). Learning calculus means exposing students to the real world of mathematicians, in which they have to experience how mathematical concepts are founded (Tall, 1992). As to the concept of function, the students are to develop a conceptual understanding indicated, at least, by their ability to state the formal definition of function and to recognize examples and non-examples (Rose & Arline, 2009). Using the formal definition of function, the students are expected to be able to justify relations as to whether they are valid functions or not. The verification of a relation must be given in the form of an argument that is derived in a deductive, axiomatic way.

My prior experience of teaching in a mathematics education undergraduate program in the last few years revealed the problems of students' understanding of the concepts in introductory calculus, including the concept of function. According to the students' achievement data, approximately a third of the student population taking introductory calculus achieved a mark of C or below–C is the lowest mark considered for passing the unit; D and E are failure marks. Those achieving A or B gained their high scores mostly from procedural items in the examinations. This trend is also evident in some other undergraduate programs of mathematics education in Indonesian universities (Gordah & Fadillah, 2014; Oktaviyanthi & Supriani, 2015; Rahmawati, 2017; Tasman & Ahmad, 2017). Teacher educators are very concerned with this phenomenon because the lack of success in calculus is very likely to incur further barriers for students requiring the subject for their studying of more advanced courses in the program.

In a cross-sectional study of mathematics undergraduate students, Carlson (1998) revealed that even the high achieving students took a long time to develop a solid comprehension of the concept of function, and many second-year students' conception was still replete with many flaws. Sabri and Minggi (2015) found that students in an undergraduate program for mathematics education experienced serious problems in deductive axiomatic reasoning, verifying the truth of mathematical statements, constructing deductive arguments, and developing mathematical proofs in the subjects in their further semesters.

In my class, many students already have a basic understanding by viewing a function as "a correspondence between two sets of elements" (Vinner & Dreyfus, 1989, p. 360). Sierpinska (1992) stated that "the concept of function can be defined in a formal symbolic way, almost without using words" (p. 29). Almost all students were able to state the definition of function either in a formal statement as given in some textbooks or in some colloquial statements, yet still bearing the correct meaning of what a function is. However, when they had relations to be examined, especially those represented in symbolic algebraic or formal expressions, they did not use the formal definition of function (Vinner & Dreyfus, 1989). They still often failed to distinguish a function from a non-function correspondence. The students' definition and images of functions determine how they justify whether a given relation is a function or not. Students not being able to verify a relation as a legitimate function could be because neither do they understand the formal definition of a function, nor have the capability to operationalize the definition to examine a valid function.

Some students have an intuitive inductive approach to examining legitimate functions (Hoyles & Healy, 2007). They just verify some points or cases from the domain of the given relations. And, having those cases satisfy the properties of function, they jump into the conclusion that the relations are functions. Problematic personal concept definition and concept image, as well as a problematic perspective of how to verify mathematical statements, should contribute to this incomplete approach. This is a problem encountered by the students in their study of the concept of function in introductory calculus. The complexity of the concept and its development suggest that students "fail to notice their developing function concept during their mathematics studies" (Hyvärinen, Hästö, & Vedenjuoksu, 2013, p. 2367).

Based upon the formal definition of function, the students are expected to be able to justify whether relations, given in various representations, are legitimate functions or not. The verification of a relation should be given in the form of a deductive formal argument. However, knowing the formal definition of a concept, such as the concept of function, does not guarantee that the students could use it to verify formally a given relation as a function (Leinhardt, Zaslavsky, & Stein, 1990; Vinner & Dreyfus, 1989). Being presented with a formal definition, students may have no understanding of how to employ a definition to prove anything (Bills & Tall, 1998) (Bills & Tall, 1998). In order to use a formal definition in proving, it must be operational, meaning that the statement of the definition could be used directly to prove the truth of a statement. The formal definition of function discussed in introductory calculus and given in the textbooks of calculus is often an example of a non-operational definition (see, e.g., Hughes-Hallett et al., 2017; Stewart, 2016; Thomas, Weir, & Hass, 2018; Varberg, Purcell, & Rigdon, 2007). Many students know the formal definition of function which they find in the textbooks. However, the definition proves to be not operational for them. The students could not use the definition to develop a deductive argumentation for verifying that a relation is a valid function. Moreover, they could not construct an operational formal definition of function. The operational formal definition of function is a working definition which details the unique properties a function has and can be directly used in developing arguments including examining relations as to whether they are functions or not. And, it is hard to find any studies specifically investigating students' constructing or understanding of the formal definition of function which is operational in nature.

1.3 The Rationale for the Research

The unsatisfactory achievement of students in calculus has become an international concern. Research has shown that many undergraduate students struggle with a conceptual understanding of calculus resulting in low achievements and high rates of failure (Patel, 2013; Szydlik, 2000; Tsamir & Ovodenko, 2013; White & Mitchelmore, 1996). Various studies (e.g., Klymchuk, Zverkova, Gruenwald, & Sauerbier, 2010; Orhun, 2012; Parameswaran, 2007; Patel, 2013; Pettersson & Scheja, 2008; Pettersson et al., 2013) have been conducted to explore students' difficulties in comprehending concepts in introductory calculus. Decades of research on these concepts have revealed problematic understandings constructed by students (see, e.g., Artigue et al., 2007). Investigations of these problems in learning calculus, such as by Tall (1993), Bridson (2002), and Tarmizi (2010) have shown that however successful a course may appear to be, there are students who struggle and will certainly need appropriate help enabling them to pursue future studies in mathematics. These students might have completed calculus courses in which they learned the accumulation of practical problem solving abilities in calculus but missing the adequate preparation needed for their achievement beyond calculus (Bressoud, Mesa, et al., 2015).

Research in undergraduate mathematics education has gained increasing relevance. There have been numerous research studies investigating university students' understanding of the concept of function (e.g., Bardini, Pierce, Vincent, & King, 2014; Bayazit, 2011; Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005; Dede & Soybaş, 2011; Hansson, 2006; Paz & Leron, 2009; Tall & Bakar, 1992; Thompson & Carlson, 2017; Weber & Thompson, 2014). However, much of the research focused on investigating students' difficulties in relation to understanding the concept of function. Some of these studies, such as Christou et al. (2005), Bardini et al. (2014), and Tall and Bakar (1992), focus on the classification of examples and non-examples of function examined within the framework of the modern definition of function. However, these studies did not require the students to provide a deductive formal verification in classifying an example as a function. None of the research specifically focused on students deductively verifying or proving a relation as representing a legitimate function.

Teaching and learning mathematics without proving, justifying, or verifying activities would be entirely unreflective of mathematical theory and practice (Hanna, 2000a). However, for students, it is exasperating to be required to prove something they consider already intuitively obvious using logical deductive arguments developed based upon the formally defined properties. The difficulty of verifying mathematical statements on the basis of the formal definition is rooted in the non-operational characteristic of the formal definitions. The definitions are not working definitions for the students (Bills & Tall, 1998). And, they encountered challenges in operationalizing the definitions; they faced problems in (re)constructing the formal definitions to be operational and then applicable to developing the intended verification. Apart from the difficulty of operationalizing a formal definition, according to Alcock and Simpson (2002), the definitions themselves are logically complicated and involving various mixed quantifiers.

Constructing a formal definition of function which is operational in nature could be approached in two ways. The first is through reconstruction, namely, analyzing the formal definition to explore its meaning, to study the consequences of its assumptions, and to identify constituting properties (Hanna, 2000b; Ouvrier-Buffet, 2015). Further, the properties are defined individually resulting in an operational formal definition of function. The second is directly identifying the individual properties in representative examples of functions and defining these properties separately. Next, the operational definition is then composed by joining all the defined properties in a conjunction stating that a relation must fulfill both properties to be a valid function. This second approach was the focus on this present study, with the expectation that the students developed and understood the operational formal definition of function, and at the same time, they learned how to define concepts.

In undergraduate mathematics education, studies on definition construction have been conducted by many researchers (e.g., Martín-Molina, González-Regaña, & Gavilán-Izquierdo, 2018; Oehrtman, Swinyard, & Martin, 2014; Ouvrier-Buffet, 2011, 2015; Swinyard & Larsen, 2012; Zandieh & Rasmussen, 2010). The research mostly focused on the construction or reinvention of the formal definition of particular mathematical concepts either involving students or mathematics researchers. The foci of research have varied from defining the Sasakian space in differential geometry (Martín-Molina et al., 2018), to defining grids in discrete mathematics (Ouvrier-Buffet, 2011), the sequence convergence in mathematical analysis (Oehrtman et al., 2014), limits (Swinyard & Larsen, 2012), and spherical triangles in geometry (Zandieh & Rasmussen, 2010). A search of the literature suggests that the defining activity to formulate an operational formal definition of a concept apparently has not been investigated.

Essentially, knowledge construction in mathematics education is centered on abstraction. Abstraction is one process through which abstract concepts are constructed (White & Mitchelmore, 1996). The development of mathematical theories and the emergence of new mathematical mental structures take place through abstraction (Epstein, 2013; Hazzan & Zazkis, 2005). Theoretically, abstraction is a dialectical process between undeveloped and established concepts (Davydov, 1972/1990). An established concept results from an undeveloped concept integrated with prior knowledge which is vertically reorganized within the existing structure of mathematical knowledge (Hershkowitz, Schwarz, & Dreyfus, 2001). In higher mathematical activities of symbol manipulation and axiomatic deduction, such as in calculus and algebra, this theoretical abstraction model is needed (Mitchelmore & White, 2007). Therefore, the process of constructing an understanding of the operational formal definition of function in this study is framed within this abstraction model.

In performing abstraction, we use representations as its effective apparatus (Damerow, 1996). Representations are the tools we use in thinking mathematics (Davis & Maher, 1990). Connecting and translating from one representation to another are essential features of meaningful mathematics learning. The inability to establish appropriate connections between representations is very likely to result in students' failure to comprehend the holistic meaning of concepts they are learning (Pantozzi, 2009). The flexibility to use representations has a clear relation to the mastery of the concept (Won, Yoon, & Treagust, 2014). The ability to manipulate the representations of a concept is an indication of the understanding of the concept (Wilkenfeld, 2013).

Therefore, by analyzing the students' representation use in the abstraction, the more holistic picture of the students' understanding can be obtained.

The explanations above solidify the reasons for conducting this present research. It is a study of the theoretical abstraction and the mathematical representations used in students' construction of an understanding of the operational formal definition of function in introductory calculus.

1.4 Research Questions and Overview of the Research Design

As identified in the previous sections, the students involved in several previous studies encountered difficulties in providing the definition-based verification of a relation as a legitimate function because the formal definition which they had understood was not operational. Formulating an operational definition of function can be implemented by reconstructing the existing formal definition or constructing a new definition. Both of these approaches were problematic for the students.

1.4.1 Research Questions

The objective of this research was to investigate the first year university students' construction of an understanding of the operational formal definition of function studied in introductory calculus. The research subjects were the students in the undergraduate program for mathematics education in one large state university located in the eastern part of Indonesia. The construction of understanding of the concept takes place through the abstraction process. Therefore, to achieve the objective of this research, two research questions were formulated.

The first question was: *How do students construct the operational formal definition of function?* This question was formulated to examine the whole understanding resulting from the abstraction processes. The students' performance on the calculus tasks designed particularly to encourage the abstraction process was analyzed. The analyses enabled me to portray the trajectories of abstraction processes the students performed in constructing their understanding of the operational formal definition of function, and to describe the level of understanding of the operational formal definition of function.

The second question was: *How do students use mathematical representations in constructing the operational formal definition of function?* The analyses of the students' solution of the calculus tasks and the interviews enabled me to describe the categories of mathematical representations and the appropriateness of the representations.

1.4.2 Overview of the Research Design

In order to seek the answer to the research questions, this study adopted a case study design (Creswell, 2013; Merriam, 2009; Yin, 2009, 2014). The case study was conducted in an undergraduate program for mathematics education in one large public university located in the eastern part of Indonesia. There were 10 students who were prospective school mathematics teachers participating in this study. They were grouped into five pairs; each pair of students was considered as a case, therefore this study was categorized as a multiple-case study (Yin, 2009). The unit of analysis investigated in this case study research was a pair of students from an undergraduate program of mathematics education who studied introductory calculus in their first year of university. The focus of the analysis was the development of students' understanding of the concept of function in introductory calculus. This focus encompassed two aspects, namely, the development of understanding of the operational formal definition of function through the theoretical abstraction, and the use of representations in the abstraction processes to develop an understanding of the operational formal definition of functions.

The present research implemented a qualitative approach using the framework of a teaching interview. The teaching interview was an adaptation of a teaching experiment (Chini, Carmichael, Rebello, & Puntambekar, 2009; Steffe & Thompson, 2000). This interview focused on the aspect of teaching, and it was conducted with particular didactic objectives (Hershkowitz et al., 2001). It was conducted with students in their learning process where it was acceptable and even expected that they could modify or change their thinking during the interview.

The students worked in pairs on the calculus tasks which were the main instrument in this research. They were specifically designed by me to encourage the students to perform the abstraction process. The tasks comprised a sequence of problems that were formulated based upon the structure of knowledge elements or concepts constituting the operational formal definition of function. The students worked on the problems by thinking aloud, namely, speaking out the thinking processes they performed while they were solving the problems. The task-based pair interviews were conducted while students were working on the calculus tasks. The data of the research were gathered from the interview transcripts as well as the results of students' pair workings. The interview transcripts were micro-analyzed using the framework of the Abstraction in Context (Dreyfus, Hershkowitz, & Schwarz, 2015; Hershkowitz et al., 2001; Schwarz, Dreyfus, & Hershkowitz, 2009) to investigate how the abstraction processes performed by the students in constructing their understanding of the operational formal definition of function. The categorization analysis was employed to obtain the level of the students' understanding of the definition of function. The framework of the Three Worlds of Mathematics (Tall, 2004a, 2004b, 2006, 2008, 2013) was used in the analysis of the mathematical representations the students used in solving the calculus problems. The representations used were categorized based upon the types of representations and the appropriateness of the representations.

1.5 Research Significance

This present study was conducted during my own professional practice as a mathematics teacher educator. It was expected to establish a better understanding of the development of students' understanding of the definition of function through the theoretical abstraction and representation use, and thus, contribute to the establishment of research-based instructional practices for calculus courses in the undergraduate program for mathematics education at my university. The findings of this study are expected to inform the planning, implementation, and evaluation of the course units in the undergraduate program for mathematics education. The calculus tasks and structure of knowledge elements produced and used in this research exemplify the components of the theoretical abstraction model which can be implemented in the teaching and learning the other concepts either in introductory calculus or in the other course units.

Mathematics literacy is still a significant challenge for Indonesian students. Two international assessment programs have revealed that Indonesian students achieve generally very low scores (Mullis, Martin, Foy, & Arora, 2012; OECD, 2014). Although they have gained an increase of average score for mathematics literacy in the Programme for International Students Assessment (PISA) 2015 compared to PISA 2012, their literacy level remained very low and included them in the group of 10 participating countries with lowest mathematical literacy (OECD, 2016a). This low performance raises the question of the quality of Indonesian mathematics teachers because, according to Ball, Hill, and Bass (2005), teachers' performance is one of the most crucial determinants of students' achievement. The teacher competence tests in 2012 revealed that certified secondary school mathematics teachers were able to attain roughly about 50% of the ideally expected score (Gultom, 2012). Indonesia absolutely needs qualified and competent mathematics teachers to improve students' mathematics achievements.

The quality of mathematics teachers is partly determined by their mathematical content knowledge developed during their undergraduate study. The undergraduate preservice education and training aim to provide prospective teachers with skills necessary to facilitate students to learn mathematics meaningfully (OECD, 2018). The content and the quality of teachers' education has an effect on student learning (Clotfelter, Ladd, & Vigdor, 2010; Ronfeldt & Reininger, 2012). This study was conducted for the students of mathematics education undergraduate program at an Indonesian university. The significance of the study is that it contributes to the preparation of the students, as prospective mathematics teachers, in terms of constructing a strong, meaningful understanding of mathematical concepts. The comprehensive understanding of the concept of function should support the prospective teachers in their advanced mathematical study at university and in their future real teaching practices. Strong mathematical knowledge is a principal competency of qualified school mathematics teachers (Mewborn, 2003; National Council of Teachers of Mathematics, 2000; National Mathematics Advisory Panel, 2008).

This research employed the thinking aloud strategy by which the participating students worked on calculus tasks specifically designed to encourage the theoretical abstraction process. While solving the problems, the students were interviewed and were asked to express the cognitive processes taking place in their minds. Still, regarding the significance of the study for participant students involved in this research, the students were exposed to thinking aloud activities, a learning strategy which is a form of cognitive activation (OECD, 2016b). Apart from the personal benefit as a learner of mathematics, their involvement in the thinking aloud activity was expected to inspire the students about the potential of this strategy which they could implement later in their professional practice as mathematics teachers.

Given that this research might be among the first studies in theoretical abstraction using the framework of Abstraction in Context in Indonesian university learning, the results should enrich the variety of learning contexts in which this abstraction model could be applied. The implementation of this abstraction method is expected to exemplify an alternative to the empirical abstraction framework which seems to still dominate the practices of teaching and learning of university mathematics in Indonesia.

1.6 My Roles in the Study

This study was considered as research into my own practice. My educational background and professional experience as a university lecturer in a mathematics education undergraduate program helped me to conduct this study. With my undergraduate and master's degrees in mathematics education, I have worked as a lecturer and teaching assistant for calculus courses for several years. This experience has shown me how students dealt with the concepts and the definition of the concepts in calculus, and it provided me with adequate knowledge about calculus concepts, particularly the concepts in the topic of function which were studied in introductory calculus. Over the same period, I also taught the course unit on the fundamentals of mathematics covering the concepts of set and mathematical logic. The mathematical logic and concepts of set are very crucial in the study of mathematics with the formal, deductive, axiomatic approach.

Regarding the familiarity with the theoretical and analytical frameworks which I used in this study, namely, the theoretical abstraction model of Abstraction in Context and the Three Worlds of Mathematics, prior to conducting this present research I trialed the implementation of the frameworks in my teaching of introductory calculus.

In terms of the researcher's role with respect to his/her relative position to the research project, Punch (1998) claims that it could be emic, namely, an insider fully participating in the research enterprise, or etic, namely, an outsider with an objective view. The researcher could begin as a member of the investigated group then turns to be outside observant, or conversely, he/she starts as an outsider then joins into the group (Punch, 1998). At the start of this investigation, my role was emic because I was conducting a teaching interview within a unit of introductory calculus in the mathematics education undergraduate program. Then, starting from the data analysis, I took the role as an outsider, a more objective analyst.

1.7 Overview of Thesis Organization

The report of the research described in this thesis consists of seven chapters. The content of each can be summarized as follows:

The present chapter has introduced the background to the research and the rationale for conducting the research. This chapter also has discussed the research questions and their potential significance. My role in this qualitative study has also been explained briefly.

A review of the literature in Chapter 2 provides comprehensive descriptions about the concept of function in introductory calculus at university mathematics. It reviews the teaching and learning of introductory calculus at a university level, particularly in the context of mathematics or a mathematics education department. The theories of abstraction and representation in mathematics learning are also discussed. In particular, the Abstraction in Context as a theoretical abstraction model and the Three Worlds of Mathematics as a framework for mathematical thinking with mathematical representations are described in detail. This chapter reveals the gaps in the existing literature basis that this study will fill and where it breaks new ground.

Chapter 3 outlines the research methodology that has been employed in investigating the abstraction processes and the representation use in constructing the understanding of the operational formal definition of function. This chapter covers eight major sections. The first section addresses the research paradigm and the justification for employing the qualitative research approach. The next sections discuss the research design, setting, and participants. The instruments used to gather the research data are outlined, followed by the explanation of data collection and the method of analysis. This chapter concludes with the discussion of the research legitimation and the ethical aspects considered in the research.

Chapter 4 and Chapter 5 present the findings resulting from the data analysis and their discussion. Each of the chapters deals with a different research question. Chapter 4 presents the research results and discussion for the question of the abstraction processes for constructing the students' understanding of the operational formal definition of function. Chapter 5 deals with the research results and discussion for the question of the representation use in constructing the students' understanding of the operational formal definition of function.

Next, Chapter 6 provides a general discussion of the research findings from the two research questions. In this chapter, the synthesis between the abstraction processes and the representation use is discussed in relation to the literature reviewed in Chapter 2.

The final Chapter 7 provides the conclusions of this research by revisiting and addressing the research questions. It also presents the recommendation based upon the findings of the study. This chapter concludes with limitations of the research.

CHAPTER 2
LITERATURE REVIEW

"The teacher … gives not of his wisdom but rather of his faith and his lovingness. If he is indeed wise he does not bid you enter the house of his wisdom, but rather leads you to the threshold of your own mind."
(Gibran, 2001, p. 51)

2.1 Introduction

As noted in the introductory chapter, the main goals for this literature review are threefold. The first is to develop an understanding of introductory calculus learning at university and the dynamics of it. The focus of the discussion is the concept of function. This is to frame the context where this present research took place. The discussion covers the review of the literature and studies on the concept of function. The second is to establish the importance of the study of abstraction and representation in the process of knowledge construction and understanding of the formal definition of function. This chapter also covers the review of the literature on defining concepts as a mathematics learning activity. The third goal involves the discussion of the theoretical and analytical frameworks used in this study.

2.2 Calculus at University

This section mainly discusses calculus as a course in undergraduate programs of mathematics education. The discussion deals with the definition of calculus, the coverage of calculus courses at university level, along with the objectives of learning calculus with an emphasis on the position of students as prospective school mathematics teachers.

2.2.1 What Is Calculus?

Calculus is a branch of mathematics. It was a systemized shortcuts to results which were found by the exhaustion method and as a method for finding the results themselves (Stillwell, 2010). It is “the crowning glory of classical mathematics” (Tall, 2013, p. 289). Calculus is one among the range of spectacular creations of human beings (Range, 2016). The historical development of calculus actually begins with integration concerning the problems of computing areas, volumes, or arc lengths (Rosenthal, 1951). The problems were discussed by ancient Greek mathematicians, especially by Archimedes with his method of exhaustion (Rosenthal, 1951; Stillwell, 2010). However, the systematic calculation of areas, volumes, or arc lengths was made possible only when algebra was available (Stillwell, 2010). The algebra-based calculus was firstly found in the work of Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century, and these two geniuses were considered as the inventor of calculus (Katz, 2009; Stillwell, 2010). In the early part of the eighteenth century, Jacob Bernoulli and Johann Bernoulli further developed calculus where it was a study of variables related by equations (Kleiner, 2012).

In the mid of the 18th century, Leonhard Euler achieved a fundamental conceptual invention which successfully positioned the function concepts as the centerpiece of calculus (Hawkins, 2001; Kleiner, 2012). Nowadays, calculus is studied in two fundamentally different ways: the theoretical calculus employed in applications, which is based upon the symbolic idea of Gottfried Wilhelm Leibniz and the infinitesimal techniques of Augustin-Louis Cauchy, and the formal mathematical analysis of Karl Weierstrass which is based upon the quantified set theory (Tall & Katz, 2014).

Essentially, calculus is the mathematics of change (Larson & Edwards, 2012; LaTorre et al., 2012). The value of calculus comes from its potential to simplify complex problems to be workable (Aspinwall & Miller, 2001). It is a method for solving problems which is based upon the exhaustion processes. The types of problem which can be solved by calculus are finding lengths, areas, and volumes of figures with curved parts and examining tangents, normal, and curvature as the local properties of figures (Stillwell, 2010). Calculus is also a formalization of the processes of finding differences and sums. Differences that measure changes have a limit called the derivative, while the processes of determining the sums lead to the idea of the integral. The two processes are inextricably intertwined: their relationship is embodied in the Fundamental Theorems of Calculus. All problems dealt with in calculus are recognized as problems of integration and differentiation. According to Kleiner

(2012), both the derivative and the integral are not simply abstractions of the idea of a tangent on curves or instantaneous velocity on the one side and of length, area, or volume on the other side. They are the fundamental concepts of calculus which are to be studied in their own right.

Calculus closely relates to other branches of mathematics. It plays a central role in the curriculum of the undergraduate program of mathematics or mathematics education (Cappetta & Zollman, 2013). Its alliance with algebra and analytic geometry expands the use of calculus to determine tangents, extreme points, monotonicity and the concavity of algebraic curves. Being integrated with the theory of infinite series, calculus becomes a complete system for the differentiation and integration of functions which can be expressed in power series. Calculus intertwined with the theory of infinite series enabled mathematicians to commence the fundamental development of number theory, combinatorics, and probability theory. Calculus as a means for understanding the behavior of functions has led to the development of new branches of mathematics including real analysis, complex analysis, and topology. The fundamental ideas of calculus have developed and matured into Analysis, which along with Algebra and Geometry, has been considered as one of the three major fields of the mathematical study (Range, 2016).

Calculus is essential in the fields of science, engineering, business, economy, and social science (Rohde, Jain, Poddar, & Ghosh, 2012a). In science and engineering, calculus utilizes mathematical functions to explicate and predict physical phenomena subject to continuous changes. In many discussions of business and economics which involve the notion of rates, such as interest, supply, and demand, all equations and functions involved necessitate the contribution of calculus theories. Calculus functions as a gatekeeper into Science, Technology, Engineering, and Mathematics (STEM) majors (Adiredja, 2014). Introductory calculus is a compulsory unit for students in the faculty of science and engineering. In the context of Indonesian universities, introductory calculus is an obligatory unit not only for students of science and engineering faculties, but also for students of faculties of Fisheries, Forestry, and Agriculture.

With respect to the field of study, the emphasis of calculus courses in those non-mathematical major undergraduate programs varies. Traditionally, calculus is learned to enable them to solve problems mathematically. Therefore, the mathematics courses for students are characterized naturally as more procedural, focusing on operative abilities and skills to apply mathematics in field-related problem solving (Engelbrecht, Bergsten, & Kågesten, 2009). The emphasis is different from that in calculus courses for mathematics education or pure mathematics undergraduate students where calculus is studied with a more rigorous and theoretical approach (Chai, Friedler, Wolff, Li, & Rhea, 2015).

The coverage of calculus which is learned in undergraduate programs is slightly different from one country to another, or even from one university to another within a country (Harel & Trgalova, 1996). The content variation ranges from informal calculus to formal real analysis including various approaches, such as a non-standard analysis approach or computer-based approaches. In general, the main topics discussed in calculus basically cover functions, limits, continuity, differentiation, integration, infinite series, geometry in space, and vector calculus. In Indonesian universities, particularly in the undergraduate program for mathematics education, calculus is usually divided into three compulsory units normally which are spread over a sequence of three first semesters (see, e.g., The Study Program of Mathematics Education, 2017). The first unit is introductory calculus comprising the study of functions along with the real number system, limits and continuity, the derivatives, and application of the derivatives. The second is intermediate calculus covering mostly the study of integrals. The topics include the definite integral, application of the integral, techniques of integration, the improper integral, and transcendental functions. The last unit is advanced calculus consisting of coordinate systems, derivative of functions of two or more variables, multiple integrals, space geometry, infinite series, and vector calculus. Several elective units of more advanced calculus, such as multivariable calculus, are sometimes offered in later semesters. A different division of calculus materials could be found in some other pure mathematics or mathematics education undergraduate programs. The calculus course is divided into only two consecutive course units which are offered to students in their first two semesters at university. The contents of such courses are dense. The chapters in the calculus textbooks such as

Varberg et al. (2007) or Thomas et al. (2018), are traditionally split into two parts, and the credit points of each course unit are greater accordingly.

2.2.2 What Is Expected From Students Learning Calculus?

Calculus is a prerequisite fundamental unit within which undergraduate students are expected to develop established mathematical thinking. It is a gateway course towards higher mathematical thinking (Jensen, 2009). Consequently, the students' understanding of calculus affects their success in some subsequent mathematics units. Calculus provides a very powerful tool not only for mathematicians, but also for people in many fields in which mathematical modeling and optimal solution are part of their foci (Smart, 2013). Its values come from the potential to simplify complex problems to be workable (Aspinwall & Miller, 2001). It is where the students are first exposed to theoretical mathematics (Adiredja, 2014).

According to Bressoud (1992), university students deal with calculus with both anxiety and expectancy. On the one hand, they are anxious because they know that calculus is not something easy to learn. Although they have studied calculus in secondary schools, they realize that university calculus will not be similar to the calculus in school mathematics. University calculus will be difficult but it is a compulsory unit which they have to pass. On the other hand, they hope that learning calculus will cover all mathematics they have previously comprehended and will then transform it into a useful means for understanding the world and solving problems.

Why is calculus taught at university? The answer to this question may vary in accordance with a mathematical departments' policy. Regarding the learning objectives, Bressoud (1992) argues that calculus is taught at university for two main goals (reasons, to be more precise). Firstly, it is because calculus is applied in various contexts by manifold disciplines; therefore if it is not taught by mathematicians, surely others would bear the responsibility to teach it. This is an answer widely accepted, and teaching and learning activities are mostly designed based upon this particular response. However, the usefulness of a course subject might be neither sufficient nor specific to calculus. Various other course units such as discrete mathematics, statistical analysis, or linear programming, to name a few, could be more beneficial to most university students. Secondly, calculus is the foundation of scientific world views. The concepts of calculus have formed the modern scientific understanding which seems to be meaningless outside the calculus context. Mathematics itself has developed into its current state thanks to calculus. The power of calculus illuminates questions in physical, biological, and social sciences, as well as engineering (Hughes-Hallett et al., 2017). Calculus ought to be studied because it is good for application and mental training and it is required by higher mathematics, other majors, and jobs (Rohde et al., 2012a).

What are the objectives of learning calculus at university? Like much of mathematics, learning calculus involves mainly two components: understanding the concepts and manipulating symbols. Understanding the concept means being able to explain the concepts and use the formulae. For example, rather than just apply the formulae or rules for finding a derivative of a function, students can expound what a derivative is, why they can use the rules, and how they derive the rules. Symbolic manipulation is part of calculus learning because calculus concepts are expressed or represented with symbols, and manipulating the expressions is needed before performing operations. Regarding the goal of learning mathematics, including calculus, Richard Askey says "mathematics is like a stool; it sits on three legs. ... All three legs are needed– problems, technique, and structure" (Jackson, 1997, p. 820). This statement clarifies the importance of a proportional emphasis between understanding the concepts (structures), mastering the skills (techniques), and using mathematics in problem solving. To a great extent, these goals are parallel to McCallum's (2000) recommendation about the objectives of learning calculus. He categorized the objectives into skills, concepts, and application, which are detailed as follows:

1. Make calculations with agility, accuracy, intelligence, and flexibility.
2. Explain the basic concepts of calculus clearly and reason mathematically with them.
3. Solve extended problems, with good judgment in the choice of tools and in checking answers.
4. Make connections between different incarnations of the same idea.
5. Use calculus to model realistic situations from engineering and from the physical, life, and social sciences (McCallum, 2000, p. 17).

Zerr (2009) recommends that learning calculus should aim more at developing conceptual understanding. He states that the emphasis on understanding the concepts will begin to raise the awareness of the students about thinking and doing like a mathematician. Having conceptual knowledge of calculus is expected to lead the students to a culture of mathematizing which is transferable into different contexts they will face in their lives. In a similar vein, Angelo (1991) claims that the conceptual understanding will enable the students to reflect on the subject, that is, to think of the 'why' of the subject. Conceptual understanding will also enable the students to portray the big picture of the subject (Kalman, 2005). Having such knowledge usually helps students reconstruct the forgotten details and also helps them solve problems although the encountered problems may be unfamiliar to them (Zerr, 2009). A conceptual understanding will also enable the students to connect their current knowledge to a new one in the future (Ma, 2010).

With a particular focus on mathematics education undergraduate programs, it is imperative that prospective teachers possess comprehensive mathematical knowledge of calculus. It is recommended by Ma (2010) that prospective teachers have a "profound understanding of fundamental mathematics" (p. 103), including calculus, that is, an understanding of the terrain of calculus with all its depth, breadth, thoroughness. They have to build a strong intuitive grasp of calculus concepts along with a robust heuristic argument for theorems or propositions such as the Fundamental Theorems of Calculus (McCallum, 2000). This is because calculus is the capstone subject for secondary school mathematics (Fothergill, 2011). The well-founded thinking of mathematics developed through the study of calculus is a great capital for prospective teachers of school mathematics to facilitate their students doing mathematics and thinking mathematically in the future. In addition, with the mastery of calculus, they will be able to facilitate instructional activities that encourage a conceptual understanding. Learning calculus is where students develop their mathematical maturity (Hare & Phillippy, 2004; Henderson et al., 2001). This maturity will inspire sustainable growth of knowledge (Conference Board of the Mathematical Sciences, 2001) which is characterized by the strong mastery of the prerequisite and concepts of calculus and the ability to use calculus in problem solving (Hare & Phillippy, 2004).

There are some aspects of calculus to be emphasized for prospective secondary school mathematics teachers. In particular, introductory calculus, considered as the first training ground for these prospective teachers, should be a course through which the students could develop the ability to connect secondary school mathematics to university mathematics (Fothergill, 2011). There are various connections that can be made between university introductory calculus and school mathematics. For example, the derivative can be used to connect the concept of stationary point where a function could attain its relative extreme values and the formula for the axis of symmetry of the graph of a quadratic function. In this case, for $f(x) = ax^2 + bx + c$, where $a, b, c \in \boldsymbol{R}$, $a \neq 0$, the axis of symmetry is $x = -\dfrac{b}{2a}$, where the function will attain its local extreme value, and for the same function, its stationary point is at $x = -\dfrac{b}{2a}$ obtained from $f'(x) = 2ax + b = 0$. More connection is in determining the concavity of the quadratic function. In school mathematics, the curve will be concave up if the value of a (the coefficient of x^2) is positive, which is actually related to the concept of concavity of the curve determined by the second derivative of the function (if $f''(x) > 0 \;\; \forall x \in D(f)$, then *f* is concave up in *D*(*f*). For the quadratic function defined above, $f''(x) = 2a > 0 \Leftrightarrow a > 0$).

Fothergill (2006) suggests that through introductory calculus, the students of mathematics education undergraduate programs should develop their capability of solving non-routine and open-ended problems. Such mathematical problems, according to Selden, Selden, Hauk, and Mason (2000), should be an integral component of the curriculum of mathematics at all levels with appropriate complexity to challenge students exercising their own strategies of problem solving. As recommended by Ganter and Barker (2004), there should be more opportunities provided for the students to build their creativity in problem solving skills rather than just imitating the methods exemplified by their instructors. Learning introductory calculus is expected to foster the students' skill of algebraic manipulation for this course provides the students with extensive opportunity to practice the skill (Conference Board of the Mathematical Sciences, 2001). The students should also develop their understanding of formal mathematical processes such as constructing definitions, making conjectures, and proving mathematical statements (Senk, Keller, & Ferrini-Mundy, 2004).

Robert Gagné's notion of the objects of mathematics learning (Bell, 1978) could also be taken into account in discussing the objectives of learning calculus. "The objects of mathematics learning are those direct and indirect things which we want students to learn in mathematics" (Bell, 1978, p. 108). The direct objects are facts, skills, concepts, and principles. The indirect ones include abilities of problem solving, theorem proving, intellectual development, learning how to learn, doing inquiry, working in groups and individually, and adopting positive attitudes. It is clear that each aforementioned goal of learning calculus could be classified in accordance with the mastery of the objects of mathematics learning. Of the three legs of Richard Askey's mathematical stool (as cited in Jackson, 1997), two are under the category of direct objects, and using mathematics to solve problems is under the indirect objects category. The ability to make conjectures, prove theorems (Senk et al., 2004), and solve non-routine problems (Fothergill, 2006) are classified as the indirect objects of mathematics learning. Zerr's (2009) recommendation on conceptual understanding is related to a direct object of concept, while mathematical maturity to inspire the knowledge growth (Conference Board of the Mathematical Sciences, 2001) and appreciation of mathematical structure are indirect objects of mathematics learning. Learning introductory calculus should result in students' understanding or mastery of all objects of mathematics learning in the course unit.

2.3 Learning Concepts in Introductory Calculus

In this section, the notion of mathematical concept is discussed as one of the direct objects of learning mathematics. In the first subsection, the discussion starts with a brief of facts, skills, and principles. The rest of the subsection expounds about the concepts. The nature of mathematical concepts, how they are defined, the notion of concept image and concept definition, and the problems of learning concepts in introductory calculus are discussed. The later three subsections deal with why students experience problems in learning introductory calculus.

2.3.1 Mathematical Concepts

Mathematical contents can be categorized under four kinds of the direct objects of mathematical learning: facts, skills, concepts, and principles (Bell, 1978). The objects progress based upon the order of complexity from facts, skills, concepts, and principles. Facts are any arbitrary convention in mathematics which are expressed in symbols or notations (Bell, 1978). Facts are the simplest object of mathematics learning. The symbol 9 is a fact symbolizing the number nine; $\boldsymbol{N}$ is mostly used to symbolize the set of natural numbers; and sine is the name assigned to a trigonometric function. Kinard and Kozulin (2008) categorize codes and symbols in mathematics into three groups. The first group is qualitative relationship symbols, for example, $\perp$ (perpendicular) and $\parallel$ (parallel). The second group is quantitative and operational relationship symbols, for example, $=$ (equal to), $<$ (less than), $+$ (addition), and $\times$ (multiplication). The last group is complex and functional relationship symbols, for example, $y = f(x)$ (a functional relationship between a dependent variable y and an independent variable x) and $\lim_{x \to c} g(x)$ (limit of $g(x)$ with respect to x approaching c). Symbols are generally used to represent not only mathematical objects, but also mathematical processes (Gray & Tall, 1994).

Mathematical skills encompass operations and procedures of organizing and manipulating mathematical information in significant ways with speed, flexibility, and accuracy that support or build mathematical ideas (Bell, 1978; Gray & Tall, 1994; Kinard & Kozulin, 2008). Here, mathematical procedures focus on stepwise nuances of solving and presenting the solution of mathematical problems that are essentially the prototypical answer to a question of how to solve a particular mathematical problem (Yuen & Clarke, 2016). For instance, the algorithm for long division and the procedures for bisecting angles. Skemp (1987) characterizes skills with the ability to translate them smoothly and reliably into action. Kinard and Kozulin (2008) consider the skills or mathematical procedures and operations, as the lowest level at which mathematical knowledge exists, preceding mathematical concepts as the second level and mathematical insight as the third level. Further, Gray and Tall (1994) explain skills or operations in terms of how they are represented. They claim flexibility as playing a fundamental role in performing the processes or procedures. For example, the process of addition could be carried out by counting, by subitizing, by deduction from given information, or by some other methods. The mastery of a skill is indicated by correctly demonstrating the skill in solving different types of problems within various situations. This mastery can be gained through drill and practices.

A mathematical concept is an abstract idea that enables us to categorize objects or events whether they are examples or non-examples of the abstract idea (Bell, 1978). A concept, as a result of abstraction (Simon, 2017; Simon, Placa, & Avitzur, 2016; Skemp, 1987), is a mental construct or representation of a category by which a person can distinguish examples from non-examples of the category (Klausmeier, 1992). It is the building block of the cognitive structure of an individual which functions as a fundamental agent in all processes of thinking. The categorization of the objects or events will apply the criteria stated in the concept. Concepts are considered by DeVaus (1991) as a tool that fulfills a useful "shorthand function" (p. 48) and are created for communication and efficiency but do not have a fixed meaning and independent existence, so for the purpose of research, they must be defined. The concept of function is an instance; through its definition, the concept conveys the conditions under which an object or event can be classified as a legitimate function or not. Similarly, the definition of limit will enable the classification of a mathematical statement as to whether it expresses a correct limit or not.

A principle is claimed to be the most complex object of mathematics learning. It is a sequence of concepts along with the relationship among these concepts (Bell, 1978). According to Kinard and Kozulin (2008), a principle, which they call mathematical insight, is derived from several conceptual understandings, formulating relationships between these understandings, and creating new ideas or applications. A theorem is an example of the principle. "If a function is differentiable at a point, then the function is continuous at that point" is a mathematical principle. This involves three concepts, that is, functions, derivatives, and continuity, together with their relationships among others. If a concept is a single mediator representing a group of specific objects, then a principle is different from concept for it represents a series of mediators, each one of which is a concept, or a principle (Gagné, 1966). Mathematical principles could be studied through processes involving indirect objects of mathematics, such as scientific inquiry, group discussion, or problem solving activities (Bell, 1978).

Mathematical concepts are formed through abstraction (Mitchelmore & White, 2000). Abstraction is a basic step in creating new mathematical concepts and in reinterpreting already existing concepts in a new context (Ferrari, 2003). According to Skemp (1987), the process of abstraction for constructing elementary mathematical concepts starts from an awareness of similarities of the underlying structure of a class of objects. In this case, objects can be concrete objects, mental objects, or relations between objects. It continues with classification by which all experiences on the basis of these similarities are collected together. The essential traits and qualities of the class of objects are detached from individual objects (Piaget, 1970; Sjögren & Bennet, 2014). This results in formation of an elementary mathematical concept which is basically a representation of the class of objects (Mitchelmore & White, 2000). Further, advanced mathematical concepts are also created through abstraction. Advanced concepts are abstracted from elementary mathematical concepts (Mitchelmore & White, 2000). This abstraction involves the processes of reinterpretation, integration, and reorganization of the elementary concepts, along with their structures, interrelations, axioms, and postulates. The constructed concepts are explicated by their definitions which capture the essence of the constituting elements from which they are abstracted. The definition of the concept provides a point from which further treatments or studies of the concepts start (Damerow, 1996).

There are three types of concepts: conjunctive, disjunctive, and relational (Elliot, Kratochwill, Littlefield, & Travers, 2000). The conjunctive concepts rely upon the joint presence of several attributes, which are abstracted from many individual experiences with the object such as boy, car, book, and orange. Disjunctive concepts are composed of a concept, any one of whose attributes may be used in the classification; one or another of the attributes enables the object to be placed in a particular category. In mathematics, we can take the concept of extreme point of a curve. An extreme point could be the lowest point of a curve, the highest point of a curve, the lowest point of a curve within a certain interval of domain, the highest point of a curve within a certain interval of domain, or the end point of a curve.

Relational concepts are formed by the relationship that occurs among the defining attributes (Elliot et al., 2000). The concept of function is an example of relational concepts. A function in mathematical terms might be a two-column table of both input and a single associated output; a graph on the Cartesian coordinate system satisfying vertical line test; or an algebraic expression $y = f(x) = x + 1,\ x \in \boldsymbol{N},\ y \in \boldsymbol{R}$. All examples have the common characteristic that every element in the first side is assigned to only one element in the second side. Further, the common characteristic is developed from some attributes such as elements of the first side (set), elements of the second side (set), and the rule of the relation. Furthermore, the set of all elements of the first side is defined as domain; the set of all elements of the second side is called codomain, the set of all assigned elements of the second side is called range, the element of the first side is called pre-image, and the element of the second side is defined as image.

2.3.2 Concept Image and Concept Definition

There are various concepts that have not been defined. We learn to recognize them by experiencing or using them in suitable contexts. Further, the meaning and interpretation of the concepts might be refined although their definitions have not been formulated precisely. With the absence of the precise definition, the concepts can still be communicated and undergo manipulation because they are named or assigned with a specific symbol (Tall & Vinner, 1981). However, a particular symbol might be too limited when compared to the whole cognitive structure which shapes the concepts' meaning and usage when they are recalled and manipulated. This cognitive structure is termed as a "concept image" (Tall & Vinner, 1981, p. 152). It covers the whole mental pictures, properties, and processes corresponding to the concept. The images could be derived from examples, schema, graphs, symbols, and some other experiences related to the concept. The images of a concept are not something static; they are influenced by experiences (Education Committee of the EMS, 2014) and subject to change through experiences, the encounter of fresh stimuli, or the maturation of the individuals. Vinner (2002) assumes that acquiring a concept means forming or generating images of the concept. This implies that understanding a concept means having developed images of the concept. But, the status of the constructed understanding is determined by the completeness of the images developed.

A different matter is the definition of a concept, if it has one. Tall and Vinner (1981) regard the "concept definition to be a form of words used to specify that concept" (p.

152). The definition is acquired by an individual and then associated with the concept. Sometimes, the definition is personally constructed by the students in the form of their own words which are used to explain their concept image. This kind of definition is called "personal concept definition" which could be different from a "formal concept definition", the one agreed upon and accepted by the community of mathematics in general (Tall & Vinner, 1981, p. 152). A personal concept definition might be either a formal mathematical definition or a non-example of mathematical definition (Zaslavsky & Shir, 2005). The formal definitions are verbal sentences that explain the concept precisely as could be found easily in the textbook of mathematics.

In the mathematics structure, all concepts, excluding the primitive ones, must have formal definitions (Ganesalingam, 2013; Vinner & Dreyfus, 1989). Many mathematical concepts appear in some form or other before their formal definitions are agreed upon. This encounter unavoidably creates a cognitive structure in the mind of individuals with its complexity yielding various related mental images when someone is exposed to a concept (Tall & Vinner, 1981). Therefore, it is necessary to differentiate between the definition of mathematical concepts and the cognitive processes taking place in conceiving the concepts. Mathematical definitions of a concept bring the concept into existence (Selden & Selden, 2008). The concepts will be exactly expressed by definitions in a way that there is nothing more and nothing less than whatever the definition expresses.

The formal definition of a mathematical concept incorporates the essence of all elementary concepts, axioms, and postulates from which it is abstracted (Mitchelmore & White, 2000). Within the mathematical community, it is agreed that the words or phrases composed in a standard formal definition do embody the essence of the concept which is defined (Edwards, 1997). The definitions in mathematics are "binding logically and [are] not as descriptions of certain aspects of an object otherwise known by senses or insight" (Sierpinska, 1992, p. 47). The words and phrases in the formal definition exactly represent the essence of and comprehensively specify the defined concept (Edwards & Ward, 2008). All attributes in a mathematical definition definitely need to be taken into account when creating an example or justifying a prospective example of the defined concept. Unlike axioms and postulates which are just taken as assumed, definitions are contested, and different from mathematical principles (propositions, theorems, lemmas, or corollaries) which are provable, definitions, as the negotiated basis for mathematical enterprise, cannot be proven (Kobiela & Lehrer, 2015).

A concept definition held by an individual will generate specific concept images. Logically, the concept images that should match the concept definition, not the other way around (Edwards, 1997). More often than not, the individual already has concept images that could be less or more developed (Viirman, Attorps, & Tossavainen, 2010). These images might not be coherent to other concept images already existing. For instance, the concept definition of a function could be "a mapping from *A* to *B* where each element of *A* is mapped into exactly one element of *B*". Yet, those students who have learned the concept of function may or may not memorize the definition of function. The concept definition somehow underlies the set of concept images; however, when the concept is used in learning, it is the concept images that are almost always evoked (Attorps, 2006; Hansson, 2006). The concept images could also cover many other aspects, for example, the idea that a function must have an algebraic formula or a specific rule; a function could have different formulae for different pieces of the domain *A*; a function is an action of mapping a in *A* to $f(a)$ in *B*; a function is a graph, or a function is a two-column table of values. Tall and Vinner (1981) remind us that there is no guarantee that these all images belong to an individual's concept image. Further, Vinner and Dreyfus (1989) argue that a certain concept image could be used differently from one student to another. They found that one image could be the reason for some students to reject a given relation as a function, whereas, for other students, it could be the basis for their reason to accept a relation as a function. For instance, the image of discontinuity, which for some students, a discontinuous graph could not be a function, but for other students, the same graph is claimed as representing a legitimate function. A similar case was found for the split-domain and exceptional point images of the concept of function. Both were used for accepting and for rejecting particular instances as valid functions.

Definitions of concept play a central role in the study of mathematics. As a theoretical system, mathematics holds definitions through which its concepts are introduced (Mariotti & Fischbein, 1997). Lakatos (1976) categorizes such definitions as naïve definitions, that is, the definitions used to specify or denominate a concept. Definitions also serve to specify a concept with certainty by synthesizing its mathematical essences (Borasi, 1992; van Dormolen & Zaslavsky, 2003). For practical purposes, definitions are used to communicate mathematical concepts within a mathematical community. The constructed concept with its essential properties is communicated by means of the so-called zero definitions (Lakatos, 1976). Definitions express the properties of the concept along with the relationships among the properties. They are used to verify or prove mathematical statements where they function to link deductive chains of mathematical argument (Freudenthal, 1973). This role is played by proof-generated definitions (Lakatos, 1976). The focus of the present study was the formal definition of function which was operable in nature, that is, the working definition which is operable for students to develop a deductive verification or proof of relations as valid functions. This kind of definition is classified as a proof-generated definition.

2.3.3 Defining Concepts

Naming the concept is essential. Skemp (1987) argues that people sometimes find it difficult to distinguish a concept from its name. The name of the concept is a sound while the concept is an idea. Naming is also important in the formation of a new concept. Skemp (1987) explains that when we hear the same name in connection with different experiences, we will be predisposed to take them together in our minds and increase our opportunity to abstract their intrinsic similarities. Furthermore, he states that assigning names to classes which have only slightly different characteristics increases the ability to classify later examples correctly. Concepts are named by words whose definitions are stated in terms of the attributes of the concepts (Klausmeier, 1992). These attributes function are criteria against which the justification of an object is undertaken to determine whether the object is an example or a non-example of the concept.

The creation of the definition of mathematical concepts is different from that of the colloquial language definition. Edwards and Ward (2008) distinguished between *extracted* and *stipulated* definitions. Stipulated definitions result from assigning a name to an object where the object has an explicit meaning relation to the word (name). The stipulated definitions are based upon the experts' advice which enables the communication of the concepts accurately and easily. Meanwhile, the extracted definitions are word-thing definitions explicating the way of using the defined word. The extracted definitions are constructed on the basis of examples of use and a body of evidence. Mathematical definitions are stipulated or analytic in nature, while everyday definitions are descriptive, extracted, or synthetic (Edwards & Ward, 2008). Students need to be aware of these different features.

Edwards and Ward (2004) recommend that instructors need to clarify the distinction between mathematical definitions and many every-day or dictionary definitions of which students may not be aware. The authors further argue that the failure to understand the distinction influences the students' comprehension of concepts. Learning activities for defining concepts are suggested ways to help students realize that distinction. For example, within a basic geometry course, students could be encouraged to focus on the usual definition of a triangle, which is useful in Euclidean and hyperbolic planes, as well as on the sphere initially. However, eventually, students will realize that certain propositions are false for all triangles constructed on the sphere, and this phenomenon necessitates a refined definition. The students can be challenged to do the refinement. Such an enculturation activity into the real field of mathematics is also to make the students aware of the fundamental role mathematical definitions play in the deductive-axiomatic structure characterizing the discipline of mathematics (Edwards & Ward, 2008).

The specific characteristic of a mathematical definition is explained by Alcock and Simpson (2002, p. 28), "a mathematical definition does have the property that everything satisfying it belongs to the corresponding category and that everything belonging to the category satisfies the definition." There are several imperative features that must be fulfilled by a formal definition of a mathematical concept (Zaslavsky & Shir, 2005). A formal definition has to be *non-contradicting*, namely, all required conditions of a definition must coexist. All terms involved in the definition must be *unambiguous*, which means all words must have a unique meaning or interpretation. A formal definition must be *invariant*, namely, the change of its representation will not change its essence or meaning. Another feature of a formal definition is *hierarchy*, which means that the definition must be based upon previously defined more basic concepts, terms, axioms, or postulates. The last feature is the non-circularity, namely, a definition does not use the defined term to define itself. In addition, van Dormolen and Zaslavsky (2003) proposed criteria for a good definition, namely, *existence* (a new concept must have an instance), *equivalence* (definitions of a new concept must be equivalent to each other), *acclimatization* (a definition has a proper place in a deductive system), *minimality* (a definition covers the minimal number of concept features), *elegance* (a definition must be simple and look beautiful), and *degeneration* (a definition must allow degenerate cases to be in or out).

Students should not be just the recipients of definitions. They should be the creators of definitions. The students' active involvement in defining activities will expose them to the natural process of the development of concept definition (Keiser, 2000). Constructing definition of concept must be positioned as a natural component of formal mathematics learning activities. Defining as a distinctive and advancing form of mathematical practice should be an integral part of learning mathematics activities (Rasmussen, Zandieh, King, & Teppo, 2005). Defining has a position equivalent to activities of exploring ideas, problem solving, conjecturing, developing notation, generalizing, specializing, and proving theorems, (de Villiers, 1998; Mason, 2005). Freudenthal (1973) highlights the importance of establishing a concept definition that is claimed to be more crucial than developing a proposition or proving a mathematical statement. Defining is closely intertwined with activities of developing and revising new concepts as well as strategies of understanding concepts (Kobiela & Lehrer, 2015).

Constructing a definition of a concept in mathematics is an activity aimed at confirming the status of an object under consideration. Formulating a definition means developing a statement which can be used to declare what is being talked about (Ouvrier-Buffet, 2011). Definitions are used by learners to comprehend a concept when it is firstly presented. Usually, there is a pervading dialectic existing between the concepts being studied and their definition being formulated (Ouvrier-Buffet, 2002), and this strengthens the legitimation of defining concepts as a strategic activity in mathematics learning focusing on a meaningful understanding of mathematical concepts (Ouvrier-Buffet, 2004). Zandieh and Rasmussen (2010) found that in the process of defining, a dialectic also exists between concept image and concept definition which is leading towards the definition of the concept agreed upon by students. Such a dialectic involves creating and using either concept images or concept definitions. Exploring the definitions as a concept holder is needed when we want to know how a mathematical concept is formed. Students often face an intrinsic problem pertaining to the definition construction in terms of which pertinent properties to include so the proper object classification could be achieved (Mariotti & Fischbein, 1997).

Defining concepts as a learning activity can be implemented simply by involving students in creating definitions in the form of a sentential statement. This is the most straightforward feature of a defining activity. However, in learning processes, Zandieh and Rasmussen (2010) have identified several activities undertaken by the students in their attempt to reach the accepted formulation of the concept definition. Negotiating both the form of the definition statement and the reason for selecting the model, revising the statement form, or refining the formulation are activities accompanying the construction of definition statement.

From the viewpoint of process, Freudenthal (1973) states that there are two kinds of defining processes, namely, descriptive (*a posteriori*) defining and constructive (*a priori*) defining. Descriptive defining is constructing a definition of a concept by providing a description of the concept completed with sufficiently characterizing properties, and constructive defining is constructing a definition of a concept by providing a statement of the concept which models it out of the already existing concepts. Further, de Villiers (1998) elaborates the difference between descriptive and constructive defining processes. With the descriptive defining, the defined concept and its properties have been previously understood by the learners, and the definition of the concept is then formulated. This defining process aims at systematizing the existing understanding. Meanwhile, in the constructive defining process, the definition of a concept has been previously known by the learners, and the definition then undergoes a change by generalizing, specializing, discarding, or including properties to the definition. This process aims at constructing a new concept.

In the present study, it was claimed that the process of constructing the definition of function took place more in the sense of descriptive defining. This is due to the activities of logically exploring basic properties characterizing a function as a special model of relation. The systematization of the existing understanding as an instructional objective of the descriptive defining process was also justified in the present study because the students constructed their definition by building-with their previous concepts (de Villiers, 1998; Hershkowitz et al., 2001). Although, there was an aspect of reconstructing the students' existing formal definition of function, it neither added new properties to the concept of function nor produce a new concept.

In calculus textbooks, mostly the definition of function is formally treated in the context of *a posteriori* descriptive defining (see, e.g., Stewart, 2016; Thomas et al., 2018; Varberg et al., 2007). The definitions are aimed at systematizing students' existing knowledge, given that the concept of function is assumed to have been learned by them in previous school mathematics. The students were assumed to have understood the concept of function and have known some properties of function. However, it seemed that their understanding, including the definition of function led to confusion for students when they were to utilize the definition to connect the logical chain in deductive argumentation, in particular, in proving the functionality of relations. Therefore, I thought that the students were in a need of a definition of function which was formally operable, and this could be formulated by singling out individual properties blended in the formal definition of function. Having these sufficient properties of function isolated individually and defined formally, they will be applicable in proving a relation as a legitimate function.

Various studies in defining mathematical concepts have been conducted in undergraduate mathematics education. Oehrtman et al. (2014) investigated how first-year university students formulate the formal definition of sequence convergence in a calculus course. The students were facilitated by tasks designed to engage them to reinvent the formal definition by leveraging their intuitive understanding of the convergence of a sequence. The students were encouraged to express their personal concept definitions (Tall & Vinner, 1981), and were challenged with problems and issues raised to create a cognitive conflict between the students' formulated definition and the standard formal definition. Through the process of guided reinvention heuristics framed within the theory of Realistic Mathematics Education (Freudenthal, 1973), it was found that the students could construct a definition of sequence convergence which was justified as equivalent to the standard formal definition. Earlier, Ouvrier-Buffet (2011) researched the defining processes at stake within an unfamiliar discrete mathematics context involving first-year university students. The students were exposed to problems where they formulated their definitions in order to solve the problems. The learning activities were set to allow them to refine their definitions in order to reach the complete solution. This study revealed the significance of the instructors' role in providing the students with necessary guidance which made the defining processes fruitful. Also, Ouvrier-Buffet (2011) found the emergence of an in-action definition, namely, "a statement used as a tool (not an object) that enables students to be operational without explicit definition" (p. 177), which complemented the three types of role-based definitions theorized by Lakatos (1976). These two studies provided at least two important aspects to consider in implementing definition construction as learning activities, namely, the importance of the tasks particularly design to encourage students to devise their definitions of concepts, and the role of the instructors in guiding them to invent the concepts through the formulation of concept definitions. The studies by Ouvrier-Buffet (2015) and Martín-Molina et al. (2018) involved university mathematicians focusing on describing the dynamics of defining processes they underwent. Their results informed the design and implementation of defining activities (resembling the professional mathematicians' ones) for undergraduate mathematics education students.

2.3.4 Between Concept Understanding and Concept Definition

Ouvrier-Buffet (2006) identified a gap evident between students' understanding of mathematical concepts and the conceptual understanding held within the community of professional mathematicians. After defining a mathematical concept comes a question of how an understanding of the concept sits with respect to its definition. Of course, being able to recite the formulated definition of the concept will never be considered as a sufficient indicator of an understanding of a mathematical concept. Sierpinska (1992) listed several indicators of an understanding of the concept. Our understanding of the concept is indicated by our observation of the examples and non-examples of the defined concept. It is the concept definition which functions to distinguish the objects of the concept from the non-objects of it (Lakatos, 1976). The ability to justify examples is one component of adaptive reasoning competency, a strand of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001). Moreover, we can verify what the concept is and what it is not; we can relate the concept to other concepts; we know the relative position of the concept within a mathematical structure or theory, and we know how to use the definition. Further, in relation to understanding definitions, Sierpinska (2005) identified several specific indicators, namely, detecting errors in ill-formulated definitions, providing instances and non-instances of the concept, and expressing the definition in symbolic conventions. Here, we can highlight the indicator of *the ability to apply the definition which means that we should be able to use the definition to solve problems*. Actually, this application firstly takes place in the observation of instances and non-instances, where the verification is required to justify whether an object is a legitimate example of the concept. In the context of formal mathematics, certainly, the verification should be in the form of a definition-based deductive argument.

Knowing a concept definition by heart is not a guarantee that the concept has been understood. The reconstruction of a formal definition could take place through rote memorization, so when students are able to construct it, this ability is only a necessary but not sufficient indicator of deep understanding (Vinner, 2002). Frequently, university students could recite formal definitions, nevertheless, they could not employ them when asked to solve problems or prove theorems (Selden, 2012). Alcock and Simpson (2004) found that some students used prototypical examples, instead of the formal definition of convergence, to argue about the convergent sequences. In a similar vein, Vinner and Dreyfus (1989) argue that students do not necessarily employ the definition in justifying a mathematical object or event whether it is an example of the concept. They further state that in most occasions, students base their decision upon a concept image resulting from their experience with examples and non-examples of the concept. Accordingly, students could have a set of examples of the concept different from the actual examples determined by the definition.

Concept images and concept definitions are two crucial aspects of an understanding of mathematical concepts. While Tall and Vinner (1981) and Vinner and Dreyfus (1989) were interested in how the set of objects determined upon the basis of the concept definition differ from that of objects determined by someone’s images of concept, Moore (1994) focused on the ways the students use their understanding of mathematical concepts. This focus is another aspect of concept understanding which is termed as *concept usage*. Concept usage is the way individuals work with concepts to generate or use examples or to construct arguments. Moore (1994) argues that in terms of understanding concepts, *concept definition*, *concept image*, and *concept usage* prevail, and further proposes a concept-understanding scheme covering those three aspects. These aspects of concept understanding form a sequence of image, definition, and usage. The students' capability to use the definition is determined by their understanding of the definition, which in turn is determined by their concept images (Moore, 1994). In practical cases, the definition is sometimes skipped in the sequence where the students just rely on their concept images when they use the concept. Although the students do know the concept definition, this shortcut takes place because they are unable to deal with the formal definition; they do not know what to do and how to do with the formal definition, and they are confused by manipulation of the logical symbols and formal mathematical language (Moore, 1994).

Concept definitions are considered as tools for gaining a more meaningful understanding of a given concept (Edwards & Ward, 2008). The concept definitions are very central in the structure of mathematical knowledge which is characterized by the axiomatic-deductive nature. They are not just a description of concepts, but also provide the logical connections necessary needed in a deductive argumentation and reasoning (Kohrman, 2018). Essentially, definitions comprise a relation of previously defined basic concepts, axioms, and postulates, along with necessary and sufficient conditions. The use of definitions necessitates the unpacking of "the definitions of the concepts involved, including their logical structure" (Harel, Selden, & Selden, 2006, p. 4). To make them applicable, the definitions should express the decomposition of the concept into its elementary features and properties (Fodor, Garrett, Walker, & Parkes, 1980).

As mentioned above, the purpose of the decomposition or reformulation of the definitions is to make them usable. Such definitions become working definitions that can be directly employed in reasoning and developing arguments including examining mathematical objects or events as to whether they are an example or a non-example of the defined concept. I called it *the operational formal definition*. This type of definition has the same meaning as the "formally operable" definition proposed by Bills and Tall (1998, p. 104). An operational formal (mathematical) definition or theorem is the one which enables individuals to create or meaningfully reproduce a formal argument.

Using the concept as the aspect of concept understanding is problematic for students because the formal definitions are not applicable; they are not operational, and reconstructing the formal definitions into an operational one is not easy for the students. Constructing the operational formal definition as a component of the development of concept understanding is the focus of the present study.

In the discussion of understanding of mathematical concepts, *to understand* is mostly associated with the meaning of *to understand well*, and the theories of understanding focus only on the types of understanding, or on the levels of understanding of Sierpinska (2005). Regarding the types of understanding, Skemp's (1987) theory is the one which is mostly referred to. He distinguishes three kinds of understanding possibly constructed in learning mathematics: instrumental understanding, relational understanding, and logical understanding. Instrumental understanding is the comprehension of mathematical methods indicated by the ability to perform the methods, algorithms, and procedures to complete a mathematical task. In contrast, relational understanding is the comprehension of mathematical contents and their justification indicated by the ability to complete a mathematical task, to explain how the methods work, and to justify why the methods work. The third type, logical understanding covers instrumental and relational understanding indicated by the ability to perform a mathematical task, to explain how the method is used, to justify why the approach works, and to demonstrate the logical necessity by a chain of inferences, from given premises along with axioms, definitions, and theorems, to logical conclusions.

Regarding the level of understanding, the focus is on evaluating the relative goodness of understanding (Sierpinska, 2005). The goodness of understanding is determined on the basis of the number of significant features covered in the process of constructing or developing the understanding (Sierpinska, 2005). The logical understanding mentioned above provides components of the expected understanding of mathematical definitions, particularly in terms of *the ability to present arguments that are logically sound and complete*. As previously stated by Tall and Vinner (1981) and Vinner (2002), *the ability to verbally state the definition of a concept* is an indicator of the understanding of the concept. The statements could vary in style and diction but they still match or have the same or equivalent meaning as described in the formal definition. The expression of the mathematical definition involves language, notations, logic symbols, and quantifiers which made it unique compared to the definitions in other disciplines (Adiredja, 2014). Therefore, *the ability to use correct mathematical terminologies and notations* also determines the understanding of mathematical definitions. A mathematical definition is formulated on the basis of the previously defined basic concepts, axioms, undefined terms, and postulates along with their interrelationships (Zaslavsky & Shir, 2005). This adds another feature of a good understanding of definition, that is, *the ability to identify all the important elements of the concept and show understanding of the relationships between them*. The four abilities are synthesized to be criteria for determining the level of understanding of mathematical definition used in this present study. Because the formal definition investigated was the operational one, these criteria were completed with *the ability to use the definition to develop a deductive argument*, particularly, a proof of a relation as a valid function.

By and large, there are two approaches that might be undertaken by researchers in investigating the notion of understanding in mathematics education. First, it can be endeavoring to discover the thinking mechanisms that lead to the construction of understanding or second, elaborating on cognitive activities that improve the quality of understanding (Sierpinska, 2005). Further, there are three basic strategies for answering the questions of understanding in mathematics education research (Sierpinska, 2005). The first strategy concentrates on the development of teaching and learning materials aimed to help the students understand better; the second strategy focuses on the diagnosis of the students' constructed understanding; and the third strategy deals with the theoretical notion of modeling the constructed understanding. In practice, there is not a strict categorization that a research project fits into one strategy and not in another. This present study could be considered as elaborating the mechanism of thought leading to the development of understanding. In terms of the strategy, to a great extent, it focused on the diagnosis of the process of constructing the students' understanding of the operational formal definition of function. It involved calculus tasks as a learning material to facilitate the understanding construction, and the study was framed within the theory of theoretical abstraction with the Abstraction in Context model (Dreyfus et al., 2015; Hershkowitz et al., 2001; Schwarz et al., 2009). This framework will be elaborated in section 2.6.

In addition, as the focus of the research was the definition of function, the biogenetic principle of Felix Klein was taken into account, namely, mathematics topics should not be presented to students as completed axiomatic-deductive structures (de Villiers, 1998). Students should, to some extent, retrace the track of the original thinking about the concept, and reinvent the concept (Freudenthal, 2002b). If the definitions are presented to students as a final product, the students lose their chance to participate in the activity of mathematization (Freudenthal, 1973). The students need to be apprentice mathematicians who are granted an opportunity to participate in the mathematical enterprise from the beginning, including the formulating or creating concept definitions (Ouvrier-Buffet, 2004). Being involved in the defining activities, students could achieve some pedagogical benefits such as a deeper conceptual understanding of the concepts involved, comprehension of the nature of mathematical definition, and understanding of the role of definitions in mathematics (Edwards & Ward, 2008).

2.4 Challenges in Learning Concepts of Introductory Calculus

In a discussion of students' problems in understanding calculus, blaming the students' prior knowledge or intelligence is never fair (Tall, 2002b). There are three aspects that certainly contribute to the phenomenon. First, introductory calculus is in the first semester in which the students experience the period of transition from school to university environment of learning; second, the preference that the students choose in dealing with introductory calculus; and third, the characteristics of learning materials covered in the course unit. Each of the three is discussed in the following subsections.

2.4.1 Change in Transition

Transition from secondary to tertiary mathematics is considered as a major issue by many mathematicians and mathematics educators (Gueudet, 2008). They are concerned because being in transition time affects the students' learning performance. The issue of transition actually comprises the whole life of students covering the aspects of their social, psychological, and cognitive well-being. From the anthropological point of view, Clark and Lovric (2008, 2009) consider the transition from secondary school to university as a modern-day rite of passage; it is a series of events and activities which university students use to cope with a crisis they encounter during the early period of their time at university. In this period of change, the didactic contract alters to the new expectation of the students' autonomy and the new responsibility of students and lecturers (Brousseau, 2002). The changes are sometimes responded to by students with strong emotions. The transition itself takes quite a long time, and it represents an essential stage in the enculturation of students into the practices of mathematicians (Nickerson & Rasmussen, 2009; Perry, Camargo, Samper, Molina, & Echeverry, 2009).

The discrepancy between learning strategies that seem to perpetuate in high schools and those strategies expected to be embraced in the university level is argued by St. Jarre (2008). He claims that when the high schools attempt to increase rigor, "more often it is more along with the lines of rigor mortis" (p. 126). More mathematics problems presented to students are apparently too rigid to encourage them to think creatively. Some students develop a belief that learning with curiosity about all things interesting does not result in the expected recognition. The recognition is in the form of grades or marks from teachers. They become aware of the importance of pleasing teachers by performing something which the teachers expect, instead of developing the culture of exploration and inquiry. Meanwhile, university calculus requires different qualities of students (St. Jarre, 2008). Students are expected to have the capability of thinking, self-correcting their thinking, adapting, and succeeding when they are situated in a new context with non-routine calculus problems. In short, universities need thinkers and learners, not performers who can only imitate and perform to get grades.

With regard to the cognitive aspect of the transition, the difference in contexts of school and tertiary education with respect to teaching styles, instructional approaches, studying and learning strategies, views about mathematics, specific mathematical concepts, the mathematical knowledge, and the objectives of learning, are potentially problematic for students (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016), A change of emphasis from more computational, problem solving strategies to more formal proof-based strategies, is a salient problem in learning mathematics (Alcock

& Simpson, 2002; Selden, 2012). A mathematically detailed study of transition by Godfrey and Thomas (2008) reveals that many students commencing their university learning still have not reached the formal thinking level as expected. In addition, Moore (1994) found that students encountered difficulties in terms of mathematical definition, language, and notations used in reasoning in the period of transition. They are not equipped with adequate argumentation skills for proving involving formal definitions and logical deduction (Biza et al., 2016; Kempen & Biehler, 2014).

In introductory calculus, the movement from learning intuition-based concepts founded on experiences, to learning concepts founded on definitions and properties which are constructed logically and deductively, is a difficult transition (Tall, 1992). Relying on certain prior information, students use their own ideas and current understanding to construct an initial framework of knowledge. Their earlier experiences of concepts exist simultaneously in the mind along with the growing body of deductive knowledge. Research shows that this produces a wide variety of cognitive conflicts which could turn to be learning obstacles (Tall, 1992). The conflict occurs because the new situation of learning exposes new mathematical knowledge which is incompatible with prior perspectives. In attempting to absorb new learning materials, they try to assimilate them into their constructed framework which finally results in a synthetic model comprising a mixture of beliefs and scientific facts. Their understanding of the materials is affected because this synthetic model is robust and apparently consists of misconceptions about the subject (Biza, Souyoul, & Zachariades, 2006; Clark & Lovric, 2009).

First-year students in mathematics education undergraduate programs come from different school backgrounds with different mathematical experiences. It is unavoidable that they might have constructed some misconceptions during their mathematics learning in schools. The misconceptions might concern a misunderstanding of mathematical concepts or mathematical practices. For example, students considered the verification of several cases was sufficient to make a generalization. They applied an intuitive inductive strategy to show the truth of a mathematical statement, whereas this statement was formulated for an infinite number of cases (Hoyles & Healy, 2007). To some extent, this present study dealt with this kind of misunderstanding of mathematical practices with a focus on the deductive verification or proof of an object as an example of a concept based upon the formal definition of the concept.

Some basic mathematics courses at university actually reinforce school mathematics aiming at students having a reasonably uniform level of fundamental understanding as the basis for more advanced mathematical courses. However, some studies found that university students fail to see university mathematics topics as a continuation, an extension, or a generalization of school mathematics topics (Biza et al., 2016; Cofer, 2015). This finding concerns university lecturers in terms of the extent to which their undergraduate instruction, covering teaching activities, syllabi, and textbooks, has been set to connect to school mathematics wherever possible and needed. Introductory calculus is one of several course units which can be clearly considered as a continuation, extension, and generalization of school mathematics topics. Such perspective of introductory calculus as a transitional unit and the need to link the unit to the school mathematics laid the basis for conducting the present study. The concept of function was taken as the focus because it was one of the topics intensively studied in school mathematics, although mostly with a practical approach, and students could see the real connection of this topic with the concepts they learned in introductory calculus. They could rely heavily on their previous knowledge. However, at the same time, their previously constructed knowledge has a potential disturbance in developing the conceptual understanding of the concept of function in introductory calculus.

2.4.2 The Tendency to Be Procedural

Understanding the abstract concepts of calculus requiring formal abstract thinking is problematic for many students. Harel and Trgalova (1996) mention the formalization of concepts as one of the students' difficulties in learning university mathematics. Encountering conceptual difficulties requires students to perform coping strategies. During their secondary school time, these strategies involve skills of computation and manipulation employed when they are to pass examinations. Therefore, it seems to be natural that when the students find the fundamental concepts of calculus difficult to comprehend, they will resort to focus on the symbolic routines of differentiation and integration (Tall, 1996), which they have been accustomed to in school mathematics learning.

Based upon my experience of teaching introductory calculus, when students are asked to find the derivative of a function using the definition of $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ or $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$, some of them just avoid the definition for it involves limits and transformation of $f(x)$ into $f(x+h)$. They rather opt to perform the evaluation of the derivative using the rules or formulae of finding derivatives, such as, $f'(x) = arx^{r-1}$, if $f(x) = ax^r$, $r \in \boldsymbol{Q}$, or $y' = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$ if $y = \frac{u(x)}{v(x)}$. Tall (1996) found that if the fundamental calculus concepts such as those mentioned above prove difficult to master and use, students will turn to focus on the symbolic routines of differentiation. The point here is that the students tend to rely upon computational and manipulative skills to answer questions and to pass the examinations. To some extent, this option is predisposed by their earlier experiences of dealing with arithmetic and algebra at schools where an answer could be obtained through a sequence of manipulations.

Of course, it is not to claim that the procedural skills are unimportant. As one of the direct objects to learn in mathematics (Bell, 1978), the skill to perform the procedures correctly and appropriately is actually among the indications of a high-level critical thinking ability (Zerr, 2009). Being able to apply the rules appropriately in solving problems is part of learning achievement. However, from the perspective of mathematical structure, the rules are just the derivative or simplification of definitions (Boyer & Moore, 1988) within specific contexts. With respect to conceptual learning therefore, the comprehension of the definition should come first, then the derivative or simplification of the definition will come later. It is important to note that the ability to determine the derivative of a function using the appropriate rules is beneficial for the students for it could help them justify the derivative function which they obtain through the definitions.

The difficulty of thinking abstractly often leads students to choose procedural understanding instead of focusing on the conceptual depth (Tall et al., 2001). While this approach may be adequate for students of non-mathematics majors, a more abstract conceptual approach is undoubtedly necessary for mathematics students in their move to upper division university courses (Swinyard, 2011). Research has revealed that university students have a considerable amount of mathematical knowledge characterized as merely procedural and lacking the types of conceptual depth with the absence of methodology for doing mathematics (Dreyfus, 2002; Ferrini-Mundy & Lauten, 1993; Zandieh, 2000). This means that the students miss the aspect of know-how which allows them to flexibly use their mathematics in solving non-routine problems. Further, Dreyfus (2002) argues that these students have been successfully exposed to the final products of mathematics without gaining insight into the processes that mathematicians have been using which result in the products. Borrowing the terminology of Skemp (1987), there is a strong indication that many university students do prefer instrumental understanding instead of relational understanding or logical understanding. In a similar vein, Harel and Trgalova (1996) argue that the students are not interested in epistemological or historical studies of calculus; they rather prefer to use ready-made solution strategies and neglect the engagement in in-depth conceptual learning.

The students' preference for procedural learning is related to the initial view of what constitutes calculus mastery. Zerr (2009) states that many students begin to see calculus as merely the mastery of procedural skills, and they spend inadequate thought upon the more significant conceptual ideas which are also part of their calculus learning. Tall (1996) reminds his readers that taking introductory calculus course based upon procedural traditions may have "limiting effect on their attitudes when they take a more rigorous course at a later stage" (p. 307). Further, Ferrini-Mundy and Gaudard (1992) state that "it is possible that procedural, technique-oriented secondary school courses in calculus may predispose students to attend more to the procedural aspects of the college course" (p. 68). Unfortunately, without a conceptual understanding, the students will face difficulties in generalizing their skills to other contexts, and consequently, they will abandon much of the power of calculus to solve problems.

2.4.3 The Characteristics of University Calculus

The other aspect claimed to contribute to the performance of the students in introductory calculus is the characteristics of learning objects (concepts) in introductory calculus. While calculus has been a dominant content in their secondary school mathematics education, calculus is of a different nature and is approached quite differently at university. Tall (2013) states that the shift from school mathematics which is dominated by embodied operations for handling practical geometry or arithmetic problems, to university mathematics which is loaded with theoretical definitions and deductions along with axiomatic mathematical structures, can be problematic. Adapting to a learning culture, where concepts are presented abstractly, nevertheless requiring rigorous definition, is problematic for many first-year university students (Leviatan, 2008; Martin, 2019).

Most of the university calculus concepts require the students to reconstruct their perspective. Being exposed to a new conceptualization of previously well-known concepts could be in conflict with the students' prior mathematical traditions (Selden, 2005). This conflict has the potential to incur cognitive and epistemological problems (Selden, 2012). For instance, in studying analytic geometry at senior high schools, the line which is tangent to a circle is basically defined as the line touching the circle only at one point and perpendicular to the radius at that point. Given the coordinates of the tangent point and the center of the circle, finding the gradient m of this tangent line involves the formulae $m = \frac{y_2 - y_1}{x_2 - x_1}$ and $m_l m_k = -1$, where m_l, m_k are real numbers, lines l and k are perpendicular to each other. Moving to introductory calculus at university, the tangent to a curve at a certain point is the limit of approximating secant lines. The notion of this tangent is the starting point of the study of derivative, where the gradient of the tangent is actually the value of the derivative of the function at the point. Moreover, the tangent at the inflection point of a cusp required further reconstructed thinking as it is no more touching the curve, but it crosses the curve at the point. Although the reconstruction of mathematical knowledge is often difficult for students, it is certainly needed in developing a cognitive means for effectively handling the concepts of calculus which are subtly different from naïve experience (Tall, 2002b).

The presentation of concepts in a formal mathematical language using an expression of symbolic logic (Clark & Lovric, 2009; Selden, 2012) is not easy for students to comprehend. Actually, in their previous mathematics learning at schools, logic has been introduced in terms of the truth value of various composite statements such as those with disjunctive or conjunctive connectives or conditional statements. The students have also learned about quantifiers and how to negate a statement along with simple valid arguments used in reasoning. However, when they are to work further with those elements of logic in learning calculus concepts, many students could not perform as expected. Selden and Selden (1995) found that many undergraduate students, even after finishing their third or fourth year, still experience problems in unpacking the logical structure of statements which are expressed informally. These students have trouble with expressing 'all children have mothers' in a logical statement 'for every child, there exists a woman who is his/her mother', and then in a symbolico-logical statement '$\forall c \in H, \exists w \in H: w = m(c)$' where c represents a child, w represents a woman, and $w = m(c)$ reads 'w is the mother of c.' These styles of expression are crucial in constructing a definition of mathematical concepts. Moreover, they find it difficult to understand the importance and the effect of interchanging the order of existential and universal quantifiers on the meaning of logical statements (Dubinsky & Yiparaki, 2000; Epp, 2009).

The introduction of thinking and reasoning in a deductive axiomatic system makes the abstract nature of calculus concepts more salient. It is a shift from merely doing, to doing with understanding, where mathematical objects are no longer considered as resulting from formulae, but as carriers of conceptual properties (Devlin, 2012). In this context, proving is a process of logical deduction from concepts, rather than a transformation of terms according to particular rules. Selden (2012) found that learning to reason from formal definitions, comprehending the meaning of theorems, applying them appropriately, and connecting concepts deductively cause difficulties for students. Epp (1998) argues that one of the best approaches to developing students' abstract thinking is through meaningful involvement in constructing and solving mathematical proofs.

There is no doubt about the central role of proof in the pedagogy of mathematics. The proof is the most essential tool of mathematics (Hanna, 2000a). Accordingly, Hanna

(2000a) strongly argues that teaching and learning mathematics without the concept of proof would be entirely unreflective of mathematical theory and practice. Actually, in practice, the students of tertiary education level could still apply informal arguments and intuitive reasoning in initial thinking, but eventually, such thinking must be formalized for the purpose of communication (Hanna, Jahnke, & Pulte, 2009). Strong rigor characterizes proofs, and proving mathematical statements emphasizes the use of formal definitions, theorems, creativity, and insight. Roh (2009) and Selden, McKee, and Selden (2010) found that students face difficulty and confusion with proving statements containing a universal quantifier because, in proving, they have to consider an arbitrary, but fixed, element to prove that the statements hold for all elements. Proving constitutes an obstacle for many first year students in learning introductory calculus (Selden, 2012).

The problems of being in the transition period of learning, the preference for the procedural approach in learning, and the abstract characteristic of learning objects are actually intertwined to each other. The students need to employ an extra effort to adapt to the new learning environment at university. At the same time, they have to change their preference because the procedural approach is no longer suitable in the university learning context where the approach is more relational, conceptual, and formal where the objects of learning were abstract in nature. Given their interrelationship, these problems should not be addressed separately. Gueudet (2008) reminded us that the accumulation of problems could discourage didactical treatments aimed to deal with them. However, didactical actions should be undertaken. Artigue (2002) ensured that the relationship and the perspective which were developed by the students with respect to learning mathematical concepts could be changed qualitatively. She argued that reconstruction, which dealt with the learning objects, was an instructional action that could be used in dealing with learning problems faced in the transition time. Focusing on the conceptual understanding of the formal definition of abstract concepts, this present study sought to address the problems partly by offering a way to develop an understanding of the formal definition which was operational in nature.

2.5 The Topic of Function in Introductory Calculus

Introductory calculus, as the first calculus course in the first semester of mathematics education undergraduate program, covers a sequence of topics: the real number system, limits and continuity, derivatives, and derivative applications. The well-known textbooks of calculus (e.g., Hughes-Hallett et al., 2017; Stewart, 2016; Thomas et al., 2018; Varberg et al., 2007) generally set these all topics in their first chapters. Under the heading of the real number system, the concept of function is discussed in terms of how functions are defined, what kind of representations are used to express functions, how functions are operated, and what are the properties of functions which make some of them special. Under this heading, the students are also introduced to some basic logic with its combinators ('and' (conjunction), 'or' (disjunction), 'if-then' (conditional statement), 'if and only if' (bi-conditional statement), and 'not' (negation)) and quantifiers ('for all' (the universal quantifier) and 'there exists' (the existential quantifier)). The concepts of equations and inequalities and the rectangular coordinate system are covered as well.

One of the most important concepts in modern mathematics is that of functions. The concept of function plays a central and unifying role in the world of mathematics (Selden & Selden, 1992). Most mathematicians and mathematics educators would concur with the claim that functions play a vital role in students' mathematical education. Attaining a broad and strong understanding of the concept of function is of significant importance. In undergraduate mathematics study, the complexity of comprehension of the function concept will be evident when the consideration goes beyond focusing only on the standard use of function concepts as a raw material in an introductory calculus course (Selden & Selden, 1992). Going beyond calculus, the concept of function is widely used, for example, in comparing the structure of abstract topological entities, that is, whether they are homomorphic. Functions themselves can also be treated as elements of abstract mathematical structures such as vector spaces, in which the operations on these structures such as the addition of vectors, are also functions.

2.5.1 Historical Sketch of the Concept of Function

The development of the concept of function goes back four millennia. The last three centuries covered the study of function in intimate connection calculus and analysis, while the previous very long period consisted of an expectation (Kleiner, 2012). The evolutionary development involved a complex network of conceptions covering the geometric representation of graphs, the algebraic formulae, the relationship between dependent and independent variables, an input-output machine of more general relationships, and the modern set-theoretic definition (Tall, 1992). It is reported that Leibniz was the first mathematician using the word “function”, however the first formal definition was stated by Johann Bernoulli in 1718 as follows: "one calls here function of a variable a quantity composed in any manner whatever of this variable and of constants” (Ruthing, 1984, p. 72). In this case, the phrase *composed in any manner whatever* is not of specific meaning, although we can infer from the context that for Bernoulli, function is a form of algebraic expressions (Kleiner, 2012).

The development of calculus which lends the significance to the concept of function was the best time for the emergence of the abstract concept of functions. In the eighteenth century, Leonhard Euler transformed calculus from a study of variables and equations into a calculus of functions. In 1755, Euler defined a function as follows “if, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities” (Ruthing, 1984, pp. 72–73). The conception of functions developed by Augustin-Louis Cauchy in 1821 was not very different from earlier definitions:

> When the variable quantities are linked together in such a way that, when the value of one of them is given, we can infer the values of all the others, we ordinarily conceive that these various quantities are expressed by means of one of them which then takes the name of independent variable; and the remaining quantities, expressed by means of the independent variable, are those which one calls the functions of this variable (Bottazzini, 1986, p. 104).

Cauchy’s definition contained shortcomings for he conceived a function as an analytic expression (a formula) or curves and did not make the notion of arbitrariness clear (Grant & Kleiner, 2015).

More than a decade later, the arbitrary property was clarified by Peter Lejeune Dirichlet with his relatively broad definition of function: “If a variable *y* is so related to a variable *x* that whenever a numerical value is assigned to *x* there is a rule according to which a unique value of *y* is determined, then *y* is said to be a function of the independent variable *x*” (Boyer & Merzbach, 2011, p. 452). This definition is considered as operationally oriented and more importantly, it is the first formulation stating one-valuedness (Viirman, 2014; Viirman et al., 2010). The concept of function has two essential features, namely, one-valuedness or univalence and arbitrariness (Even, 1990; Freudenthal, 2002a). The arbitrariness no longer required a function to be well-behaved. This character was illustrated by Dirichlet with his pathological function defined as: $y = c$, if *x* is rational, and $y = d$, if *x* is irrational, $d \neq c$, and $c, d \in \boldsymbol{R}$ (Boyer & Merzbach, 2011). The property itself was strengthened by Georg Bernhard Riemann who also explained the uniqueness of value assigned to each member of the independent variable (Kleiner, 2012).

The development of the concept of function continued until the twentieth century when the conception of function as a mapping between two arbitrary sets became dominant. In 1939, the ordered pair definition of function was introduced by Bernhard Bourbaki. He defined a function from *E* to *F*, which both are sets, as a special subset of the Cartesian product $E \times F$.

> Let *E* and *F* be two sets, which may or may not be distinct. A relationship between a variable element *x* of E and a variable element *y* of F is called a *functional relation in y* if, *for all* $x \in E$, *there exists a unique* $y \in F$ *which is in the given relation with x*.
> We give the name of *function* to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with *x; y* is said to be the *value* of the function at the element *x*, and the function is said to be *determined* by the given functional relation (Bourbaki, 2004, p. 351).

Because such a definition is presented in terms of sets, its logic must be scrutinized in the same way as that of set theory. *This is the formal definition of function normally used in the study of calculus nowadays.* In general, calculus textbooks often present the definition of function in this formal style (see, e.g., Hughes-Hallett et al., 2017; Stewart, 2016; Thomas et al., 2018; Varberg et al., 2007).

2.5.2 Function as a Core Concept in Introductory Calculus

Calculus is the mathematics of change (Rohde et al., 2012b). The notion of change in mathematics is the core of the concept of function. “One purpose of the function is to represent how things change” (Tall, 1996, p. 289). Function is the study of how the change of certain variables influences the change of the other variable. Therefore, to understand calculus necessitates the mastery of the concept of function. Calculus in the first year of university deals with the study of the concepts of limits, derivatives, and integrals. Monk (1994) states, “the concept that the subject is built out of, the one that lies behind such notions as limit, derivative, and integral is that of function” (p. 21). It is obvious that an understanding of functions plays a determinant role in students’ success in calculus.

The centrality of the concept of function in many course units in mathematics undergraduate program is unquestioned, and the students’ mastery of the concept of function is expected to be one of the learning objectives of studying mathematics at undergraduate level (Bagley et al., 2015; National Research Council, 1989). Not only does it do as in calculus, the concept of function also plays an important role as a prerequisite concept in more advanced course units in the undergraduate program for mathematics education or pure mathematics, such as real analysis, abstract algebra, discrete mathematics, and numerical methods. The demand for calculus courses has always been increasing. Consequently, the mastery of calculus concepts or prerequisite concepts, among them is the concept of function, gains relatively increasing importance.

In introductory calculus, the concept of function is often discussed as part of the preliminary chapter. This chapter is generally considered as pre-calculus (Larson & Edwards, 2012; Varberg et al., 2007). The chapter covers other concepts such as real numbers, absolute values, equations and inequalities, the rectangular coordinate system, and the graphs of simple equations. The topic of functions itself discusses their definition, representations, operations, and their properties. The concept of function is material in the discussion in the topics of limits and continuity. An understanding of the behavior and properties of functions along with the properties of real numbers determines the strong grasp of the concepts of limit and continuity.

This structure of the preliminary book chapter is the one used in the university where this present research took place. However, in other universities, pre-calculus sometimes is separated from introductory calculus and treated as a different course unit. In such a unit, there are various basic topics included besides functions. Equations and inequalities are set as a single chapter, and so are the rectangular coordinate system and graphing equation. Some additional topics are matrices, probability, basic analytic geometry, and basic trigonometry.

The concept of function presents a cognitive challenge for the students. In particular, switching between global and local perceptions of function is problematic for the students (Artigue, 2009; Vandebrouck, 2011). The problem occurs because the tradition of secondary school mathematics learning mostly focuses on only pointwise, where functions are viewed as a relationship between two sets of numbers, and a global viewpoint, where representations are tables of variation. On the other hand, the focus of university mathematics learning is on a formal axiomatic world requiring the adoption of a local perspective. The challenge of transition from global to local perspectives is also highlighted by Vandebrouck (2011) and Artigue (2009). They mentioned the need of evolution of thinking of functions when students move from secondary school to university.

This thinking evolution involves a complex conceptualization of the notion of function in terms of duality property of a function as both an object and a process (Bressoud, Ghedamsi, Martinez-Luaces, & Torner, 2016). The conceptualization necessitates students to develop variational thinking in an early stage (Dooley, 2009; Warren, Miller, & Cooper, 2013). The dual nature of the concept of function has been discussed by some researchers, for instance, Breidenbach, Dubinsky, Hawks, and Nichols (1992), Sfard (1991), and Zandieh (2000). Sfard (1991) states that the understanding of many concepts in mathematics could be either operational or structural. Operational understanding is dealing with functions as processes, algorithms, and actions, while structural understanding is treating functions as static, integrative objects. The two aspects of this conceptual understanding are intertwined reflexively, which means that a process necessitates an object to operate on, but the process itself could be eventually mastered as an object on which other processes will operate (Bagley et al., 2015).

2.5.3 Problems in Learning the Concept of Function

The concept of function exemplifies the simplicity in mathematics, for its definition is very simple: There are two sets and each member of the first set is corresponded to exactly one member of the second (Akkoc & Tall, 2002). Such a simple concept actually provides access to the vast complexity of the world of interrelated mathematical notions. For some students, the combination of the simplicity and the complexity results in a complicated array of personal meanings that hinder their comprehension of the concept of function (Akkoc & Tall, 2002). There are several aspects in learning the concept of function where the students experience problems and encounter difficulties.

2.5.3.1 Representations of Functions

The first problem experienced by the students concerns the representations of a function. The difficulty in understanding the concept of function is partly because of the variety of representations assigned to the concept (Christou et al., 2005). Being exposed mainly to recognizable formulae and graphical representations apparently make some students believe that such functions are the only ones existing (Jensen, 2009). The limited view of representations may prevent students to flexibly move from one representation to another. Meanwhile, meanings of mathematical concepts are constructed from properties that are distributed in a number of representations (Thomas, 2008a). A meaningful understanding of the concept can be developed through the representation linkage (Noss & Hoyles, 1996). Within the linkage, students have to have representational versatility which enables them to translate between representations as well as to interact with each representation either procedurally or conceptually (Thomas, 2008b). Christou et al. (2005) found that students' comprehension of graphical representations of functions requires the fluency of symbolic or procedural manipulations of functions. However, some students still experience mental restrictions on functions, where their view of a graphical representation of a function is distracted by the continuity feature of the function (Jensen, 2009). For these students, if a function is discontinuous at a point, then they think it is undefined at that point.

2.5.3.2 Operations on Functions

With regard to operation on functions, Kimani (2008) reported that students encountered problems with the composition of functions in terms of how they obtain and treat the composite functions. Regarding composition, students could compose two functions procedurally, but they do not know the conditions the functions must have with respect to the relation between the domain and the range of the functions. Therefore, the composite functions produced sometimes look weird. For instance, if $f(x) = -1 - x^2$ and $g(x) = \sqrt{x}$ with their natural domain respectively, then they will make $(g \circ f)(x) = g(f(x)) = \sqrt{-1-x^2}$ as the first composite function, and $(f \circ g)(x) = f(g(x)) = -1 - x$ as the second. Here, they still consider $(g \circ f)(x) = \sqrt{-1-x^2}$ as one composition, although this function is undefined for any real number. This example could indicate that the students are still in the action stage of the conception of function and cannot go beyond it (Dubinsky & Harel, 1992). On this stage, the students operate with functions by simply performing calculations on specific numbers. In the context of composing functions, the calculation is carried out on specific formula of functions expressed algebraically. The example also shows that the students might not think of the composite functions as a function itself (Even, 1990, 1993), which has to be defined in its domain.

The students considering $g \circ f$ above as a legitimate composition may neither care nor know about the conditions for the existence of composite functions. Kimani (2008) found that students have an operational understanding of function composition, which is indicated by their tendency to reason that a composite function is a function resulting from evaluating the second function using the output values of the first function. This model of reasoning is sometimes still acceptable if students are working on the point-to-point composition. It turns out to be problematic when they work on composing functions with given formulae if they ignore the condition of the existence of composite function, that is, the range of the first function and the domain of the second function are not mutually exclusive. Kimani (2008) reported that because composing functions are executed procedurally by composing equations, the students considered the composite function as of no relationship to the original functions.

2.5.3.3 Formal Verification of Legitimate Functions

The formal definition of functions is very central in calculus. Most students are able to determine whether a mathematical expression, a relation, is a function or not intuitively. Moreover, they understand the definition of functions which, for some authors of calculus textbooks, is considered as the formal definition, for instances, "a function f from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$" (Thomas et al., 2018, p. 1), or a slightly more explanative one, "a function f is a rule of correspondence that associates with each object x in one set, called the domain, a single value $f(x)$ from a second set. The set of all values so obtained is called the range of the function" (Varberg et al., 2007, p. 29). Students use these definitions inductively to justify a relation to be a function by taking few points, $x \in D$, and show that they are assigned to only one element in another set *Y*. Given the domain of the relation is generally an infinite subset of a real set, few points are very far from covering the whole elements of the domain. This inductive approach is neither valid nor accepted. The aforementioned formal definitions actually could not be directly used to justify relations as legitimate functions. Students do not use the definition when they are to decide whether an object is an example of the concept (Vinner & Dreyfus, 1989). The definition is inoperative and students just rely on characteristics of familiar examples (Tall & Bakar, 1992). Students tend to intuitively distinguish examples from non-examples of function.

As long as students do not reconstruct the definition into an operational version, they will experience difficulty in employing the definition to justify a relation as mathematically valid function in a deductively accepted manner. The situation is different when they are to justify that a relation is not a function, because they just need to provide counterexamples which means that one case will be sufficient to show the falsehood of a statement. To make the formal definition of function a working definition requires a reconstruction of the definition. Although the reconstruction of mathematical knowledge is often difficult for students, it is certainly needed in developing a cognitive means for effectively handling the concepts of calculus which are subtly different from naïve experience (Tall, 2002b).

The formal definition of function considered as operational is one detailing the unique properties of a function and can be directly employed in proving the functionality of relations. If a relation is given, to show that it is a function means to prove that each property is satisfied. The following definition is quoted from Stewart and Tall (2015) which I consider as an operational formal definition of function.

> Let A and B be sets. A *function* $f : A \to B$ is subset f of $A \times B$ such that:
> (F1) For each $x \in A$ there exists $y \in B$ such that $(x, y) \in f$.
> (F2) Such an element y is unique: in other words, if $x \in A$ and $y, z \in B$ are such that $(x, y) \in f$ and $(x, z) \in f$, it follows that $y = z$ (p. 97).

This definition is actually a detailed expression of Bourbaki's definition. Similar definition construction can be found in some textbooks introducing the mathematical analysis (see, e.g., Bartle & Sherbert, 2011; Daepp & Gorkin, 2011; Manfrino, Ortega, & Delgado, 2015). F1 is a property expressing the existence of partners for each element of the domain of a relation *f*. F2 is another property stating the uniqueness of the partner of elements of the domain. Proving a relation as a function can be done by showing that both F1 and F2 are satisfied by the relation. Such a definition is a working definition, namely, a meaningful definition that is strategic for use in proving mathematical statements (Bills & Tall, 1998).

In examining a relation, the students will first show whether each element of *A* has an image in *B*, and they will then verify whether the image existing in *B* is unique or not. The first check is to examine whether all elements of the domain are paired off with elements of the codomain. This is a special property that has to be satisfied by a relation if it is a legitimate function. I call this characteristic the exhaustive domain property. The second check is to verify whether the partner of the elements of the domain, if any, is no more than one. I call this characteristic the unique image property. The definition formulated by Stewart and Tall (2015) above is claimed to be operational in nature as it provides clear statements which explain what 'a unique image' means or how to show this uniqueness.

2.5.3.4 Special Properties of Functions

A function is a relation that satisfies special properties. Further, certain properties could be assigned to a function that make it a special function. A function must satisfy the exhaustive domain property, but it is not necessary for the codomain to be exhaustive. A function whose the codomain is exhaustive is called a surjective (onto) function (Stewart & Tall, 2015). Similarly, a function must fulfill the unique image property, namely, the image or the partner of the elements of the domain must be unique, but the partner of the elements of the codomain need not be unique. A function whose elements of codomain have a unique partner in the domain is a special function which is called an injective (one-to-one) function (Stewart & Tall, 2015).

Verifying the surjective and injective functions is also challenging for students. They tend to employ the inductive approach by testing some points either in the domain or the codomain. They determine the properties after examining only some points or cases from either domain or codomain of the functions. Similar to the problem of the deductive verification of a functional relation, verifying a function as an injective or surjective function is problematic when the students did not know how to use the formal definition of injective or surjective function to develop a deductive argument of proof.

Students' inability to classify certain functions as legitimate and to justify the specific properties of functions may result in misinterpreting or overgeneralizing the concept of function (Jensen, 2009). Such students are precluded from having a correct conceptual understanding. They believe that a function must involve a process of manipulating its inputs, must have numbers as both its inputs and outputs, must be defined explicitly by an algebraic formulation, or must have only one rule instead of being defined in a piecewise style (Vinner & Dreyfus, 1989). In terms of properties of function, some students cannot differentiate between the uniqueness of the image for each pre-image as a property of a function and the uniqueness of pre-image for each image as a property of an injective function (Dubinsky & Harel, 1992; Vinner & Dreyfus, 1989). This overgeneralization leads the students to a misconception that all functions are one-to-one in nature (Jensen, 2009). Oehrtman, Carlson, and Thompson (2008, p. 27) observe that "students continue to emerge from high school and freshman college courses with a weak understanding" of the concept of function.

As mentioned previously in the section of the rationale for this research in Chapter 1, the problem encountered by the students in verifying an object as an example of a concept based upon the formal definition is rooted in the form of the definition itself. The students do not know how to use the definition to devise a deductive argument proving that the object was a legitimate example of the concept. To be more particular, the students cannot provide a deductive argument showing that a relation is a legitimate function using the formal definition of function because they do not know how to use the formal definition. They cannot interpret the essence of the formal definition of function to find out that the formal definition itself is based on two individual properties, namely, the exhaustive domain property and the unique image property. Further, the isolation of the two special properties will not be usable if they are not expressed in a working or operational version of the expression.

An operational formal definition of function could be formulated in two different ways. It could be done by analyzing the formal definition. In general, the students have known this formal definition of function. Isolated properties resulting from the analysis were then formally defined in an operational version of expression. The other way is by directly identifying the properties, not from the formal definition, but from the representative examples of functions, and then formulating the formal definition of the properties in an operational version of the expression. This present research chose to focus on the latter way. So far, it appears that there has been no particular research conducted on constructing an operational formal definition of function. The plethora of extant research on the understanding of function has not particularly dealt with this focus. Previous studies, such as those by Bardini et al. (2014), Christou et al. (2005), and Tall and Bakar (1992), investigating the students' understanding of the definition of function only focused on justifying a range of various relations with different representations as to whether they were valid function, without a particular emphasis on the use of definition to develop a deductive justification. On the other hand, constructing definition of a mathematics concept has been the focus of various research studies in undergraduate mathematics education, such as those carried out by Martín-Molina et al. (2018), Oehrtman et al. (2014), and Ouvrier-Buffet (2011), but they generally focused on formal definitions, without specifically emphasizing on the construction of an operational formal definition.

2.5.4 Research on the Concept of Function

There is a vast body of literature examining the nature of students' understanding of the concept of function. Within the theoretical framework of the Three Worlds of Mathematics (Tall, 2004a, 2004b, 2008), the conceptual understanding of 236 first year university students was investigated by Christou et al. (2005) to understand how they operated with relations and functions expressed in different worlds of the embodied, proceptual, and formal. The students were currently undertaking their calculus course. One of the questions asked the students to give the definition of function. This question was followed by two other questions requiring the students to reflect on their abstract definition in order to identify the properties of given functions. That study revealed that the students from the top class could solve problems set in the highest level of the formal world. They could provide the definition of function and could explain why a certain relation given in a word problem was categorized as a function. However, students involved in the study by Christou et al. (2005) were not specifically required to provide a formal deductive justification for a function using their definition. As the students could not perform symbolic manipulations or extract graphical information, they might have produced only an ordinary, intuitive explanation stating that the given relation is a function because each element of the first set will have only one partner in the second set. From the perspective of the world of mathematical thinking, the study revealed that in the context of understanding the concept of function, the students' progress followed the sequence of proceptual, embodied, and formal. The results of the analysis indicated that the students only could deal with the embodied relations after successfully dealing with the symbolically represented relations. The theory of the Three World of Mathematics, which is reviewed in section 2.7.2 (p. 78), was the framework of this present study to analyze the mathematical representations used by the students.

Investigating university students' understanding of examples and non-examples of function, Tall and Bakar (1992) asked 109 respondents to identify which of a number of graphically and symbolically expressed relations were representing y as a function of x. In general, it was revealed that the students still experienced problems in identifying functions expressed in an algebraic, symbolic representation. Even for the constant function, 70% of the students incorrectly justified it as non-function when it was in symbolic expression, and 55% of the students correctly justified it a function in its graphical representation. With the symbolically represented relation, although 38 of the 109 respondents mentioned in their answer that there must be one y for each x, or that the function has to be 'many-one' or some other equivalent statements, those students did not go further to verify deductively that the given relations really fulfilled those statements considered as their definition of function. There was not reported evidence showing that any student could provide a deductive, definition-based verification of a relation claimed as a valid function. Another group of students aged 16–17 years also participated in the study conducted by Tall and Bakar (1992). These students had learned the concept of function in their previous year and had studied function as part of calculus courses, but only a little emphasis was put on the notion of domain and range. Surprisingly, no satisfactory answers were given by all the students when they were asked to provide a definition of function.

Similar findings were reported by Bardini et al. (2014). Despite the majority of the first-year university students recognized the family of functions, many still could neither provide an appropriate definition of function nor deductively verify given relations as a legitimate function. The fact that the students were categorized as in the top 80% of achievement in their final year of senior high school, Bardini et al. (2014) argued that they had a misconception about, and limited understanding of, the concept of function. The students were asked to respond to the question: *Explain, in plain English, what a function is*. The question was not aimed particularly at seeking a formal definition of function. The results showed that although 63% of those who responded to the question could give valid descriptors of a function, most of their explanations were categorized as incomplete or incorrect. Here, the descriptors of functions were the same as those called concept images of functions (Tall & Vinner, 1981). Only a few students could provide a correct explanation. Further, the authors mentioned that students face problems in determining a graph or a rule expressing relations represented a valid function. They found many participating students hold naïve ignorance and misconceptions of the concept of function.

Based upon the research findings, Bardini et al. (2014) concluded that the school mathematics, indeed, emphasized mathematical skills instead of focusing on the deep, conceptual comprehension. Their suggestion strengthened the importance of connecting the concepts being studied to the previously constructed concepts and reinforced the previous concepts within new contexts. They raised the importance of a conceptual understanding of function developed upon the identification of aspects of the definition of function which were critical in examining the functionality of given relations. This notion was based upon their findings showing that the students could not appropriately use their personal definition of function to justify relations representing a valid function. In other words, the students' definition of function proved to be inoperative. And, the identification of the aspects of the definition aimed at constructing a definition which was operative in nature was in line with the focus investigated in this present study.

Many studies on students' conceptual understanding of the concept of function covered the aspect of representations as an additional, complementary focus of investigation. This was because different representations of a function could confuse a student. The ability to justify a particular relation given in various representation was a crucial indicator of the students' comprehension of the concept of function. The findings of the study conducted by Tall and Bakar (1992) previously mentioned was an example of the essential role played by the representations regarding the understanding of the concept of function. The constant function expressed in a graphical representation was correctly justified as a function, but it was justified as not a valid function in its algebraic representation. Again, this phenomenon revealed the problem experienced by students resulting from their problematic concept image and concept definition of function.

2.6 Abstraction in Mathematics Learning

2.6.1 General Overview

Abstraction has been studied since the times of Aristotle and Plato. Plato's theory of forms is the basis of the ancient Greek philosophy of abstraction. Plato theorized an abstraction as discovering the essence of each material thing and the process of knowing or knowledge acquisition took place upon the basis of perception levels (Schwarz et al., 2009). The influence of Plato's notion is evident in Piaget's genetic epistemology which forms the basis for the theory of psychological development (von

Glasersfeld, 1991). In essence, genetic epistemology discusses two things, namely, the contents of knowledge, and the way of the development of knowledge. These two things, the process and the product of knowledge construction, are the central concerns in the notion of abstraction as a cognitive process of constructing knowledge in mathematics learning.

To the community of mathematicians, an abstraction is a product. It is a mathematical object incorporating a structure that is common to various examples appearing in many contexts (Dreyfus, 2014). The structure covers various basic elements along with the interrelations among them. The object is obtained from the process of ignoring the contextual differences, and focusing, instead, on the common core structure of the mathematical things within various contexts. The similar core features from different-looking objects are recognized, pulled out from the contexts, and tied together (Hiebert & Lefevrfe, 2009). An example of an abstraction is vector space, a mathematical object obtained from the core structure of Euclidean 3-space, the complex plane, the solution set to a linear equation system with real coefficients, and the state space of a system of mechanical quantum. Vectors as the elements of the vector space are context-dependent. They could be matrices, functions, ordered pairs of real numbers, complex numbers, or 3-dimensional points. However, by ignoring the contextual variety, various algebraic operations can be performed on the vectors. Therefore, in mathematical communities, the notion of decontextualization is a crucial feature of abstraction (Dreyfus, 2014; Mitchelmore & White, 2007).

To the community of mathematics educators, abstraction is a process rather than a product, and the focus of interest is more on the processes through which the learners develop their understanding of the structure of a mathematical object (Dreyfus, 2014). Abstraction is the process through which abstract concepts are constructed (White & Mitchelmore, 1996). The development of mathematical theories and the emergence of new mathematical mental structures take place through abstraction (Epstein, 2013; Hazzan & Zazkis, 2005). Essentially, knowledge construction in mathematics education is centered in abstraction. The processes of concept construction attract mathematics educators because learners attempt to achieve a comprehension of the concept structure through these processes (Dreyfus, 2014). Here, the structure means that the components as well as the interconnection between them, and the term ‘concept’ could also be a strategy or a procedure (Dreyfus, 2014). The investigation of abstraction by mathematics educators also focuses on circumstances and activities that facilitate the processes. Therefore, in the community of mathematics educators, context is a crucial variable in the study of abstraction (Dreyfus, 2014).

In general, concepts themselves are an abstraction (Woolfolk & Margetts, 2007). The concepts, as the outcomes of the abstraction, are called constructs by Dreyfus et al. (2015). Although there is no consensus amongst the experts in terms of the unique definition of abstraction (Hazzan & Zazkis, 2005), all concur based on two principles, namely, an abstraction process results in a new mental object, and this object isolates some relevant attributes from others considered irrelevant (Hassan & Mitchelmore, 2006). Where they differ is on the abstraction process itself.

2.6.2 Empirical and Theoretical Abstraction

Within the community of mathematics educators, currently, there are mainly two views dominating the discourse of the abstraction process. The first account is empirical abstraction. To this view, abstraction is the sequence of the processes of identification of similar properties shared by a collection of particular objects, classification of experiences based upon the identified properties, and generalization of the properties to other objects (Mitchelmore & White, 2007). In general, empiricist view of abstraction is characterized by three essential features: (a) abstraction is drawn from commonalities which are recognized across a set of specific examples; (b) abstraction involves the separation of the contexts from the focused examples; and (c) abstraction is ascension from the concrete (perceptual-material entity) to the abstract (mental-conceptual entity) (Ozmantar & Monaghan, 2007).

Piaget (1977/2001) is considered as among the first experts attending to the issue of abstraction in the empirical view. He expounded empirical abstraction concerned with the aspects of object which were noticeable to our senses. A concept was formed by empirical abstraction where the essential qualities of a set of objects were detached from the individual objects (Piaget, 1970); pseudo-empirical abstraction concerned with the action on objects; and reflective abstraction concerned with the relations between the actions on objects. Empirical abstraction is the model that bears on physical entities which are external to individuals (Steffe, 1991). It is constituted by the isolation of similar sensory properties of experiences of an object and preserving them as a combination for recognizing further examples (von Glasersfeld, 1991).

According to Skemp (1987), the similarity identified is not from superficial appearances of the objects, but from the underlying structures which are purposefully focused on. For example, children learn about lines as a concept by recognizing the similarities between roughly linear objects resulting from various actions such as folding paper, pulling rope, and planning surface (Mitchelmore & White, 2007). This abstraction approach apparently has colored most of our teaching practices in mathematics lectures in Indonesia, which frequently follow the natural series of "theorem–proof–application" (Dreyfus, 2002, p. 27). In learning activities, when students constructed their understanding through examining several worked exercises, then they are learning by empirical abstraction (Schwarz et al., 2009).

The notion of context (decontextualization) in abstraction has been the focus of the criticism by the proponents of situated cognition and the dialectical materialists (Ozmantar & Monaghan, 2007). In mathematics instruction, they observed that the concern seems to be on the transfer of "abstract, decontextualized concepts" (Brown, Collins, & Duguid, 1989, p. 32), which is framed by the theory that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). Meanwhile, context is actually a feature of learning processes which are always relative to the learners. Decontextualization, therefore, results in the exclusion of the individual and this can impoverish the constructed knowledge instead of enriching it (van Oers, 1998b, 2001). According to van Oers (1998a), knowledge construction is a continuous process of progressive recontextualization, not decontextualization.

The second account is theoretical abstraction which is distinguished from empirical abstraction with respect to the three characteristics of empirical abstraction mentioned previously. To this view, abstraction is associated with the construction of mathematical knowledge and the vertical reorganization of the knowledge into a new structure (Hershkowitz et al., 2001). The product of abstraction is not *a decontextualized entity*, but *a structured developed one*. In Davydov's (1972/1990) view, this abstraction begins with a first *abstract* form that is undeveloped, vague and often lacking in consistency. It develops dialectically between the concrete and the abstract, and ends with a *concrete* form that is a more elaborated and consistent concept. Here, it is clear that the perception of abstractness and concreteness is different from that in empirical abstraction. The theoretical concepts resulting from an analysis of the connections among objects are adequate not only to identify and classify objects or phenomena, but also to explain the qualities manifesting from the objects (Davydov, 1972/1990).

In learning activities, this abstraction takes place by deeply *analyzing* a problem to identify its critical components and relationships (Mitchelmore & White, 2007), which results in *synthesizing* an elaborated, consistent concept (Dreyfus et al., 2015). For example, someone learns about the asymptotes starting from an understanding that an asymptote is a line that will never be touched but continuously be approached by the graph of a function. Being exposed to new functions of which their asymptotes are crossed by the graphs not only once but many times, the student may reconstruct, refine, and finally reorganize his or her concept of asymptotes by taking the notion of the limit of the function at infinity (Kidron, 2011).

Theoretical abstraction offers an approach worth implementing in calculus learning. Mitchelmore and White (2007) argue that empirical abstraction is more appropriate in the initial stage where students construct basic mathematical concepts, such as number and space, while in higher mathematical activities of symbol manipulation and axiomatic deduction, such as, in calculus and algebra, theoretical abstraction is needed. Therefore, this present study adopted the theoretical abstraction method in order to help students construct a stronger understanding of the concept of function in introductory calculus. In particular, Abstraction in Context (AiC) (Hershkowitz et al., 2001; Schwarz et al., 2009) as an exemplification of theoretical abstraction was implemented as a theoretical and analytical framework.

2.6.3 Abstraction in Context (AiC): The Framework

Abstraction in Context (AiC) was developed by Rina Hershkowitz, Tommy Dreyfus, and Baruch Schwarz (Hershkowitz et al., 2001; Schwarz et al., 2009). It is a theoretical framework for investigating the processes of constructing abstract mathematical knowledge as it occurs within a context including particular mathematical, curricular, and social aspects along with a learning environment (Dreyfus et al., 2015). In practice, the knowledge constructed through abstraction processes is analyzed and described by means of observable epistemic actions. Epistemic actions are mental actions by which the knowledge is constructed (Pontecorvo & Girardet, 1993). In the AiC theory, abstraction is defined as "a process of vertically reorganizing some of the learner's previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner" (Dreyfus et al., 2015, p. 117).

The notion of vertical reorganization in AiC is based upon the theory of vertical mathematization (Freudenthal, 2002c). Through vertical mathematization, mathematical knowledge is reinvented by the learners, where the activity of reinventing mathematics involves a process of reorganizing previously constructed mathematical knowledge. This results in a higher-level abstract concept. Because the students' mathematization sometimes does not coincide with the intended one, so a reorganization is in order, therefore, abstraction uses the term 'vertical reorganization' (Dreyfus et al., 2015). Another basis of AiC is the theory of ascension to the concrete knowledge (Davydov, 1972/1990) which was discussed previously. AiC moves from undeveloped to more developed knowledge. In addition, AiC is developed based upon the activity theory in terms of learning contexts and artifacts (Leont'ev, 1981). According to this theory, context is all the interconnected factors framing the structure and meaning of learners' actions. The knowledge constructed as an outcome of learning activities naturally turns to be an artifact of the following learning activities.

AiC takes place through a three-stage process which consists of the need for a new concept, the emergence of the new concept, and the consolidation of that concept (Dreyfus et al., 2015). The need is provoked by a task that presents vagueness for the students. The tasks are designed to constrain the actions which might be performed by the students and to afford the ones seeming desirable in the process of constructing an understanding. The processes of abstraction lead from initial unrefined abstract entities (previous knowledge constructs) to a novel structure (concept, strategy, or procedure). The new knowledge construct emerging is analyzed in mathematical, social, curricular, and historical contexts, using a model of three epistemic actions identified as relevant to abstraction processes. The actions are *Recognizing*, *Building-with*, and *Constructing*, hence abbreviated as RBC (Dreyfus et al., 2015; Hershkowitz et al., 2001; Schwarz et al., 2009). Recognizing is an action of realizing or knowing a previous mathematical construct or structure as relevant in the encountered mathematical situation, problem, or context. It means that it is not the first time the recognized structure occurs in the learner's mind, and it could be used whenever needed. Building-with is an action of using and combining recognized constructs or concepts in order to reach specific goals (a specific method, a justification of statements, or a solution to problems). This action may take place when learners are solving a problem, explaining a situation, or reflecting on a process, where they are not enriched with new, more complex knowledge, but capitalized on their available structural knowledge. This action has a connotation with an application, that is, it utilizes their current structural knowledge to build-with it the aforementioned goals. Constructing is an action of assembling the available artifacts of knowledge to produce a new mental structure (concepts, methods, or strategies) vertically reorganized within the body of knowledge.

Constructing covers actions of recognizing and building-with, meaning that the constructing process combines all the epistemic actions (Hershkowitz et al., 2001). According to Schwarz et al. (2009), the constructing action links recognizing and building-with actions as its building block to create a unity. Therefore, the actions of recognizing and building-with constitute and are nested in the action of constructing. Meanwhile, the actions of recognizing are nested in the actions of building-with and constructing, and the actions of building-with are nested in those of constructing. Moreover, a constructing action can be nested in a more global action when the former is constitutive of the latter. This mechanism ensures that the epistemic actions do take place in a nested way and not in a chained one. In other words, the constructing does not simply come after recognizing and building-with in a linear pattern but at the same time necessitates recognizing and building-with constructs that are already developed. This way of functioning is called "dynamic nesting of the epistemic actions" (Hershkowitz et al., 2001, p. 218).

Of particular importance, constructing as the highest level of the epistemic actions does not imply that students have constructed the new concepts once and forever, because they might have not fully comprehended their new concepts which are often frail and context-dependent (Dreyfus et al., 2015). Consolidation is the next process through which the students develop their awareness of the concepts, confidence, and flexibility in using them, as they become more immediate and context-independent (Dreyfus & Tsamir, 2004). Consequently, the concepts are consolidated through further abstraction, especially in the stage of building-with. The three actions and consolidation pertaining to the activity of knowing in which the learners are engaged are observable by others who might question, share, negotiate on, or even further construct on what is exposed in a learning context.

The AiC theory emphasizing learners' mental activities has "the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice" (Dreyfus & Gray, 2002, p. 113). It is a theoretical lens that enables an analysis of the dynamic of the construction of understanding at a micro-analytic level (Dreyfus et al., 2015). The potentials were the reasons for choosing AiC to frame this present study. Another reason was that AiC reflected a renewed perspective of mathematical abstraction compared to the classical, empirical perspective.

As a theoretical framework, AiC puts an important emphasis on the aspect of design that will constrain the types of epistemic actions which will be conducted by the learners (Dreyfus et al., 2015). The design should allow the process of continuous transformation of understanding. It consists of a series of activities offering the learners opportunities to learn well defined mathematical ideas through vertical reorganization of their knowledge. In this present study, calculus tasks were specifically designed to encourage the students to perform abstraction in developing their understanding of the intended mathematical structures. The development of the calculus tasks was based upon the structure of knowledge elements. The knowledge elements are the concepts or knowledge that the lecturers or the researchers expect the students to construct (Dreyfus et al., 2015). This structure is a trajectory of students' learning resulting resulted from an *a priori* analysis (Dreyfus & Kidron, 2014; Ron, Dreyfus, & Hershkowitz, 2010). In this analysis, the previous knowledge of the learners is assumed. Then, the knowledge constructs required to develop the intended understanding are determined. It also considered the constructs which may be helpful but unnecessary in developing the intended understanding. The operational definition for each construct is formulated which clarifies under what circumstances the students' epistemic actions for each construct will be claimed to correspond to the intended ones.

The theory of Abstraction in Context has been used in numerous studies since its first development (see, e.g., Bikner-Ahsbahs, 2004; Kidron & Dreyfus, 2004; Ozmantar & Monaghan, 2007; Schwarz, Dreyfus, Hadas, & Hershkowitz, 2004; Stehlíková, 2003; Tsamir & Dreyfus, 2002). In particularly, AiC has been employed in various studies in undergraduate mathematics education. Investigating the construction of first-year university students' understanding of limits by the definition of horizontal asymptotes, Kidron (2011) used AiC as a framework, where she clarified the importance of cognitive conflicts as a source of the need to do constructing, the most crucial epistemic action of AiC. This conflict was triggered by the difference between the students' concept image and the formal definition of the concept. AiC was used to explain the processes through which a student reconsidered his/her concept images and developed her conceptual understanding of the definition of the horizontal asymptotes. Kidron (2008) has also used the Abstraction in Context framework in her study of first-year college students' understanding of the concepts in differential equations, where she combined AiC with the Three Worlds of Mathematics, another framework of this present study.

Kouropatov and Dreyfus (2014) have also employed AiC in studying theoretical abstraction processes performed by the students in learning to comprehend the Fundamental Theorem of Calculus. These researchers exemplified the role of the design and the *a priori* analysis in implementing AiC as both theoretical and methodological framework. That study, which involved students in their period of transition to university learning, has very much inspired the presentation of the results of this present study. In the Indonesian university context, so far, the research of Nurhasanah, Kusumah, Sabandar, and Suryadi (2017) is the only study employing the AiC theory. They investigated the construction of students' understanding of the non-conventional mathematical concept of Parallel Coordinates. Another is a study of Budiarto, Rahaju, and Hartono (2017), however, it investigated the abstraction profile of secondary school students in constructing their understanding of geometry concepts. The investigation of abstraction in the present study was conducted alongside the investigation of the use of mathematical representations. The notion of mathematical representations is discussed in the next section.

2.7 Representations in Mathematics Learning

2.7.1 Mathematical Representations

Mathematical knowledge is a product of mathematical thinking which takes place in the process of abstraction (Damerow, 1996). According to Wartofsky (1979), human knowledge, including mathematical knowledge, is achieved by means of representations. The process of mathematical knowledge construction uses mathematical representations which are claimed as the powerful apparatus of mathematical abstraction (Damerow, 1996). The higher the level of mathematical abstraction is, the more complex the representation is involved (Kamii, Kirkland, & Lewis, 2001).

Generally, a representation is something (the representing) that stands for or models of something else (the represented) (Duval, 2006; Goldin & Shteingold, 2001; Palmer, 1977). This is a dyadic structure of representations where a signifier has a direct and simple relationship with a represented object (Duval, 1995). A complete specification of a representation, according to Palmer (1977), requires the clarification of five components: the thing that represents; the thing that is represented; the aspects of the represented thing which are modeled; the aspects of the representing thing which involve in the modeling; and the correspondences between the representing and represented things. With these five features, a representation is perceived as a representational system. Goldin (2008) proposes the notion of a representation as a configuration which means that in a representational system, the elementary components could be configured or recombined to produce another representation form. The representing configuration could "act in place of, be interpreted as, connect to, correspond to, denote, depict, embody, encode, evoke, label, link with, mean, produce, refer to, resemble, serve as a metaphor for, signify, stand for, substitute for, suggest, or symbolize" the represented configuration (Goldin, 2008, p. 179). A representation might perform one or more of these actions.

Specifically, mathematical representations are visible or tangible productions that encode, express, or symbolize mathematical objects, notions, or relationships (Goldin, 2014). The products could be in the forms of diagrams, charts, number lines, graphs, physical models, mathematical symbolic expressions, models, formulae, equations, or pictures. Mathematical representations not only refer to the products but also to the process of inventing or making the products; in other words, representations refer to "the act of capturing a mathematical concept or relationships in some form or to the form itself" (National Council of Teachers of Mathematics, 2000, p. 67). Representations are justified as mathematically conventional if they are based upon standards and conventions prevailing within the community of mathematics (Goldin, 2014).

Mathematical representations are used in thinking about and communicating mathematics in all learning contexts (Davis & Maher, 1990; Hiebert & Carpenter, 1992). Different representations may describe or convey different properties of a represented mathematical idea. For example, a function can be expressed as an algebraic equation, a curve on a Cartesian coordinate plane, a set of ordered pairs, or a two-column table of values. Traditionally, each representation represents a different perspective of the concept of function (Goldin, 2003; Sierpinska, 1992). Considered as the language of mathematics, a different mathematical representation conveys different information (Smart, 2013). Representations play an important role in the study of mathematics. All kinds of mathematical enterprises, including learning, teaching, research, and development, would be impossible without mathematical representations (Hiebert & Carpenter, 1992; Vergnaud, 1987).

Mathematical representations are categorized based upon the characteristics and functions of the representing configuration. Bruner (1968) classifies representations on the basis of which they are formed. He proposes enactive representations that are based upon actions, iconic representations that are formed from images, and symbolic representations that are based upon languages. Goldin and Kaput (1996) categorize the representations into visual/spatial, auditory/rhythmic, and tactile/kinaesthetic representations. Visual representations are quasi-pictorial representations; auditory representations are shown by learners when they recite multiplication tables in rhythm; and kinesthetic representations refer to physical actions for expressing mathematical ideas. Goldin and Kaput (1996) also mention concrete or abstract representations which are based upon how the representations are embodied, and static or dynamic representations which depend on the properties of media used in the representations.

Of particular interest is the classification distinguishing external representations from internal ones. External representations refer to those which are external to a learner producing them, physically sensed, can be worked with and be directly accessed by others for discussion, interpretation, and manipulation (Goldin, 2014; Smart, 2013). External representations are developed based upon shared assumptions; their structures are based upon underlying conventions; and their correct use is determined by the conformity with conventional standards (Goldin, 2008). Examples range from the logical system of axioms, postulates, definitions, and theorems, to notational/symbolic systems for arithmetic, algebra, geometry, calculus, and so forth. Goldin (2014) also includes spoken language and interjection, gestures, facial expressions, and movements as external representations as they may carry mathematical meanings. According to Mesquita (1998), in some mathematical problems, the external representations play a descriptive role, namely, they only show the relationship and properties involved in the cases, without transformation leading to solutions. In other problems, they play a heuristic role, namely, they function to support intuition and suggest solution procedures.

On the other hand, internal representations refer to those existing in mind when a learner is thinking of external mathematical reality or interacting with external representations, which cannot be directly observed, and therefore can only be inferred from what the learners perform or are able to perform within various contexts (Dreyfus, 2002; Goldin, 2014; Smart, 2013). Internal representations may be different from individual to individual. Examples are personal mathematical symbolization, visual and spatial imagery, and problem solving strategies. The distinction between external and internal representations is emphasized because the communication of mathematical ideas is only possible with external representations (Dreyfus, 2002). External representations can be constructed individually or socially (Goldin & Shteingold, 2001). They are the only representations that are visually observable and analyzable through research (Smart, 2013). Based upon the description of representations, in this present study, mathematical representations are defined as systems, configurations, designs, or arrangements that express mathematical entities and describe those entities into thinkable and communicable forms.

Scientists, including mathematicians, employ different representations to express every aspect of their work to develop a research design, describe the findings, and explain their ideas in general (Kozma, 2003). As the learning of mathematics, to some extent, is a kind of training students to work as professional mathematicians, therefore the notion of variety is also addressed whenever the mathematical representation use is discussed within the context of teaching and learning mathematics. Certain mathematical entities can be best expressed with a particular representation, and multiple representations can facilitate the connection between aspects of the mathematical idea, the development of a more holistic, deeper comprehension, the use of a rich set of operations, and the more effective communication of the idea (Ainsworth, Prain, & Tytler, 2011; van Someren, Boshuizen, de Jong, & Reimann, 1998; Yerushalmy & Schwartz, 1993). Different representations can be linked for the purpose of teaching and learning mathematics (Goldin, 2014). The link between the forms establishes a basis for deep, meaningful learning and understanding as well as fluent working with mathematical entities (Wood, Joyce, Petocz, & Rodd, 2007). Rather than being restricted by the form of representations, a free move from one representation to another enables an understanding and expression of the substantial features of ideas (Kozma, 2003; Savelsberg, de Jong, & Ferguson-Hessler, 1998). The ability to flexibly coordinate multiple representations is a salient indication of expert level of understanding of mathematical ideas (de Jong et al., 1998).

Connections and translations among representations are an important feature of mathematical representations. According to Janvier (1987a), the connections and translations are psychological processes taking place among representations. Pimm (1995) argues that the two processes are not only a one-to-one relation but also one-to-many, many-to-one, and many-to-many. These processes involve the manipulation of mathematical entities either within or between different representations (Thomas, 2008b). The connections between representations are two ways in nature and the translations between representations take place back and forth (Goldin & Shteingold, 2001). The fluent translations between representations and procedural and conceptual interactions with representations constitute representational versatility (Thomas, 2008b). A representation system is not a static thing, but a dynamic one (Vergnaud, 1998). It develops along with the cognitive development of an individual learner. Representations of mathematical concepts are made at different levels (Juter, 2006), and the notion of representation translations also covers the process of moving between representations of different levels.

The Three Worlds of Mathematics (Tall, 2004a, 2004b, 2006, 2008, 2013) is really a framework for the development of mathematical thinking. In this present study, this theoretical framework was used in analyzing the use of representations. The theory positions representations as the objects of study, presenting three distinct domains of how students represent their understanding of mathematical concepts. In essence, this theory reflects the three fundamentally different worlds where the operations on mathematical objects take place. This theory is discussed in the next section.

2.7.2 The Three Worlds of Mathematics: The Framework

2.7.2.1 The Theory of Three Worlds of Mathematics

The theory of the Three Worlds of Mathematics originated from a reflective observation of the ways learners operate on mathematical objects. Gray and Tall (2001) speculated that mathematical objects, including the corresponding concepts, are cognitively constructed and developed in particular ways and languages. Taking the concept of vector as an example, the conception starts from a geometric object represented with an arrow embodying physical concepts of force or velocity, to a symbolic object represented with operations on matrices, and to a formal object as a deduced element of an axiomatic vector space (Tall, 2004a). These led to the awareness of the existence of not only three different types of mathematical objects: geometric, symbolic, and axiomatic, but also the *three distinct worlds* of cognitive development of mathematical thinking: embodied-conceptual, symbolic-proceptual, and formal-axiomatic (Watson, Spyrou, & Tall, 2003). For the sake of simplicity, the three worlds will be respectively referred to as embodied, symbolic, and formal worlds. The theory emphasizes the construction of mathematical object representations and is inspired by several theories on concept development, such as abstraction processes of Piaget (1977/2001), representations of Bruner (1968), and the development of geometric objects of van Hiele (1986). The term 'world' expresses not a single representation or group of representations, but the development of particular ways of mathematical thinking of individuals that sophisticatedly grow as they construct new conceptions (Tall & Mejia-Ramos, 2010).

The development of a learner's thought builds on "met-before" (current mental capacities based upon prior experiences) and "set-before" (inborn mental structures maturing as we make connections in our thinking) (Tall, 2013, p. 133). There are three fundamental mental structures involved in the construction of the three interconnected worlds of mathematics, namely, recognition of patterns, similarities, and differences, repetition of series of actions to achieve automaticity, and language for describing and refining the ways of thinking (Tall, 2008, 2013). There is no one-to-one correspondence between the three aspects and the three emerging worlds of thinking. Tall (2008) further explains that recognition of categorized geometrical figures and shapes supports the thinking of geometry and graphs in the embodied world, and repetition of actions symbolized as thinkable concepts result in arithmetic and algebraic thinking in the symbolic world. These two processes further develop through mathematical language by which the mathematical ideas along with their relationships are described, defined, and deduced to reach set-theoretic, deductive thinking in the formal world. The detailed description of the three worlds is as follows.

The *embodied-conceptual* world is "a world of (conceptual) *embodiment* building on human perceptions and actions developing mental images verbalized in increasingly sophisticated ways to become perfect mental entities in our imagination" (Tall, 2013, p. 133). This world results from a combination of enactive and iconic representations theorized by Bruner (1968). The embodied world covers the enactive and iconic representations resulting from actions on visual or image representations of mathematical objects which are observed or sensed in the real world or imagined in the mind (Tall, 2004a). The enactive representations express actions such as putting together or breaking apart, while the iconic representations signify visual or image organizations. The cognitive development within the embodied world starts through the understanding of actions and perceptions. As individuals reflect on their sensory experiences, their conceptions develop and become more sophisticated. Eventually, the ideal mathematical concepts, which cannot be seen in the real world, could be constructed, such as a perfect geometrical figure. Students are considered as working within the embodied world when they are able to visually observe or work on mathematical ideas correctly using observable mathematical facts (Smart, 2013).

The symbolic-proceptual world is "a world of (operational) *symbolism* developing from embodied human actions into symbolic procedures of calculation and manipulation that may be compressed into procepts to enable flexible operational thinking" (Tall, 2013, p. 133). The symbolic world is constituted by mathematical symbols which mathematical processes and concepts act upon. A procept is an amalgamation of a process, a concept which is produced by the process, and the symbol that evokes both of them (Gray & Tall, 1994). For example, a function represented symbolically as $f(x) = 2x + 1$ expresses the process of calculating the value of the function *f* for a specific *x* value, and signifies the complete concept of the function for a general *x* value. The mathematical objects in the symbolic world are developed from the flexibility between mathematical processes of doing mathematics and concepts for thinking mathematics. The cognitive development in this world begins with the symbols used for manipulation and computation, such as numerals and arithmetic operators, where the learners focus on the properties of and the relationship between symbols, not on the physical appearance of them (Tall, 2008). The justification of students' thinking in the symbolic world is based upon their performance in symbolic manipulation and computation (Smart, 2013).

The formal-axiomatic world is "a world of (axiomatic) *formalism* building formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof" (Tall, 2013, p. 133). The formal world is constructed from the properties of mathematical objects and the related concepts used to develop mathematical structures (Tall, 2008). Here, the cognitive activities work on the properties of embodied and symbolic objects along with the logical deductions aiming at stipulating axioms, formulating formal definitions, and deducing theorems. According to Tall (2002a), the cognitive development of the formal world starts with generalizing considered as drawing conclusions from specific instances. Further, definitions are formulated from objects with their generalized properties, and some of the axioms are also formed. Furthermore, theorems as logical-deductively proven properties are derived from established definitions and axioms. All processes related to axioms, definitions, and theorems are to develop the structure of mathematical systems (Tall, 2004a, 2004b). The justification of students' thinking in the formal world is based upon logical proofs (Smart, 2013). Here, the truths in the embodied and symbolic worlds which are based upon observations, manipulations, and computations are no longer considered valid. Some of them transform into the axioms as self-evident truths which are later applied in proving theorems (Tall, 2002a).

For example, in learning the concept of function, examining graphs or tables as to whether they are functions or not is considered as in the embodied world, while manipulating an equation to identify whether it can be justified as a function is considered as in the symbolic world. When students develop a specific function formulated from specific sets with particular properties of a relation, they are working in the formal world. Although the overall objective of mathematical development is to achieve the formal level, the strict hierarchy does not characterize the theory (Smart, 2013). The emphasis in this theory is on learners' flexibility to move back and forth between worlds, rather than only progress to the other worlds.

2.7.2.2 Representations in and Connections Between Mathematical Worlds

Mathematical representations are objects in the worlds of mathematical thinking (Tall, 2004a, 2008). External mathematical representations provide a communication means for describing the concepts within the different worlds. The communication of the concepts can be achieved through the categorization of mathematical representations in accordance with the descriptions and the definitions of embodied, symbolic, and formal world, which allows an analysis of the development of a concept understanding (Smart, 2013). With respect to the concept of function, an arrow diagram, or a graph is a representation considered as an object in the embodied world; and an equation of two variables is an algebraic representation of function in the symbolic world, and a deductive proof of a relation as a legitimate function based upon the formal definition of function is an object of the formal world.

The Three Worlds of Mathematics puts an important emphasis on the interconnection between the mathematical worlds. According to Tall (2008), the development of mathematical cognition begins with the embodiment of concepts and actions, to developing those actions into a routine which are then expressed by symbols. The symbols enable mathematical operations to be performed. Next, it develops to formulating formal definitions and proving theorems deductively. The transition from the embodied world to the symbolic world can move back and forth, so the development path is not necessarily linear. Similarly, neither the transition from the symbolic world to the formal world nor the transition from embodied world to the formal world is linear in nature. The dynamic movement between two mathematical worlds strengthens the cognitive development. The transition from embodiment to symbolism means that the embodied objects are transformed into symbolic ones, hence symbolizing-embodiment, is a new world, while the opposite direction, embodying symbolism, transforms the symbolic objects into embodied ones (Tall, 2008). The three worlds and their connections are modeled in Figure 2.1. This model is inspired by and partly adapted from Tall (2005, 2008) and Smart (2013). The connection between the formal world and the embodied world results in the worlds of embodying-formalism and formalizing-embodiment. And, the worlds of symbolizing-formalism and formalizing-symbolism result from the connection between the formal world and the symbolic world.

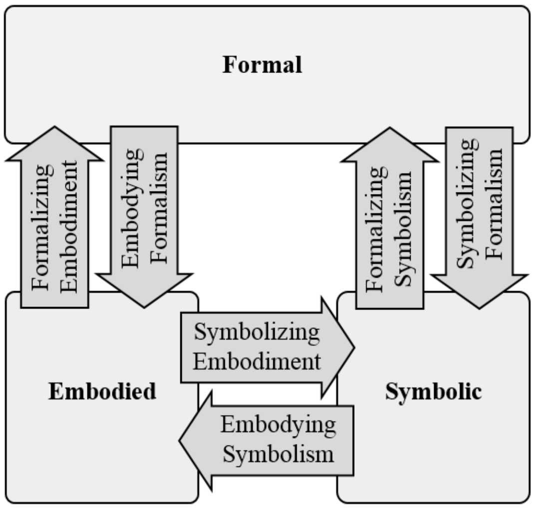


Figure 2.1 Connections Between the Three Worlds of Mathematics

Further, because representations are the objects of each world of mathematical thinking, and mathematical representations are connected to or transform between each other, as previously discussed (e.g., Goldin & Shteingold, 2001; Janvier, 1987a; Pimm, 1995; Thomas, 2008b), it follows that the connected representations become the objects of the connected worlds of mathematical thinking. For example, a graph of a function is transformed into an algebraic expression is the object of the symbolizing-embodiment world, and conversely, a function given in a symbolic representation is drawn on the Cartesian coordinate system produces an object of the embodying-symbolism world.

The Three Worlds of Mathematics theorized the cognitive development of mathematical thinking which focuses on the mathematical representations expressing the mathematical objects. It was used to analyze the use of mathematical representations in this present study. It provides a theoretical ground for both mathematics and its cognitive development. The theory explains explicit categories and their interconnections, as well as clear descriptions of development and understanding of mathematical objects (Smart, 2013). It also allows for incorporating representations as objects of study which enables the recognition of students' understanding within the different categories.

Various studies have employed the Three Worlds of Mathematics to frame their analysis. Smart (2010, 2013) used this theory in investigating the development of university students' understanding of the concepts of limits and derivatives in introductory calculus. She focused specifically on how the students made connections between mathematical worlds in developing their understanding of the concepts. Similarly, Juter (2006) employed the theory to describe the comparison between first-year university students' understanding development and mathematicians' historical development of the concept of limits. In conjunction with the APOS (Actions, Processes, Objects, and Schemas) theory of Dubinsky and McDonald (2002), the Three Worlds of Mathematics was applied by Stewart (2008) and Stewart and Thomas (2007, 2009) in examining the learning of linear algebra concepts by cohorts of first and second year university students. The research identified the students' difficulties in comprehending concepts and proposed potential trajectories for preventing them. The combination of the Three Worlds of Mathematics with the Abstraction in Context has been used by Kidron (2008) to analyze first year college students' understanding of the concept of differential equations. Another study of Christou et al. (2005) examined the first year university students' understanding of the concept of function as situated in the different worlds of mathematical thinking. In particular, the studies of Christou et al. (2005) and Smart (2010, 2013) were influential in how I designed the instrument and how I analyzed the representation use in the present study.

2.8 Conclusion

In this chapter, a review of literature has been conducted on learning introductory calculus at university, in particular, focusing on the concept of function. Based upon the review, learning introductory calculus at university is still problematic, particularly for students in mathematics education or pure mathematics majors. With the high expectation of conceptual, formal understanding of concepts, first-year university students encounter challenges because they are in the transition from school mathematics to university mathematics and they still rely very much upon the practical approach which contrasts with the formal, deductive-axiomatic approach of introductory calculus at university. The formal concept of function as a central concept in the introductory calculus is problematic for the students. They experience difficulties with the formal definition of function which is very essential in the formal study of introductory calculus. This brings to the surface the significance of constructing definitions and using definitions as a mathematics learning activity, which is expected to assist the students in developing their understanding of the formal definition of function. Focusing the investigation on the abstraction processes and the representations use in understanding the operational formal definition of function, the present study used the frameworks of the Abstraction in Context and the Three Worlds of Mathematics.

When considering this review, it is argued that the investigation of the abstraction processes and the use of mathematical representations is significant to the instructional practice of university mathematics learning. The investigation of how the students performed the abstraction processes in developing their understanding of the operational formal definition of function and how they represented their understanding revealed findings which may provide university instructors with insights about students' profile of abstraction and understanding of the formal definition of function as well as the students' dynamic of the representation use. The investigation of these foci was conducted through particular research methods that are discussed in the next Chapter 3.

CHAPTER 3
RESEARCH METHODOLOGY

All intelligent thoughts have already been thought;
what is necessary is only to try to think them again.
(Goethe, 1982)

3.1 Introduction

This chapter presents the approach which is used in order to reveal the students' understanding of the definition of the concept of function in introductory calculus. It consists of eight major sections. The first section focuses on the research paradigm and the justification for employing the qualitative research approach. The second section discusses the design of the research. The third section deals with the research setting and the participating subjects. The instruments used in the study are described in the fourth section which is followed by the sections describing the data collection and analysis. The ethical consideration is addressed in the penultimate section, which is followed by the closing section discussing the legitimacy of the research.

3.2 The Nature of the Research

The discussion of the nature of the research comprises an insightful comprehension of the research paradigm illuminating the research. The research paradigm is the basic belief system or worldview covering different assumptions about the world and the way to understand it, which guides, but not necessarily drives, the researcher in choosing a research method (Cohen, Manion, & Morrison, 2018; Guba & Lincoln, 1994). The importance of the research paradigm resides in its central role as a guidance for the investigator in aspects of a research undertaking, first from the intent of the research, to selecting the methods of gathering and analyzing the data, to reporting the findings and outcomes (Leavy, 2014; Ling & Ling, 2017; Treagust, Won, & Duit, 2014). Without an explicit paradigmatic position, a researcher still could conduct an investigation on a particular topic of interest, yet, a paradigmatic stance held by the researcher will always influence the research project, albeit the implication of the paradigm's assumptions will not be critically examined (Ling & Ling, 2017).

However, it is particularly important to realize that the questions and purposes of the research are more influential on the selection of methods than the philosophical stance or paradigm of the researchers (Tashakkori & Teddlie, 2003).

This study investigated the development of university students' understanding of the concept of function in introductory calculus. To construct their understanding of the calculus concepts through abstraction processes, calculus tasks were presented to the students in the learning activities, which were specifically designed to encourage students to perform theoretical abstraction processes. In this study, the students as research subjects were regarded as anticipative, meaning-making individuals who developed a personal meaning of the encountered situations, namely, calculus tasks and, their calculus understanding was developed through intentionally creative actions and interpretations (Cohen et al., 2018). The interpretations took place within the social, cultural, and temporal contexts (Marshall & Rossman, 2016). The students were taught to construct their own mathematical knowledge which was referred to as the students' mathematics (Steffe & Thompson, 2000). The students' mathematics was identified from what they uttered, did, or performed as responses to the mathematical tasks during learning activities. The processes of constructing mathematical understanding were considered from the students' current perspective, and not from the expert mathematicians' perspectives (Dreyfus et al., 2015).

The researcher should suspend or forgo the assumptions about the students and their contexts in favor of examining the phenomenon and its situation within its own characteristics (Hammersley, 2013). By looking behind what students did and expressed, their mathematics, developed through the processes of abstraction, was explained and translated into an appropriate model, the mathematics of students (Steffe & Thompson, 2000), which is considered as legitimate mathematics as long as its mathematical ground could be found. Here, mathematics is considered as a human activity (Freudenthal, 2002c) and is itself a product of human intelligence (Piaget, 1980), which is, therefore, ontogenetically justified based upon the history of its invention and development by individuals (Steffe & Thompson, 2000). With such characteristics, this study is framed in the interpretivist paradigm (Treagust et al., 2014).

The central feature of interpretivist research is to develop an understanding of “the localized meaning of human experience” (Treagust et al., 2014, p. 7) which is attempted through identifying and interpreting the processes by which people acquire or construct various meanings over time (Stinson & Bullock, 2015). Rooted in relativist ontology and constructivist epistemology (Taylor, 2014; Treagust et al., 2014), interpretive researchers believe the students’ mathematical understanding to be dependent on their experiences, culture, contexts, beliefs, as well as values, and be shaped by their social interactions (Bikner-Ahsbahs, 2006). The interpretivist researchers share common assumptions that frame the implementation of their investigations. According to Candy (1989), research with an interpretivist paradigm is value-laden, and such values will affect the framework, the focus, as well as the design of the study. Moreover, any phenomenon being investigated is characterized as being comprised of various factors, processes, events, causes, and effect which are mutually interdependent. The phenomenon is multifaceted in nature which requires a holistic investigation instead of a fragmented approach separating dependent variables from independent ones. Furthermore, developing an understanding of individual cases is the main purpose rather than constructing generalized interpretations. This last assumption is emphasized by Ponterotto (2005) as the idiographic-emic feature of interpretive research, that is, focusing on the understanding of the unique individuals with their non-generalizable context.

The nature of a research study depends on the research questions and objectives. Considering the research questions, it could be realized that an inquiry needed to be conducted on how students develop their understanding of the calculus concepts through abstractions and how they use representations in the abstractions. To research and gain insight into these issues, an investigation was carried out on the students’ actions, utterances, written works, as well as interactions taking place in the processes which lead to construction and constitution of intended understanding (Hammersley, 2013). The study focused on how students think about the calculus problems presented to them, which were expected to lead them to construct their understanding of the calculus concepts. It inquired the way the students construct their world of knowledge (Merriam & Tisdell, 2016), which they performed through abstraction processes. It also focused on how they take into account their prerequisite mathematical knowledge to support the construction of understanding of the new concepts. The data are then analyzed by focusing on the understanding constructed and the processes underwent by the students, from which the inferred findings along with the processes are described in the thick description (Geertz, 1973). In order to attain this objective, a qualitative approach was the obvious option.

Regarding the research objective, according to Robson (2002), one's aims in exploratory research are to identify what is happening and to seek for a new insight of the phenomena by examining them in a new perspective, while in descriptive research, the phenomena were portrayed with an accurate profile. Considering the focus this research sets out to investigate, this study was explorative in nature, as it aimed at exploring the processes of abstraction undertaken by the students in developing their understanding of calculus concepts and the use of representations in the processes. This exploration was expected to provide new insight about the abstraction for constructing understanding by assessing the abstraction in the framework of the Abstraction in Context perspective (Dreyfus, Hershkowitz, & Schwarz, 2001; Dreyfus et al., 2015; Schwarz et al., 2009). This perspective is considered as a new angle of seeing the abstraction theoretically, rather than empirically. This study also has a descriptive flavor because it presents a more detail profile of the development of mathematical understanding of introductory calculus concepts.

3.3 Research Design

After clarifying the nature of the research, the design of the research was defined. Relying on the selected research paradigm (Creswell, 2009), the research design is basically the logic linking the problems and purposes of the research to the procedures of gathering and analyzing the data (Yin, 2009). Based upon the nature of the research, the case study was the selected research method. It was implemented within the teaching interview framework as an adaptation of the teaching experiment (Steffe & Thompson, 2000).

3.3.1 Case Study

Case studies are among the most preferred research methods in interpretivist studies which are qualitative in nature (Cohen et al., 2018; Garrick, 1999; Treagust et al., 2014). Creswell (2013) defines a case study as a qualitative method where the researcher investigates "a bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving multiple sources of information (e.g., observations, interviews, audio-visual material, and documents and reports), and reports a case description and case themes" (p. 97). To frame the case study within the interpretivist paradigm, the focus of inquiry should be on the intrinsic values of cases (Stake, 2003) from which the emic meaning or knowledge constructed by people within the case is sought (Stake, 1995) to achieve a profound comprehension of what is happening with the case in its natural context (Robson, 2002). The nature of the case investigated should be contemporary (Yin, 2014). As the purpose of this present research was to gain insight into the theoretical abstraction processes and the representation use in developing an understanding of the introductory calculus concepts, the case study was thought to be the appropriate method.

Case study research could be further defined by its special properties. Merriam (2009) characterized the qualitative case study as being particularistic, descriptive, and heuristic. Particularistic means that the case study concentrates on a specific situation, event, program, or phenomenon. The case itself is of importance for the information it could reveal about the focused phenomenon and for the things it might signify. The particularity of the research focus makes the case study an appropriate method for investigating the practical problems which occur in daily practices including teaching and learning situations. The case study is descriptive because it produces a richly thick description of the phenomenon being investigated. The thickness of the description, which could also be prosaic and literary in nature, results from the complete, literal description of the case under study which comprises many variables with their all interactions portrayed over a specific time period. The case study is heuristic because it can illuminate the understanding of the phenomenon being investigated. It can facilitate the discovery of new meanings, the extension of experiences, and the confirmation of knowledge.

Implementing a case study necessitates five important components (Yin, 2009). The first component is the research questions. What the investigator seeks to understand is crucial in deciding to use the case study method in the research (Merriam, 2009). In a case study aiming at understanding what is happening to a particular phenomenon, the "what" question could be used to guide the research strategy (Merriam, 1988), and this kind of question is "a justifiable rationale" for an exploratory study (Yin, 2009, p. 9). The complete profile of the phenomenon which is the answer to the question assigns the descriptive feature to the case study (Robson, 2002). In practice, a case study is mostly directed by the how and why questions as these types of questions lead the researchers toward clarification of a phenomenon that may require a longer process of inquiry instead of just a mere incidental one (Burns, 2000; Yin, 2009). In this present research, the question of how students constructed their understanding of introductory calculus concepts through the theoretical abstraction processes which covered what their achieved understanding looked like, and how they used mathematical representations in the processes confirmed the research as a descriptive exploratory case study.

The second component is the study proposition. However, as mentioned earlier, this present study is an explorative case study; therefore, the propositions are not required to be specified. Instead of relying on the proposition to direct attention to the aspects which should be scrutinized within the study, the direction is based upon the research study purpose (Yin, 2009). The purpose of this study was discussed in the previous section. The third component is the unit of analysis. The identification of the unit of analysis (i.e. case or phenomenon being investigated) and the kind of the case is very crucial in a case study (Creswell, 2007). It is the unit of analysis that determines whether a study is categorized as a case study or not (Merriam & Tisdell, 2016). The fourth and fifth components, the logic linking the data to the research proposition (purposes) and the criteria for interpretation of the findings, represent the method of data analysis and the perspective for interpreting the research results (Yin, 2009).

The unit of analysis in this present study was a pair of students. They were first year students from an undergraduate program of mathematics education who learned introductory calculus. The aspects to focus on the analysis were the development of the understanding of the operational formal definition of function through the theoretical abstraction and the use of representations in the abstraction processes to develop an understanding of the operational formal definition of functions. In addition, instead of using one case, this study took several pairs of students to investigate the development of understanding. Therefore, it was categorized as a multiple-case study (Yin, 2009). The use of more than one case was to allow the production of either similar findings or different findings for predictable reasons (Yin, 2009). The case study is considered as a challenging method of inquiry, especially in terms of how to capture adequate data from the cases under study (Robson, 2002). The implementation of this case study was framed within the teaching interview setting, which considered some aspects of the Abstraction in Context theory (Dreyfus et al., 2015).

3.3.2 Teaching Interview: An Adaptation of Teaching Experiment

Interviewing is claimed to be the best method for a qualitative case study (Merriam & Tisdell, 2016). This present case study research was conducted within the methodological framework of a teaching interview. The teaching interview is not a mere conversation for gathering information from the interviewees without affecting them. It is different from the traditional clinical interview for two reasons. First, it is a methodology consisting of and focusing on not only the traditional interview, but also on the aspect of teaching. The interview is conducted with didactic objectives (Hershkowitz et al., 2001). It aims at understanding the effect of certain interventions on the students' thinking and the development of that thinking. The interview takes place while the students are doing the activity of learning. Second, because it is done with students in their learning process, it is acceptable and even hoped that the students will modify or change their thinking during the interview. The teaching interview is an adaptation of the teaching experiment (Chini et al., 2009) which is a methodological tool that is exploratory in nature (Cooper & Warren, 2011). It is originally derived from Piaget's clinical interview to be a means for exploring students' mathematics (Steffe & Thompson, 2000).

The teaching experiment is an experimental method for seeking answers to questions on the characteristics of learning mathematics and the development of mathematical knowledge as well as the role played by learning interactions (Czarnocha & Maj, 2008). The experiment is for the researcher to directly experience students' mathematical learning as a way to understand the powerful mathematical concepts the students construct (Steffe & D'Ambrosio, 1996; Steffe & Thompson, 2000). It covers a sequence of teaching episodes (or interactions) involving the researcher (as a lecturer or a teaching agent), participant students, an observer (as a witness of the instruction processes), and interviews (as a method of documenting what transpires in the processes) (Steffe & Thompson, 2000). By the sense of this definition, a teaching experiment is guided by hypotheses which are generated based upon the students' action and aimed at modeling the students' schema. The hypotheses are tested in a sequence of learning cycles. On the other hand, Confrey and Lachance (2000) define a teaching experiment as a planned intervention, which is implemented over a period of time in a learning context in which teaching processes go on involving a dialectical relation between conjecture and the instructional components such as curriculum, methods of instruction, and assessments. In this definition, the conjectures are the crucial aspect deriving the implementation of the teaching interview. The teaching experiment was designed to eliminate the separation of research practices from teaching practices (Steffe & Thompson, 2000) by conducting an investigation and teaching simultaneously, aiming at improving the quality of learning (Czarnocha & Maj, 2008).

The teaching experiment is a living methodology for exploring students' mathematics and it has not been standardized (Steffe & Thompson, 2000). This means there is no strictly prescribed structure of a teaching experiment. Therefore, according to Czarnocha and Maj (2008), changing the structure of a teaching experiment is possible in order to adjust it to the orientation of the research. In the teaching experiment, traditionally, there were several teaching episodes that were implemented within certain periods of experiment. The investigation could be focused on studying the development of students' thinking over the period. The structure of the teaching experiment proposed by Steffe and Thompson (2000) was adapted in this present research. In this study, the aspect of teaching episodes was modified: Instead of using a sequence of several episodes, this research comprised only one teaching episode. Therefore, rather than focusing on the students' mathematics developing over the period covering several consecutive teaching episodes, it focused on the students' understanding which developed and changed over a sequence of interactions within a teaching episode. These interactions were triadic ones, that is, an interaction among three components: lecturer, students, and the mathematical understanding under consideration (Koichu & Harel, 2007). An observer was involved within the teaching process. He was a cooperative colleague who was a lecturer in the department of mathematics in which the present study was conducted. The task-based interview (Goldin, 2000) was used to gather data from students who worked on calculus tasks in pairs with thinking aloud. This modified version of the teaching experiment was termed the teaching interview (Chini et al., 2009; Hershkowitz et al., 2001).

In the implementation of the teaching interview, Steffe and Thompson (2000) emphasize several characterizing elements. The first is the importance of *exploratory teaching* prior to conducting the teaching interview. Being involved in exploratory teaching is a prerequisite for researchers who are new and want to implement the teaching interview. This teaching is to build a good rapport and establish communication with the students by being acquainted with the way the students learn mathematics. In the exploratory teaching, the researcher should train himself to set aside his own mathematical knowledge and let the students freely construct their mathematical knowledge (Norton & D'Ambrosio, 2008). In conducting this present research, I was in charge of teaching a class of Introductory Calculus course unit in which the research took place. This allowed me to engage in several exploratory teaching sessions that accustomed myself to the students' way of learning. Besides, those sessions provided an opportunity to train the students to think aloud while they were working on learning activities as thinking aloud was not usually practiced in their learning.

Conjecture or hypothesis is the second element emphasized in the teaching interview. Most often, the conjecture is motivated by a dissatisfaction with the results of traditional, typical practices of teaching and learning. The conjecture is an inference drawn from inconclusive evidence and functioning as a tool for reconceptualizing the approach to both the content and the pedagogy of certain mathematical concepts (Confrey & Lachance, 2000). It has two dimensions. The mathematical dimension of a conjecture concerns the mathematical content, as an answer to the question: What should be taught? Meanwhile, the pedagogical dimension deals with the way the teaching (tasks, activities, and resources) should be organized, as an answer to the question: How should the content be taught?

In this research, the topic of function was the focus because of the kind of dissatisfaction experienced by me, the researcher, in terms of the way the students understood the conceptual definition of function. The common approach widely implemented in Indonesia seemed unable to help students develop their understanding of the formal definition of function and what role the definition should play. The use of the formal definition of concepts in a mathematical proof or proving was a critical ability to succeed in learning university mathematics that was mostly approached conceptually and formally, not practically. As for the concept of function in introductory calculus, however, the formal definition of function proved to be inoperable for students. As a consequence, they experienced difficulties in using the definition to develop a deductive argument to verify, justify, or prove that a relation was functional in nature. Then, it was conjectured that there was a need for learning how to formulate an operational version of the formal definition of function. This constituted the content dimension of the conjecture. At the same time, given their cognitive level, university students should learn mathematical concepts through the process of theoretical abstraction. One of the models of the theoretical abstraction process was the Abstraction in Context. This formed the pedagogical part of the conjecture. The Abstraction in Context, as a theoretical framework, offered an approach. This model requires an *a priori* analysis resulting in the structure of knowledge elements, which could be considered as a learning trajectory. The trajectory was treated as a predicted learning path used to guide the design of calculus tasks. Hence, the trajectory served only as a working hypothesis that might be confirmed or modified based upon the data analysis (Dreyfus et al., 2015).

The third element is *modeling students' mathematics* as the core of the data analysis. In developing a model of students' mathematics, the researcher's understanding of what the students could not understand was not less important than the understanding of what the students could understand (Steffe & Thompson, 2000). The focus of the modeling was to comprehend the originality, the coherence, and the robustness of the students' thinking (Ackerman, 1995). The primary challenge encountered by the researcher concerned the treatment of the students' errors in thinking. The errors or misconceptions should be considered as the ways students dealt with learning opportunities that were provided to them (Dreyfus et al., 2015). Instead of directly justifying the errors as absolutely wrong, the researcher should attempt to comprehend the students' performance by trying to construct a referential framework as a rational basis for that performance (Steffe & Thompson, 2000).

Further, Steffe and Thompson (2000) emphasized the importance of attributing mathematical realities to students which are independent of the researcher's mathematical realities when attempting the modeling. Here, modeling is constructing an explanation of students' mathematics that refers to all mathematical things which might constitute the students' mathematical realities. The students' mathematics results from mathematizing activities, that is, constructing mathematics or reinventing mathematics, facilitated by working on realistic mathematical problems (Freudenthal, 2002b). The notion of mathematization was one of the philosophical bases of the Abstraction in Context. The explanatory model is called the mathematics of students. In constructing the model, it is necessary to examine the thinking behind the students' statements and actions. The mathematics of students is considered as legitimate mathematics if we can identify "rational grounds for what students say and do" (Steffe & Thompson, 2000, p. 269). The mathematics of students is established to be a conceptual foundation for the students' mathematics education (Steffe, 2013; Steffe & Wiegel, 1992).

The implementation. The teaching interview was conducted as a part of teaching and learning activities in the unit course of Introductory Calculus. This was a unit for first-year university students in their first semester. The lecture for this unit started on Monday, September 5, 2016. Actually, the credit for the unit was 3, which meant that in each week, there was only one lecture which lasted for 3×50 minutes. However, I offered the students an additional tutorial lecture each week. The students agreed upon having the tutorial class every Tuesday starting on the second week. Practically, since the second week, there were two meetings for this unit each week. As explained earlier, the students were not familiar with thinking aloud processes. So, the tutorial meeting each week was devoted to further discuss problems they found on Monday lectures, and to train them to think aloud while working on calculus problems. In the first lecture, I introduced the research that I would implement later which involved only some students in the class. I also discussed the research with the partner lecturer who I asked to be an observer in the research implementation. In the fourth week, I reminded the students about the research, and distributed the consent form to them in the class. I collected the completed forms at the tutorial meeting, and all students signed for their readiness to be involved in the research. There were 10 students selected to participate in the study.

While the teaching interview was intended to resemble a natural learning environment (Chini et al., 2009), I set up a room different from an ordinary classroom. In the room, I placed a big table with four chairs. An audio recorder and a video handy camera were set which were visible to students. The 10 students were divided into five pairs. The teaching interview for each pair involved a 150-minute session and there was only one pair in each session. The teaching interview was scheduled according to the availability of each student pair and the interviews were conducted on October 6, 7, 8, 12, and 14, 2016. One pair of students was only available on Saturday, October 8, 2016. In the teaching interview, the students discussed and worked in pairs on calculus tasks with thinking aloud and I asked them to probe or clarify the thinking or reason behind their utterances, statements, or actions, while they were discussing and working on the tasks. I posed probing or clarifying questions whenever necessary, especially when I found them occasionally falling in silent work or saying statements unclearly. The observer was engaged in the teaching interview to observe the processes and to help focus the video camera on the students' work.

The thinking aloud method could reduce the problem of memory failure which possibly occurred when the verbal data were collected at the end of the learning activity (Wade, 1990). To ensure that the students performed thinking aloud in this study, following the suggestions of Young (2005), the learning activity being undertaken was set to be appropriate for eliciting verbal data, the students were informed about who was listening to them and what the purposes of the data were, and they were provided with adequate chances to practice thinking aloud prior to the research implementation. Further, Wade (1990) argued that engaging the students in real learning activities would produce more reliable think-aloud data than if asking them to report on hypothetical situations. Here, the key was the engagement of the students within activities where they were immersed and allowed to expose their ongoing thoughts toward the end point of the activities (Young, 2005).

3.4 Research Setting and Participants

3.4.1 Research Setting

The research took place in an undergraduate program for mathematics education at a large public university situated in the eastern part of Indonesia. This program was conducted under the Department of Mathematics in the Faculty of Mathematics and Natural Science. It was designed as a four-year (eight-semester) Bachelor of Education course for prospective secondary school mathematics teachers. Another program run under the Department of Mathematics was the Bachelor of Science program for those studying the pure mathematics. With an average intake of 120 students each year, the Bachelor of Education program hosted around 480 students. In the academic year 2016/2017, there were three classes of freshmen (first-year students); each class had 35 students with an average age of 18 years. Two of the classes were regular Bachelor of Education program, and one class was International Bachelor of Education program. The language of instruction in the regular programs was Indonesian language. In the international program, the instructional languages were both English and Indonesian language. The grouping of the students into regular and international programs was based upon their preferences made in their enrolment to the study program. This present research was conducted in one freshmen class of the regular program in their first semester studying introductory calculus.

The students entered this study program of mathematics education through three different admission schemes; two national scale schemes and one university-based scheme (Ministry of Research Technology and Higher Education of the Republic of Indonesia, 2015). The first scheme was the National Selection of State Universities Admission (NSSUA). This scheme was conducted only for upper secondary school students graduating in the respective academic year. They organized their admission during the last semester in their upper secondary school. The selection considered several criteria including the unit course performance, special academic achievement, the stream or class type, the accreditation status of host schools. The second path was the Joint Selection of State Universities Admission (JSSUA). In this track, students sat a national scale entrance test, and the selection was based upon the test scores and the rank of the scores among all applicants in the study program where the students enrolled. The two selection schemes were administered by the cluster of Indonesian state universities. The third scheme was the Local Selection of State Universities Admission (LSSUA) set up independently by the respective university. The selection for this scheme was based only on the result of the test administered by the university.

The International Bachelor of Education program was to prepare prospective secondary school mathematics teachers to teach in international schools in Indonesia or in schools in English speaking countries. The language of instruction in this international program was English. The contents of the curricula for both regular and international programs were mostly the same. The difference was in a few additional units, such as English for Teaching Mathematics and Multicultural Mathematics Education, for the international program.

There are three compulsory units of calculus spread over the first three semesters of the program. Introductory Calculus was offered in Semester One, Intermediate Calculus in Semester Two, and Advanced Calculus in Semester Three. The contents of Introductory Calculus unit for all classes were the same. At the university, Introductory Calculus was a compulsory unit for all students in the Mathematics and Natural Science Faculty and Engineering Faculty. However, in the mathematics education bachelor program, there was a different emphasis put more on a conceptual relational approach to calculus, compared to a more practical approach to calculus in other non-mathematics study programs. For students in the program, Introductory Calculus covered not only how to apply the mathematical concepts, but also, with more proportion, how to think mathematics formally, develop an axiomatic argument, a formal verification, and a proof of mathematical statements. Introductory Calculus was considered as a fundamental unit course within which the students were trained to develop established mathematical thinking and prepared for the higher advanced mathematical course units.

In detail, the Introductory Calculus unit covers the topics of the real number system, limits, the derivatives, and the applications of the derivatives. The topic of real number system consists of real number properties, fundamental logic, equations and inequalities, absolute values, the rectangular coordinate system, and functions (formal concept, properties, graphs, and operations). Limits cover an introduction to limits, rigorous study of limits, limit theorems, limits involving trigonometric functions, limits at infinity, infinite limits, and continuity. Meanwhile, the topic of the derivatives comprises the origin of the derivative concept, the rules for derivatives, the derivative of trigonometric functions, the chain rule, higher-order derivatives, and implicit differentiation. And, the applications of the derivatives cover maxima and minima, monotonicity and concavity, practical applications, graphing functions using the derivatives, and the Mean Value Theorem for derivatives. Regarding the topic of functions, the students have learned it in their high schools. They already have encountered the basic notion of function, namely, each element of the domain is paired off with only one element of the codomain. They also have learned some other aspects of function, such as the graph of basic linear, quadratic, cubic, simple rational, and simple trigonometric functions, as well as the composition of functions. They also have learned the derivative and the integral of functions, although not formally as done at university.

3.4.2 Research Participants

The selection of the research participants was based upon the convenience sampling method which depended on students' willingness to take part voluntarily in the research (Johnson & Christensen, 2017). Ten students participated in this research. They were drawn from the class of 35 first-year students studying Introductory Calculus in their first semester at university. As they were all freshmen, it was assumed that their mathematical knowledge based tended to be of the same level, that is, their mathematical knowledge developed in their upper secondary school. There were no mathematical knowledge-based criteria specifically used in the selection of the participant students.

After conducting several lectures in the class, one week before implementing the research, I delivered the consent forms to all students without stating an obligation to take part in the research. However, the next day, I collected all the consent forms, and all 35 students signed their readiness to participate in the research. Therefore, as the participants would be working on calculus tasks with aloud thinking, I decided to select the participants who were talkative or had a high verbalization ability and showed a good habit of working together (Dreyfus et al., 2001). These characteristics were identified during the previous lectures when they were assigned to discuss and solve problems within their pairs. In the classroom, all students but one sat in a paired seating arrangement with two attached desks. Because they were to work in pairs in the study, the students were selected in their pairs, instead of selecting them individually. The 10 students are listed in Table 3.1. In the research implementation, the odd-numbered student was paired with the following even-numbered student, namely, Adel was paired off with Tanti, Amzi was with Naya, and so on.

Table 3.1 The Research Participants

No.	Names*	Gender	Secondary School	Secondary School Study Stream	University Admission Scheme
1.	Adel	Female	General Vocational School	Agriculture	LSSUA
2.	Tanti	Female	General Upper Secondary School	Science	NSSUA
3.	Amzi	Female	Islamic Upper Secondary School	Science	NSSUA
4.	Naya	Female	General Upper Secondary School	Science	NSSUA
5.	Dina	Female	General Upper Secondary School	Science	JSSUA
6.	Yuni	Female	General Upper Secondary School	Science	NSSUA
7.	Dyn	Male	General Upper Secondary School	Science	NSSUA
8.	Sam	Male	General Upper Secondary School	Science	NSSUA
9.	Iful	Male	Islamic Upper Secondary School	Science	JSSUA
10.	Vito	Male	General Upper Secondary School	Science	JSSUA

*Pseudonyms

Based upon the table, nine participant students took the Science stream in their upper secondary schools. There are two other streams in the upper secondary school system in Indonesia, namely, Social Science and Language. The coverage of mathematics subject in the Science Stream is wider than that of Social Science and Language stream. One student had a vocational secondary school background within an Agriculture stream. In terms of mathematics subject in secondary school, the coverage of calculus material for the Science stream in the Upper Secondary School was the same as that in the Agricultural Vocational Secondary School (Ministry of Education and Culture of the Republic of Indonesia, 2016). Two students were from Islamic Upper Secondary Schools and seven students were from the General Upper Secondary School. The university admission scheme of the participants varied. Of 10 respondents, six were accepted at the university through the National Selection of State Universities Admission, three were through the Joint Selection of State Universities Admission, and one was through the Local Selection of State Universities Admission.

3.5 Research Instruments

There were two instruments that were employed to gather the data for this present research. The first involved calculus tasks and the second was the interview guide. These instruments were developed by myself. The calculus tasks were used as learning material for the students. The interview guide was used with the students while they were discussing and solving the problems in the calculus tasks.

3.5.1 Calculus Tasks

In order to facilitate the abstraction process for developing the participant students' understanding of the operational formal definition of function, a series of calculus problems were designed. This was the first instrument of this present research. The development of this instrument was based upon *the structure of knowledge elements* of the concept of function under investigation. The structure of knowledge elements was a foreseen trajectory of students' learning. It resulted from an *a priori* analysis of the learning tasks to be carried out by the students in the study, by which the intended knowledge elements, their constituting elements, and the interrelations were identified (Ron et al., 2010). To start developing the structure of knowledge elements, some assumptions regarding the previous knowledge of the students were first made. The next was identifying and examining the potential learning path the students would undergo in learning activities. This trajectory comprised the concepts or strategies in the domain of mathematical contents which the students were predicted to develop or understand along the learning process leading them to develop or understand the target concept being studied (Dreyfus et al., 2015). The structure of knowledge elements covered the knowledge elements (concepts or strategies) completed with their mathematical definition as well as the operational definition, that is, the description of the circumstances under which the students could be justified as using, expressing, or constructing the concepts or the strategies. The elements were also structured based upon the hierarchical relationship among them. Some were contained in others and some were prerequisites for others.

I first developed the structure of knowledge elements based upon the analysis of the topic of function in Introductory Calculus for the first semester in the Mathematics

Education Bachelor Program. There were seven concepts identified as constituent concepts in the development of an understanding of the operational formal definition of function, as listed in Table 3.2. A few other concepts, such as the membership of sets and subsets, were also identified as the prerequisite concepts for these elements, which were assumed to have been developed or understood by the students. They were grouped into two overarching concepts as seen in Table 3.2 below. The structure was then verified by a mathematics education academic in an Indonesian university. It was also verified by my supervisors. The complete structure of knowledge elements is presented in Table 4.1 in Chapter 4 (see p. 124).

Table 3.2 The Knowledge Elements for the Operational Formal Definition of Function

No.	Knowledge Elements (Concepts)	Overarching Knowledge Elements
1.	Association between sets' elements (ASC)	General Relation (GRE)
2.	Ordered pairs (ORP)	
3.	The Cartesian product of two sets (CAR)	
4.	Relation (REL)	
5.	The exhaustive domain property (EXH)	Special Relation (SRE)
6.	The unique image property (UNI)	
7.	Operational formal definition of function (FUN)	

The calculus tasks were then designed by the guidance of the structure of knowledge elements. For each concept, I designed a problem or question which was expected to enact the cognition of the students to construct or develop an understanding of the concept. The tasks were organized in a sequence to facilitate the continuous transformation of the concepts. The tasks were designed to include building new connections between the students' previous concepts, hence improving the depth of their understanding, and leading to vertical reorganization of their mathematical knowledge (Dreyfus et al., 2015).

For example, regarding the concept of association between sets' elements, the task required the students to make associations between the elements of two sets. The task was as follows:

1. The followings are pairs of two sets. What kinds of association between members of the two sets could you make?

(a) $M = \{a, b, c\}$ and $K = \{x \mid 2 < x < 4, x \in \boldsymbol{N}\}$.

(b) $P = \{z \mid z^2 + 2z + 2 = 0, z \in \boldsymbol{R}\}$ and $R = \{6, 7\}$.

This might be an easy task if the elements from two given sets had a very clear relationship that could be made, such as corresponding numbers to numbers. In this case, one set contained alphabetic letters, and another set contained numbers. This was to encourage the students to develop their understanding that an association could be made arbitrarily between elements of any two sets without having to follow a specific rule. Also, another pair of sets were given in which one of them had no elements. This was to prompt the students to develop their understanding that one way to associate the elements between two sets was by making no association between them. This is the concept of empty association. The next task was for developing the understanding of the concept of ordered pairs, which was designed based upon the concept of association between sets' elements. In this task, the students were challenged to compare the results of two associations, which were different in the order of the elements. The results the first association were expressed in a direction opposite to the results of the second association. This task was intended to lead the students to understand that the order of the elements in an association was significant. Several items in the tasks were adapted from Feeley (2013), Christou et al. (2005), and Nicholson (2012).

The tasks were firstly developed in an English version. These tasks were verified by my supervisors. The tasks were also validated by a proficient Indonesian researcher of undergraduate mathematics education. After verification, the tasks were translated into Indonesian language. This translation process aimed at avoiding students' misunderstanding and misinterpretation of the tasks for the language differences. For this purpose, the back-translation strategy was employed because it could minimize translation imprecision with the results that preserve the equivalence between Indonesian language and English (Liamputtong, 2010). This procedure involved two translators. One translated the tasks from English to Indonesian language, and the other translated them back from Indonesian language to English without referring to the original English version (Brislin, 1970). As suggested by Sutrisno, Nguyen, and Tangen (2014), considering myself as a bilingual researcher, I took the role as the first translator to translate the English version to Indonesia language. The Indonesian version of the tasks was reviewed by a calculus lecturer of an Indonesian university. The back translation was performed by a bilingual lecturer of Mathematics Education Study Program at Universitas Negeri Makassar, Indonesia. The back-translated instrument was compared to the original English version. Few inconsistencies were found, and I made a necessary revision to the Indonesian version accordingly. Further, I trialed the instrument with a pair of students from another class. Based upon the trial, it was found that the students had to spend more time than predicted, particularly in the items justifying the relations given in different representations. Hence, the number of examples of relations was reduced. Several sentences were refined to make them more readable. The two versions of this instrument can be found in Appendix A1 and Appendix A2.

3.5.2 Interview Guidance

This research employed a task-based interview (Goldin, 2000) to gather its data. For the interview, I prepared a guidance sheet which was useful in framing the interview questions. It consisted of prompts and probes possibly uttered within the interview. Prompts enabled the interviewer to clarify topics being discussed, particularly when the interviewee misunderstood them, or asked for more clarification (Cohen et al., 2018). Probes enabled me to ask the interviewees to extend, elaborate, add to, exemplify, provide detail for, clarify or qualify their response (Patton, 2002; Wellington, 2015). They helped me, the interviewer, understand more the cognitive processes of the respondents (Priede, Jokinen, Ruuskanen, & Farrall, 2014). This guidance was first developed in English. After reviewed by my supervisors, I translated it to Indonesian language. The interview guidance was also trialed along with the calculus tasks instrument, and practically, there was no change made in the guidance after the trial. The interview questions were not given ahead of time to the participants. The two versions of the interview guidance can be found in Appendix B1 and Appendix B2.

3.6 Data Collection Methods

In terms of data, Yin (2006) explicitly mentioned six sources by which research data for a case study could be gathered. They included documents, archival records, interviews, direct observations, participant-observations, and physical artifacts. The use of various data sources was intended to increase the credibility of the research. Thus, the following section provides basic descriptions about some methods which were used in the data collection process of this study. Selecting appropriate and accurate information was a crucial issue in implementing a case study (Cohen et al., 2018). The basic description of the methods which were employed to gather the data for this particular research is provided in the following sections.

3.6.1 Interviews

One of the components of the teaching interview was the method of documenting what transpired in the processes, namely, interviewing. The interviews were conducted with the students while they were working on the calculus tasks. The students were asked to think aloud (Tarmizi, 2010), that is, to verbalize their thoughts while they were solving the problems in the calculus tasks. They were set up to work in pairs to create a social context for students' interaction. A social context was one of the contexts considered in the Abstraction in Context theory (Dreyfus et al., 2015). The social interaction was expected to provide the students with more opportunities to think of, discuss, and solve the calculus tasks (Kouropatov & Dreyfus, 2014). Research has confirmed that interaction in pair working enables mathematical argumentation (Hershkowitz et al., 2002). Moreover, working in pairs could allow the establishment of an atmosphere of confidence for the students. Within the interaction, the students could benefit from the possibility to "fill in gaps for each other" thinking (Houssart & Evens, 2011, p. 66).

The majority of the interviews actually consisted entirely or mostly of discussion. The notion of asking the respondents to work on a certain task was an alternative for the common interview practices. This type of interview, called a task-based interview, has diagnostic and explorative functions (Goldin, 2000; Houssart & Evens, 2011). It aimed at expanding the information which is available to the researcher via the task. The questions depended on the individual's responses to the task, so the interview was semi-structured in nature, which was appropriate for an in-depth analysis of a particular issue (Creswell, 2012). Because it involved a pair of students, therefore it was called a task-based paired interview. In the interviews, I used a voice recorder and a handy video camera to record the thinking aloud, the conversation, as well as the gestures.

3.6.2 Written Documents

Merriam and Tisdell (2016) considered written material documents as a source of qualitative research data. Public and personal documents were two among various forms of documents commonly used in case study research. Personal documents which refer to the individuals' written text of the whole or part of their reflections on a specific event or topic (Taylor, Bogdan, & DeVault, 2016). The documents could be already available in the research setting or they could be ones that were generated by the researchers. The researcher-generated documents were those prepared either by the investigator or by participants when the research has started (Merriam & Tisdell, 2016). The documentary record should be used carefully for it might contain inaccurate information. Yin (2009) suggested that the documents should not be accepted as a literal recording of events that have taken place" and they were used "to corroborate and augment evidence from other sources" (p. 103).

In this study, the documents used were generated by the researcher and prepared by the participants for the researcher. The documents referred to written responses (answers or solution) which were given by the participants to the calculus tasks. The calculus tasks which were designed in the form of worksheets were delivered to the participating students. They were set to work on the tasks in pairs, of which the responses were written down on the same worksheets. These worksheets were collected for another source of data for the present study.

3.6.3 Observations

One requirement of the teaching interview is the presence of an observer. In implementing the teaching interview, the researcher should expect to experience the situation in which students were operating within an unanticipated way, making unexpected mistakes, or unable to move forward (Steffe & Thompson, 2000). It is in this situation an observer could be of help to keep the processes running. In the teaching interview, the observer should be a non-participant observer (Cohen et al., 2018). The observation was conducted in a less structured style focusing on the activities the students were undertaking, that is, the abstraction processes taking place while they were working on and solving the problems in the calculus tasks. The observation was conducted to gather the data which complemented the data from interviews and students' work (Simpson & Tuson, 2003). The observer's field notes during the processes were considered as a source of additional data.

3.7 Methods of Data Analysis

For a case study, there is no certain method specifically prescribed for analysis the data. The data analysis method is one of the least established aspects of case studies (Yin, 2014). The case study researcher's primary concern is the data analysis plan (Burns, 2000) in which the extent of the data analysis and the method of the analysis will be determined. Defining the analysis strategy is essentially needed in a case study to guide the researcher to manage and analyze the data as well as to interpret the results of the analysis (Yin, 2006).

The processes of data analysis for this research were divided into two main phases. The first phase was the management of the data which had been gathered. This phase covered the steps of transcribing the audio interviews, synchronizing the transcriptions with the video recording, member checking, and translating the transcriptions. For the data of students' worksheets, the management covered digitalization and clarification. The second phase was the analyses of the data. This phase covered the analyses of the abstraction processes, the level of understanding constructed through the abstraction, the category and the appropriateness of the mathematics representations.

3.7.1 Data Management

The management of the data started with transcribing the audio interviews. The transcriptions were not made in a naturalistic way where each utterance was written down in as much detail as possible (Oliver, Serovich, & Mason, 2005). Instead, the interviews were transcribed in a denaturalized way where the concern was on the accurate substance, meaning, and perceptions created and shared in the conversation (Oliver et al., 2005). The interviews were still transcribed in verbatim in order to have an evidential complete account of the conversations. The transcriptions were carried out by research assistants.

Every word recorded was typed in the transcription including linguistic fillers, namely, a part of speech which is referred to as non-silent pause which has been filled by words or phrases such as ‘amm’, ‘you know’, or other vocalizations (Crystal, 2008). Although all peculiar components of speech such as stutters and nonverbal, involuntary vocalization should be discarded in denaturalized transcriptions (Oliver et al., 2005), I still included the pauses in the transcriptions, which were indicated by ellipses. Common response tokens such as ‘amm’, ‘OK’, ‘yeah’, ‘nuh’ were also covered in the transcriptions because they could provide additional insight of the conversation (Gardner, 2001). In my case, the token ‘yeah’ was replaced with a proper word ‘yes’, and ‘nuh’ with ‘no’. In the context of Indonesian language, ‘yeah’ is similar to *iyo* or *iye*, and ‘nuh’ is similar to *ndak*.

After being transcribed by the research assistants, I read all the transcription texts for checking the accuracy. This check was done by listening to the interview tapes. After checking the accuracy, the grammatical errors in the texts were also rectified. Improving the grammar was performed because the participant students were engaged in member checking. The participants might feel respected if they found their utterances were written in better grammar (Oliver et al., 2005). Moreover, the focus of the analysis was on informational content (Gardner, 2001), instead of the specifics of communication such as repairs or accent (Oliver et al., 2005).

The next step was the synchronization. The interviews contained the utterances expressing the students’ ideas, reasons, and thinking or reasoning processes while they were working on the calculus tasks. Besides recording their voices, during the interview, a video camera was set to focus on the students’ worksheets. It was to capture the objects on the worksheets which were pointed at or referred to when the students uttered the words/phrases such as ‘this’, that’, ‘these’, ‘those’, here, there, and ‘like this’. The transcriptions were synchronized with the video recordings in terms of these words/phrases. The objects were inserted in the utterances which were framed within square brackets [ ].

In this present study, I played the role as the researcher, the data collector, as well as the data analyst, and this gave the potential for my bias (Miles, Huberman, & Saldaña, 2014). There is a potential chance for researchers to impose their beliefs in the research implementation in which their voices could dominate the interview conversation (Mason, 2002). Therefore, the participant students were involved in member checking the transcriptions (Birt, Scott, Cavers, Campbell, & Walter, 2016). The transcripts were returned to each pair of students. They were asked to verify, confirm, and validate the correctness of the utterances (Doyle, 2007). In addition, in terms of the students' worksheets, they were scanned to save them in a digital version. I also asked the students to clarify some parts after preliminary examining their answers.

All interviews were conducted in Indonesia language. The transcriptions which were checked by the participant students were written in Indonesia language. After member checking, the transcriptions were translated into English. The translation procedure employed the back-translation approach (Brislin, 1970; Liamputtong, 2010; Sutrisno et al., 2014) which was similar to the one used in the translation of the instrument of calculus tasks. I acted as the first translator to translate the Indonesian version to English. The back translation was performed by a bilingual lecturer in Universitas Negeri Makassar, Indonesia. The back-translated transcriptions were compared to the original Indonesian version. From the comparison, a few inconsistencies which were found were revised accordingly.

3.7.2 Data Analyses

The content analysis approach was adopted to analyze the data of this present study. The content analysis mainly aims at providing insights into the phenomena under investigation (Fraenkel, Wallen, & Hyun, 2012; Hsieh & Shannon, 2005; Merriam, 2009). This is in line with the purpose of this investigation of the abstraction for constructing an understanding of the concept of function. In practice, the content analysis was an approach to examining text-based data in order to develop a description of the components in the texts or an interpretation of the abstract ideas of the qualitative data (Graneheim, Lindgren, & Lundman, 2017).

There were three methodological approaches to the content analysis, namely, conventional, directed, and summative (Hsieh & Shannon, 2005). I adopted the directed approach to analyzing the qualitative evidence being collected. This approach starts with predetermined categories. These categories were developed based upon the research questions, the previous knowledge, and the theoretical framework of the study (Elo & Kyngäs, 2008; Graneheim et al., 2017; Hsieh & Shannon, 2005).

Based upon the theory and the existing research, three kinds of categories were formulated. The first was for the levels of the students' understanding of the definition of function which was developed through the abstraction. The details of this category can be found in section 4.3.1 (p. 129). The second was for the types of representations used by the students in developing their understanding of the definition of function. The last was for the appropriateness of the representation used by the students. The details of the second and the last categories can be found in Chapter 5.

To proceed with the content analysis, the unit of analysis was selected (Guthrie, Petty, Yongvanich, & Ricceri, 2004). Within a case study, the unit of analysis or the case must be defined and bounded to consider the research problems that have been formulated (Yin, 2014). The unit of analysis is a bounded system which could be an individual, a program, or a group based on the extent which the research findings will be explicated and reported (Merriam & Tisdell, 2016). The unit of analysis of this case study was a pair of students for the aspect that encompass: the theoretical abstraction for developing an understanding of the operational formal definition of function and the use of mathematical representations in the abstraction. As explained in section 3.4.2 (p. 99), there were five pairs of students from an undergraduate program of mathematics education involved in this study. Following the suggestion of Graneheim and Lundman (2004), the analysis was conducted on the whole interview transcripts and the students' worksheets. In the analysis, I focused only on the manifest content of the data. Following the steps explained by Hsieh and Shannon (2005), with the categories on hand, I read all the data and highlighted the parts that appeared to represent the categories. Next, the highlighted parts were coded based upon the corresponding categories.

Specifically for the abstraction process, the analysis employed was the Abstraction in Context analytical framework. It was a micro-analysis tracing the epistemic actions which were windows for evaluating the main building blocks of the process of abstraction. The epistemic actions are recognizing, building-with, and constructing. Recognizing is the student seeing or expressing the previously constructed concept or construct as relevance to the situation or problem being solved; building-with refers to the student using and combining the recognized concepts in order to devise a strategy, a justification, or a solution to a problem at hand; and constructing comprises the assemblage and integration of previous concepts to produce a new concept or construct (Dreyfus et al., 2015). The analysis of the mathematical representations used in the abstraction processes was employed the Three Worlds of Mathematics framework.

As mentioned in the section of the research design, an *a priori* analysis was conducted to identify elements of knowledge which were expected to be constructed by the students. They were the expected or anticipated constructs that emerged when the students were dealing with the tasks and performing abstraction processes. This *a priori* analysis produced a predicted learning trajectory (Ron et al., 2010), namely, the structure of knowledge elements for the concept. The results of this *a priori* analysis are presented in Chapter 4 (see Table 4.1, p. 124). In the context of the content analysis, this structure represented the categories that were used in analyzing the data. The learning trajectory was used as a basis for the Abstraction in Context analysis of the abstraction processes performed by the students in constructing their understanding of the operational formal definition of function.

The implementation of Abstraction in Context analysis followed the step-by-step procedures which were described by Dreyfus et al. (2015). The interview transcripts were read to identify the sections where the epistemic actions of constructing have occurred. The actions or utterances were marked as the end of the constructing action. Further, being led by the need to understand and interpret, the analysis moved backward and forward to identify the utterances expressing the recognizing and building-with actions which contributed to the identified constructing action.

3.8 Ethical Considerations

The following describes the ethical concerns that I addressed in this study. The first issue concerned *access to and acceptance in the institution* where the research was conducted (Cohen et al., 2018). In this case, I found no significant problems in obtaining permission to conduct the research from the head of the department as he knew me professionally. Because the structure of the institution was clearly hierarchical in authority, I used the top-down approach to gain assent and cooperation (Cohen et al., 2018). After receiving an approval from the dean of the Faculty of Mathematics and Natural Science, I met the head and the secretary of the Mathematics Department as well as the head of the Mathematics Education Study Program as the second level of authority. They approved the research to be implemented in a regular class of Mathematics Education program. They decided to put me in charge of teaching the unit of Introductory Calculus in that class, partnering with another lecturer. In implementing the research this lecturer helped as an observer/co-operative colleague.

The second issue was *privacy*. Privacy concerned various aspects of a research project such as respondents, instrumentation, data collection and analysis, and reporting (Cohen et al., 2018). Privacy dealt with the sensitivity of the given information, the observed setting, and information dissemination (Diener & Crandall, 1978). Students in the class of Introductory Calculus had the right not to participate in the research, and if they did participate, they had the right to not respond to certain questions they felt threatening to their privacy. Privacy had three corollaries, that is, anonymity, confidentiality, and informed consent (Cohen et al., 2018).

Anonymity was one way of dealing with privacy. Anonymity was to guarantee information gained from the participants would not reveal their identities. Confidentiality was a way to protect the participants' right to privacy. Confidentiality referred to avoiding revealing information from the respondents that could identify them or could make their identity traceable. Anonymity and confidentiality were dealt with in reporting the research. They were maintained by deleting the identifiers of the subjects (Frankfort-Nachmias & Nachmias, 1992). I made an agreement with the interviewees that I would use pseudonyms instead of using their real names.

The next issue was *the informed consent*, that is, the procedures for individuals to decide whether to be involved in the research after being informed about all aspects which could affect their decisions (Diener & Crandall, 1978). This informed consent, according to Diener and Crandall (1978), included four components. First, *full information* meant that the consent was adequately informed. Second, *comprehension* entailed the fact that all information regarding the research project was understood by the participants. Third, *competence* entailed the ability of the respondents to make appropriate decisions after being informed about the research. Fourth, *voluntarism* implied the freedom of the respondents to choose to participate on the ground that they knew the possible risks of their involvement. In addition, this component assigned the right of self-determination by the participants to renegotiate and withdraw without any risk once the research had begun (Frankfort-Nachmias & Nachmias, 1992; Miller & Bell, 2002).

Earlier, in the first lecture, I introduced and explained the research which would be implemented. This introduction covered the activities of the research, the commitment expected from the participants, and the possible risks and benefits of being involved in the research. I discussed the research with the students again three weeks later at which time I delivered the consent form to students. The explanation also comprised the benefits the students could have by their involvement in the project. However, it was clarified to them that their participation would not affect in any way their final mark in the unit of Introductory Calculus. Their work on the tasks would not be counted for determining their final marks, but their involvement was likely to benefit them in the way that they would exercise their ability to clearly express their thinking. Also, when they decided to quit the research, there would not be any consequence for them. I encouraged their confidence so that they could participate without any anxiety. I explained to them about what they would do in the interview and what kinds of responses I needed from them. I also told them that they could quit from this research whenever they felt if they needed to do so. With all explanation and discussion, I considered that they understood and were well informed about the research. The students were all adults who were mature enough to make correct decisions.

Another ethical issue was *consequence* (Kvale & Brinkmann, 2009). In this case, I was fully aware of avoiding putting the respondents in a harmful situation. I told them that whatever responses I obtained from them, they would not reflect on their current study. I believed that their involvement in this study brought something different for them. Although the change, which resulted from their engagement, was hard to measure, I hoped that they acquired a new insight into and an awareness of how to think mathematically and how to do mathematics. In addition, thinking aloud should have made them aware of the importance of having structured thinking and how to explain this to others. To a certain degree, I preserved a "deliberate naiveté" attitude (Kvale & Brinkmann, 2009, p. 30) in order to provide a broader space for them to express their responses. During the fieldwork, the respondents appeared to feel a great involvement in this research. They were impressed when the transcript of their interview was given back to them for member checking. It was a valuable experience for them, although some of them expressed the view that they actually did not need to check the transcript as they believed that the transcript was already fine and validly contained their correct utterances.

3.9 Legitimation

There is a general agreement on the need to demonstrate the legitimation of the qualitative interpretive inquiry. The judgment of whether a study is legitimate is performed against specific quality standards. Taylor (2014) mentions the twin criteria of trustworthiness and authenticity as fundamental standards for the quality of interpretive research. However, he reminds us that they are not meant to serve as a prescriptive straitjacket. The researcher is to adapt and adopt those criteria according to the epistemological and practical considerations of each investigation. For this present interpretive case study research implemented in the teaching interview framework, I followed the suggestion of Confrey and Lachance (2000) who adopted elements of trustworthiness criteria proposed by Guba and Lincoln (1989), namely, credibility, dependability, and confirmability, along with the suggestion of Steffe and Thompson (2000) and Kvale and Brinkmann (2009) concerning the generalizability, regarding standards of quality.

3.9.1 Credibility

Credibility is argued as one of the most determining aspects in establishing trustworthiness (Lincoln & Guba, 1985). A judgment of credibility was based upon how well the perspectives constructed by the respondents matched the versions constructed by the researcher (Confrey & Lachance, 2000). In other words, credibility is justified by the answer to the question of “How congruent are the findings with reality?” (Merriam, 2009, p. 213). The mathematics of students which was constructed from the data in this present study was considered credible if it correctly modeled the students’ mathematics. The model which credibly representing the constructions of the respondents should be developed from credible data (Confrey & Lachance, 2000). The credibility of the study lies the “the confidence in the truth value or believability of the study’s findings” (Jeanfreau & Jack, 2010, p. 616). From the respondents’ viewpoint, the constructed model or interpretation described by myself in the research results was credible when participants could recognize the model as their own (Beck, 1993). There are various ways proposed by Shenton (2004) to justify the credibility of the research, some of which were adopted in this research. They are presented in the following paragraph.

The credibility of the study was established through prolonged engagement with the research context. Before conducting the research, I had several lectures in the class of Introductory Calculus. These were considered as adequate length to become familiar with the students’ learning culture in the class. During the lectures, I built a good rapport with the students (Cohen et al., 2018). I expected that they would be more confident to be involved in the research and be open with their thoughts during the interviews. Thus, the data gathered would be more credible and more congruent with the real knowledge of the students (Merriam, 2009). At the same time, the students familiarized with the thinking aloud strategy. In addition, the students were encouraged to be frank in expressing their thoughts and were convinced that their work on the problems presented during the research would not be counted into determining their mark on the unit of Introductory Calculus. A peer who was a mathematics education academic of an Indonesian university was invited as a cooperative colleague to scrutinize the research project. I discussed with him the research design, the instruments, and particularly the interpretation of the data, from which the feedback he provided was taken into account in refining some aspects of this research project. Further, the feedback was received from the thesis supervisors. Furthermore, one supervisor focused on validating the data analysis and interpretation. During the teaching interview, some probing and iterative questions were employed to gain more clarification of the responses, which resulted in more detailed and accurate statements or utterances. In addition, member checking was also performed to improve the degree of credibility. In this case, the transcriptions of the interviews were sent to the respective pair of respondents to check whether the statements or utterances were correctly transcribed and accurately matched what they actually intended to express during the teaching interview.

3.9.2 Dependability

Dependability is closely tied to the notion of credibility (Lincoln & Guba, 1985). The relation between the two is that justification of credibility goes some distance in establishing the dependability. The concept of dependability speaks to the stability of the data (Confrey & Lachance, 2000). It is a conceptualization of the reliability which is embraced in quantitative research. The focus on the dependability was not on expecting the consistent research results found by an outsider conducting the same research, but rather, on expecting the outsider to concur that, given the data which had been gathered, the results were consistent and made sense (Merriam, 2009). In replicating a qualitative study it is hard to obtain the same results, because the object of investigation is highly contextual and multifaceted in nature. However, as Merriam (2009) argued, the different results do not discredit the findings of the previous or the next research. All studies are dependable if there is a consistency between the findings and data presented.

To ensure the dependability of this present study, the methodology covering the research design and its implementation, as well as the operational detail of data collection and analysis, were reported in detail to a level which enabled other researchers to replicate the work (Confrey & Lachance, 2000; Shenton, 2004). The detailed reports were expected to allow the outsiders to follow and justify the methodology, analysis, and interpretations. By all these details, the research report was also expected to enable the readers to develop a thorough comprehension of the research methods and their effectiveness. As previously mentioned, the dependability and the credibility were closely connected to each other. Member checking and peer scrutiny were also contributors to address the dependability of the research.

3.9.3 Confirmability

The standard of confirmability refers to the degree to which the findings of the research are derived from the data, and not from the whims of the researcher (Confrey & Lachance, 2000; Tobin & Begley, 2004). The quality of the research results is justified on how well outcomes were grounded on the data from respondents who were independent of the researcher. The independence of the findings of the researcher meant that they were not influenced by the biases, viewpoints, motivations, and interests of the researcher (Guba, 1981). If the results were confirmable, then an outsider or other researchers should be able to reconstruct them using the available data. The confirmable results meant that they were objectively interpreted from the respondent' data, which were free from the subjectivity of the researcher. The confirmability was achieved when the clear link of the research outcomes to the conclusion could be demonstrated and followed (Moon, Brewer, Januchowski-Hartley, Adams, & Blackman, 2016).

The confirmability of this research was established by providing an explanation of decisions which were made within the research processes. The methodology was described thoroughly in order to allow the reader to determine the extent to which the data, the findings, and their interpretation could be justified. The clarity of the research processes presented was to enable an outside observer to audit the implementation of the research. In order to reduce the potential effect of the researchers' bias on the research findings, the researcher's predisposition underpinning the decision made in terms of the methods was acknowledged in the report. It was completed with an explanation of this in the limitations of the study.

3.9.4 Generalizability

Specifically talking within the context of the teaching experiment which was adapted as a teaching interview in this research, Steffe and Thompson (2000) claimed that the generalizability of the mathematics of students as a model, developed for explaining the students' mathematics, is indicated by the usefulness of the model to organize and guide our experience of students doing mathematics in other contexts beyond the one where the model was originally developed. In this sense, the generalizability was not that the results of the research would be true for the context population from which the sample was drawn. The readers of the research appreciated the results and transferred them intuitively in their context. Through the naturalistic generalization, the readers' decision to use the findings was made based upon their understanding (Tracy, 2013). The decisions were facilitated by the detailed research description which allowed the readers to judge about the extent to which the findings could be generalized analytically in their situation (Kvale & Brinkmann, 2009). These particular notions of generalizability were equivalent to transferability, namely, the degree of the applicability of the research findings to other contexts (Lincoln & Guba, 1985).

To enable the generalization, the report of the present study was presented in a thick and rich description. The description comprised the detailed depiction of the setting, context, and respondents, along with a detailed account of the findings completed with adequate supporting evidence (Merriam & Tisdell, 2016). The thick description of the phenomenon under study allowed the readers to develop a proper understanding of the phenomenon (Shenton, 2004). The sufficient description which adequately contextualized the study enabled the readers to determine the match, or assess the similarity between their situation and the research context (Guba & Lincoln, 1989), and then to decide whether the findings could be transferred.

3.10 Conclusion

The aim of this study was to investigate the students' performance in constructing the operational formal definition of function in introductory calculus. For this objective, the inquiry was focused on the abstraction and mathematical representations used by the students in the process of formulating the definition of the concept of function. In order to answer the research questions, the present study was situated within the interpretivist paradigm which employed the explorative qualitative approach. The research was designed as a multiple-case study. It was implemented in the setting of the teaching interview which was an adaptation of the teaching experiment method.

The qualitative data were gathered from interviews, written documents, and observations. The students were interviewed while they were solving the calculus tasks particularly designed to facilitate the abstraction processes for constructing the operational formal definition of function. The ethical matter was addressed by ensuring the protection of the privacy and anonymity, as well as obtaining an informed consent. The research legitimation was established against the quality standard of trustworthiness covering the criteria of credibility, dependability, confirmability, and generalizability. The data were analyzed within the frameworks of the Abstraction in Context and the Three Worlds of Mathematics. The research results and the discussion are presented in the next two consecutive chapters. Chapter 4 is devoted to the first research question concerning the abstraction processes, and Chapter 5 is devoted to the second question concerning the use of mathematical representations.

CHAPTER 4
ABSTRACTION FOR UNDERSTANDING THE OPERATIONAL FORMAL DEFINITION OF FUNCTION

> "Thus, the reason most people have trouble with mathematics is not that they don't have the ability but that they cannot apply it to *mathematical abstractions*."
> (Devlin, 2000, p. 11)

4.1 Introduction

This present chapter deals with the data analysis and discussion in response to the research question focusing on students' abstraction in constructing the definition of function. The question addressed within this chapter is as follows: *How do students construct the operational formal definition of function?* As reported in Chapter 3 on the research methods, this research was a multiple case study implemented within a teaching interview framework which was adapted from a teaching experiment. The focus of the research question was the development of an understanding of the operational formal definition of function through the theoretical abstraction. The calculus tasks were the main instrument for data gathering. The tasks were used in the task-based interviews conducted while the students were working in pairs on the task from which students' works were also collected. The study involved five cases. Each case was a pair of students. The epistemic actions of Abstraction in Context were applied as the method of analysis to describe the abstraction processes from each case. For the product of the abstraction, namely, the understanding of the operational formal definition of function, the category analysis was used to describe the level of the constructed understanding.

The rest of this chapter covers three sections. The following section 4.2 presents the knowledge elements of the operational formal definition of function as a result of the *a priori* analysis of the concept under investigation. The next section, section 4.3, elaborates the abstraction performed by the pairs of students in constructing the operational formal definition of function. This elaboration will start with the general overview of the product of the abstraction which continues to a more detailed process of abstraction of a selected pair of students as a focused case. The abstraction processes performed by this pair is described in an episodic style (Dreyfus et al., 2015); each episode deals with the construction of the understanding of knowledge element. This section concludes with a summary of the results of all pairs of students.

In the presentation of analysis results, the interview excerpts quoted are completed with the number indicating the order of the utterances in the interview transcripts. Within an utterance, there is sometimes an insert indicated by [ ]. This is to show an object a student points at or refers to when he or she says the demonstrative pronouns such as ‘this’ or ‘that’. Some utterances in the interview segments have a shaded part at their right-side end. This part lists the interpreted epistemic actions the utterances contribute to the process of constructing knowledge element under investigation. The following is an example.

[0279] Sam	Different, different. For this [(2,4)], its origin point is 2; for this [(4,2)], its origin point is 4. The origin points are different.	R-ORE B-ORE

This was the 279th utterance in the interview transcript which was expressed by Sam. While uttering the sentences, he pointed to (2,4) and (4,2) on the worksheet. R-ORE and B-ORE indicated the epistemic actions interpreted from the utterance. R-ORE meant Sam recognized the knowledge element ORE and B-ORE meant he also built-with it.

This chapter ends with section 4.4 devoted to a discussion of the research with regard to the abstraction in constructing the understanding of the operational formal definition of function. This section provides an interpretation of all units of analysis in order to develop a holistic understanding of the abstraction as an answer to the focused research question.

4.2 Knowledge Elements for the Operational Formal Definition of Function

The study of the concept of function in university mathematics, particularly in introductory calculus, is based upon a deductive approach. It means that students have to justify whether a given relation is a function by employing a verification method which is deductive axiomatic in nature. The formal definition in Bourbakian style per se does not help the students adequately when they are to examine relations. Showing through a mathematical proof that *for each element x of the first set in the given relation, there exists a unique element y of the second set to which x is assigned* is not an easy thing. This proof requires a transformation of the formal definition into statements which separately clarify individual attributes of a function. The first attribute is that each element x of the first set is associated with at least one element y in the second set. The second one is that each element x that has partner(s) in the second set, has exactly only one y as its partner. In other words, each associated element of the first set has only one partner. This transformation results in an operational formal definition of a function. This is a kind of working definition by which we could formally verify whether a given relation is a legitimate function. The operational formal definition of function that follows is adapted from the definitions formulated by Bartle and Sherbert (2011), Daepp and Gorkin (2011), and Stewart and Tall (2015).

Let X and Y be sets and $f \subseteq X \times Y$. $f : X \to Y$ is a function if and only if:
(P1) $\forall x \in X, \exists y \in Y$ such that $(x, y) \in f$ and
(P2) $\forall x \in X, \forall y, z \in Y,\ (x, y) \in f \wedge (x, z) \in f \Rightarrow y = z$.

P1 can also be expressed: $\forall x \in X, \exists y \in Y,\quad f(x) = y$. P2 can also be stated: $\forall x \in X, \forall y, z \in Y,\ \ f(x) = y \wedge f(x) = z \Rightarrow y = z$. Condition P1 ensures that each element in X is corresponded to some element of Y, and condition P2 ensures that none of the elements of X are assigned to more than one element of Y. Logically, it implies that certain elements of Y might not be related to any element of X, or an element of Y is assigned to more than one element of X (Daepp & Gorkin, 2011). In this study, P1 is called the exhaustive domain property, while P2 is called the unique image property.

Given its presentation in terms of sets, the logic of the definition must be approached in a similar way to that of set theory. The concept of function is developed based upon the concept of relation. The concept of relation is developed hierarchically from the concepts of association between sets, ordered pairs, and the Cartesian product. The general relation is then made special with the two specific conditions mentioned above. These concepts, along with some other prerequisites for them, constitute the formal concept of function.

Based upon the above concept hierarchy, there are two overarching conceptual components necessary in the construction of the operational formal definition of function:

1. Knowledge about the concept of relation in general in which the students are expected to construct their conceptual understanding of relation based upon the concept of the Cartesian product of two sets.
2. Knowledge about the special properties of relation which classify it as a function. The students are expected to construct their conceptual understanding of the two special conditions of relation: exhaustive domain and unique image. Defining these two properties will make the operational formal definition of function. To deal with this knowledge, the students need to build-with their knowledge about relation.

The knowledge elements or the constituent concepts for each of the overarching components were identified and they were presented in Table 3.2 in Chapter 3 (p. 102). The aggregate of association between sets' elements (ASC), ordered pairs (ORP), the Cartesian product of two sets (CAR), and relation (REL) is named general relation (GRE). The students are expected to construct ASC as the basis for constructing ORP. Then, CAR is developed on the basis of ORP. Further, REL is established upon the concept of CAR. These paths reflect the hierarchical nature of elements of knowledge subsumed under GRE. ASC subsumes the membership of set (MEM) and Arbitrariness of Association (ARB). These two elements are assumed as previous constructs, which means that the elements are considered to have been already constructed by the students before working the designed calculus tasks. Under ORP, there could be ASC and another assumed previous construct, Order of Elements (ORE). Similarly, CAR is constructed upon ORP, and REL is build-with CAR and subset of a set (SUB).

Similarly, three knowledge elements form the component of special relation (SRE). They are the exhaustive domain property (EXH), the unique image property (UNI), and the operational formal definition of function (FUN), with the elements of EXH and UNI at the same level. Nevertheless, in practice, we mostly check EXH first before proceeding to examine UNI. They are combined to form a special relation called a function. Both EXH and UNI are developed based upon GRE. Other featuring elements of a relation, that is, domain of relation (DOM), codomain of relation (COD), and range of relation (RAN), also form the basis for, and therefore, are subsumed under EXH and UNI. GRE, DOM, COD, RAN are assumed as previously constructed elements for SRE. Adopting the style of presentation in Kouropatov and Dreyfus (2014), the structure of the elements of knowledge is depicted in Table 4.1 which includes a description and operational definition of all elements.

Table 4.1 The Structure of Knowledge Elements for the Operational Formal Definition of Function

			Intended Elements of Knowledge		
			Acro-nyms	**Description**	**Operational Definition**
1.	Knowledge about the concept of relation in the context of given sets	General relation (GRE)	ASC	"Association between sets' elements": Elements of a set can be associated with elements of other sets.	I will say that students have constructed this element, if, by any kind of representation, they associate, assign, or link elements of a set to elements of other sets, or they explain the association of elements of a set with elements of other sets.
			ORP	"Ordered pairs": An ordered pair (a,b) is a pair of mathematical objects, that is, members of a set. The order in which the objects appear is significant for that $a \neq b \Rightarrow (a,b) \neq (b,a)$. In other words, $(a,b) = (c,d) \Leftrightarrow a = c \wedge b = d$.	I will say that students have constructed this element, if, by any kind of representation, they identify or explain that the order of the elements involved in (a,b) as a result of an association is significant.
			CAR	"The Cartesian product of two sets": The Cartesian product of two sets, A and B, written as $A\times B$, is a set of all ordered pairs which associates each element of A with each element of B, or, $A\times B=\{(a,b)\vert a\in A,\ b\in B\}$	I will say that students have constructed this element, if, by any kind of representation, they list, identify or explain all possible ordered pairs resulting from associating the elements of any two sets.
			REL	"Relation": Let A and B be sets. R is a relation from A to B if R is the empty set or there exist elements of A assigned by R to elements of B. Or, $R: A\rightarrow B \Leftrightarrow R \subseteq A\times B$.	I will say that students have constructed this element, if, by any kind of representation, they identify or define a set of ordered pairs as a subset of the Cartesian product of two sets.

Table 4.1 The Structure of Knowledge Elements for the Operational Formal Definition of Function (continued)

		Intended Elements of Knowledge		
		Acro-nyms	Description	Operational Definition
		Assumed previous constructs		
		MEM	“The membership of set”: An object is an element of a set if it fulfills the definition of the set; otherwise it is not an element of the set.	I will say that students have recognized this element, if, by any kind of representation, identify or point out objects which are members of the sets. Otherwise, they can identify objects which are not in the sets.
		SUB	“Subset of set”: If all elements of *A* are elements of *B*, then *A* is a subset of *B*, or, $A \subseteq B \Leftrightarrow \forall x, x \in A \Rightarrow x \in B$.	I will say that students have recognized this element, if, by any kind of representation, they explain or identify that all elements of the first set are elements of the second set.
		ARB	“Arbitrariness of association”: An association between two sets could be arbitrary in nature, without a specific rule to follow.	I will say that students have recognized this element, if, by any kind of representation, they explain, show, or identify that an association between elements of two sets could be made without following a rule.
		ORE	“Order of elements”: The elements in pair (a,b) are ordered in that way different from those in the unordered pair $\{a,b\}$.	I will say that students have recognized this element, if, by any kind of representation, they explain, show, or identify the difference between ordered elements and non-ordered elements.
2. Knowledge about the concept of special relation in the context of given sets	Special relation (SRE)	EXH	“The exhaustive domain property”: Each element of the domain of relation is assigned to element(s) of the codomain. Let R: $A \rightarrow B$. R is an exhaustive domain relation $\Leftrightarrow$ $\forall a \in A$, $\exists b \in B$, $(a,b) \in R$.	I will say that students have constructed this element, if, by any kind of representation, identify, explain, or show that all elements of the domain of a given relation are assigned to element(s) of the codomain, and they could express the logic statement: Let R: $A \rightarrow B$. R is an exhaustive domain relation $\Leftrightarrow$ $\forall a \in A$, $\exists b \in B$, $(a,b) \in R$, or other equivalent statements.

Table 4.1 The Structure of Knowledge Elements for the Operational Formal Definition of Function (continued)

	Intended Elements of Knowledge		
	Acro-nyms	**Description**	**Operational Definition**
	UNI	"The unique image property": An assigned element of the domain is associated with only one element of the codomain. Let R: $A \to B$. R is a unique image relation $\Leftrightarrow$ $\forall a \in A$, $\forall b$, $c \in B$, (a,b), $(a,c) \in R \Rightarrow b = c$.	I will say that students have constructed this element if, by any kind of representations, they explain or show that each of all assigned elements of the domain of a given relation is associated with only one element of the codomain, and they could express the logic statement: Let R: $A \to B$. R is unique image relation $\Leftrightarrow$ $\forall a \in A$, $\forall b$, $c \in B$, (a,b), $(a,c) \in R \Rightarrow b = c$, or other equivalent statements.
	FUN	"The operational formal definition of function": Let A and B be sets, and f is a relation from A to B. If f fulfills the properties of exhaustive domain and unique image, then f is a function. Let f: $A \to B$. f is a function $\Leftrightarrow$ $\forall a \in A$, $\exists b \in B$, $(a,b) \in R$ $\wedge$ $\forall a \in A$, $\forall b$, $c \in B$, (a,b), $(a,c) \in R \Rightarrow b = c$.	I will say that students have constructed this element if, by any kind of representations, they define a function with as relation which each element of the domain has a partner in the codomain, and there is only one partner for each element of the domain, and they could express the logic statement: Let f: $A \to B$. f is a function $\Leftrightarrow$ $\forall a \in A$, $\exists b \in B$, $(a,b) \in R$ $\wedge$ $\forall a \in A$, $\forall b$, $c \in B$, (a,b), $(a,c) \in R \Rightarrow b = c$, or other equivalent statements.
	Assumed previous constructs		
	GRE	"General relation": As detailed in Number 1 above.	As detailed in Number 1 above.
	DOM	"Domain of relation": The set of all first elements of the ordered pairs (a,b) which belong to the relation. The set A as the first set in a relation R: $A \to B$ is the domain of the relation.	I will say that students have recognized this element if, by any kind of representations, they identify, make a set of, or point out all assigned elements of the first set, or they explain the properties of those elements as the member of the first set in a given relation.

Table 4.1 The Structure of Knowledge Elements for the Operational Formal Definition of Function (continued)

	COD	"Codomain of relation": The whole set B as the second set in a relation R: $A \rightarrow B$ is the codomain of the relation.	I will say that students have recognized this element if, by any kind of representations, they identify, make a set of, or point out all elements of the second set, or they explain the properties of those elements as the member of the second set in a given relation.
	RAN	"Range of relation": The range of the relation is the set of all second elements of the ordered pairs (a,b) which belong to a relation R.	I will say that students have recognized this element if, by any kind of representations, they identify, make a set of, or point out all elements of the second set to which the elements of the first set are assigned, or they explain the properties of those elements fulfilling the given relation.

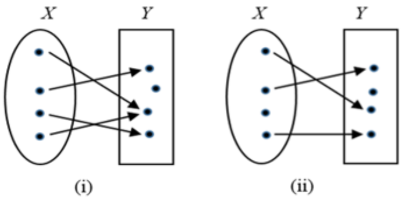
Notes: The definitions of the Cartesian product, relation, domain, codomain, range, and function are adapted from Gallier (2011), Rosen (2012), and Stewart and Tall (2015).

The calculus tasks were designed so as to give the students chances to construct the knowledge elements described in the table. Working on the tasks was expected to lead the students to construct the operational formal definition of function. The tasks are presented in the following figures. In Figure 4.1, the tasks are for the construction of the concept of general relation. In Figure 4.2, the tasks consist of several items continuing from Number 2 in Figure 4.1. They are designed for the construction of special properties of relations, that is, the exhaustive domain property and the unique image property.

1. The followings are pairs of two sets. What kinds of association between members of the two sets could you make?
(a) $M = \{a, b, c\}$ and $K = \{x \mid 2 < x < 4, x \in \boldsymbol{N}\}$.
(b) $P = \{z \mid z^2 + 2z + 2 = 0, z \in \boldsymbol{R}\}$ and $R = \{6, 7\}$.
2. Let A and M be two non-empty sets, $k \in A$ and $w \in M$. If we associate k to w, we can express the result (k, w) as an element of the association. Given two sets: $L = \{2, 3\}$ and $C = \{2, 4, 6\}$.
(a) Write all the results of the association if each element of L is associated with C's elements whose values are greater than that of the element of L.
(b) Write all the results of the association if each element of C is associated with L's elements whose values are less than that of the element of C.
(c) Are the results in (a) the same as in (b)? Why? Explain your reason.
(d) Write all the results of the association if each element of L is associated with C's elements which are even numbers.
(e) Compare the results in (a), (b), and (d). What is the uniqueness of the results in (d) compared to those both in (a) and (b)?
(f) Let us group the results in (a) and (d), respectively, into different sets. Take into account the two initial sets involved, that is, L and C. What relationships could be identified between the two sets? What are these two sets? If you can define this special mathematical object, what will be your definition?

Figure 4.1 Tasks for Constructing the Concept of General Relation

2. Let A and M be two non-empty sets, ….
(g) Look at the results in (a) and (d). Group them into different sets, respectively. In (f), as a special mathematical object you have named these two set, and have defined the object as well. What commonality do they have which does not belong to the set of results in (b)? Express this special property. How do you define the object which has this particular property?
(h) Look at and compare the pictures beside. What differences could you identify? If you focus on (ii), what is its special property? Express this special property. How do you define the object which has this particular property?



(i) Consider the two special properties, each you have discussed in (h) and (g). Construct a relation which fulfils both of the properties.
(j) How do you define the special relation you have constructed in (i)?

Figure 4.2 Tasks for Constructing the Concept of Special Relation

4.3 Abstraction for Constructing the Operational Formal Definition of Function

In this section, the abstraction processes carried out by students for constructing the operational formal definition of function are presented. The processes took place while the students were working on the calculus tasks designed for the topic of association between sets. They worked in pairs solving the problems as presented in Figure 4.1 and Figure 4.2. The abstraction the students performed is analyzed in general, which is then followed by a detailed analysis of one pair. Each of the seven knowledge elements (four from general relation and three from special relation) is described in a different episode following the hierarchical order as explained in the previous section. All acronyms which are used refer to those described in Table 4.1 above.

4.3.1 A General Overview of the Students' Performances

In abstraction for understanding the concepts of introductory calculus, the performances of the students in answering the calculus tasks were classified into three categories: complete understanding, partial understanding, and limited understanding. In order for a pair of students to be considered as constructing complete understanding (CU) of the concept, they should (a) state, express, show, identify, explain, or perform things which match or have the same or equivalent meaning as described in the operational definition of the concept or knowledge element; (b) use correct mathematical terminologies and notations; (c) identify all the important elements of the concept and show understanding of the relationships between them; and (d) present supporting arguments which are logically sound and complete.

To be justified as having constructed partial understanding (PU), the students should (a) state, express, show, identify, explain, or perform things which partly match the operational definition of the knowledge elements; (b) use some incorrect mathematical terminologies and notations; (c) identify some important elements of the concept and show general understanding of the relationships between them; and (d) present some arguments which are logically sound and contain some gaps. And, for the category of limited understanding (LU), they should (a) state, express, show, identify, explain, or perform things which mostly do not match the operational definition of the knowledge elements; (b) misuse or fail to use mathematical terminologies or notations; and (c) fail to identify important elements or places too much emphasis on unimportant elements.

The categorization of constructed understanding above is used in the analysis of the whole data of all concepts under investigation. The students' performances regarding their understanding of the concepts or knowledge elements are presented as in the following table. In this table, the cells are filled with the status of the students' understanding, namely, CU, PU, or LU, judged based upon their responses to the problems presented in the calculus tasks.

The overall performance of all students' abstraction in constructing the operational formal definition of function is presented in Table 4.2. As could be observed in the table, by and large, all participating students have constructed their complete understanding of the concept of general relation (GRE), which covers the association of sets' elements, ordered pairs, the Cartesian product, and relation. In essence, all students could follow the hierarchy of the concepts, which started from the associating the elements of two sets, to understanding the difference between an ordered pair and a set, to the Cartesian product as a set of ordered pairs, and culminated in a relation as a subset of the Cartesian product.

Table 4.2 The Performance of Participating Pairs of Students in Constructing the Operational Formal Definition of Function

Knowledge Elements	**Student Pairs**				
	Iful–Vito	Dina–Yuni	Dyn–Sam	Naya–Amzi	Adel–Tanti
Association between sets' elements (ASC)	CU	CU	CU	CU	CU
Ordered pair (ORP)	CU	CU	CU	CU	CU
The Cartesian product of two sets (CAR)	CU	CU	CU	CU	CU
Relation (REL)	CU	CU	CU	CU	CU
The exhaustive domain property (EXH)	CU	CU	PU	CU	PU
The unique image property (UNI)	CU	CU	PU	PU	PU
Operational formal definition of function (FUN)	CU	CU	PU	PU	LU

Note: CU: Complete Understanding PU: Partial Understanding LU: Limited Understanding

The difference among their performance, which could be identified in the data, covers aspects such as the approaches they used in working on the problems, the use of symbolic logic expression, and the time they spent to finish the problems. Iful and Vito were the only pair implementing the random approach in working on the problem for the association of sets' elements, while all the other pairs used a systematic approach.

Iful and Vito were also the pair that finished their work more quickly than the others, while Adel and Tanti spent the longest time, with prolonged discussions, in working on the problems. Although there were few variations in the way the students expressed statements in symbolic logic style, the variations could still be accepted for they did not influence the essential meaning of the statements.

In terms of the special properties of relation, students performed differently. Iful–Vito and Dina–Yuni successfully completed the construction of the operational formal definition of function. Naya–Amzi could develop a complete understanding of the exhaustive domain, but facing difficulties in the concept of unique image property, and consequently they could not devise the operational formal definition of function. Dyn–Sam identified the two special properties of relation correctly, but they could not express the properties in an appropriate formal logic statement. Adel–Tanti encountered problems in understanding both special properties on relation, so they did not proceed to define a function in the operational formal style.

4.3.2 Focused Pair – Dyn and Sam: How Did They Construct the Elements of Knowledge?

I will particularly focus on Dyn and Sam on the abstraction for the operational formal definition of function. There were two reasons to choose this particular pair as the focused one. First of all, this pair was the most dynamic in their discussion while working the calculus tasks. As the first experience of thinking aloud practice, the conversation between the students flowed more easily compared to the other four pairs. Dyn and Sam actively asked each other for clarification of expressions, statements, or steps each of them performed, and therefore the data were richer. Secondly, they did not succeed in developing a complete understanding of all targeted concepts, however, essentially they identified all concepts appropriately. This pair of students attempted to express the concepts formally in logical symbols which resulted in non-standard style quantified statements. The quantification of the statement was worth focusing on. The presentation of the results is organized into episodes. Each episode deals with a different element of knowledge with some possible overlapping which might be found between two episodes.

4.3.2.1 Episode 1: Constructing Associations Between Sets' Elements

Working on the task Number 1 in Figure 4.1 (see p. 128), Sam first read the question. After Sam read the question, Dyn suggested that elements of the two sets must be identified. Dyn also found that the element of *K* was clear. Given the second set *K* was expressed in the set-builder notation instead of listing its element, Sam easily found that *K* contained only one element, that is, 3. With the element of *K* determined, Sam went further by stating that they just needed to make associations and questioned how they should represent the associations they made [0004, in the following excerpt].

[0003]	Dyn	First, we determine the elements of each set, respectively, because the elements of set *M* are there already, that is, *a*, *b*, and *c* … and that set *K* … its element is clear, but we need to write it first.	R-MEM
[0004]	Sam	So, the element of set *K* is 3. So … the association … make an association between elements …. So, we just make an association, should we use a diagram?	B-MEM R-ARB
[0005]	Dyn	An arrow diagram.	

Based upon the operational definition of MEM, it is interpreted that Dyn and Sam recognized MEM for they knew that the two sets had at least one element, hence non-empty sets [0003–0004]. The recognition of MEM was evident in [0003], where Dyn uttered that the element of *K* was clear and just needed to be written. A need for the ASC to be constructed could be identified in Sam's utterance in [0004], "So … the association … make an association between elements …." Assuming that Sam also recognized MEM, it could be found that in [0004] he built-with MEM by ascertaining the element of *K*. The non-emptiness of the sets allowed them to make an association between the elements of the two sets. The statement of Sam [0004] "So we just make an association …" prior to questioning the possible use of an arrow diagram to represent the association indicated that he also recognized the arbitrariness (ARB) of the association they were asked to make.

[0010]	Sam	The first association is only *a* and *K*, isn't it?	
[0011]	Dyn	It could be only *a* with 3, *b* has no association with 3, and *c* has no association with 3.	B-MEM B-ARB
[0012]	Sam	So, the first association is *a* with 3.	B-MEM B-ARB

Next, in [0011–0012], the constructing of ASC was taking place when Sam offered the first association that was from *a* in *M* to the only element of *K*, which was agreed on by Dyn. He then elaborated more about this association: “It could be only *a* with 3, *b* has no association with 3, and *c* has no association with 3” [0011]. Here, Dyn built-with MEM and ARB, which had been recognized earlier in [0003–0004]. Sam clearly mentioned the elements of the first and the second sets, and he made an arbitrary association between them. Further, he mentioned the association between only *b* with 3 and, next, only *c* with 3. Sam went with the second model where only 2 elements of *M* were associated with 3 in *K*. Building-with the concept of arbitrariness, both were not being trapped in thinking about what rule they had to follow in assigning the elements of the first set to the element of the second set. Dyn and Sam looked for arbitrary linkages possibly made between the elements of *M* and *K*.

The next discussion portrays the process Dyn and Sam undertook in constructing the association between sets’ elements which built-with MEM and ARB right since they make the first association. Dyn and Sam employed a systematic approach by building-with MEM and ARB. This helped them see all possible models of association easily. They started by pairing 1 element of *M* to the element of *K*, then pairing 2 elements of *M* to the element of *K*, and the last is pairing the whole 3 elements of *M* to that of *K*. It could be observed that they shared their understanding of associating sets’ elements. On one utterance, Dyn offered an example of possible association which was justified by Sam, and on another utterance, it was Sam suggesting an association which was approved by Dyn [0017–0023].

[0017]	Dyn	Only *b* having an association with the set … having an association with the set *K*, that is, 3.	B-MEM B-ARB
[0018]	Sam	Yes … an association of *M*. Amm … associating *a*, *b*, and *c* … *b* …. Amm … the third association ….	B-MEM B-ARB
. . .			
[0022]	Sam	The fourth association has *a* and *b* only, and not *c*. The fifth association could *b* and *c* only, not *a*, or could *a* and *c*, *b* has no association. OK.	B-MEM B-ARB
[0023]	Dyn	*a* and *c* only, *b* has no association. OK.	B-MEM B-ARB

Finally, they concluded that there existed seven possible associations could be made.

From the excerpts, it can be concluded that Dyn and Sam constructed their understanding of the association between sets' elements, based upon recognizing and building-with MEM, and ARB. They listed 7 models (Figure 4.3). The illustrations made by Dyn and Sam in the form of an arrow diagram indicate that they already apply the idea of direction in their answer. Even though it is hard to find the symbol of arrowhead in the figures, but it could be interpreted that they make those associations from *M* to *K*, for they put *M* in the left side.

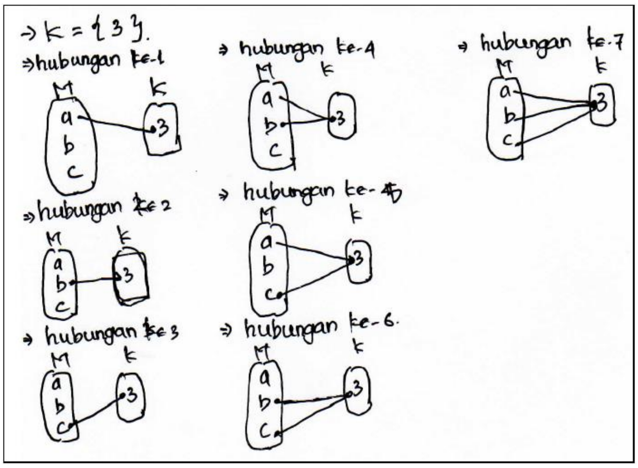


Figure 4.3 Models of Association Between the Elements of
M and *K* Made by Dyn–Sam

How did Dyn and Sam deal with the empty set? Moving to Part 1b (Figure 4.1, p. 128), Dyn and Sam expressed that there was no association that could be made between elements of the two sets because one of them is the empty set. Dyn and Sam had a prolonged discussion when working on the second pair of sets, *P* and *R*. Set *P*, expressed in a set-builder notation, is the empty set. Its elements are defined as real numbers z that are roots of a quadratic equation, $z^2 + 2z + 2 = 0$. With this set, the students are expected to contrast two conditions for the membership of *P*, and then decide the emptiness of *P*. Dyn and Sam could find the roots of the quadratic equation using the root formula, and also realized that the roots are imaginary numbers. After obtaining the roots of the equation, Sam further examined them because they were imaginary numbers, while the elements of the set must be a real number.

Initially, Dyn showed confusion in terms of the membership of set *P*. Dyn thought that an additional property of z would not restrict z itself; instead, it would expand the coverage of z. In this case, satisfying the given quadratic equation is the first condition of z. Here, z is found to involve imaginary numbers. But, the second condition states that z is a real number. So, Dyn thought that this condition would change z from imaginary to be a real number [0097], a claim which was confirmed by Sam [0098].

[0097]	Dyn	It means that it is real, because it is z, while, z is a real number.
[0098]	Sam	So, it means that these [the roots] are real numbers.

After being reminded to reflect carefully on the conditions of the elements of P, that they are in a conjunctive statement.

[0063]	Dyn	They are not suitable … here its z is a real number and what we get is an imaginary number.
[0064]	Lec	So, how is it?
[0065]	Sam	Amm … how?
[0066]	Dyn	Nothing, there are no elements of set P … because ….

They realized that the second condition matters [0063]. He considered this condition which then leads him to conclude that "Nothing, there are no elements of set P …" [0066]. They finally could deal appropriately with the two conditions for the elements of P, and both arrived at the conclusion that P was the empty set [0132–0133], because no numbers could be in P.

[0132]	Sam	It means P is … the empty set.
[0133]	Dyn	It means, these [the roots of the equation] are not its members. These [the roots of the equation] are not P's members.

With regard to the association, Dyn and Sam concluded that no correspondence could be made between the elements of P and R, because there were no elements in P to associate with.

[0231]	Dyn	It means no association, because what we can associate with are its elements. If it has no element, meaning it can't be related, right?
[0233]	Sam	Is that so? No association?
[0233]	Dyn	So, the association between set P and set R ….
[0234]	Sam	So, the association between set P and set R, they have no association.

From the excerpts, it could be observed that Dyn and Sam responded quite straight to the point being discussed. Collaboratively, they thought about the kinds of association they could make with an understanding of implicit arbitrary nature of the requested associations. In terms of the existence of a set's members, Dyn and Sam could decide whether a set is empty or not after determining the existence of an element of the set. Dyn and Sam realized that whenever two or more conditions are contradictory to each other, it means that nothing will satisfy them. And, for a set, it implies that the set has no elements, hence the empty set. It could be found that Dyn and Sam constructed the element of the association between sets' elements, which was built-with the elements of membership of set and the arbitrariness of association. They have constructed *a complete understanding of the association between sets' elements.* Therefore, it could be concluded that the element of MEM, along with ARB, was nested in ASC.

4.3.2.2 Episode 2: Constructing Ordered Pairs

The direction is an important consideration in associating elements of sets. Sam's question [0004] about the use of a diagram was considered as the utterance expressing a need for the element of order, which was responded to by Dyn with his suggestion to use an arrow diagram [0005]. In Number 2 of the calculus tasks (Figure 4.2, p. 128), the students set to work further on associating the elements of two sets, with particular rules and direction for the association. A way of expressing the results of the association of sets' elements was introduced to the students, that is, an ordered pair. Working on questions 2a–2c was expected to encourage them to construct the knowledge element of ordered pairs (ORP). Based upon the operational definition of this element, the students' construction of the ordered pair concept could be justified when they showed an indication of understanding that the order of the elements in a pair (a,b) does matter, which means $(a,b) \neq (b,a)$ if $a \neq b$ and $(a,b) = (c,d)$ means $a = c$ and $b = d$. The knowledge elements ASC, MEM, and ORE could be the bases upon which the ordered pair concept was constructed.

Recognizing the elements of ASC and MEM, Dyn and Sam made a correct list of pairs resulting from the associations between two given sets, *L* and *C*, following the rule for each association.

[0251]	Sam	It means … yes, right … it means 2 sets, right? We relate each … set *L* to set *C* which is greater, which the elements of *L* are greater than those of *C*. It means 2 goes to 4 and 6, 3 … gets 4 and 6. Right?	R-MEM R-ASC

Sam drew an arrow diagram and then writes a list: (2,4), (2,6), (3,4), and (3,6). For the second association, Dyn mentioned the results:

[0258]	Dyn	It means 4 comma 2, 4 comma 3, 6 comma 2, 6 comma 3.	

Constructing the concept of ordered pairs was expected to take place when they worked on the question asking about whether the two lists of results of associations were the same or different. Dyn and Sam had an identical conclusion that the results of the two associations were different from each other. They mentioned that the results were different because they were from different rules of the association.

[0262]	Dyn	To me, they are different.	
[0263]	Sam	They are different, right? Different because of the condition of its association. In Part a, it associates elements of *L* with elements of *C* whose values are greater than those of *L*. In Part b, elements of *C* to elements of *L* whose values are less than those of *C*. It means they are different.	B-ASC B-MEM
[0264]	Dyn	Different. The types of associations are different.	B-ASC
[0265]	Sam	Because the types of associations, the types of associations from set *L* to *C* and *C* to *L* are different.	

Sam considered the rules of the association and the sets involved in his reason for the conclusion [0262]. Moreover, Sam recognized the concept of order (ORE) in associating the elements of two sets [0265]. Dyn only mentioned the difference between the rules of the association [0264]. This was interpreted that he built-with the concept of association (ASC). They did not take 'order' into account in their explanation. The expectation in this episode is that the students could understand the difference between (a,b) and (b,a) as the order in the former pair is inverted in the latter one.

[0280]	Dyn	If it is written like this, 4 comma 2 and 2 comma 4, they are different, right?	
[0281]	Sam	Different, different. For this [(2,4)], its origin point is 2; for this [(4,2)], its origin point is 4. The origin points are different.	R-ORE

Further, it is apparent that Dyn [0280] had not fully understood this concept of order. The element of ORP which he had just constructed was still fragile. Conversely, in Sam's response [0281], it was evident that he understood the concept of ORP. Sam recognized the order of the elements (ORE), where he called the first element as the origin point, and two pairs were different if their origin points were different.

The exchange between Dyn and Sam continued. Dyn had another thought of inverting the order of elements in a pair.

[0319]	Dyn	If they could be inverted, the same.
[0320]	Sam	Of course they are different. 2 comma 4, 4 comma 2.
[0321]	Dyn	But the relationship just involves 2 and 4. I can also invert them.

Sam [0323] firmly built-with the concept of order for his reason, and even mentioned 'order' specifically, "… So, we couldn't write 4 comma 2 as 2 comma 4, because if we do it … orderly, the position shows which one is the origin, which one is the result." Responding to this reasoning, Dyn still looked at the result to be the same. After being asked about the sameness of the ordered pairs, Dyn attempted to think about the association again [0326]. Next, he concluded that the results were different. Actually, there was no utterance showing the exact reason for Dyn's conclusion. It just could be interpreted that the reason clarified by Sam [0327] was agreed on by Dyn, when he restated his conclusion [0328], and this was considered the end of the process of constructing the ordered pairs. Following the flow of the constructing process of ORP, it could be concluded that, in essence, ASC and ORE were nested in ORP. The element of MEM was recognized and built-with at the beginning of the process, and it was nested in ASC.

[0323]	Sam	To me, 2 comma 4 couldn't be written as 4 comma 2 because this writing shows which one is the origin, which one is the result. So, we couldn't write 4 comma 2 as 2 comma 4, because if we do it … orderly, the position shows which one is the origin, which one is the result.	B-ASC B-ORE
[0324]	Dyn	But, in the end, they are the same. The result, they are just the same.	
[0325]	Lec	Which ones are the same?	
[0326]	Dyn	The association between 2 and 4 … if it is an association between 4 and 2 …, oh, different.	
[0327]	Sam	Different, because, for example, like this, for example, *C* to *L* where the association is less than … the elements of *L* which are less than the elements of *C*. It means, 2 is less than 4. If it is inverted, 4 is less than 2, it is wrong.	B-ASC B-ORE
[0328]	Dyn	They are not the same.	

A phenomenon is identified in terms of the words used. The confusion experienced by Dyn about the sameness between (2,4) and (4,2) could be due to his view of an association which was focusing only on the elements involved, not including the rule for the association. In [0321], Dyn said "But the relationship just involves 2 and 4. I could also invert them." And, he concluded "But, in the end, they are the same. The result, they are just the same" [0324]. Such a thought might also be related to the use of the phrase 'association between' [0326] which could lead someone to think that the association is two ways in nature. This is correct in particular for some associations which satisfy the commutative property. However, in the general context of ordered pairs, the *direction* does matter; hence, the order of elements is significant, and $(a,b)=(b,a)$ if and only if $a=b$.

4.3.2.3 Episode 3: Constructing the Cartesian Product of Two Sets

In the question items 2d and 2e in Figure 4.1 (see p. 128), the students worked on an association which *pairs each element of the first set to each element of the second set*, and then compare the results of the association to those resulting from the association in items 2a and 2b. These problems were designed for facilitating the construction of the element of the Cartesian Product of Two Sets. The students are expected to be able to identify the uniqueness of the association, compared to the other previous associations. The set of all ordered pairs resulting from this association is the Cartesian product of the sets involved.

In his response to the question [0267], Sam built-with MEM, ACS, and OPR. In early utterance [0267], Sam built-with both MEM and ASC. When he mentioned that "Set *L* is 2, 3 ...," it was building-with MEM, and he continued "It means, the elements of *L* to all, right?" which indicated that he knew the membership of *C* of being all even, so the elements of *L* must be associated with them all. It could be observed that Dyn and Sam did not explicitly mention the concept of ordered pairs which had been constructed previously. Sam wrote the result of the association as a list of ordered pairs [0269]. This was not rejected, and therefore interpreted and agreed on by Dyn. I interpreted that they were using or building-with the concept of ORP in their answer.

[0267]	Sam	(Reading the question) Write all results of the association where each element of *L* …. Set *L* is 2, 3. For *C*, it is 2, 4, and 6. It means, the elements of *L* to all, right? So, 2 to 2, 2, to 4, 2 to 6, then 3 to 2, because all are even, right?	B-MEM B-ASC B-ORP
[0268]	Dyn	The elements of *C* are all even.	B-MEM

[0269]	Sam	If each element of *L* is associated with elements of *C* which are even numbers … so, it could be written … the association of the elements of *L* with the elements of *C*, that is, 2 comma 2, 2 comma 4, 2 comma 6, 3 comma 2, 3 comma 4, and 3 comma 6.	B-MEM B-ASC B-ORP

The following excerpt is considered as the constructing process of CAR. Dyn and Sam seemed to rather easily identify the uniqueness of the results from Part d, that all elements of the first set went to each of the elements of the second set [0272–0276], and those of a and b did not have this special property [0277–0278]. However, their utterances actually did not correctly express the uniqueness they had found. This was a problem of how to describe a situation properly. For example, in [0276], Dyn said, "Elements of set *L* and elements of set *C* have a partner," of which the meaning was more general than that of the Cartesian product of *L* and *C*. The list of results of such an association could be exemplified simply by the following: (3,2), (2,4), (3,6), which is different from that being discussed in Part d: *Associating each element of L to each element of C*.

[0272]	Dyn	It [Part d] covers all.	
[0273]	Sam	All elements of *A* have partners …. How? What is unique?	
[0274]	Dyn	What is unique in d is all elements of the sets have a partner either in *L* or in *C*. All elements of *L* and *C* have a partner.	B-MEM B-ASC
[0275]	Sam	Oh, yes, yes. In the results of association in d, each element of *L*, what?	
[0276]	Dyn	Elements of set *L* and elements of set *C* have a partner.	B-MEM B-ASC
[0277]	Sam	While, in Parts a and b ….	
[0278]	Dyn	Not all elements have a partner.	

Regarding the dynamic of epistemic actions, it could be inferred that in the process of constructing CAR, there existed MEM, ASC and ORP which were all nested in CAR. Hierarchically, MEM was subsumed under ASC, and also ASC was under OPR. However, some utterances pertained to the elements individually, such as in [0272], where Dyn built-with MEM, and then in [0274] where he built-with both MEM and ASC.

After this segment, Dyn raised again a question about the sameness of ordered pairs. When they returned to the discussion of CAR, Dyn still held a similar statement regarding the uniqueness of the association in Part d.

[0365]	Dyn	In Part d, each element of set *C* and set *L* all them have partners. If Part a, set *C* has an element that has no partner; all elements of *L* have a partner. In b, for set *C*, not all elements have a partner. So, only in Part d, all elements have a partner.	B-MEM B-ASC

Dyn and Sam then moved to Part 2f (Figure 4.1, p. 128). They formed two sets the results of the associations in Parts a and d. They denoted the sets as *A* and *D*: $A = \{(2,4),(2,6),(3,4),(3,6)\}$, and $D = \{(2,2),(2,4),(2,6),(3,2),(3,4),(3,6)\}$. Before they dealt with the construction of the concept of relation, I introduced the special mathematical term for a set of all results of the association in Part d (set *D*), that is, the Cartesian product of two sets, along with its symbol. The discussion was then continuing on how to define the Cartesian product itself. I provided an explanation of the term which was to help Dyn and Sam define the Cartesian product [0399]. Continuing my explanation, Sam [0400] expressed an appropriate statement describing the Cartesian product of *L* and *C*.

[0397]	Lec	*D*. *D* is a set, and it is called the Cartesian product.
[0398]	Sam	*D* is the Cartesian product.
[0399]	Lec	Yes. It could be written *D* equals *L* times *C*. That is the symbol of the Cartesian product. So, for the Cartesian product of *L* and *C*, *L* times *C* and *C* times *L*, they are different. In this case, because if it is from *L* to *C*, then we write *L* times *C*, which results in this [*D*]. It pairs all off ….
[0400]	Sam	All elements of *L* with all elements of *C*.

Further, I asked *what the Cartesian product is* to clarify Sam's idea and to encourage Dyn and Sam to define the Cartesian product of two sets. Apparently, part of the definition stated by Sam [0400] was lucid adequately to inspire Dyn's construction of the definition of the Cartesian product:

[0419]	Dyn	The Cartesian product is a set of pairs of each element of a set to each element of another set.

Dyn and Sam only uttered "a set of pairs" instead of "a set of ordered pairs" in their definition of the Cartesian product. However, this was considered as a sufficiently accurate definition. After a discussion about the technical matters in terms of logical symbols, Dyn and Sam successfully expressed the definition of the Cartesian product of two sets in its symbolic logic statement.

[0453]	Dyn	*k* is an element of *A*, meaning all elements of *A*. Then, *w* is also for all elements of *B*.

[0454]	Sam	(Writing) The Cartesian product $A \times B = \{(k,w) \mid \forall k \in A \text{ and } \forall w \in B\}$ Each element *k* … element *k* of … each element of *A*. How is it?	B-MEM B-ASC B-ORP

The use of the different font in [0454] is to distinguish the written sentence from the uttered sentences in the transcription. This sentence is the translation of the Indonesian version written during the interview. Expressing the Cartesian product $A \times B$ as $\{(k,w) \mid \forall k \in A \text{ and } \forall w \in B\}$ in which the universal quantifier $\forall$ is explicitly used for the membership of *A* and *B* is not standard in daily practices of mathematics, for traditionally, we do not write that quantifier. Here, *A* and *B* were arbitrarily chosen sets. *A* was not the one denoting the set of the results of the association in Part a mentioned earlier.

4.3.2.4 Episode 4: Constructing Relations

After constructing the knowledge element of the Cartesian product of two sets, students were ready for the next construction, that is, the concept of relation. According to the operational definition of a relation, if students could explain, identify, or determine that certain set is a subset of the Cartesian product of two sets, then they are considered to have constructed the notion of a relation. Further, they just need to be informed about the term "relation" as a special mathematical term used in the mathematical study. In the task (Item 2f in Figure 4.1, p. 128), the students were asked to identify the relationship between the sets of all ordered pairs resulting from two different associations.

The process of constructing the concept of relation (REL) was observed as taking place in the following segment [0371–0379]. Dyn correctly identified the relation between the sets, "What I see is all elements of *A* are in *D*" [0371]. Then, Sam responded "So, *A* is a subset of *D* … right?" [0372]. They recognized SUB. The element of SUB was assumed to have been constructed by them beforehand. This is an example of a basic relation between two sets. They were already on the right track with their utterances [0371–0372], however, later, Dyn seemed to misunderstand the question which was asking the relation which could be identified between the two sets. He apparently interpreted a relation between the sets as the commonality between the sets' elements. This was the effect of daily language in the mathematical study. In the everyday colloquial discussion, the relation between two things could sometimes be interpreted as the commonalities existing between the two things. Soon, after being reminded about what was being asked in the question, Dyn realized that it was the subset considered as the relation between the sets.

[0371]	Dyn	What I see is all elements of A are in D.	R-SUB
[0372]	Sam	So, A is a subset of D … right?	R-SUB
[0373]	Dyn	And for their relation. The relation is between elements of A which are the same as elements of D.	
[0374]	Sam	What? The relation of elements of A which are the same as elements of D?	
[0375]	Dyn	Yes, elements of A which are the same as elements of D.	
[0376]	Lec	What is asked there?	
[0377]	Sam	The relation. What relation we could identify between the two sets.	
[0378]	Dyn	It means, A is a subset of D. Each element of set A is an element of set D.	R-SUB
[0379]	Sam	So, A is a subset of D.	R-SUB

Further, Dyn remembered, or recognized, that D was the Cartesian product $L \times C$ [0479]. Based upon their conclusion, that is, $A \subseteq D$, I then introduced them to 'relation' as a mathematical term for set A given it is a subset of the Cartesian product. This was to lead Dyn and Sam to define the concept of relation. It could be inferred from [0481] that Sam actually understood the concept of relation. He just could not express the statement he wrote and then reiterated, using an appropriate style such as starting with "A relation from L to C is …."

[0479]	Dyn	D is the Cartesian product L times C.	R-CAR
[0480]	Lec	D is the Cartesian product. According to mathematics concept, because A is a subset of D while D is the Cartesian product of two sets, that is, L and C, so D equals L times C, then A is called a relation from L to C.	
[0481]	Sam	(Writing) Because $A \subseteq D$ and D is the Cartesian product $L \times C$, then A is a relation from L to C.	B-SUB B-CAR

Instead of using the term 'subset' directly, Dyn apparently tried to make a definition by explaining the meaning of subset in his own words. He chose the word 'part' to explain subset: "A relation is a part of the Cartesian product L times C or a relation is some of pairs in the Cartesian product L times C" [0486]. This utterance might be influenced by the fact that a relation is a subset, which bears the meaning 'a part', so

Dyn attempted to state that a relation covered only some (could neither be all nor none) ordered pairs of the Cartesian product $L \times C$. Such a statement was confirmed by Sam, "Some pairs relating *L* to *C*" [0489]. After being reminded that a relation was basically a set, then Dyn concluded that "A relation from *L* to *C* is a set of some pairs of the Cartesian product" [0491].

[0486]	Dyn	A relation of *L* to *C*. A relation is a part of the Cartesian product *L* times *C* or a relation is some of pairs in the Cartesian product *L* times *C*.	
[0487]	Sam	So, a relation is just some pairs?	
[0488]	Dyn	Because *A* is a subset of *D*, right?	B-SUB
[0489]	Sam	Some pairs relating *L* to *C*.	R-ORP
[0490]	Lec	If you just say pairs, you miss the essence … a relation is a set.	
[0491]	Dyn	A relation from *L* to *C* is a set of some pairs of the Cartesian product *L* times *C*.	B-ORP B-SUB

Building-with ORP, CAR, and SUB, it could be interpreted that Dyn and Sam had constructed the element of relation (REL). They were being trapped in defining a relation by trying to expound 'subset' in their own words, such as 'part' and 'some'. However, using the word 'some' is not suitable for 'subset', because in terms of coverage, 'some' only implies the meaning of 'at least one' but 'not all'. Whereas, a subset in mathematics context also covers 'nothing' because a null set is a subset of each set, and 'all' because a set is a subset of itself.

Proceeding to the formal logical expression of the definition of a relation, Dyn and Sam collaborated to develop a logical statement representing their idea of what relation was. The construction of the definition was started by Dyn offering an ordered pair as an arbitrary element of the set and determining the membership of the elements of the ordered pair. Sam's idea was that they had to use 'or' as the connective for the membership of the first and the second sets. I challenged them to explain why they used such a connective [0505].

[0498]	Sam	Relation *A* from *L* to *C* … like this? It is equal to … is equal to?
[0499]	Dyn	*k* comma *w* where … some ….
[0500]	Sam	Where some ….
[0501]	Dyn	Some elements *k* of *A* or ….
[0502]	Sam	Should we say 'or'?
[0503]	Dyn	Yes.
[0504]	Sam	Or some elements *w* of *B*?
[0505]	Lec	Why do you use 'or'?

The reason provided by Dyn seemed to change the focus from the use of a connective to the use of a quantifier.

[0506]	Dyn	Because, to me, it could be that here [*A*] some of its elements have no partner or it could be some of the elements of here [*B*] have no partner as well.

When I questioned Dyn's explanation, Sam changed his mind and took 'and' as a connective to use in the statement, "'And' ... because there are only some, it could be 1, could be 2" [0508], although he did not provide a relevant explanation for using 'and'. Dyn agreed on this choice.

[0507]	Lec	How could you pair them off if there is nothing in there [*B*]?
[0508]	Sam	'And' ... because there are only some, it could be 1, could be 2.
[0509]	Dyn	Oh, yes. It is right, 'and' ….

Further, Sam and Dyn could elaborate the correct meaning of using 'or', and this led Sam to arrive at the logical expression of the definition of relation.

[0510]	Sam	If we use 'or', it means … it means it could be only this [$k \in A$] ….
[0511]	Dyn	Yes, could be this [$k \in A$] only, could be this [$w \in B$] only
[0512]	Lec	If it is for that [$k \in A$] only, what would be the result?
[0513]	Dyn	There would be no partner. It means it doesn't result in a pair. It means, we have to use 'and'.
[0514]	Sam	(Writing) Relation A from C to L = $\{(k,w) \mid \exists k \in A$ and $\exists w \in B\}$.

This ended the process of constructing the element of relation. Their formulated definition was: "Relation A from C to L = $\{(k,w) \mid \exists k \in A$ and $\exists w \in B\}$" [0514]. Although the definition did not correctly use the symbols for the sets involved, such a statement was still acceptable. In defining the concept of relation, Sam named the relation as relation A which associated set C to set L. However, in the formulated definition, they used different sets in the set notation. It should consistently use C and L for the sets in the right-hand side of the definition and should express: Relation A from C to L = $\{(k,w) \mid \exists k \in C$ and $\exists w \in L\}$. Formally, the definition of a relation is simply written as: A relation $R : C \to L \Leftrightarrow R \subseteq C \times L$. Dyn and Sam were considered to have developed their complete understanding of the concept of relation. In conclusion, Dyn and Sam have constructed the concept of a general relation.

After successfully finishing the part of the calculus tasks for constructing the concept of a general relation, Dyn and Sam continued to work on the problems as presented in Figure 4.2 (see p. 128). The questions are to facilitate the abstraction processes for constructing the concept of special relation, which leads to the operational formal definition of function.

4.3.2.5 Episode 5: Constructing the Exhaustive Domain Property

The focus in the construction of this concept was on three different sets, each contained all the ordered pairs resulting from the association in question items (a), (b), and (d) in Figure 4.2 (see p. 128), respectively. Dyn and Sam made the following sets, named according to the numbering of the items in the tasks: $A = \{(2,4),(2,6),(3,4),(3,6)\}$, $B = \{(4,2),(4,3),(6,2),(6,3)\}$, and $D = \{(2,2),(2,4),(2,6),(3,2),(3,4),(3,6)\}$. Then, three sets were to be examined in order for them to identify the unique or common property belonging to both *A* and *D* in comparison to *B*. Through this activity, the students were expected to understand the exhaustive domain property which would form one attribute for a relation to be a special one, that is, a function. According to the operational definition of the concept of the exhaustive domain property (EXH), students were considered to have constructed this concept if they identified, showed, or explained that all elements of the domain of a given relation are assigned to at least one element of the codomain, and they could express the logic statement: Let $R: A \to B$. R is an exhaustive domain relation $\Leftrightarrow \forall a \in A, \exists b \in B, (a,b) \in R$, or other equivalent statements.

An early response from Dyn was considered as superficial, as he said that both *A* and *D* were sets [0519]. This was not unique for *A* and *D* because *B* itself was also a set. By focusing only on *D*, on the other hand, Sam stated that *D* was a relation from *L* to *C* [0520], while *A* itself was also a relation from *L* to *C*.

[0519]	Dyn	It is just the same. The common properties … it means both of them are sets.
[0520]	Sam	It could also be that *D* is a relation from *L* to *C*.
[0521]	Lec	How could it be?
[0522]	Sam	Because *D* relates *L* to *C*.

Apparently, this was the moment in which they consolidated their understanding of the concept of relation. Instead of discussing the uniqueness of relation represented in sets A and D, Dyn and Sam still focused on the peripheral aspects of the relations, such as domain and codomain. Further, Sam rewrote the definition of relation, based on which Dyn clarified that D was actually a relation as well.

[0534]	Sam	(Writing) A relation from L to C, that is, $\subseteq$ of the Cartesian product $L \times C$.	
[0535]	Lec	OK, now, is D also a relation?	
. . .			
[0542]	Dyn	Because D is also a subset of D.	R-SUB B-SUB

Recognizing and building-with the elements of domain (DOM) and codomain (COD), Dyn and Sam identified the commonality of the domain and codomain of A and D, that both were relations from L to C [0544], which distinguished them from B, a relation from C to L [0545].

[0544]	Dyn	A and D are relations from L to C.	R-DOM R-COD B-DOM B-COD
[0545]	Sam	B is a relation from C to L.	

Apparently, they still could not shift the focus of their attention from the difference between the domain and the codomain of the relations. I suggested that they wrote the domain and codomain with all their elements at the end of each corresponding relation. Next, Dyn and Sam were encouraged to focus on the essential property of the relation.

[0561]	Lec	Besides that, if you think again … what is the unique property in A and D, which can't be found in B, B doesn't have that uniqueness? By taking into account that sets involved in relation A, that is, L as the first set and C as the second set. It similar to D. For B, the first set is C and the second set is L. And, you got the relations. What is unique in A and D, which is not in B?

The reminder was responded to by Sam and Dyn by further examining whether all sets A, B, and D were subsets of the Cartesian product of two sets. Afterward, Dyn apparently shifted his focus and succeeded in identifying the uniqueness common to A and D that all the elements of their domain had partners [0573]. This was considered as the start of constructing EXH. Sam showed his constructing action by clarifying that it was 2, an element of the domain of B, which had no partner. This was to correct Dyn's utterance mentioning "For B, not all them … there is 6 which has no partner"

[0573]. It is interpreted that in all utterances in this constructing processes [0574–0580], Dyn and Sam built-with the knowledge elements of ASC, DOM, and COD.

[0573]	Dyn	The elements of the domain. For *A* and *D*, we focus on the elements of the domain, that is, *L*, they have partners. For *B*, not all them … there is 6 which has no partner.	B-DOM B-COD B-ASC
[0574]	Sam	Not 6. It is 2 … 2 which has no partner in *B*, it is from *C* to *L*.	B-DOM B-COD B-ASC
. . .			
[0579]	Lec	OK. So you have seen the essence, something beyond the order of *C* and *L* in the sets.	
[0580]	Sam	For *A* and *D*, each element of their domain has partners in codomain … while for *B*, not all elements of its domain have partners.	B-DOM B-COD B-ASC

I then introduced a name for the identified unique property of *A* and *D*. A relation with such a property is considered special and the property is called an exhaustive domain property. Dyn and Sam attempted to define the exhaustive domain property of a relation. Dyn directly referred to the way the Cartesian product of two sets was defined which involved all elements of the two sets. Being inspired by the Cartesian product, Dyn thought of a modified definition suitable for the exhaustive domain property. He changed the quantifiers for the membership of the sets involved in the definition of the Cartesian product which used a universal quantifier for the members of either the first or the second set.

[0583]	Lec	Yes, they are unique, exhaustive relations, their domains are exhaustive, right? Now, how do you define such a property?
[0584]	Dyn	It seems similar to the Cartesian product, because each element of *A* … amm … oh only the domain. It means, it could be some, some elements *w* of *B*.
[0585]	Sam	And for the other … for *k* … for each *k*.
[0586]	Dyn	For each element *k* of *A*. Yes.

In [0584], Dyn proposed that only in the domain did it use a universal quantifier and for the second set, it used an existential quantifier to express that only some of its elements were considered. Sam had a similar thought and he offered *k* to symbolize the elements of *A*, the first set [0585]. These two symbols of set elements were also used in their definition of the Cartesian product of two sets. Sam concluded the discussion by stating and then writing the definition of the exhaustive domain property [0590]. He set the definition in the form of a set expressed with the set-builder notation, which was agreed on by Dyn.

[0587]	Sam	A and D are special relations.
[0588]	Dyn	A special relation …..
[0589]	Lec	You don't need to say A and D anymore, but now for a special relation like A and D, how do you define it?
[0590]	Sam	A special relation from A to B, k comma w where for each element k of A and there exists an element w of B. (Writing) A special relation A to B = $\{(k,w) \mid \forall k \in A$ and $\exists w \in B\}$.

Dyn and Sam expressed the exhaustive domain property as a set. They defined the exhaustive domain property as a special relation from set A to set B. The expression showed peculiarities especially in terms of the quantifier use. Combining the quantifiers in a conjunctive expression is rather confusing because traditionally, the quantifiers were used following an order of which the meaning turned to be very different if the order was reversed. With such an expression as $\forall k \in A, \exists w \in B$, there must be a predicate or a propositional function immediately following it. In this case, the relation R from A to B with the exhaustive domain property could be defined as $\forall k \in A, \exists w \in B, (k,w) \in R$. Here, $(k,w) \in R$ is the predicate explaining the properties of k and w. Alternatively, it could be written as $R = \{z \mid \forall k \in A, \exists w \in B, z = (k,w)\}$. Here, the order of the quantifier signifies that meaning that all elements k of A has a partner in B. "Having a partner in B" means that there is at least one element in B, in this case, w, which is paired off with k. So, the comma in "$\forall k \in A, \exists w \in B$" is not of the same meaning as 'and' or '$\wedge$' (this is the symbol of 'and' used in logic), as being usually assumed in the expression of several conditions that apply for the membership of a set. The expression above signifies an order, that indicates which comes first and which comes second. If the quantifiers are the same, either universal or existential, the order does not matter. Reversing the quantification order will preserve the same meaning. Based upon their way of logical quantification expression, it is concluded that Dyn and Sam constructed only a partial understanding of the exhaustive domain property.

4.3.2.6 Episode 6: Constructing the Unique Image Property

For constructing the element of unique image property, the students worked on Part 2h (Figure 4.2, p. 128), where two illustrations were presented for them. They were to identify the essential property of relation common to the two diagrams. The target property here is the unique image property, that is, a relation is also a special relation if each assigned element of the domain is associated with only one element. Dyn and Sam had experienced a similar task in the previous item which was for identifying the exhaustive domain property of relations. By working on this item, they should have learned how to focus on the essential properties instead of accessorial components.

It could be the exhaustiveness of the domains in the previous task which inspired Dyn's initial response to the given task. Examining the domains of the relations illustrated in the task, the property of exhaustiveness was not found because one element of the domain in the second relation had no partner. So, Dyn turned his focus on the codomains. However, neither of the codomains were exhaustive. This examination led Dyn to make his conclusion in [0599]: "It means, in the codomains, only some elements having partners." Sam seemed to disagree with this observation and, instead, he turned to focus on the domain counting the number of points in the domains which had partners.

[0599]	Dyn	Only several elements of the codomain which have partners, because for this [codomain in Illustration (i)], it is for each element, for this [domain of in Illustration (ii)] some elements. Then, for this [codomain in Illustration (i)] some elements, for this [codomain in Illustration (ii)] also some elements. It means, in the codomains, only some elements having partners.	R-ASC B-ASC R-COD B-COD
[0600]	Sam	The special property?	
[0601]	Dyn	Do you see other things? Yes, only that.	
[0602]	Lec	Is there any other, besides that property?	
[0603]	Sam	That is 4 [the number of elements of the domain in Illustration (i) which have partners], that is 3 [the number of elements of the domain in Illustration (ii) which have partners].	B-ASC R-DOM B-DOM

Reminding Dyn and Sam not to focus on the accessories, but rather to look for the more essential property of the relations apparently was enough to inspire Dyn to shift his attention and finally to identify one commonality of the two relations. Dyn's utterance [0605] was where he started constructing the property of unique image in the relations. Sam was focusing on the codomain, where he found in one of the illustrations that one of the points on the codomain had two partners in the domain [0605]. Dyn affirmed his idea by emphasizing that he looked at the elements of the domain [0607]. This affirmation assisted Sam to identify the intended property which considered the elements of *X*, the domain of the relations [0608]. The construction process of the unique image property was built-with the elements of DOM, COD, and ASC.

[0604]	Lec	That could be considered just as accessories, if you turn to focus on that. Could you find an essential property in the relations?	
[0605]	Dyn	The elements of *X* that have partners, have only 1 partner.	
[0606]	Sam	1 … 1 ... 2. There are two in this [domain].	
[0607]	Dyn	I mean, these [elements of the domain of relation (i)] have only one partner.	B-DOM B-ASC
[0608]	Sam	Each element of *X* which has partners, has only 1 partner.	

Dyn and Sam succeeded in verbally defining the property of unique image of the illustrated relations. They knew the meaning of an entity whose existence was only one, that is, it existed but it was only one. The next challenge was to express the definition using logical symbols. Dyn firstly made a reference to the logical symbolic expression of the definition of the exhaustive domain relation. However, he soon realized that the point to focus on was the expression of the only one existence of the partner of the domain's elements. Taking the form of an ordered pair (k, w), Dyn and Sam discussed which one was *only one* in the ordered pair (k, w), whether it was *k* or *w*. A rather long exchange took place between them, after which they concluded that it was *k* which was only one in the definition [0624–0625]. Sam was apparently confused between 'is the only one' (the partner of the *k*) and 'has only one' (*k* itself), but he finally got the notion of where the only one element was [0671–0673]. Dyn was also aware of the location of the only one element [0677].

[0624]	Sam	It is only *k*, right? Because *w* could be ….
[0625]	Dyn	Amm … *k* is only 1 … *k* is only 1.
. . .		
[0671]	Sam	*k*. *k* is the one which is the only one. Set *X* has only one partner in set *Y*.
[0672]	Lec	So, where is that which is only one?
[0673]	Sam	In *Y*.
. . .		
[0677]	Dyn	It is *y* as the only one, it means that *y* ….

Up to this point, Dyn and Sam understood characteristic of the elements of the ordered pair (k, w). They proceeded to formulate the expression of the unique existence of *w* as the partner of *k*. Sam firstly proposed that they had to bring something up.

[0699]	Sam	We have to bring something up ….
[0700]	Lec	We have to bring something up … show another element of *Y*, which is not *w*, and in fact ….
[0701]	Sam	It is the same.

Dyn and Sam realized that there must be two partners of k, that is, $w1$ and $w2$, and further, it needed to be shown that $w1 = w2$. However, in the end, Sam wrote the symbolic definition of the unique image property as follows.

[0740] Sam (Writing)
Relation from A to B = $\{(k,w1) | \exists k \in X$ and $\exists w1 \in Y$
= $\{(k,w1), (k,w2) | \exists k \in X, \exists w1 \in Y, \exists w2 \in Y$ and $w1 = w2\}$.

Sam started his definition with an incomplete statement: $\{(k,w1)|\exists k \in X$ and $\exists w1 \in Y$. in this expression, Sam missed the sign "}" at the end. Then, the definition was expressed as a set: $\{(k,w1), (k,w2)|\exists k \in X, \exists w1 \in Y, \exists w2 \in Y$ and $w1 = w2\}$. This expression was complicated and it was not formulated as a conditional statement. It is unusual to write two entities $[(k,w1), (k,w2)]$ signifying the model of the element of a set. The symbol of the element must represent the general form of the elements of the set. In this case, the set contains ordered pairs, so it should be sufficiently symbolized as (k,w). The conditions of the element do not state the unique image property of a relation. The statement must use an implication instead of a conjunction.

In essence, Dyn and Sam constructed their understanding of the property being investigated. However, they stumbled on how to express the concept of the property in a correct symbolic language. In the illustrations, it was shown that not all elements had a partner; but, if an element had partners, its partner was the only one. Dyn and Sam succeeded in seeing this property and stated that 'an element, that had partners, had only one partner'. The situation that not all elements had a partner as being depicted in the illustrations might affect the students to turn to the existential quantifier. This was used to negate the universal quantifier. They thought that the property should be interpreted as 'there exists an element which has only one partner', which in logical symbols is expressed as $\exists k \in A, \forall w1, w2 \in B,$ $(k,w1),(k,w2) \in R \rightarrow w1 = w2.$

The unique image property could also be stated as 'the paired elements each has only one partner' or 'no elements of the domain have more than one partner'. To express the statement using logical symbols requires a reconstruction. This will be translated into 'for each element, if it has partners, then its partner is only one' which is then expressed as $\forall k \in A, \forall w1, w2 \in B, (k,w1),(k,w2) \in R \rightarrow w1 = w2.$

4.3.2.7 Episode 7: Constructing the Operational Formal Definition of Function

The construction of the operational formal definition of function simply involved combining the two special properties identified previously. A function is a relation which has both an exhaustive domain and a unique partner of the domain's elements. For this construction, Dyn and Sam rewrote the conjunction of the two special properties they have defined.

[0751]	Dyn	It means a relation from *A* to *B* is equal to ….	R_GRE
[0752]	Sam	Relation from *A* to *B* … *k* comma *w* again?	B-GRE
[0753]	Dyn	Yes. For each … amm … *k* comma *w*1, *k* comma *w*2 … for each *k* ….	
[0754]	Sam	For each element *k* of *A*, there exist *w*1 and *w*2 elements of *B*, and *w*1 equals *w*2. Is it correct? Because for each *k*, so, all *k* in *A* has only 1 partner … the domain has only one partner. (Writing) Relation A to B = {(k,w1), (k,w2)|∀k∈A, ∃w2, w2∈B and w1=w2}	R-EXH R-UNI B-EXH B-UNI

Dyn and Sam still could not define a function in formal operation style. In this final stage, they were just able to express the formal definition of function. However, when they wrote the definition in operational style, they just covered the second special property, that is, the unique image property. And, again, they did not make an improvement for the statement expression. They just rewrote the statement from the previous episode of constructing the unique image property. And, this expression remained unclear.

4.3.3 Summary of the Students' Performances

4.3.3.1 Dyn–Sam

The abstraction performed by Dyn and Sam showed that they succeeded in developing their understanding about the concept of general relation following the trajectory from associating the elements of two sets, to the concept of ordered pairs, the Cartesian product of sets, and culminated in defining a relation as a subset of the Cartesian product. In the part of the special relation, Dyn and Sam could only identify and define the exhaustive domain property, and expressed the property in a symbolic logic statement. However, Dyn and Sam could not reach the complete level of understanding of the unique image property. For this property, they did not take the subdomain to apply the property of unique image. Instead, Dyn and Sam thought that each element of the domain has only one partner in the codomain which was in fact, the formal definition of function. Whereas, it showed that each paired element of the domain has only one partner, not all the domain's elements. Dyn and Sam did not achieve the complete level of understanding of the unique image property of relation. Consequently, Dyn and Sam did not develop the operational formal definition of function to the category of complete understanding.

4.3.3.2 Iful–Vito and Dina–Yuni

Based upon the data analysis presented in Table 4.2 (p. 130), both of the pairs Iful–Vito, and Dina–Yuni successfully constructed the sequence of the intended elements of knowledge which concluded in developing the operation formal definition of function. The abstraction processes they performed, in general, followed the trajectories of the knowledge element structure of which the hierarchical order is described in Table 4.1 (p. 124). Because these pairs have the same level of understanding of the concepts that are being investigated, a summary of their results is jointly presented.

In terms of the construction of the concept of a general relation (GRE), both groups of students started the abstraction processes by identifying the possible associations between the elements of two given sets. It was observed that the students understood the problem and proceeded to find possible associations between the elements of the first pair of sets. After finding the element of the second set, Dina and Yuni questioned the rule to follow for making the association.

[0023]	Yuni	They [*M* and *K*] have no association, so they are mutually exclusive. But, if we want to make up the association?
[0024]	Lec	Make up? It asks what association you could make.
[0025]	Yuni	These [*M*] are alphabet and this [*K*] is number, what is the association?

When Dina and Yuni were reminded about the arbitrary property of the association, [0029] "It doesn't need to be … *a* is something of 3," they proceeded to list the possible associations they could identify. Dina and Yuni employed a systematic approach like the one used by Dyn and Sam, while Iful and Vito, on the other hand, made the associations between the elements of two sets without following a certain pattern. Iful and Vito developed possible associations randomly and checked their list continuously to ensure that all possible associations were already included. However, both pairs of students missed the empty association. Just like Dyn and Sam, they did not consider the empty association as a possible one to make. In addition, none of the pairs attempted the other direction of the association, namely, from the second set to the first set. In the end, both had seven models of association.

In constructing ASC, Iful–Vito and Dina–Yuni could recognize and build-with the elements of MEM and ARB. Unlike Dyn and Sam, both pairs experienced no difficulty in understanding the properties of the conditions given in a set-builder notation. They directly identified the contradiction between the two conditions expressed in one set and concluded that the set must be the empty set. Finding that the first set in the second pair of sets was empty, both pairs concluded that there was no association they could make between the two sets. In this case, they did not consider the empty association as one model of possible associations. However, given the operational definition of the construction of ASC, Iful–Vito and Dina–Yuni constructed their complete understanding of the concept.

Regarding the element of ordered pairs (ORP), both pairs of students realized that the order of the elements in the expression (a,b) was different from the one in (b,a). These two pairs were similar in terms of seeing the expressions as an inversion of the other. Dina and Yuni first observed that if the order was not considered, then the two expressions were the same. Both pairs kept their focus on the different associations producing these ordered pairs, where (a,b) resulted from an association from set L to C, while (b,a) was from C to L. Moreover, Dina and Yuni also recognized that the expression was different from a mere set $\{a,b\}$ because of the use of brackets, so the order of the elements did matter. For this element of ORP, both pairs recognized and built-with ASC and ORE, and they have constructed their full understanding of the element.

Moving to the element of the Cartesian product of two sets (CAR) which ended at the construction of the concept of relation (REL), Iful–Vito and Dina–Yuni succeeded in their abstraction processes. It was easy for them to identify an association which resulted in a list of all possible ordered pairs from the first to the second set. Dina–

Yuni agreed on a conclusive statement about the Cartesian product of L and C as "All elements of *L* are associated with all elements of *C*" [0456, Yuni]. Iful–Vito defined the Cartesian product as: $A \times B = \{(k,n)\}\ k \in A,\ n \in B.$ This symbolic statement was slightly different to the traditional set builder notation: $A \times B = \{(k,n) \mid k \in A,\ n \in B\}$, where the definitions of the elements involved in the ordered pair were put inside the set builder notation. However, it does not change the meaning of the statement.

The construction of the element of relation (REL) performed by these two pairs completed the construction of the general relation. They experienced no difficulties in recognizing that a set of ordered pairs was a subset of the Cartesian product of the two sets. By this recognition, they were introduced to the specific mathematical term for the concept, that is, relation.

The abstraction performed by Iful–Vito and Dina–Yuni in constructing the special relation took place successfully. Similar to the performance of Dyn and Sam, these four students could recognize and built-with the concept of general relation, domain, and codomain in constructing the exhaustive domain property of a relation. Given three relations in the form of sets of ordered pairs, they could identify the specific property of two sets which did not belong to the third set. Both pairs could express the property in a symbolic logic statement.

Iful and Vito identified the property of exhaustive domain that each element of the first set (domain) had partners in the second set (codomain) [0481–0483].

[0481] Iful   Each element of the first set ….
[0482] Vito   Each element of the first set has partners in ….
[0483] Iful   The second set.

Then, they attempted to express the property in an operational style of definition.

[0539] Iful   We take k as an element of *L*.
[0540] Vito   Let … there is an element *k* of *L*, right? *k* has 'for each' ….
[0541] Iful   We can write, *k* is an element of *L*.
[0542] Vito   Yes, for each element *k* of *L*.
[0543] Lec   And, then?
[0544] Vito   It has partners ….
. . .
[0551] Iful   It is paired off with ….
[0552] Lec   Why do you pair it off directly?
[0553] Iful   Oh, it must exist first.
[0554] Lec   Yes, it must exist first. What is it?

[0555]	Vito	The partner to be.
. . .		
[0559]	Vito	How to symbolize it? It has partners, so, if it has partners, it must be paired off ….
[0560]	Iful	*k* is an element of *L*, then it is paired off. And, *m* is an element of *C*.
[0561]	Vito	For each element *k* of *L*, there exists an element *m* of *C*.

From the conversation above, it could be observed that the quantification of the definition was initially challenging for them because they just focused on the universal quantifier for the element of the domain [0539–0542]. Afterward, they directly proceeded to pair the elements of the domain without specifying the thing with which these elements will be paired off. Then, I asked them why they directly paired the element off [0552]. This reminded them to firstly specify the partner to be. For this kind of context, there must be a coordination between two quantifications, that is, the universal and the existential quantifiers. Iful realized that the partner to be must firstly exist [0556], which he symbolized as *m* as an element of *C* [0560]. Vito then constructed a universal existential statement [0561], which resulted in the definition as follows: Relation between *L* and C, $\forall k \in L$, $\exists m \in C \mid (k,m)$. Iful and Vito considered the expression "(k,m)" as symbolizing *k* was related to *m* or (k,m) was an element of the relation. Actually, the complete expression should be $(k,m) \in R$, where *R* was the relation. Similarly, the definition provided by Dina–Yuni was: Let domain = *L*, codomain = *C*, let $x \in L$, let $y \in C$, $\forall x \in L, \exists y \in C, x \rightarrow y.$ They had different ways of expressing an element of a relation, one used an ordered pair to signify that *k* is related to *m*, and the other used an arrow to indicate that *x* is associated with *y*.

Regarding the property of unique image, Iful–Vito and Dina–Yuni succeeded not only in stating the property appropriately, but also expressing the property in a symbolic logic statement. Both pairs were involved in a rather long discussion before arriving at the acceptable logical expression of the definition of the special relation property. Initially, Iful and Vito were only referring to the first illustration where each element of the domain has only one partner, and Vito prematurely concluded: "It is that each element of *X* is tied to only one in *Y*" [0599]. However, Iful tried to examine Vito's idea against the second illustration, where he found that some domain's elements had no partner. Vito then tried to speculate by excluding the quantifier: "The elements of *X* have only one element in *Y*. It means the elements of *X*, oh …, don't use 'each'"

[0604]. It seems that Vito thought that the statement "One element of *X* has only one element in *Y*" [0606] bore no more universal quantification meaning. However, in a mathematical statement, it still does. Iful refused this suggestion and defined the unique image property correctly by taking the subdomain: "Each element of *X* which has partners in *Y*, is only tied to one element in *Y* …" [0609]. A similar thought was observed in the conversation between Dina and Yuni when they worked on the same problem. Focusing firstly on the first illustration, they drew a conclusion that each element of the domain was assigned to only one element of the codomain. They were then reminded that it was for the first illustration. Yuni realized that their conclusion does not apply in the second illustration. In the end, Dina and Yuni agreed upon the following statement: "Each element of the domain which is assigned … is associated with only one element of the codomain" [0903].

After identifying the property of unique image, both pairs attempted to define the property and stating it in the symbolic-logic style. For this purpose, Iful and Vito understood the principle of the unique existence of an object [0665–0667].

[0665] Vito	If there exists another one, it is the same.
[0666] Lec	How is its existence?
[0667] Iful	Every time it exists.
. . .	
[0675] Iful	For each element *k* of *X* …. And, *m* is an element of *Y* ….
[0676] Vito	The quantifier must be there, we can't just say *m* is an element of *Y*, what is *m*? *m* … does it exist or is it for all?
[0677] Iful	For all.
[0678] Lec	Just pay attention to the quantifiers.
[0679] Vito	There exists another one, but it is the same. It should be like this, I think. For each element *k* of *X* and for each element *m* of *Y*, *n* is an element of *Y* ….
[0680] Lec	What about *n*, is it for each or not?
[06811Iful	It is also for each.

The focus of their discussion was on how to symbolize the unique existence property. Iful offered the universal quantification for the element of the domain and the symbol of the element of the codomain [0675]. Recalling the principle of the unique existence, Vito then suggested another element of the codomain [0679]. From this point, they proceeded to devise the predicate for those quantified elements and resulted in the definition of unique image property: $\forall k \in X, \forall m, n \in Y \mid (k,m) \wedge (k,n) \Rightarrow m = n.$ The definition developed by Dina–Yuni was just similar to that expressed by Iful–Vito. In the end, Dina–Yuni and Iful–Vito concluded the abstraction processes for the special relation with the operational formal definition of function. This definition is simply a conjunction of the two special properties. The definition formulated by Dina–Yuni can be seen in Figure 4.4.

★ Definisi fungsi:

Domain : X , misal $a \in X$
kodomain : Y misal $b \in Y$
misal relasi R
R adalah fungsi jika $\forall a \in X \ \exists b \in Y, a \to b$
dan $\forall a \in X, \ \forall b, c \in Y, a \to b \cap a \to c \Rightarrow b = c$

★ Definition of function:

Domain = X, let $a \in X$
Codomain = Y, let $b \in Y$
Let relation be R
R is a function if $\forall a \in X, \exists b \in Y, a \to b$
and $\forall a \in X, \forall b, c \in Y, a \to b \wedge a \to c \Rightarrow b = c$

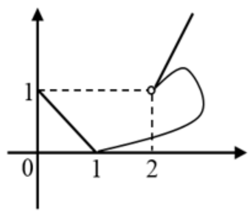
Figure 4.4 The Operational Formal Definition of Function by Dina–Yuni

3. The special relation you have defined in the previous question is called a *function*. Now, look at the following list of expressions. Examine the following expressions, and state if they are function or not. Give an explanation to justify your answer.

(a)[1]

Days	Things
Sunday	Orange
Monday	Computer
Tuesday	Assignment
Wednesday	Jakarta
Thursday	Aeroplane
Friday	Jakarta
Saturday	Orange

(b)[2]



(c) $x = 5$

(d) $x(y) = \begin{cases} y & \text{if } y \text{ is rational} \\ -y & \text{if } y \text{ is irrational} \end{cases}$

(e) $\boldsymbol{R}$ is the set of real numbers. Relation β is from $\boldsymbol{R}$ to $\boldsymbol{R}$; $\beta(y) = \log(2 - y)$, for each $y \in \boldsymbol{R}$.

(*f*) Let $\boldsymbol{N}$ be the set of natural numbers and $\boldsymbol{Z}$ be the set of integers. Relation L is from $\boldsymbol{N}$ to $\boldsymbol{Z}$ where $L = \{(a, b) \mid a + b = 0, a \in \boldsymbol{N}, b \in \boldsymbol{Z}\}$.

Notes:
1. Adapted from Christou, et al. (2005).
2. Adapted from Feeley (2013).

Figure 4.5 Additional Tasks for the Concept of Special Relation

Having the operational formal definition of function constructed correctly, the students were provided with a list of relations given in different representations to be justified whether they were a legitimate function or not (Figure 4.5). The operational formal definition of function could be used to verify the relations in questions 3c, 3d, 3e, and 3f. They consolidated their constructed definitions by working on, particularly, question 3f. This item expressed a relation which was a legitimate function stated in a formal representation.

$L: a+b=0$

$a \in N$; $a+b=0$

$b \in Z$; $b=-a$

⬆ * Ambil sebarang $a \in N$ maka terdapat $b=-a$

sehingga ~~a+b=0~~ $a+b = a+(-a) = 0$

▲ * Ambil sebarang $a \in N$

Ambil sebarang $m, n \in Z$

$(a,m) \wedge (a,n)$

$\Rightarrow a+m=0 \Rightarrow m=-a$

$a+n=0 \Rightarrow n=-a$

~~karena~~ $(m=-a$ dan $n=-a) \Rightarrow m=n=-a$

$L: a + b = 0$

$a \in N$; $a + b = 0$

$b \in Z$; $b = -a$

⬆ * Let $a \in N$ arbitrarily, then there exist $b = -a$

such that $a + b = a + (-a) = 0$

▲ * Let $a \in N$ arbitrarily

Let $m, n \in Z$ arbitrarily

$(a,m) \wedge (a,n)$

$\Rightarrow a + m = 0 \Rightarrow m = -a$

$a + n = 0 \Rightarrow n = -a$

$(m = -a$ and $n = -a) \Rightarrow m = n = -a$

Figure 4.6 A Verification for a Functional Relation by Iful–Vito

Both Iful–Vito and Dina–Yuni could devise a deductive argument showing that the relation was a legitimate function using the operational formal definition. Figure 4.6 presents the formal proof devised by Iful–Vito for a relation in Part 3f (Figure 4.5), which verified that $L = \{(a,b)|a + b = 0, a \in \boldsymbol{N}, b \in \boldsymbol{Z}\}$ was a legitimate function. In the proof, Iful and Vito first showed that the relation satisfied the property of exhaustive domain. They proved that each natural number a had a partner $b = -a$ which was an integer such that $a + b = 0$. Next, they showed that the relation also satisfied the property of unique domain. They showed that the partner of a was only one in the set of integers.

4.3.3.3 Naya–Amzi

As could be seen in Table 4.2 (p. 130), Naya and Amzi performed well in the construction of the concept of general relation. They could achieve the level of complete understanding of each element to be constructed, from association between sets' elements, ordered pairs, the Cartesian product of two sets, to the last, relation.

As to the association between the elements of two sets, Naya and Amzi employed a systematic approach to list the possible models. Instead of using arrow diagrams to represent the association, uniquely they directly formed a set of ordered pairs resulting from the association. For example, the first two associations they made were: $R = \{(a,3),(b,3),(c,3)\}$ and $R = \{(a,3),(b,3)\}$. Recognizing and building-with the elements of MEM and ARB, Naya–Amzi completed a list of seven models of association and they did not include the empty association. For the concept of ordered pairs, Naya and Amzi identified the characterizing feature of the order of the elements (ORE). They distinguished (2,4) from (4,2) based upon the position of their elements.

[0223]	Naya	Its elements ….
[0224]	Amzi	Their position ….
[0225]	Naya	Their position is different.
[0226]	Lec	The position is different.
[0227]	Amzi	Because the position of their elements is different.

Regarding the Cartesian product of two sets, it was evident that Naya–Amzi successfully constructed their complete understanding of the concept. Recognizing and building-with, particularly, ORP, they could easily identify the uniqueness of the results of three different associations, one of which had the determinant property of the Cartesian product of two sets. However, initially, they could not appropriately express that unique property. Amzi stated, "Yes, the elements of set *L* which are paired off with the elements of set *C*, all them have partners?" [0266]. In other words, they stated that all elements of the first and the second sets have partners, which in fact, this does not accurately reflect the concept of the Cartesian product. Apparently, Naya and Amzi did not realize that their statement was too general and that it also covered many results which were not the Cartesian product of two sets. The problem with this kind of statement is that although all elements of the first and the second sets are paired off, and therefore they all have partners, it does not always result in a situation where all elements of the first set are paired off with each element of the second set–as required in the Cartesian product. An example was offered to Naya and Amzi to incur a conflict in their minds.

[0352]	Lec	If they are all paired off, you could look at an example 2 to 2 … 3 to 4, and then 3 to 6, all them are paired off.

This inspired them to reword their statement, and Naya stated: "The Cartesian product is a set of pairs resulting from two associated sets where all elements of the domain are paired off with all elements of the codomain" [0428]. Pairing all elements of the domain to all elements of the codomain is the essence of the Cartesian product, and it is already covered in Naya's statement. Further, they continue their discussion on finding the relationship between two sets, that is, *a* and *d*, one of which is the Cartesian product of two sets, *L* and *C*. Naya and Amzi could identify that one set was a subset of the other set, the Cartesian product. By this, the core of the relation concept had been known. The term "relation" was introduced to them for a set which is the subset in the previously identified relationship. These students then attempted to define the Cartesian product, and at the same time, they also defined the concept of relation. The definitions were as follows: "Let *A* and *B* be two sets, $x \in A$ and $y \in B$, the Cartesian product, namely, $A \times B = \{(x,y) | \; x \in A \text{ and } y \in B\}$, $a \subseteq$ the Cartesian product" [Naya, 0447]; in their work: *a* was the subset, *d*, the Cartesian product, was the superset; they wrote those two letters symbolizing sets in lower case, and they expressed "$a \subseteq$ the Cartesian product" instead of "$a \subseteq d$." This is exactly the standard expression of the definition of the Cartesian product of two sets. To conclude, this completed the construction of their understanding of the concept of general relation.

Regarding the special relation, Naya–Amzi could construct their understanding of the first special property of relation, that is, the exhaustive domain. They could identify the characterizing feature of two relations compared to another relation, that the two had all the elements of their domains paired off with elements of the codomain. By assuming that the symbol of an ordered pair (a,b) bears the meaning that *a* is related to *b*, they defined the property as follows:

[0628] Amzi (Writing)
Let A and B be two set, $x \in A$ and $y \in B$ and
relation, namely, F which relates A to B
$\forall x \in A\ \exists y \in B\ (x,y)$

Working on the second special property of relation, Naya and Amzi apparently could not perceive the idea of confining some elements of the domain in two given illustrations where the property of unique image applied. They simply found the property common to two illustrated relations as "the elements of the domains have only one partner in the codomain" [0667]. They actually realized that in the second illustration, not all elements of the domain satisfy this property. Prior to stating their conclusion, Amzi commented: "Yes, not each of them" [0665] which was agreed on by Naya: "Not each, right?" [0666]. This seemed to be their reason for not using the universal quantification, each, at the beginning of their conclusive statement. However, in expressing the statement symbolically, they did use the universal quantifier for the element of the domain: "$\forall x \in A, \exists y, z \in B,\ (x,y)=(x,z)$". The use of existential quantifier for the elements of the codomain was inappropriate because it covered only the selected elements. Meanwhile, to state that the partner of *x* was unique, it had to be that for each *y* and *z* as possible partners of *x*, it would be implied that they were the same. The predicate "$(x,y)=(x,z)$" should not stand alone. It should be a conclusion from the hypothesis that both (x,y) and (x,z) were the elements of the relation.

4.3.3.4 Adel–Tanti

Adel and Tanti worked on the calculus tasks to construct their understanding of the concept of general relation. They successfully developed their understanding of the concept to the complete level, although they took a longer time to complete the questions compared to the other pairs of students. I involved myself frequently to remind them about the assumed previous constructs they should recognize and build-with during the processes.

Similar to the other students' performance, Adel and Tanti made seven models of associations, in which they employed a systematic approach to identifying all possible associations by recognizing and building-with MEM and ARB. They did not also include the empty association in their list. Regarding the concept of ordered pairs, initially, they found it difficult to understand that (2,4) is different from (4,2), given that both ordered pairs resulted from two different associations. Adel considered relating the expression of an ordered pair to the representation of points on a coordinate plane [0457]. Through this point of view, Adel convinced Tanti that the expressions (2,4) and (4,2) are not the same; hence, they found the essence of the concept of ordered pairs.

[0456]	Tanti	This is 2 with 4, 4 with 2.
[0457]	Adel	The point, what point is it?
[0458]	Tanti	Yes.
[0459]	Lec	Well, when you are talking about points, what do mean by points?
[0460]	Tanti	The relation.
[0461]	Adel	Yes, yes.
[0462]	Tanti	It means they are not the same.

As to the concept of the Cartesian product, Adel–Tanti could identify the defining feature of a set to be the Cartesian product of two sets. However, they initially experienced a difficulty in expressing the statement appropriate for the concept. Adel firstly expressed: "It takes all, all, associating all elements in L and C" [0710], and she further rectified: "It means a set of all results of associating all elements of L with all elements of C" [0798]. For this last expression, Tanti defined it symbolically as follows: "Let $k \in L$ and $w \in C$. Then this set can be written $HP = \{(k,w)\}$." By recognizing and building-with the elements ORP and SUB, Adel and Tanti also succeeded in constructing the concept of relation, "A is a subset of D. D is the Cartesian product" [Adel, 0934]. The term "relation" was introduced to them to assign to set *A*, a relation, which was a subset of the Cartesian product. Overall, they completed their construction for the general relation to the level of complete understanding.

It was observed that Adel and Tanti actually could identify the special feature targeted from comparing three relations. They made a conclusion "All elements of the domain are associated with elements of the codomain" [1057]. However, they had no idea of how to express that property in a symbolic logic statement. Therefore, their understanding of the concept could not be justified as complete. Adel and Tanti also encountered a significant challenge in identifying the special property of unique image of the paired elements of the domain in a relation. Similar to the problem faced by Naya and Amzi, Adel and Tanti could not see the need to confine the paired elements of the domain into a subdomain and applied the unique image property to all elements of the subdomain. All they could conclude, which was expressed by Tanti, was "The commonality of the two illustrations is elements of the domains are associated with only one element in the codomain, respectively" [1163]. For this statement, they also did not have any idea how to express it in a symbolic logic style.

4.4 Discussion

Principally, this chapter was written to answer the first research question: *How do students construct the operational formal definition of function?* The construction of the definition took place in the abstraction processes. To be more specific, the abstraction which frames this present study is the theoretical abstraction that takes the model of Abstraction in Context (Dreyfus et al., 2015). With this model, abstraction is defined as a process of vertical reorganization of some previously constructed student's knowledge within mathematics and by mathematical tools aiming at the construction of new knowledge. Three epistemic actions, Recognizing, Building-with, and Constructing, dynamically interact in a nested manner during the abstraction. Therefore, the discussion will be based upon these epistemic actions, which focuses on how the elements of knowledge are constructed leading to the formulation of an operational formal definition of function. The students who succeeded in constructing the intended definition are considered as having developed a complete understanding of the concept. The understanding itself is categorized into three levels, that is, complete understanding, partial understanding, and limited understanding.

Working on the calculus tasks designed specifically to encourage abstraction processes for formulating an operational formal definition of function, two of five pairs of students succeeded in defining function in the operational formal style. Although not all participating students could reach the intended definition, it still could be claimed that the calculus tasks contributed to their understanding as expected. All students could perform well in constructing their understanding of the general relation concept encompassing four hierarchically ordered elements of knowledge. Three pairs who did not achieve the complete understanding of some knowledge elements leading to the construction of the operational formal definition of function did not perform well in the concept of special relation. Their performance in developing their understanding of special properties of relation varied.

Concerning the concept of general relation, the students could recognize and build-with the knowledge elements of membership (MEM) of a set along with the notion of arbitrariness (ARB) in an association of the elements of two sets (ASC). How to associate the elements of *M* with the elements of *K* taking a direction from *M* to *K*? The first model should demonstrate that no element of *M* is associated with the single element of *K*. The second model should be that each one of the elements of *M* is associated with the element of *K*, respectively. The third model should associate each two elements of *M* to the element of *K*. The last model is an association of all three elements of *M* to the element of *K*. For Dyn and Sam, as an instance (see Figure 4.3, p. 134), although they did the association following this systematic way, they missed the first model above. This empty association could be illustrated with a diagram of two sets, *M* and *K*, without any arrow connecting their elements. This trivial empty association could be justified in a formal logical perspective to show the truth of the statement ‘each element of the empty set is an association between elements of *M* and *K*’. Because there are no elements of the empty set, so the negation of that statement is always false, hence the statement itself is always true. This association was from one direction only.

In discussing possible associations between elements of two finite sets, we deal with the formal concept of relation, which is the overarching conceptual component targeted in the present context. In terms of the number of possible associations that could be made between the elements of two finite sets, from the point of view of a mathematical expert, we have a specific formula: Let $n(A)$ and $n(B)$ be the number of elements of sets A and B, respectively, then the number of different models of association from A to B is $2^{n(A)\times n(B)}$ (Rosen, 2019). It means that for the case given in the task, there should be eight models if they make them from M $(n(M)=3)$ to K $(n(K)=1)$. As no notion of direction which mentioned in the question item, associations made from K to M should be considered different from the seven ones made by Dyn and Sam. Apart from the empty association, there are seven different associations which can be made between the elements of M and K. They are the associations from K to M, whose direction is the opposite of those illustrated in Figure 4.3 (p. 134). So, in total, there should be 15 models of association that could be made between the elements of M and K.

The arbitrariness is an important aspect in the concept of relation, and moreover in the concept of function. The developmental history of the concept of function reveals that this essential characteristic of function was realized not from the beginning, but rather, it was clarified many years later by Peter Lejeune Dirichlet (Grant & Kleiner, 2015). The arbitrariness property allows the correspondence made between elements of two sets without following specific rules. Misunderstanding this notion causes students to confine the space of examples of functions by thinking that a function must include a specified formula governing the association between the elements of two sets. Hence, the students decided that a constant function $y = c$ is not a function because y is independent of the values of x (Carlson, 1998; Tall & Bakar, 1992). Provision of set K containing a single number and M containing three letters was deliberate to encourage the students to employ arbitrary property in the association. One pair initially had a concern about what relationship between those three letters and the number could be, "They [M and K] have no association, so they are mutually exclusive. But, if we want to make up the association? …. These [M] are alphabet and this [K] is number, what is the association?" [Yuni, 0023–0025]. They finally realized that they were asked to correspond the elements of the sets, and thus, they just made them following no rules.

Regarding the concept of ordered pairs, the very notion the students need in constructing the general relation is that the order of the elements, which is not invertible, distinguishes a pair (a,b) which is ordered, from an ordinary set $\{a,b\}$ which is unordered. Here, the form (a,b) was introduced to students, and identifying that $(a,b) \neq (b,a) \Leftrightarrow a \neq b$ or $(a,b) = (c,d) \Leftrightarrow a = b$ and $b = d$ was considered adequate for them. They did not need to go further to deal with the various definitions of an ordered pair, such as, the one proposed by Kazimiers Kuratowski defining $(a,b) = \{\{a\},\{a,b\}\}$ (Gallier, 2011; Halmos, 1974). One pair of students tried to relate the ordered pair to the coordinate of a point on the Cartesian plane, which was exactly an effective way to clarify that the order of the elements is significant. Some students discounted the rules concerning the ordered pairs, and instead, focusing on the elements in the pair: "But the relationship just involves 2 and 4. I can also invert them" [Dyn, 0321]. This notion of anti-order has also been found in the study of Dreyfus et al. (2015). Developing the concept from associating elements of two sets following certain rules was apparently effective to lead the students to realize the significance of the order. The rules of the association became a context in which the students reflected that changing order of the elements will accordingly change the object represented by the ordered pair. This result was in alignment with those from the study undertaken by Akar and Şener (2014).

Given several sets containing the results of the association between two sets, the students could identify the unique feature of one set among the others that the set covered the most complete results, characterizing the Cartesian product of two sets. However, having such an identification, some students initially faced difficulty in expressing the feature in a verbal statement. They just defined the set as one in which "[a]ll elements of L [domain] and C [codomain] have a partner" [Dyn, 0274]. To make them realize the overgeneralized meaning of this statement, a non-example was given to incur a conflict in their mind that such a premature definition was also satisfied by the example which is not the Cartesian product, hence too broad. Historically, revising a definition is often needed when it is not accurate enough to separate examples from non-examples (Lakatos, 1976). The data showed that the students could rectify their defining statement and further express it in a precise symbolic form, such as, "… $A \times B = \{(a,b) \mid x \in A \text{ and } y \in B\}$ …" [Naya, 0447]. This study confirms similar results with regard to the significance of non-examples (e.g., de Villiers, 1998; Lin & Yang, 2002; Zaslavsky & Shir, 2005) in encouraging students to modify their definitions.

Next, with their prior understanding of subset and the Cartesian product, the students could define relation as an arbitrary subset of the Cartesian product. All of them could state symbolically, but Dyn and Sam did not stop with the definition that "*R* is a relation if $R \subseteq A \times B$." Apparently, Dyn and Sam's definition of the Cartesian product: $\{(k,w) \mid \forall k \in A \text{ and } \forall w \in B\}$, in which the universal quantifier $\forall$ was explicitly used, inspired them to think about an equivalent expression of which the meaning is translated from "a relation as a subset of the Cartesian product." They found that if it is a subset, then it does not need to use universal quantifiers as it is used in the Cartesian product, and, instead, it is adequate to use existential quantifiers: $\{(k,w) \mid \forall k \in A \text{ and } \exists w \in B\}$. This expression bears similar meaning to Halmos' (1974) definition: A relation *R* is a set of ordered pairs, which is explicitly stated symbolically as: $R = \{z \mid \forall k \in A, \exists w \in B, z = (k,w)\}$. Such an attempt to explain a concept of subset from different viewpoints confirms the findings of Kobiela and Lehrer (2015) that defining activity is inseparable from the development of an understanding of concepts.

The final objective of the abstraction in this study of function is the construction of an operational formal definition of function. This definition is formulated from the conjunction of two different properties which are special for a relation. For a given relation *R*, it is not necessary that all its domain's elements have at least one partner in the codomain. Therefore, if all the elements of the domain are assigned to at least one element of the codomain, then this property forms the first featuring character of a functional relation, that is, the exhaustive domain property. Recognizing and building-with the concepts of domain, codomain, and general relation, all students could identify and define this property, but only three pairs could express the definition in an appropriately quantified statement. The challenge in expressing the property using logical quantification lays on rephrasing statements in equivalent ways (Epp, 1999) to produce a new formal statement of which the logical features are explicitly expressed (Selden & Selden, 1995). Having a new equivalent formal statement resulting from rephrasing, the next step is translating the statement into one using logical symbols.

In practice, the identification of the property first produced a rather informal statement. The aforementioned two steps determine the formulation of an operational formal definition of a mathematical concept. The problem of explicitness as found in the study of Selden and Selden (1995) was also evident in this study. Dyn and Sam could not rewrite "… each element of their domain has partners in codomain …" [Sam, 0580] into, for instance, "for each element of the domain, there is an element of the codomain such that the latter one is the partner of the former one." From this universal existential statement, as discussed by Epp (2011), naming all involved variable will ease the way to express the statement with logical notation, such as, "for each element *k* of domain *A*, there is an element *w* of codomain *B*, such that *w* is the partner of *k*." Instead, Dyn and Sam referred back to the definition of the Cartesian product upon which they based a quantified definition of the exhaustive domain property: $\{(k,w) \mid \forall k \in A \text{ and } \exists w \in B\}$. From its appearance, this was intended to define a relation satisfying the exhaustive domain property, instead of defining the property to be satisfied by a relation. It is like defining 'kind people' instead of defining 'kindness.' However, the expression only defined a set with an incomplete condition for its element.

The second special property characterizing a functional relation from a general relation is that there exists only one partner for the elements of the domain. The basic notion of this property is the unique existence. Only two pairs of students could complete the formal statement of this property in an appropriate quantified expression. One pair could identify the property and expressed it in an informal statement, but failed to translate the statement into a correctly quantified sentence. The other two pairs preserved their notion which was suitable for one illustrated case: all elements of the domain were associated with exactly one element of the codomain, and they were not able to negotiate with the second illustration where the property of unique image applied only for a proper subset of the domain. The difficulty was not only in identifying the property as unique in the given relations, but also, and in defining the property in a quantified statement. The main challenge in identifying the property concerned the notion of the subdomain. Some students did not realize the need to restrict the coverage of the domain to allow the use of the universal quantifier (Devlin, 2003). In terms of the unique image property, it does not necessitate all elements of the domain to have a unique partner. Rather, the property only applies to the paired elements. Those elements form a subdomain in which the property could be defined using the universal quantifier. As a result, it is defined: all paired elements of the domain are associated with only one element in the codomain. Epp (2011) calls it *a universal conditional statement*. It has some equivalent statements which could be a purely conditional statement: If the domain's elements are paired off, then their partner is only one, or a purely universal one: All paired elements of the domain have only one partner. What is the meaning of the unique existence in mathematics? If y is a unique image of x, it means every other z is not an image of x. Or, if w is also an image of x, it must be equal to y. It could also mean that every two a and b, if they are an image of x, then they are the same. The two pairs of students succeeding in quantifying the definition of the unique image property took the last translation, and they formulated the definition as: $R: X \to Y, \ \forall a \in X, \forall b, c \in Y, a \to b \wedge a \to c \Rightarrow b = c.$ These pairs concluded the abstraction process with an operational formal definition by combination the properties of exhaustive domain and unique image. And, they finally realized that when the two properties were combined, it resulted in the formal definition of function: f is a function from X to Y if each element of X is assigned to exactly one element of Y.

Two pairs of students achieved the complete level of understanding of the operational formal definition of function (see Table 4.2, p. 130). It means that these students succeeded formulating the intended definition of function that is operational formal in nature. The definition which they formulated can be seen in Figure 4.4 (p. 159). The expression was not precisely the same as the one adapted from Bartle and Sherbert (2011), Daepp and Gorkin (2011), and Stewart and Tall (2015) (see the operational formal definition on p. 122). The definition was formulated as a conditional statement using the logical connective "if p, then q" or "q, if p", while the definition should be formulated as a bi-conditional statement using the connective "if and only if". However, following the notion of the students' mathematics and the mathematics of students (Steffe & Thompson, 2000), the formulated definition was considered as the students' legitimate mathematics because its mathematical ground was identifiable and all of the essential components of the accepted definition were covered. This indicates that being set up to work on calculus tasks, the students could perform appropriately to construct a definition of mathematical concepts. The tasks were designed based upon the structure of knowledge elements with which the definition. This structure could then be claimed as one trajectory providing a conjectured route (Clements & Sarama, 2004) to be a cognitive path describing the progression of learning processes of a mathematical concept (Arnon et al., 2014). The nested relationship among the epistemic actions performed on the elements of knowledge has been confirmed in the abstraction processes carried out by the students. In the processes, the vertical reorganization of all constructed concepts has taken place starting from the concept of association of sets' elements towards the concept of general relation, and proceeding to two special properties of relation which constituted the operational formal definition of function.

Most definitions of mathematical concepts are logically complicated and involve various mixed quantifiers (Dubinsky, Elterman, & Gong, 1988). Sierpinska (1992) stated that "the concept of function can be defined in a formal symbolic way, almost without using words" (p. 29). The problem encountered by some participating students in their abstraction for constructing the definition was mostly related to the knowledge of logic. They had a rather low ability to unpack the structure of informal statements and then to express them in a formal statement with an explicit logical structure. The logical syntax which covers what symbols to use and how to use them appeared to be problematic for them. The same difficulty has been reported by Selden and Selden (1995) and Ngansop (2018). The role of logic and linguistic aspects in developing a definition has been investigated by Ouvrier-Buffet (2006), where she found that poor logic and language competency would block the progress in constructing and refining a definition. Concerning the competency of mathematical logic, Ngansop (2018) has suggested the need to focus more on improving the ability to unpack the logical structure of mathematical statements. He recommends that students should be trained to perform a two-way translation between formal and informal statements. The students participating in this present research actually have studied the foundation of logic, including practicing to restate an informal statement in an equivalent formal statement, and vice versa. They also have learned how to quantify mathematical statements using logical notations. Being involved in this research became an opportunity for them to consolidate their understanding of mathematical logic. That they encountered problems in the defining activities was an indication that some of them still lacked logical competency.

Developing an operational formal definition involves substantial cognition (Alcock & Simpson, 2017). The explicit request to formulate a definition induces reflective cognitive action which encourages the evolution of the proposed definition (Ouvrier-Buffet, 2006). In this research investigation, it appears that my role as a lecturer was significant, especially in facilitating the reflection during the process of constructing definitions. According to Kobiela and Lehrer (2015), the lecturer plays their role as destabilizer for definitions that have been formulated by proposing counterexamples. Such counterexamples, called by Lakatos (1976) as monster-barring, are to provoke a definitional argument leading to revise the definitions (Kobiela & Lehrer, 2015). Besides providing counterexamples, the lecturer also offered some hints (Ohlsson & Regan, 2001) necessary to break the barrier whenever the students faced difficulty in their problem solving. Similarly, the role of the teachers as a knowledgeable agent has been investigated by Ozmantar (2005), where the agent acted as a scaffolder inducing the students to reflect on what they are doing so they might progress to the optimal level of cognitive potency (Hershkowitz et al., 2001).

The traditional conception of mathematics definition stipulates that the definitions should be concise, minimal, elegant, and precise (Ouvrier-Buffet, 2006; Vinner, 2002). The formal definition of function appearing in various calculus textbooks is one that satisfies those criteria. To some extent, it is good, for it is rather easy for students to memorize. All participating students in this research did memorize the main part of the formal definition of function stating, at least, *that each element of the domain is assigned to only one element of the codomain*. The rote memorization has been reported by Vinner (2002) as an insufficient indicator for a deep understanding of the concept of function. The definition was found to be inoperative for students (Bakar & Tall, 1991). They did not know what and how to show that each domain's element has a unique partner. Frequently, university students could recite formal definitions, yet, Selden (2012) found that they could not employ them when they were asked to solve problems or prove theorems. Instead, they used methods based upon their concept images to verify functional relations (Elia & Spyrou, 2017; Vinner & Dreyfus, 1989), or used prototypical examples in their arguments (Alcock & Simpson, 2004). How is the definition of function formulated by the students? For a definition to be operative means it could be used or we could work using it, for instance, to verify examples or non-examples, or to prove theorems. The students who succeeded in developing an operational formal definition of function (Figure 4.4, p. 159) used the definition to verify a given relation with a deductive argument (Figure 4.6, p. 160).

4.5 Conclusion

The investigation has been conducted on the students' performance in abstraction for developing their understanding of the operational formal definition of function. The results showed a variety of the levels of understanding achieved by the students. Some of them successfully constructed their complete understanding of the whole concepts under investigation. They could also implement the formulated definition of function to verify the functional status of a relation with a formal deductive argument. For the processes of abstraction, it was found that the Abstraction in Context as a model of the theoretical abstraction was performed by the students. The three epistemic actions were identified to be evident in the process of constructing the understanding of the operational formal definition of function which the students underwent. The learning trajectory for the operational formal definition of function was confirmed in the abstraction processes. It means that to formulate the definition of function which was operational formal in nature could be started, from the very concepts of sets, namely, the association between sets' elements for which the other concepts of set, such as the membership of sets was a prerequisite, to the concepts of ordered pairs, the Cartesian product of two sets, relation, and the special properties of relation.

Some students encountered problems in their abstraction which resulted in the non-complete level of understanding. They experienced difficulties in identifying and defining the special properties of relation, namely, the exhaustive domain property and the unique image property. Their difficulties were partly caused by their lack of understanding of mathematical logic. Besides the processes and the results of the abstraction, the mathematical representations used by the students were also investigated. The next Chapter 5 presents the results and discussion of the research concerning the mathematical representations.

CHAPTER 5
REPRESENTATIONS IN CONSTRUCTING THE OPERATIONAL FORMAL DEFINITION OF FUNCTION

"Representation is a crucial element for a theory of mathematics teaching and learning … because the use of symbolic systems is so important in mathematics, the syntax and semantic of which are rich, varied, and universal …."
(Vergnaud, 1987, p. 227)

5.1 Introduction

This chapter presents the findings and discussion in response to the research question dealing with the use of representations in the abstraction processes of constructing the definition of function. The question addressed within this chapter is as follows: *How do students use mathematical representations in constructing the operational formal definition of function?* As discussed in Chapter 3, the mathematical representations were the second aspect to focus on this present study, namely, the use of representations in the abstraction processes to develop an understanding of the operational formal definition of functions. This multiple case study involved five cases and each case consisted of a pair of students. The data were gathered from five pairs of students who worked on calculus tasks by thinking aloud. The students' solutions to the calculus tasks, which were provided in the worksheets, were the main data. The task-based interviews were conducted while they were working on the calculus problems. The interview transcripts were used as complementary data. The Three Worlds of Mathematics representation theory of David Tall (2004a, 2004b, 2006, 2008, 2013) was used as an analytical framework. There were two aspects of the use of the representations focused upon in this research, that is, the categorization and the appropriateness.

This chapter covers several sections. The first section presents the results of the analysis of the categorization of representations. This is followed by the section presenting the findings of the aspect of the appropriateness of used representations.

The chapter concludes with the discussion of the study regarding the representations used in developing an understanding of the operational formal definition of function.

5.2 The Category of Representations Used in Constructing an Understanding of the Operational Formal Definition of Function

Categorizing representations used by the students in the processes of abstraction for developing an understanding of the operational formal definition of function was based upon the framework of the Three Worlds of Mathematics. In the simple form, this theory classified mathematical representations into three different categories, namely, embodied representation, symbolic representation, and formal representation. In analyzing the data, however, a difficulty was encountered when dealing with the students' works in which the representations used were not unique. The students used one mathematical representation type in connection to the other type. For such a case, the categorization could not fall in either one of the three possible worlds of representations. The representations category could not be simplified by examining the proportion of each of the representations used and then further classifying them into the category based upon which of them had the greatest proportion. The criterion of the proportion would only be appropriate to apply to the students' responses where different worlds of representations were used, but were in isolation from each other. In this case, the category was determined based upon which representation was used dominantly or with the greatest proportion.

The connections between representations was another aspect to focus on in the analysis of representations use in the students' works. Initially, connections between representations which were used by the students in their works would be analyzed separately from the representation categorization. The connection between two worlds of representations was observed in two directions. For example, the connections between the embodied representations and symbolized representations could be in either embodying-symbolism representations or symbolizing-embodiment representation. Thus, there were six connections of representation worlds, that is, embodying-symbolism, symbolizing-embodiment, embodying-formalism, formalizing-embodiment, symbolizing-formalism, and formalizing-symbolism (Smart, 2013).

Regarding the connected representations, as mentioned earlier, they could not be classified into either one of the Three Worlds of embodied, symbolized, and formal representations. They resulted in other representation types. This led to a combination of the initial categories of representations, which were based upon the Three Worlds of Mathematics, with the new six categories, which were based upon the connections between representations. Therefore, instead of using only three categories of representations, another six categories were added to cover and enrich the classification of the representations. They were: embodied representations (ER), symbolic representations (SR), formal representations (FR), embodying-symbolism (ES), symbolizing-embodiment (SE), embodying-formalism (EF), symbolizing-formalism (SF), formalizing-embodiment (FE), and formalizing-symbolism (FS).

For each of the nine categories, a description was developed to help determine the types of representations. The following are the descriptions of the representation categories. The descriptions of the first seven categories were adapted from Smart (2013).

1. Embodied Representations (ER). Mathematical objects are expressed in action-image representations that take the forms or properties of objects which are either observed in the real world or imagined in the mind. The representations may take the forms of shapes, tallies, diagrams, charts, curves not on graphical axes, lines, graphs.
2. Symbolic Representations (SR). Mathematical objects are expressed in symbolic representations that signify both processes or procedures and concepts. It involves symbolic, step-by-step procedures of calculation and manipulation on equations, inequalities, functions, formulae, whether using algebra or just numbers and operators.
3. Formal Representations (FR). Mathematical objects are expressed in representations that use formal statements, axioms, definitions, theorems, or formulae along with logical arguments within the framework of a deductive axiomatic mathematical system.
4. Embodying-symbolism Representations (ES). Mathematical objects that are firstly expressed in symbolic representations are stated in embodied representations. For example, an equation or a function which is expressed in symbolic representations is represented as curves on a coordinate plane. Also, it

could be any graphical representation that does not represent a particular equation or function.

5. Symbolizing-embodiment Representations (SE). Mathematical objects that are firstly expressed in embodied representations are stated in symbolic representations. For example, a curve which is firstly described in a picture is then expressed as a symbolic function/formula/equation. It could also be a procedural algebraic solution to a question which is originally expressed as a graph.
6. Embodying-formalism Representations (EF). Mathematical objects that are firstly expressed in formal representations are stated in embodied representations. Formal mathematical ideas are underpinned by embodied representations (Tall, 2008). For example, a formal definition is expressed as a picture or diagram. It could also be a diagram that refers to some formal property of function.
7. Symbolizing-formalism Representations (SF). Mathematical objects that are firstly expressed in formal representations are stated in symbolic representations. For example, a mathematical theorem which is firstly expressed in a formal representation is then stated in a symbolic expression. Also, it could be a symbolic manipulation explaining some formal mathematical property.
8. Formalizing-embodiment Representations (FE). Mathematical objects that are firstly expressed in embodied representations are stated in formal representations. For example, a curve which is firstly represented as a picture is then explained, stated, or expressed in a formal definition. It could also be a set firstly expressed in an arrow diagram, and is then represented in a set-builder notation.
9. Formalizing-symbolism Representations (FS). Mathematical objects that are firstly expressed in symbolic representations are stated in formal representations. For example, a function which is firstly represented in a symbolic expression is then explained, stated, or expressed in a formal definition.

The categorization of the representations was analyzed based upon the knowledge elements or concepts which were constructed in the abstraction processes for understanding the operational formal definition of function. These concepts were described in Table 4.1 in Chapter 4 (p. 124).

5.2.1 A General Overview of the Categories of Representations Used by the Students

Based upon the data analysis, the categorization of the representations used by the students in abstraction processes for developing their understanding of the operational formal definition of function is presented in Table 5.1. It is necessary to state here that for this analysis of the categorization of representations, the focus was purely on the types of representations used by the students and not on the appropriateness of the representations.

Table 5.1 The Categories of Representations Used by the Students in Constructing the Operational Formal Definition of Function

Knowledge Elements	**Student Pairs**				
	Iful–Vito	Dina–Yuni	Dyn–Sam	Naya–Amzi	Adel–Tanti
Association between sets' elements (ASC)	ER	ER	ER	SR	ER
Ordered pairs (ORP)	SE	SR	SE	SR	SE
The Cartesian product of two sets (CAR)	FS	FS	FR	FS	FS
Relation (REL)	FS	FS	FR	FS	FS
The exhaustive domain property (EXH)	FR	FR	FR	FR	ER
The unique image property (UNI)	FR	FR	FR	FR	ER
Operational formal definition of function (FUN)	FR	FR	FR	FR	ER

In the table above, it can be observed that there were some variations of the types of representations the students used in their responses to the calculus tasks. Of nine possible categories, there were only five categories of representations that were used by the students dominantly in answering the calculus tasks, namely, embodied representations, symbolic representations, formal representations, symbolizing-embodied representations, and formalizing-symbolic representations.

Table 5.1 above shows what types of representations the students used in their responses to the calculus tasks with respect to each concept or knowledge element. For the first concept of association between sets' elements, four pairs of students used embodied representations in their responses dominantly. Naya and Amzi were the only students who expressed their responses to the concept of association between sets' elements in symbolic responses. They used ordered pairs to represents the associations the made, while the other pairs of students used arrow diagrams.

For this concept, there were two parts of the question the students had to solve. The first part required the students to directly make all possible models of association between the elements of two given sets. The second set, $K = \{x \mid 2 < x < 4,\ x \in \boldsymbol{N}\}$, was given in the set-builder notation which could be considered as formally represented. For this particular set, all students could easily determine the element of K by considering its properties. Looking at the properties of K's element, the students then wrote $K = \{3\}$ and this was considered as a symbolic representation. This was because the expression represented a set whose element was obtained through a process of determining a natural number which was greater than 2 and less than 4. Further, the association models made by four pair pairs of students were expressed mostly in embodied representations, namely, arrow diagrams of associations. The second part of the question also involved a pair of sets, one of which was given in the set builder notation, namely, $P = \{ z \mid z^2 + 2z + 2 = 0, z \in \boldsymbol{R} \}$. To determine the elements of this set, all students tried to find the solution of the quadratic equation which involved symbolic procedures of using the formula of finding the roots. The successful procedures led them to determine that P was an empty set. Considering this property of set P as a null set, all students who succeeded in determining the emptiness of P then expressed that there was no association possibly made between the given sets. Although there were symbolic procedures presented by the students in their responses to this part, they were not the dominant part of the whole responses of Iful–Vito, Dina–Yuni, Dyn–Sam, and Adel–Tanti. Instead, the concept of association between the elements of two sets was dominantly expressed in embodied representations. Naya and Amzi, who responded to the first part of the question with symbolic representations, had their whole responses to the concept of association between sets' elements categorized as symbolic representations.

All responses to the concept of ordered pair contained symbolic representations. However, the symbolic representations were found to be dominant only in the responses of two pairs, namely, Dina–Yuni and Naya–Amzi. Meanwhile, the responses of Iful–Vito, Dyn–Sam, and Adel–Tanti were dominated by the connection between embodied and symbolic representations. Therefore, their responses were categorized as symbolizing-embodiment representations. For this particular knowledge element, the expected understanding was that the students could identify the essential property of an ordered pair that distinguished it from an ordinary set. In the question items (see Number 2 Part a, b, and c in Figure 4.1, p. 128), the expression of the result of an association was introduced as an ordered pair of two elements. This part could be considered as a symbolically represented concept. The students were asked to list the results of two different associations, and then compared those results. Two pairs of students, Dina–Yuni and Naya–Amzi, directly made a list of all the results in the form of ordered pairs. Three pairs, Iful–Vito, Dyn–Sam, and Adel–Tanti, started the answers by drawing an arrow diagram of the associations, which were embodied representations. From the arrow diagram, they then made a list of pairs of set elements as the results of the associations. Further, they compared the properties of the pairs resulting from the two associations and concluded that the pairs were not the same although the elements of the pairs were found to be the same. Actually, the students could go further to attempt to define the concept of ordered pairs which certainly required formal representations, however, none of them took the challenge.

The category of representations used in expressing the concepts of the Cartesian product of two sets and relation were generally in common. Four pairs of students, Iful–Vito, Dina–Yuni, Naya–Amzi, and Adel–Tanti, used mostly the connected formalizing-symbolism representations. In their responses, the four pairs started with the identification of the Cartesian product by comparing the results of three associations which were expressed in symbolic representations. Based upon this comparison, they wrote a statement of the Cartesian product, from which they moved to define the Cartesian product which was expressed in formal representations. This chain of the representations was justified as connected representations. The mathematical objects were firstly expressed in symbolic representations, then were stated in formal representations, hence formalizing-symbolism representations.

On the other hand, Dyn and Sam used formal representations dominantly for responding to the two concepts. For the concept of the Cartesian product, they expressed their responses dominantly in formal representations. Actually, Dyn and Sam did the processes rather similar those done by the other pairs of students in responding to the concept of the relation. They also started their responses with symbolic representations. However, this kind of representation was only a small portion in their responses. They focused more on defining the concept of the Cartesian product formally. The defining part, which used formal representations, appeared to dominate their responses. Therefore, the responses of Dyn and Sam was categorized as formal representations.

For the three knowledge elements of special relation, namely, the exhaustive domain property, the unique image property, and operational formal definition of function, the formal representations were the dominant representations. Four pairs, namely, Iful–Vito, Dina–Yuni, Dyn–Sam, and Naya–Amzi, responded to the questions mostly with formal representations, while only Adel and Tanti used the embodied representations dominantly in their responses.

The four pairs of students firstly identified the intended properties from the given relations which were presented in either a set of ordered pairs or an arrow diagram. The properties identified were then stated using embodied representations. Further, they defined the properties in formal representations. Although they started to answer the questions with embodied or symbolic representations and then connect to defining the properties in formal representations, the proportion of the formal representations in their responses was dominant. Therefore, instead of categorizing their responses into formalizing-embodiment or formalizing-symbolism representations, their overall responses were considered as more appropriately classified in the formal representations category. The same approach was implemented by Adel and Tanti. However, they could only provide embodied responses to the calculus tasks pertaining to the three concepts. Their responses contained only the identification of the properties and there were no further definitions which should be expressed in formal representations.

Another point of view is how the representation categories vary in the responses of each individual pair of students. In Table 5.1 (p. 179), it could be observed that both pairs of Dina–Yuni and Iful–Vito had four types of representations which were dominantly used in their responses. The dominant representation in Iful and Vito's responses to the beginning part of calculus tasks concerning the associations between sets' elements. For the concept of ordered pairs, they used symbolizing-embodiment representations. When they moved to the next two concepts of the Cartesian product of two sets and relation, they connected the symbolic representations to the formal representations. In this case, these students attempted to make a formal definition which was expressed in symbolic logic statements. This connected representation of formalizing-symbolism type appeared dominantly in their responses. Further, in their responses to the tasks concerning the special properties of relations and the operational formal definition of function, Iful and Vito could provide answers expressed mostly in formal representations. This pattern of representation use was also evident in the works of the other two pairs, namely, Dina–Yuni and Naya–Amzi, especially in their responses to the concepts of the Cartesian product of two sets, relation, and the special properties of function, as well as the operational formal definition of function. Dina and Yuni responded to the concept of association between sets' elements in embodied representations and to the concept of ordered pair in symbolic representations. Nevertheless, Naya and Amzi used symbolic representations dominantly in their responses to the concepts of ordered pair and associating sets' elements. Therefore, they used only three representations as dominant in their responses the whole seven knowledge elements.

A different pattern was shown in the responses provided by Dyn and Sam as well as Adel and Tanti. Earlier, for the knowledge element of associating sets' elements, these students did use embodied representations dominantly. They expressed their responses to the concept of ordered pair in symbolizing-embodied representations. However, in their work concerning the rest of five concepts, Dyn and Sam used formal representations dominantly. Adel and Tanti expressed their responses in formalizing-symbolism representations for the next two concepts of the Cartesian product and relation as did by the other three pairs of students, namely, Iful–Vito, DinaYuni, and Naya–Amzi. When Adel and Tanti moved to work on the last three concepts, they used mostly embodied representations. Both Dyn–Sam and Adel–Tanti had only three categories of representations which were used in their responses to the whole seven knowledge elements.

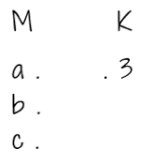
5.2.2 Focused Pair – Iful and Vito: What Kind of Representations Did They Use?

Three pairs of students had a rather similar pattern in terms of the categories of representations used in the abstraction processes of constructing the operational formal definition of function. They were Iful–Vito, Dina–Yuni and Naya–Amzi. These three pairs responded to the first two knowledge elements differently, although their responses to the concept of ordered pairs contained a symbolic representation component. The patterns of their responses to the other five knowledge elements were the same. For the five knowledge elements, they used formalizing-symbolism representations for the concepts of the Cartesian product and relation, and used formal representations of for the last three knowledge elements (see Table 5.1, p. 179). I took the pair of Iful–Vito as the focused case to represent these three pairs. Other reasons for choosing Iful and Vito were that their responses showed four categories of representations and two of them were connected representation categories. The presentation of the analysis results is organized based on the knowledge elements as depicted in Table 5.1 (p. 179).

5.2.2.1 Associations Between Sets' Elements

Iful and Vito started their solution to the questions concerning the association between sets' elements with a discussion. It seemed that Iful could easily determine the element of set K based upon the conditions of its member given in set-builder notation. Then, he wrote down $M = a, b, c$ and $K = 3$ to represent the two sets, but they were expressed without braces. Further, Iful wrote a diagram of two sets without arrows connecting their elements and concluded that it was not a model of association possibly made between M and K [0015]. The discussion proceeded. They identified all possible models of association [0018–0019]. In this discussion, it could be observed that they responded to the question by explaining the possible associations. The explanation was expressed in imagined actions of pairing the elements between the two given sets. These imagined actions were categorized as embodied representations.

[0015] Iful — Here, it is about an association, there is an association … all are associated … it is about the association between the elements of sets, which certainly *M* and *K* do not stand alone, *b*, *c*, 3.
(Writing)



This one has no association.

[0016] Vito — Yes.

[0017] Lec — The request is whatever the models you think of, just take them all.

[0018] Iful — So, we write them all. It means there are lots, it means this *c* to 3.

[0019] Vito — Then *ab*, *ac*.

[0020] Iful — *K* 1, 2, 3, amm … *a*, *b*, *c*, means *ab*, then *bc*, *ac*?

Next, Iful wrote down all diagrams (Figure 5.1) which illustrated the possible models of the association.

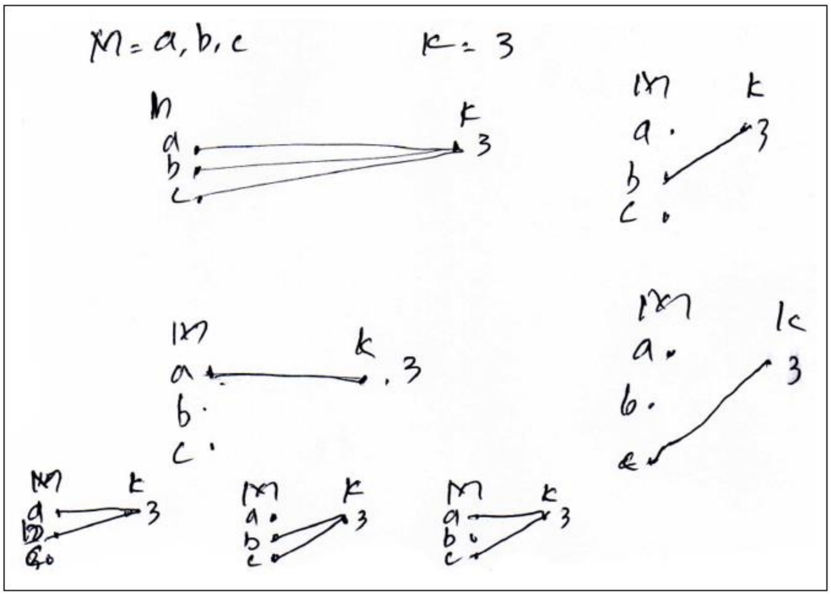


Figure 5.1 Models of Association Between the Elements of *M* and *K* Made by Iful–Vito

These diagrams were considered as image representations expressing the characteristics of an association between the elements of two sets. These diagrams were used dominantly in Iful and Vito's answer to the task. Therefore, the responses were classified as embodied representations.

In their worksheets, actually there was a procedure presented by Iful and Vito, namely, finding the elements of set $P = \{z \mid z^2 + 2z + 2 = 0, z \in \boldsymbol{R}\}$. The procedure involved the formula of finding the roots of a quadratic equation as could be seen in Figure 5.2.

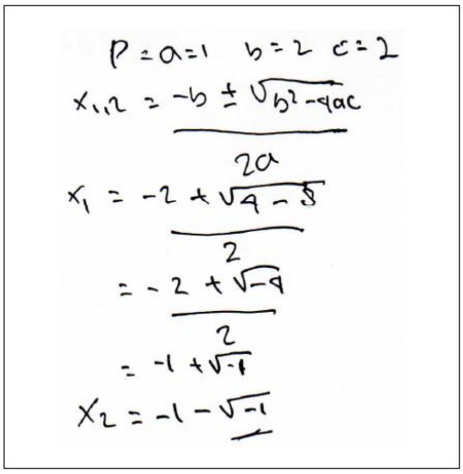


Figure 5.2 Finding the Roots of $z^2 + 2z + 2 = 0$ by Iful–Vito

This part was considered as symbolic representations. From this procedure, they found that P was an empty set, and then concluded that there was not an association could be made from or to the elements of P. This could be considered as connecting symbolic representations to embodied representations. However, this connected representation was just a small part of the whole responses. Therefore, the responses to the knowledge element of association between sets' elements were categorized as embodied representations.

5.2.2.2 Ordered Pairs

Mostly, the representations used in their responses were symbolizing-embodiment. Although the expression of the results of an association had been introduced earlier in the question, Iful and Vito still used arrow diagrams in the beginning of their responses. Next, they represented the diagrams as a set of ordered pairs. The expression of the sets by extension, namely, listing each member of the set, was categorized as symbolic representations.

[0138] Vito They are different.

[0139] Iful And the second one, that is 2 to 6, while here 6 to 2, the reversal of the first.

[0140] Vito Certainly, they are different, right?

[0141] Iful Yes.

[0142] Vito Because … because ….

[0143] Ifil Part a ….

[0144] Vito Part a.

[0145] Iful Reversal.

[0146] Vito Based upon the elements, they are different, we made an association of 2 sets whose element is 2 … to 3; it is different. But, from their rules, they are the same, only notation … the results of the associations are different. The results of the associations are different. Everything is different.

[0147] Iful Because … they are not the same, right? If the notation is written, 4 comma 2, while in Part a, 2 comma 4 … in Part b, 4 comma 2.

[0148] Vito Yes, the results of the associations are different … one is the reverse of the other.

[0149] Iful It's different because … reversal.

After listing the results of the associations, Iful and Vito moved to identify the difference between the results of the two associations. They focused their attention on the order of the elements in the expression (a, b). Iful stated, "And the second one, that is 2 to 6, while here 6 to 2, the reversal of the first" [0139]. Looking at the notation, Vito agreed on the difference identified by Iful, "Yes, the results of the associations are different … one is the reverse of the other" [0148]. With this agreement, they wrote the conclusion that the results of the associations were different because one was the reversal of another. This conclusion was based upon the property or the characteristic visible in the expression of ordered pairs. This conclusion was completed with examples of (2,4) and (4,2). All were categorized as symbolic representations, and therefore, symbolizing-embodiment representations were considered as the dominant in the responses to the concept of ordered pairs (Figure 5.3).

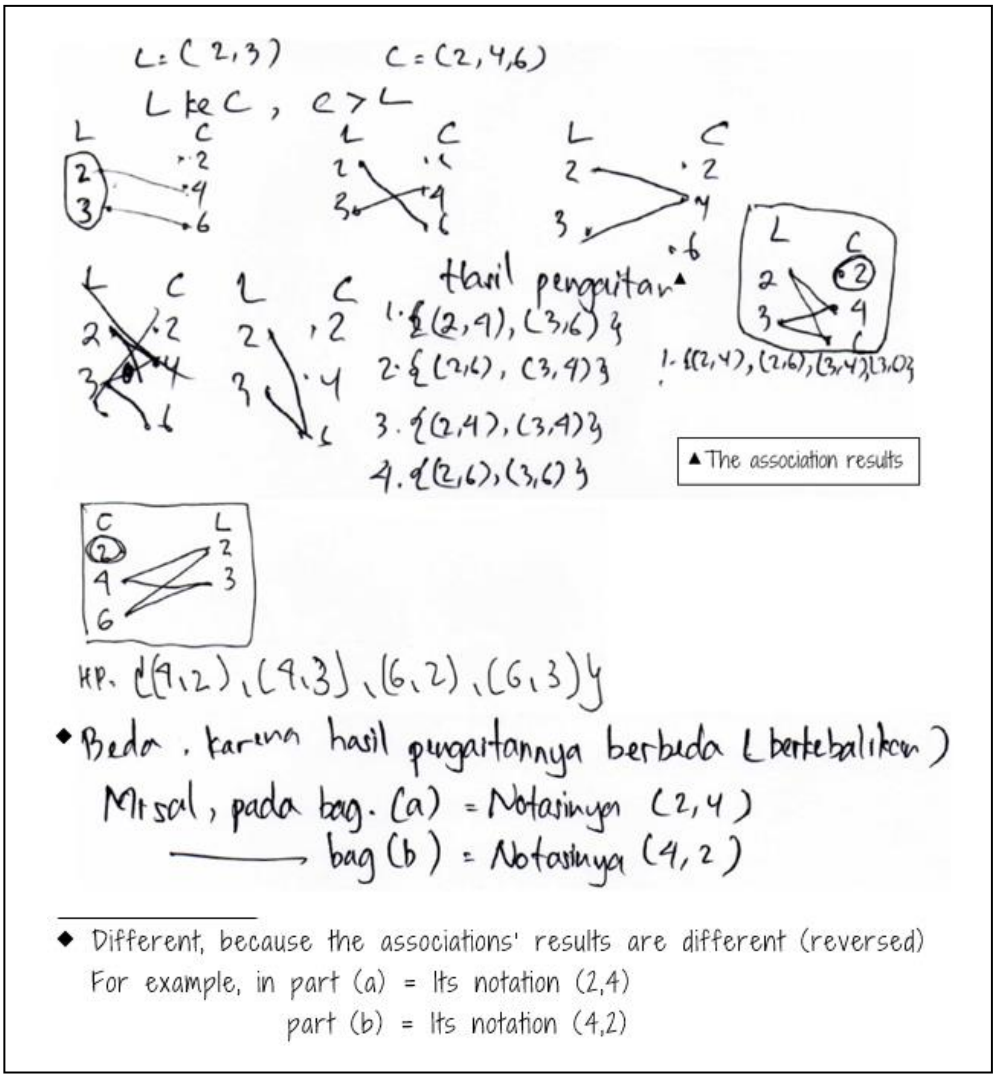


Figure 5.3 The Responses of Iful–Vito to the Concept of Ordered Pairs

5.2.2.3 The Cartesian Product of Two Sets

For this particular concept of the Cartesian product of two sets, Iful and Vito rewrote the results of three different associations. These all were expressed in symbolic representations. From the lists, they tried to characterize each list with a kind of general form of pairs not existing. For Part (a), they assigned a model of the ordered pair $(x,2)$ as the one not existing in the list, while for Part (b), they assigned the ordered pair $(2,y)$. These forms were considered as formalizing the characteristic identified in the lists, hence categorized as formalizing-symbolism representations.

[0176] Iful   In a, there is not 2, right? In Part a, there is not.
[0177] Vito   Here [Part b], there is not 2 comma *x* or just say 2 comma *y*.

[0178]	Iful	Yes, here [Part b], 2 comma *y* doesn't exist.
[0179]	Vito	Here [Part d], all exist; it is complete here.
[0180]	Iful	Meanwhile, here [Part d] is complete.
[0181]	Lec	What is the question?
[0182]	Vito	What is unique, all in d could be concluded as all pairs which could be formed, all could have partners from *L* to *C*, all could have partners.

Iful and Vito tried to understand the concept of the Cartesian product of two sets by observing the properties of the sets containing ordered pairs resulting from associating the elements of two sets.

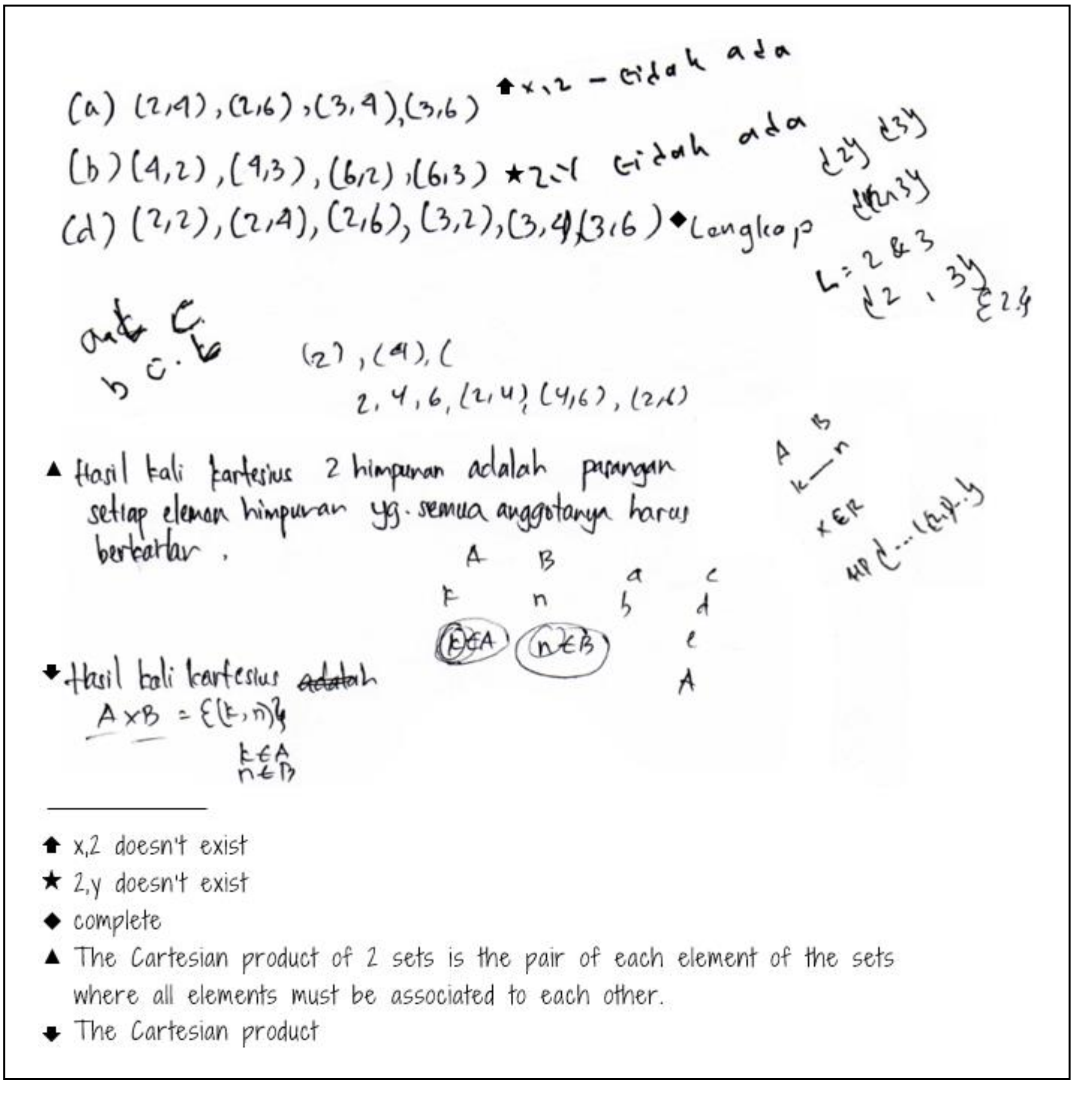


Figure 5.4 The Responses of Iful–Vito to the Concept of the Cartesian Product of Two Sets

Iful and Vito identified the list in Part (d) as complete, because all possible ordered pairs existed. To them, the term the Cartesian product was introduced, and they made a definition of this concept, based upon the property they identified in Part (d). Iful and Vito defined that *the Cartesian product of two sets is the pair of each element of the sets where all elements must be associated with each other* (see Figure 5.4 above). In terms of representation use, it could be observed that from the lists of ordered pairs that were expressed in symbolic representations, they identified and defined the Cartesian product of two sets which were stated in formal representations. The responses flowed from the symbolic representations to the formal representations. Therefore, the formalizing-symbolism representations were claimed to dominate the answer of Iful and Vito for the concept of the Cartesian product of two sets.

5.2.2.4 Relations

The trend in the responses to the concept relation was similar to the one in the concept of the Cartesian product of two sets. They used the connected representations from the symbolic to the formal, namely, formalizing-symbolism representations. Iful and Vito developed their responses to this concept based upon the Cartesian product. In Figure 5.5, it could be seen that the responses began with expressing two sets of ordered pairs, set *A* and set *D*. These sets were developed from the associations the same as one dealt in the concept of the Cartesian product. Responding to the question of what the relationship between the two sets they could identify, Iful and Vito found that *A* was a subset of *D* [0297–0304]. They expressed this relationship formally as $A \subseteq D$. The term relation was introduced to Iful and Vito, and they proceeded to make a formal definition of relation. They finished their responses with formally representing the definition of relation as a subset of the Cartesian product of two sets.

[0297]	Vito	*A* is a subset of *D*.
[0298]	Iful	*A* is a subset of set *D*.
[0299]	Lec	What is *D*?
[0300]	Vito	The Cartesian product of *L* and *C*.
[0301]	Iful	The product of set *L* and set *C*.
[0302]	Lec	So, what is *A*?
[0303]	Iful	The Cartesian product.
[0304]	Vito	A subset of the Cartesian product.

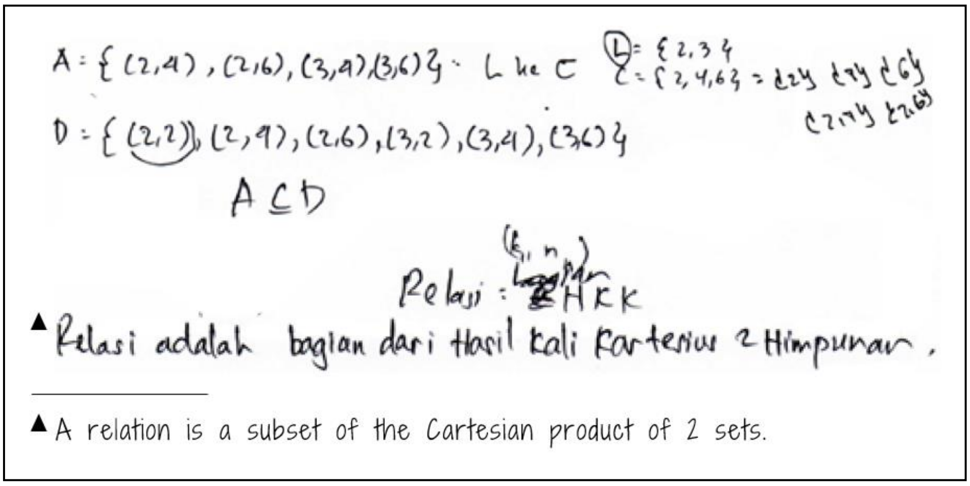


Figure 5.5 The Responses of Iful–Vito to the Concept of Relation

5.2.2.5 The Exhaustive Domain Property

In identifying the property of the exhaustive domain, Iful and Vito discussed it using embodied representations. Initially, they focused on the peripheral aspects of the relations, such as the relationship between the sets [0410–0413] or the equality of the domains or codomains [0425–0426]. All of them were considered as embodied representations. After agreeing upon the unique property of the exhaustiveness of the domains [0432–0434], they wrote down statements of the unique property identified.

[0410]	Vito	Oh, there is an element in *B* which is not in *D*, meaning that *B* is not a subset of the Cartesian product.
[0411]	Iful	Yes, *A* is its subset, while *B* is the reverse of *A*, meaning that *B* is not a subset of *D*.
[0412]	Lec	Here, you have to see *A* and *D*, and find what their unique property is; it is not about the relationship between *A* and *D*.
[0413]	Vito	Oh, yes, *B* is not a subset of *D*.
. . .		
[0425]	Iful	Yes, A is L to *C*, and *A* is the same as *D*.
[0426]	Vito	*B* … *C* to *L*.
[0427]	Lec	Is it unique?
[0428]	Vito	No, let's see the sets. This is *A*, this is *B*, this *D*, *A*, *B*, and *D*. They are common in … *A* and *D*, 2 and 3 have partners.
. . .		
[0432]	Iful	Yes, the first group … their elements must have partners.
[0433]	Lec	Yes, what about here [*B*]?
[0434]	Vito	Not, it is not.

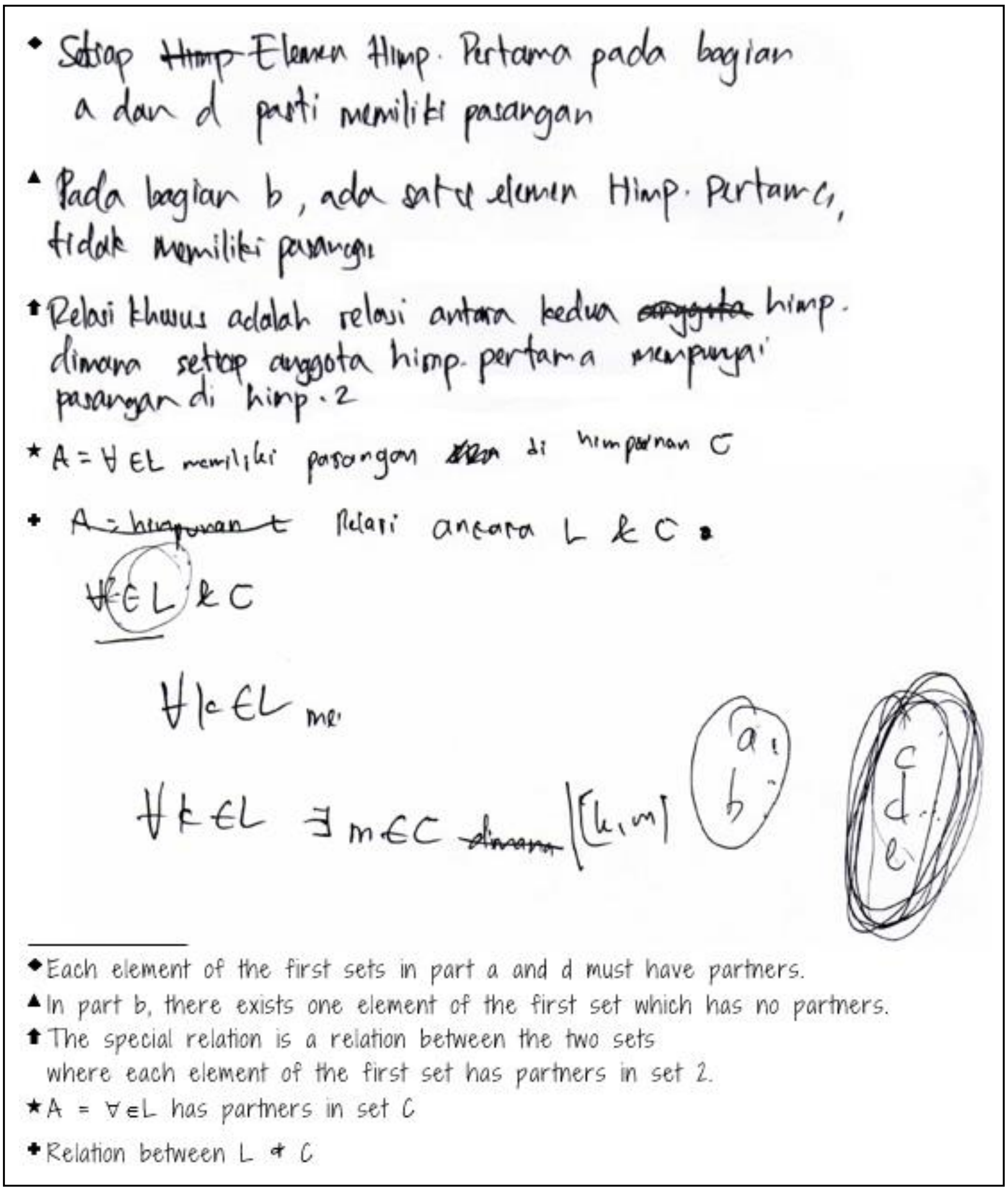


Figure 5.6 The Responses of Iful–Vito to the Concept of the Exhaustive Domain Property

Next, they made a connection to formal representations where they stated the special property of the relations, namely, *the special relation is a relation between the two sets where each element of the first set has partners in the second set*. As required in the task, they further developed the expression of this special property in a symbolic logic statement: $\forall k \in L \ \exists m \in C \mid (k, m)$. Here, (k, m) is to express that k is partnered with m (Figure 5.6). Based upon the whole responses provided, the formal representations were used dominantly.

5.2.2.6 The Unique Image Property

The category of representations used for the second special property of relation was the same as that in the previous exhaustive domain property. Iful and Vito mostly used formal representations in their responses.

[0598]	Iful	The commonality of *X* and *Y* … number one and number two.
[0599]	Vito	It is that each element of *X* is tied to only one in *Y*.
[0600]	Iful	Hang on.
[0601]	Vito	Correct, right? Each in *X*, one, has one.
[0602]	Iful	Each element of *X* only have one partner, because this [Illustration i] is like ... as you see.
[0603]	Lec	What about this [Illustration ii]? It is interesting.
[0604]	Vito	The elements of *X* only have one element in *Y*. It means the elements of *X*, oh …, don't use 'each'.
[0605]	Iful	Each element of *X* only …, yes ….
[0606]	Vito	One element of *X* has one element in *Y*.
[0607]	Iful	No. Each element of *X*, which has partners in *Y*. Because this [an element of the domain in Illustration ii] right, it doesn't have partners in *Y*; it means it is not covered.

Iful and Vito started with a discussion of the property of the relations given in the form of an arrow diagram. They focused on the embodied characteristics of the relations in order to identify the essential commonality of the two relations. Vito proposed the identified commonality [0599] which actually applied to the first illustrated relation. The discussion moved and Iful finally saw the essence of the commonality [0607], which then was expressed in a formal representation that *each element of X, which has partners in Y, only ties one element*.

Iful and Vito then tried to define that special property in formal representations which used symbolic logic statements (see Figure 5.7). They had several attempts to express the unique existence of the partner of each element of the domain of the relation. First, they wrote: $\forall k \in X, \exists m \in Y, n \in Y \mid (k,m) = (k,n)$. Here, they thought that it would be sufficient if there existed two elements of the codomain *Y* which both were the partners of *k*, and they equated (k,m) with (k,n). Then, they realized that the existence of the two partners of *k* was insufficient. For the unique image property, all elements of the codomain which were the partners of *k* had to be all the same. They expressed: $\forall k \in X, \forall m \in Y, \forall n \in Y \mid (k,m) = (k,n)$. Iful and Vito changed the quantifier for the elements of the codomain *Y*, but they still preserved the equality of the two ordered pairs, $(k,m) = (k,n)$. After a discussion, they finally realized that it was not that $(k,m)=(k,n)$, but it was $m=n$, if k was paired off with both m and n. Iful and Vito wrote: $\forall k \in X, \forall m,n \in Y \mid (k,m) \wedge (k,n) \Rightarrow m=n.$ This completed their responses to the concept of unique image property of relation.

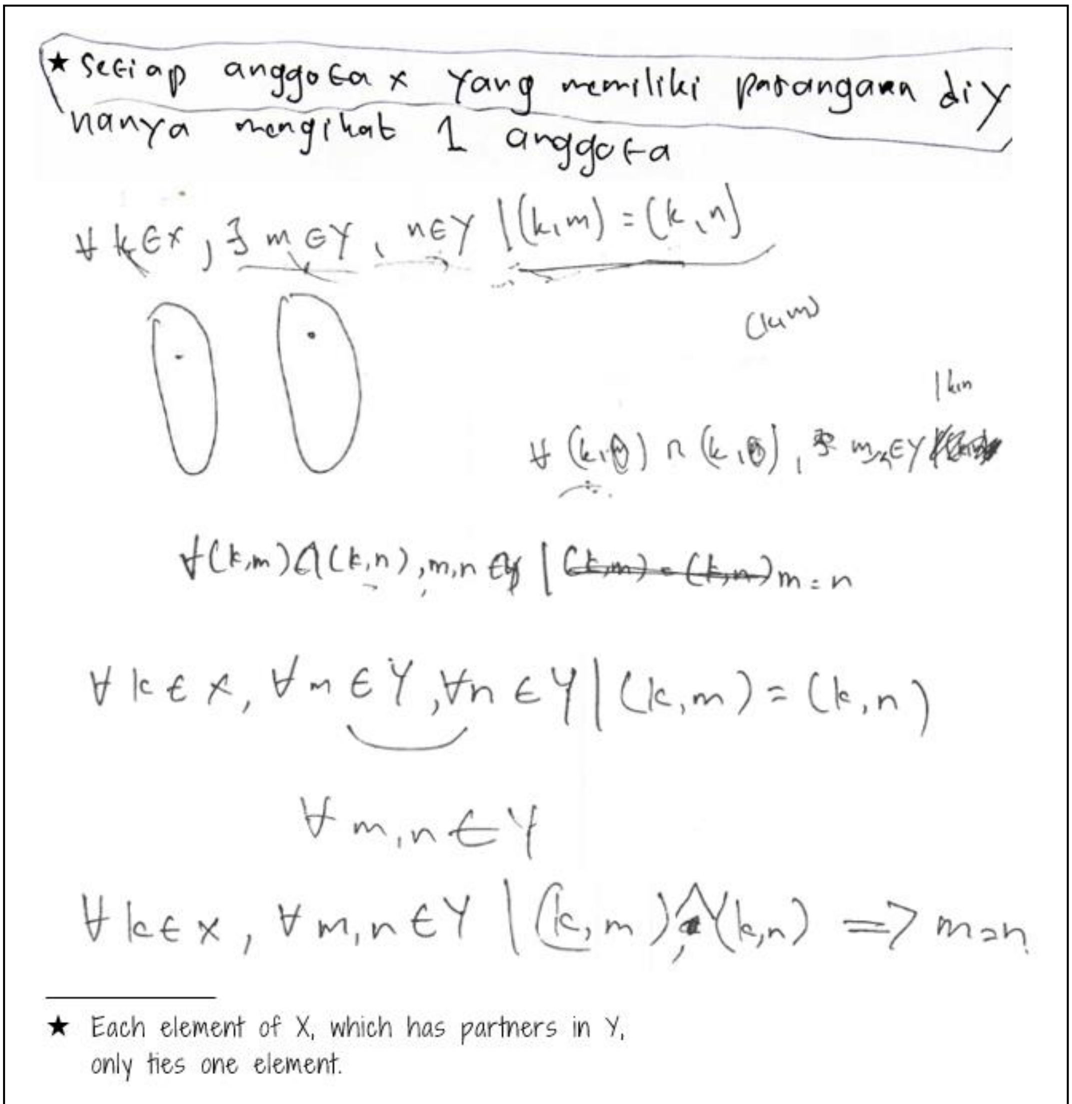


Figure 5.7 The Responses of Iful–Vito to the Concept of the Unique Image Property

5.2.2.7 The Operational Formal Definition of Function

For the last knowledge element, namely, operational formal definition of function, it was expected that the students would use responses that were expressed in formal representations. This was because the operational formal definition is a definition that was basically formulated in formal mathematics language using symbolic logic. In the calculus tasks, the students were asked to make an example of relation satisfying both the properties of the exhaustive domain and the unique image. It was intended to lead the students to define the concept of function in formal and operational form, which is actually as simple as combining the definition of the two special properties of relation.

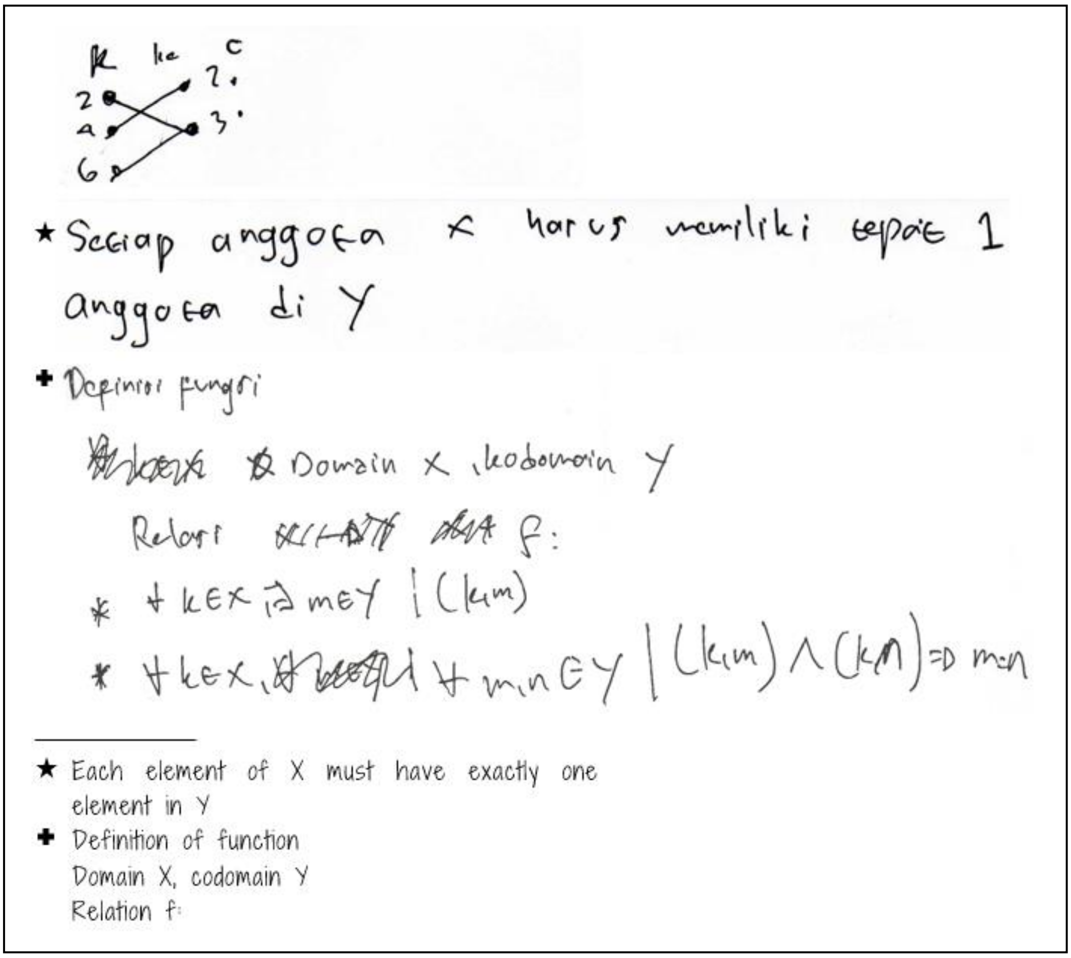


Figure 5.8 The Responses of Iful–Vito to the Operational Formal Definition of Function

In their responses (Figure 5.8), the example made by Iful and Vito was expressed in embodied representations, namely, an arrow diagram function. With this example, they made a definition of function in a formal form, each element of *X* must have exactly one element in *Y*. Further, they developed the operational formal definition of function and this was expressed in formal representations.

5.2.2.8 The Representations Used in Justifying Relations

Besides the seven knowledge elements, Iful and Vito also solved problems of several relations as to whether they were a functional relation or not (see Figure 4.5 in Chapter

4, p. 159). There were 6 relations expressed in three different representations. In the solution of the problems, the categories of representations used by the students were also analyzed.

The first two relations were given in embodied representations. One relation was presented as a graph and another one as a table. The verification for the graphed relation is illustrated in Figure 5.9. Initially, Iful and Vito worked directly on the graph to prove that the graph did not represent a functional relation. In this case, they used the vertical line test, by which they could find that one line intersected the graph more than once for one point in the domain (the horizontal axis). Iful and Vito made a test line for $x = 2,$ where they found that the line crossed the graph at two points, namely, (2, p) and (2,1) [in the original version, the graph was not defined at point (2,1)]. This test led them to conclude that the graph was representing a legitimate function. In addition, they drew another arrow diagram to describe the non-functional nature of the graph. In the end, they realized that the given graph discontinued at (2,1). They made another test line on the right-hand side of $x = 2,$ and they made the same conclusion that the graph was not a relation.

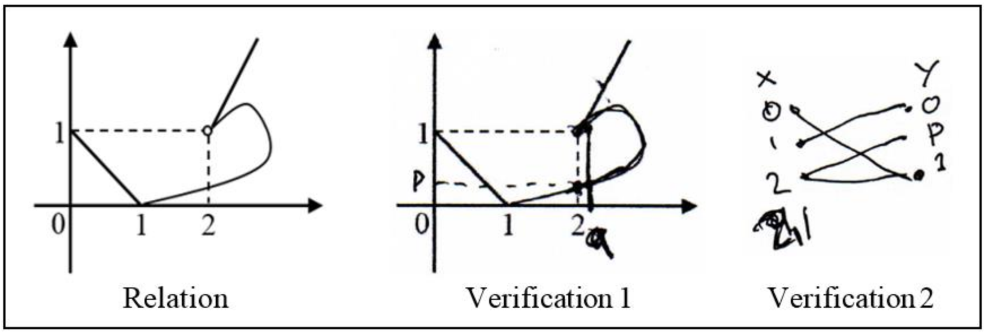
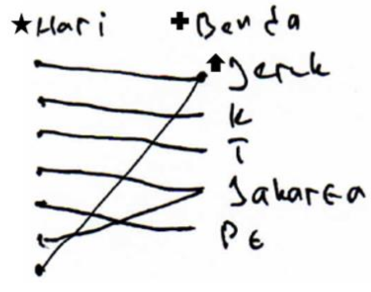


Figure 5.9 A Verification of Embodied Relation (Graph) by Iful–Vito

Expressing the relation in another embodied representation was also used in verifying the second relation, which was given as a two-column table (Figure 5.10). The table was represented in an arrow diagram. The diagram showed that each element in the domain (the left side set) was paired off with only one element in codomain (the right side); the relation, therefore, was a function. Thus, for the two embodied relations, Iful and Vito verified them in embodied representations.

Days	Things
Sunday	Orange
Monday	Computer
Tuesday	Assignment
Wednesday	Jakarta
Thursday	Aeroplane
Friday	Jakarta
Saturday	Orange

Relation



Verification

★ Days ✚ Things ⬆ Orange

Figure 5.10 A Verification of Embodied Relation (Table) by Iful–Vito

The next two relations were given in symbolic representations. Iful and Vito represented the first relation $[x = 5]$ in a graph on the Cartesian plane and an arrow diagram. The verification process expressed the symbolic relation in embodied representations. This exemplified the use of the embodying-symbolism representations (Figure 5.11).

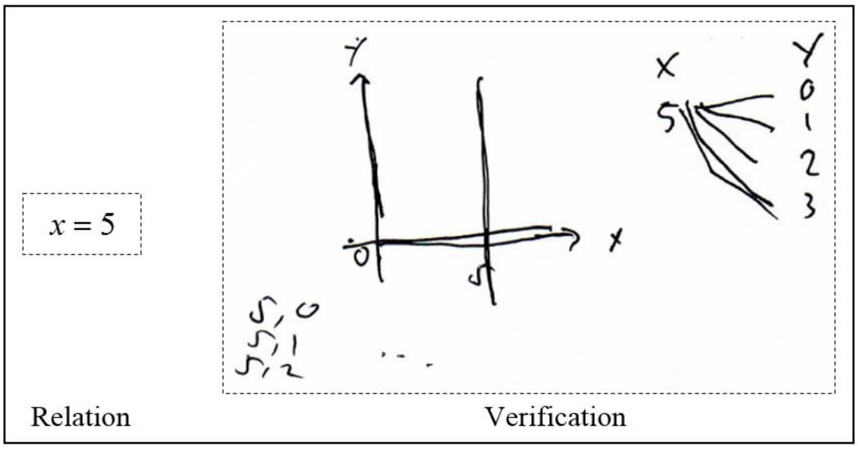


Figure 5.11 A Verification of Symbolic Relation (Algebraic Equation) by Iful–Vito

For the relation $x(y) = \begin{cases} y & \text{if} \quad y \text{ is rational} \\ -y & \text{if} \quad y \text{ is irrational} \end{cases}$, this symbolically represented relation was verified by an inductive case method. As depicted in the following excerpt, Iful and Vito used symbolic calculations by taking two values of *x*, where they found that the image of the values were different. They decided the status of the relation as a function using the results of the calculations. This procedure was categorized as symbolic representations.

[0889]	Iful	If y is rational, meaning if its y equals 2, then its xy also equals 2; if its y equals the root of 2, meaning ….
[0890]	Vito	For example, let's try x2, meaning it is 2 … because if for the root of 2, meaning it is minus the root of 2.
[0891]	Iful	It means they are different.
[0892]	Vito	It means it is a function.

The last two relations were formally represented. The first relation [$\boldsymbol{R}$ is the set of real numbers. Relation β is from $\boldsymbol{R}$ to $\boldsymbol{R}$; $\beta(y) = \log(2 - y),$ for each $y \in \boldsymbol{R}$] was not a function. Instead of verifying the relation by formal proof using the negation of the definition of function, Iful and Vito verified the non-functional nature by counterexamples. This involved calculations based upon which they found that there existed elements of the domain with no image in the codomain. Iful and Vito took 9 as an example element of the domain to show that the relation was not a function [0960–0988]. The justification for this formal relation used symbolic representations, and therefore the responses were categorized as symbolizing-formalism representations.

[0960]	Lec	Because the domain must cover all real numbers, what y could you try? Just mention it.
[0961]	Iful	9.
[0962]	Lec	9, ok, what do you think?
[0963]	Vito	Log minus seven.
[0964]	Iful	Log minus seven.
[0965]	Lec	Yes, what is it?
. . .		
[0981]	Iful	Minus seven is to which power of ten?
[0982]	Lec	Right, what is the power of ten resulting in minus seven?
[0983]	Vito	None. There is not such a power.
[0984]	Iful	No, no, it could not be.
[0985]	Lec	Why?
[0986]	Iful	Because if a positive is raised to whatever power, the result must be a positive.
[0987]	Lec	So?
[0988]	Vito	It means there is no partner for 9.

For the last formal relation [Let $\boldsymbol{N}$ be the set of natural numbers and $\boldsymbol{Z}$ be the set of integers. Relation L is from $\boldsymbol{N}$ to $\boldsymbol{Z}$ where $L = \{(a, b) \mid a + b = 0, a \in \boldsymbol{N}, b \in \boldsymbol{Z}\}$], Iful and

Vito provided a deductive argument to verify it as a legitimate function. They proved the functional nature of the relation using the operational formal definition of function. The responses were clearly categorized as formal representations (Figure 4.6 in Chapter 4, p. 160).

5.2.3 Summary of the Categories of Representations Used by the Students

5.2.3.1 Iful–Vito

The representations used by Iful and Vito in their responses to the calculus tasks concerning the seven knowledge elements were classified into four different categories, namely, embodied representations, symbolizing-embodiment representations, formalizing-symbolism representations, and formal representations. The representations of mathematical objects Iful and Vito used in their responses to the problem concerning the association between sets' elements were dominantly embodied representations. Some part of their responses showed symbolic representations which still had the connection to the embodied representations. However, at most their responses were embodied representations. Iful and Vito expressed their responses to the concept of ordered pair mostly in connected symbolizing-embodiment representations.

For the concepts of the Cartesian product of two sets and relation, the formalizing-symbolism representations were used by Iful and Vito dominantly. The connected representations were justified by their answers which were started with expressing the results of the associations in symbolic representations. They changed to the formal representations to express the properties identified in the results of the association which characterized the Cartesian product of two sets and the relation, respectively. The concepts of the Cartesian product and relation were then defined in formal logic statements. Iful and Vito expressed their responses to the knowledge elements of the special properties of relation and the operational formal definition of functions mostly in formal representations.

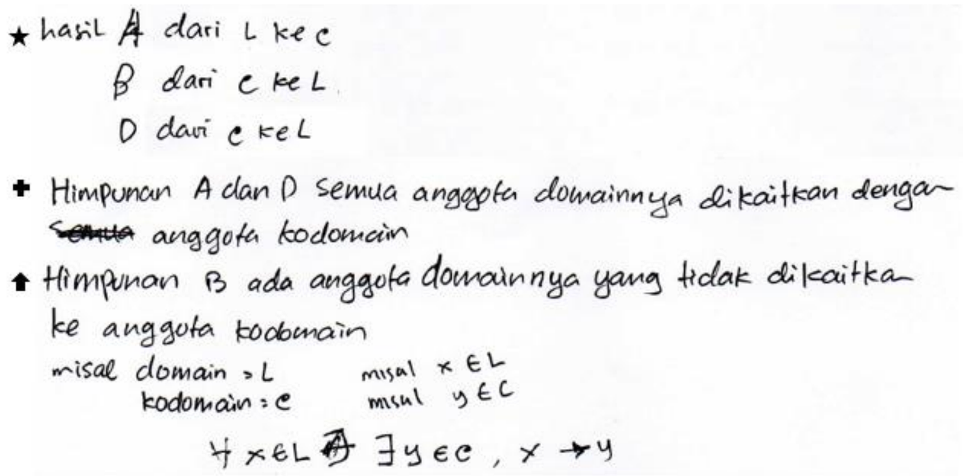
In justifying relations, Iful and Vito dealt with embodied relations using embodied representations. They represented the relations in other embodied representations and examined the legitimacy of the relations as to whether they were functions or not. In verifying a symbolic relation specifically given in algebraic expression, they also represented the relation in embodied representations. For this response, the representations they used were classified as embodying-symbolism. Another connected representation model was used by Iful and Vito in verifying a formal relation. Their responses to this particular relation were then categorized as symbolizing-formalism representations. For the last given formal relation, Iful and Vito employed a formal proof to verify that it was a legitimate function, and the proof was all in formal representations.

5.2.3.2 Dina–Yuni

Overall, the representations used by Dina and Yuni in their responses to the calculus tasks concerning the seven knowledge elements were categorized into four types, namely, embodied representations, symbolic representations, formalizing-symbolism representations, and formal representations. Dina and Yuni used representations with a pattern similar to those used by Iful and Vito, expect in their responses to the concept of ordered pair. For the concepts of associating elements of two sets, the representations used in their response were embodied representations. These representations were dominant with a small part of the responses expressed in symbolic representations. For the concept of ordered pair, Dina and Yuni's responses were dominated by symbolic representations.

Regarding the concept of the Cartesian product of sets, Dina and Yuni expressed their responses mostly in formalizing-symbolism representations. These connecting representations were justified by the expression of the results of the associations in symbolic lists, from which Dina and Yuni identified the unique property characterizing the concept of the Cartesian product. The identified property was then stated in a formal definition. The same category of formalizing-symbolism representations was also found to be dominant in the responses of Dina and Yuni to the calculus tasks concerning the concept of relation

Dina and Yuni's answer to the tasks pertaining to the special properties of relations and the operational formal definition of functions were dominated by formal representations. For the first special property, namely, the exhaustive domain, the responses provided by Dina and Yuni started with symbolic representations. They stated the domain and the codomain of each relation, namely, *A* was from *L* to *C*, *B* was from *C* to *L*, and *D* was from *L* to *C*. The rest of the responses were all in formal representations which expressed the property and its formal definition (Figure 5.12). For the second special property of the unique image, all responses were classified as formal representations. A small portion of embodied representations was used in the responses to the question concerning the operational formal definition of function. This part represented an example of function. The responses mostly concerned the definition of function expressed in symbolic-logic statements which were all categorized as formal representations.


★ hasil A dari L ke C

B dari C ke L

D dari C ke L

✚ Himpunan A dan D semua anggota domainnya dikaitkan denga ~~semua~~ anggota kodomain

⬆ Himpunan B ada anggota domainnya yang tidak dikaitka ke anggota kodomain

misal domain = L misal $x \in L$

kodomain = C misal $y \in C$

$$\forall x \in L\ \exists y \in C,\ x \to y$$

★ Results A from L to C

. B from C to L

. D. from C to L

✚ Set A and D all elements of their domains are associated with elements of the codomain

⬆ Set B there exists an element of its domain which is not associated with elements of the codomain

Let domain = L let $x \in L$

codomain = C let $y \in C$

$$\forall x \in L\ \exists y \in C,\ x \to y$$

Figure 5.12 The Responses of Dina–Yuni to the Concept of the Exhaustive Domain Property

Dina and Yuni’s use of representations were also analyzed in terms of their justification of relations as to whether they were functions or not. It was found that for embodied relations, they also used embodied representations for examining the relations. The relation expressed in a graph was represented in an arrow diagram, based upon which they decided that the graph did not represent a legitimate function (Figure 5.13).

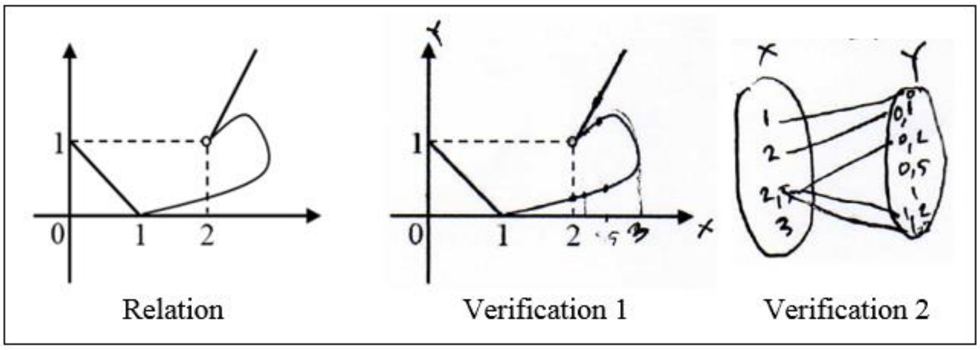


Figure 5.13 A Verification of Embodied Relation (Graph) by Dina–Yuni

A relation given in symbolic algebraic form: $x(y)=\begin{cases} y & \text{if} \quad y \text{ is rational} \\ -y & \text{if} \quad y \text{ is irrational} \end{cases}$ was verified by calculations. Dina and Yuni applied the inductive case approach to show that the relation was a legitimate function. The whole responses for this particular relation was categorized as symbolic representations. The symbolic procedure is partly depicted in the following discussion.

[1485]	Yuni	Minus *y*, if *y* is irrational.
[1486]	Lec	What irrational number did you choose?
[1487]	Dina/ Yuni	The root of two.
[1488]	Lec	The square root of two, so what is the result?
[1489]	Yuni	Minus the root of two.
[1490]	Dina	Yes, minus the root of two.
[1491]	Lec	Now, is that a function?
[14921	Yuni	Yes, amm … yes.
[1493]	Lec	Because?
[1494]	Yuni	Only one exactly is its element in the codomain. One, one. Two, two. Minus two, minus two.

The formally represented relation [$\boldsymbol{R}$ is the set of real numbers. Relation β is from $\boldsymbol{R}$ to $\boldsymbol{R}$; $\beta(y)=\log(2-y)$, for each $y\in\boldsymbol{R}$] which was not a function was verified by calculations as described in Figure 5.14. This verification was classified as symbolizing-formalism representation. Actually, there should be a formal proof showing that the relation was not a function. The proof was an argument developed from the definition of non-functional relation. This definition was the negation of the operational formal definition of function. However, Dina and Yuni did not prove the relation in this way. Instead, they used the counterexample approach.

★ karena ada anggota ~~domain~~ y yang tidak memiliki kaitan di anggota R.

✚ misal $y = 2$

$f(2) = \log(2-2)$

$= \log 0$

= ~~∞ tidak hingga~~

⬆ = tidak ada

$^{a}\log b = c$

$a^{c} = b$

$^{10}\log 0 =$

★ Because there exists an element y whose no association with an element of R.

✚ Let $y = 2$

⬆ = none

Figure 5.14 A Verification of Formal Relation by Dina–Yuni (1)

$L = \{(a,b) \mid a + b = 0,\ a \in N,\ b \in Z\}$

★ misal

Domain = N, $a \in N$

kodomain : Z, $b \in Z$

✚ L adalah fungsi jika $\forall a \in N, \exists -a \in Z,\ a \to -a$

karena $a + (-a) = 0$ dan

$\forall a \in N, \forall b, c \in Z,\ a \to b \wedge a \to c \Rightarrow a + b = 0$ dan $a + c = 0$ maka $b = c$

$a + b = 0 \Rightarrow a = -b$

⬆ $a + c = 0 \Rightarrow a = -c$ karena $a = a$

maka $-b = -c,\ b = c$

★ Let

Domain = **N**, $a \in$ **N**

Codomain = Z, $b \in Z$

✚ L is a function if $\forall a \in$ **N**, $\exists -a \in Z$, $a \to -a$

because $a + (-a) = 0$ and

$\forall a \in$ **N**, $\forall b, c \in Z$, $a \to b \wedge a \to c \Rightarrow a + b = 0$ and

$a + c = 0$ then $b = c$

$a + b = 0 \Rightarrow a = -b$

⬆ $a + c = 0 \Rightarrow a = -c$ because $a = a$

then $-b = -c$, $b = c$

Figure 5.15 A Verification of Formal Relation by Dina–Yuni (2)

Another formal relation [Let ***N*** be the set of natural numbers and ***Z*** be the set of integers. Relation L is from ***N*** to ***Z*** where $L = \{(a,b) \mid a+b=0, a \in \boldsymbol{N},\ b \in \boldsymbol{Z}\}$] was proved by Dina and Yuni as a valid function. Their proof was expressed in formal representations (Figure 5. 15).

5.2.3.3 Naya–Amzi

Naya and Amzi’s responses to the calculus problems were expressed in mathematical representations that were classified into three different categories. They used symbolic representations in their answers to the problems concerning the association between the elements of two sets and the concept of ordered pairs. With a similar pattern to the responses provided by both Iful–Vito and Dina–Yuni to the other five concepts (refer to Table 5.1, p. 179), Naya and Amzi dealt with the problems of the Cartesian product of two sets and the concept of relation dominantly using formalizing-symbolic representations. Meanwhile, the representations used in their responses to the tasks concerning the special properties of relations and the operational formal definition of function were mostly classified as formal representations.

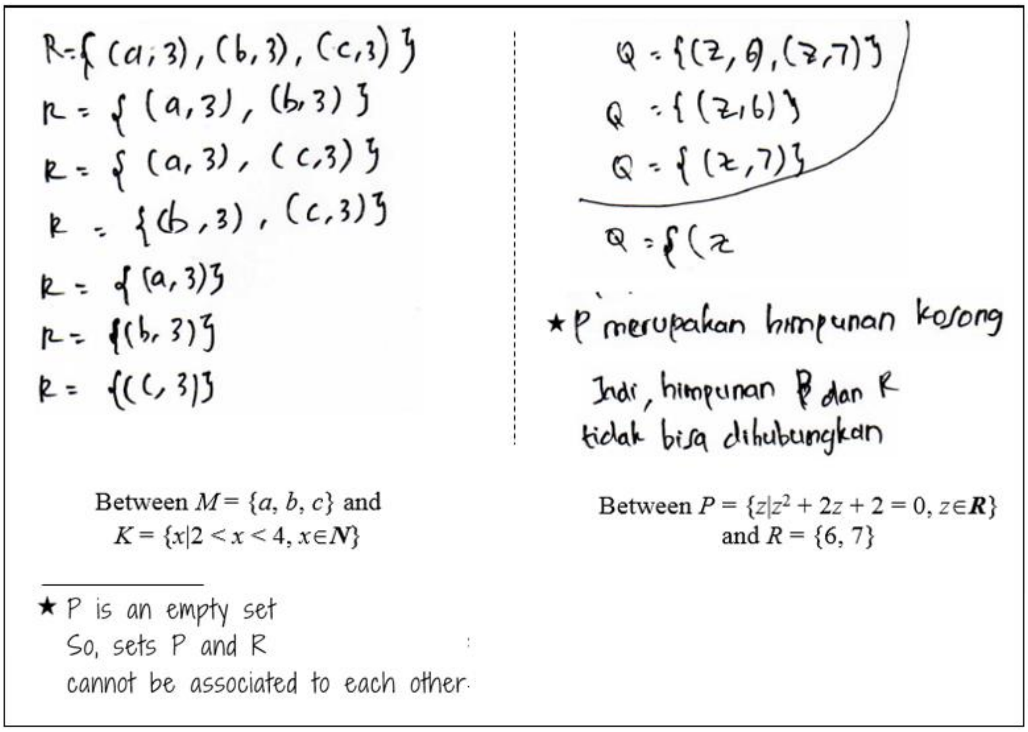


Figure 5.16 The Responses of Naya–Amzi to the Concept of Association Between Sets’ Elements

Instead of expressing the models of association in arrow diagrams like those in the responses of all the other participating students, Naya and Amzi used sets of ordered pairs. There were algebraic components involved in the responses, such as, the process of determining the element of *K* and the emptiness of *P*. The responses were dominated by the set of ordered pairs, which resulted from pairing the elements of two sets. They consistently used that model of associating the elements of two sets in the second pair of sets, *P* and *R*, although at the end they concluded that no association could be made between the two sets because one of them was a null set (Figure 5.16). Therefore, the representations used dominantly in the responses were categorized as symbolic representations. In their responses to the concept of ordered pairs, it was observed that their answers were also expressed mostly in symbolic representations. The characterizing property of an ordered pair was identified through comparing lists of the results of two associations. They stated the essence of the concept of ordered pairs, that $(2,4) \neq (4,2)$ because the position of their elements was different.

With regard to the concept of the Cartesian product of two sets and relation, Naya and Amzi's responses were categorized as formalizing-symbolism representations. For the Cartesian product, they listed the results of three associations. The lists were compared to each other from which Naya and Amzi identified the uniqueness of one list. This particular list was claimed as the Cartesian product of two sets (Figure 5.17). For this first part of the responses, the lists of the ordered pairs and the explanation of the identified uniqueness could be categorized as formalizing-symbolism representations.

The second part of the responses could be seen in Figure 5.18. Here, Naya and Amzi made a connection to the formal representations to define the Cartesian product of two sets. The representations used in the whole responses, both Part 1 and Part 2, were categorized as formalizing-symbolic representations. This trend was also observed in Naya and Amzi's responses to the concept of relation. From the responses provided for the problem regarding the exhaustive domain and the unique image properties of relations, Naya and Amzi's responses were dominated by formal representations. The formal representations were also found to be dominant in their responses to the operational formal definition of function.

a = (2,4), (2,6), (3,4), (3,6)   C = {2, 4, 6}

b = (4,2), (4,3), (6,2), (6,3)   L = {2, 3}

d = (2,2), (2,4), (2,6), (3,2), (3,4), (3,6) ⇒ ★ hasilkali cartesius

✚ pada pengaitan di (d), semua anggota himpunan L ~~dan~~ yang dikait-kan denggn anggota himpunan C semuanya memiliki pasangan,

⬆ pada bagian(a) dan (b), ada satu anggota yang tidak dipasangkan sedangkan pada bagian (d), semua anggota ~~memiliki~~ domain dipasangkan dengan semua anggota kodomain

---

★ The Cartesian product

✚ In association (d), all elements of set L which are associated with elements of set C all have partners,

⬆ In part (a) and (b), there exists an element which is not paired off, while in part (d), all elements of the domain are paired off with all elements of the codomain

Figure 5.17 The Responses of Naya–Amzi to the Concept of the Cartesian Product of Two Sets (Part 1)

a = {(2,4),(2,6),(3,4),(3,6)}

d = {(2,2),(2,4),(2,6),(3,2),(3,4),(3,6)}

★ Hasil kali cartesius adalah himpunan pasangan yg berasal dari dua himpunan yang saling dikaitkan dimana semua anggota domain dipasangkan dengan semua anggota kodomain.

✚ Misalkan ada 2 himpunan yaitu himpunan A dan B, misalkan $x \in A$ dan $y \in B$

Hasil kali cartesius yaitu $A \times B = \{(x,y) \mid x \in A$ dan $y \in B\}$

★ The Cartesian product is a set of pairs resulting from two sets associated to each other where all elements of the domain are paired off with all elements of the codomain.

✚ Let be two sets, namely, set A and B, let $x \in A$ and $y \in B$

The Cartesian product, namely, $A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$

Figure 5.18 The Responses of Naya–Amzi to the Concept of the Cartesian Product of Two Sets (Part 2)

Regarding the justification of relations, it was found that Naya and Amzi responded to the given embodied relations with embodied representations. The embodied representations were also used to verify to a symbolic relation and a formal relation. They represented the symbolic relation $[x=5]$ in an arrow diagram (Figure 5.19). Therefore, the responses were classified as embodying-symbolic representations. In addition, one formal relation [$\boldsymbol{R}$ is the set of real numbers. Relation β is from $\boldsymbol{R}$ to $\boldsymbol{R}$; $\beta(y)=\log(2-y)$, for each $y\in\boldsymbol{R}$] was verified by calculation and another formal relation [Let $\boldsymbol{N}$ be the set of natural numbers and $\boldsymbol{Z}$ be the set of integers. Relation L is from $\boldsymbol{N}$ to $\boldsymbol{Z}$ where $L=\{(a,b)\mid a+b=0, a\in\boldsymbol{N},\ b\in\boldsymbol{Z}\}$] was verified by its embodied property. For these responses, the former was categorized as a symbolizing-formalism representation, and the latter was an embodying-formalism representation.

Figure 5.19 A Verification of Symbolic Relation (Algebraic Equation) by Naya–Amzi

5.2.3.4 Dyn–Sam

In general, the representations used by Dyn and Sam in their responses to the calculus tasks concerning the seven knowledge elements were categorized into only three types, namely, embodied representations, symbolizing-embodiment representations, and formal representations. The embodied representations were used mostly in their responses to the association of sets' elements. The embodied representations for the association of sets' elements were in the form of arrow diagrams. The symbolizing-embodied representations were used in the responses to the concept of ordered pairs, where they first made an arrow diagram of the associations, and then formed a list of pairs resulting from two different associations. From the lists, they identified the property characterizing an ordered pair that distinguished it from an ordinary unordered set.

★ pada hasil Pengaitan di (d) setiap anggota himpunan L dan himpunan C memiliki pasangan

✚ hasil kali kartesius adalah himpunan pasangan setiap anggota ke ~~himpunan S A dengan setiap~~. suatu himpunan ke setiap anggota himpunan lain.
hasil kali kartesius sama dengan $x \mid x$ pasangan dari setiap anggota (x)

⬆ Hasil kali Kartesius $A \times B = \{$ ~~x | x~~ ~~pasangan~~ (k, w) (x)
Hasil kali kartesius $A \times B = \{(k, w) \mid$ ~~adalah~~
$A \times B = \{(k, w) \mid \forall k \in A$ dan $\forall w \in B\}$.

★ In the results of the association in (d) each element of set L and set C has partners

✚ The Cartesian product is a set of pair of each element of a set to each element of another set.

⬆ The Cartesian product equals $x \mid x$ is a pair of each element (x)
The Cartesian product $A \times B = \{(k,w)$ (x)
The Cartesian product $A \times B = \{(k,w) \mid$
$A \times B = \{(k,w) \mid \forall k \in A$ and $\forall w \in B\}$

Figure 5.20 The Responses of Dyn–Sam to the Concept of the Cartesian Product of Two Sets

$A = \{(2,4), (2,6), (3,4), (3,6)\}$ dan $D = \{(2,2), (2,4), (2,6), (3,2), (3,4), (3,6)\}$

★ hubungan A ke D yaitu $A \subseteq D$.
himpunannya yaitu $A \cup D = D$.
D = adalah hasil kartesius. ⇒ $D = L \times C$.

✚ karena $A \subseteq D$ dan D hasil kali kartesius $L \times C$
maka A relasi dari L ke C

⬆ relasi adalah himpunan beberapa pasangan dari hasil kali kartesius.
~~relasi L ke C~~ = {
relasi A dari L ke C = $\{(k, w) \mid \exists k \in A$ dan $\exists w \in B\}$.
relasi dari L ke C yaitu $\subseteq$ dari hasil kali kartesius $L \times C$.

★ The relation of A to D, namely, $A \subseteq D$.
The sets, namely, $A \cup D = D$.
D is the Cartesian product $D = L \times C$.

✚ Because $A \subseteq D$ and D is the Cartesian product $L \times C$ $L \times C$
Then A is a relation from L to C

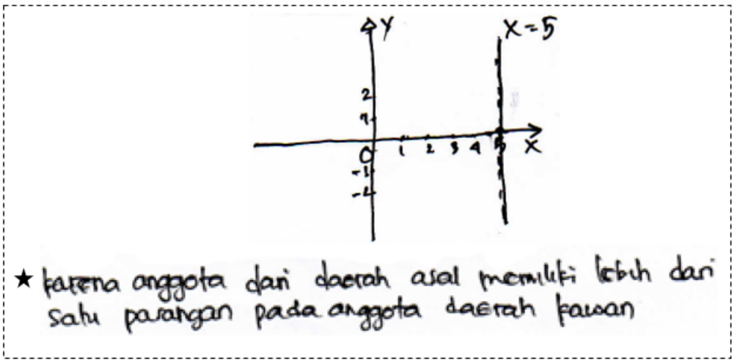
⬆ A relation is a set of some pairs of the Cartesian product.
A relation from L to C = $\{(k,w) \mid \exists k \in A$ and $\exists w \in B\}$.
A relation from L to C, namely, $\subseteq$ of the Cartesian product $L \times C$.

Figure 5.21 The Responses of Dyn–Sam to the Concept of Relation

Dyn and Sam used formal representations dominantly in their responses to the other five concepts. For the concepts of the Cartesian product of two sets and relation, their responses showed a representation category different from that showed by the previous pairs of students, namely, Iful–Vito, Dina–Yuni, and Naya–Amzi. For these two concepts, the responses of Dyn and Sam were mostly expressed in formal representations. As can be found in Figure 5.20, Dyn and Sam stated the property of the Cartesian product and defined it. Similarly, with regard to the concept of relation, they identified the relationship between two sets, namely, the first set was a subset of the second set, and the second set was the Cartesian product. They then developed a definition of a relation that was expressed in formal representations (Figure 5.21).

In terms of verification of relations, Dyn and Sam provided a justification expressed in embodied representations for two embodied relations. The two formal relations were verified with formal representations. One symbolic relation was verified in symbolic representations. Another symbolic relation $[x=5]$ was verified in embodying-symbolism representations, where the algebraic relation was represented in an embodied graph based upon which they made a decision that the given relation was not a legitimate function (Figure 5.22).

$x=5$



Relation

Verification

★ Because an element of the domain has more than one partner in the codomain

Figure 5.22 A Verification of Symbolic Relation (Algebraic Equation) by Dyn–Sam

5.2.3.5 Adel–Tanti

Adel and Tanti’s responses to the knowledge elements were expressed generally in only three categories of representations, namely, embodied representation, symbolizing-embodied representations, and formalizing-symbolic representations. They answered the problems of the association between the elements of two sets in embodied representations. The representations in their responses to the special properties of relation and to the operational formal definition of function were also mainly in embodied representations. Adel–Tanti responded to the concept of ordered pair with symbolic representations. Only in their responses to the concepts of the Cartesian product of two sets and relation, the representations used were dominantly formalizing-symbolism representations.

Similar to the responses provided by the other students, Adel and Tanti’s expressed the models of the association between the elements of two sets in arrow diagrams which were categorized as embodied representations (Figure 5.23). The responses they provided to the problem concerning the concept of ordered pairs were also mainly in embodied representations.

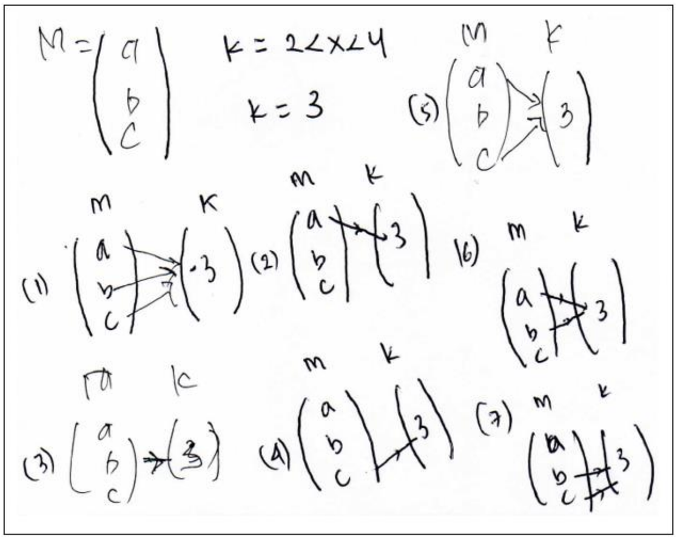
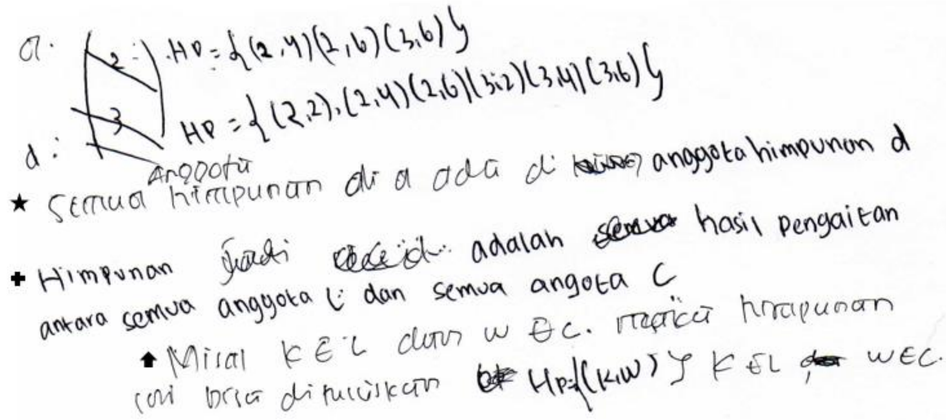


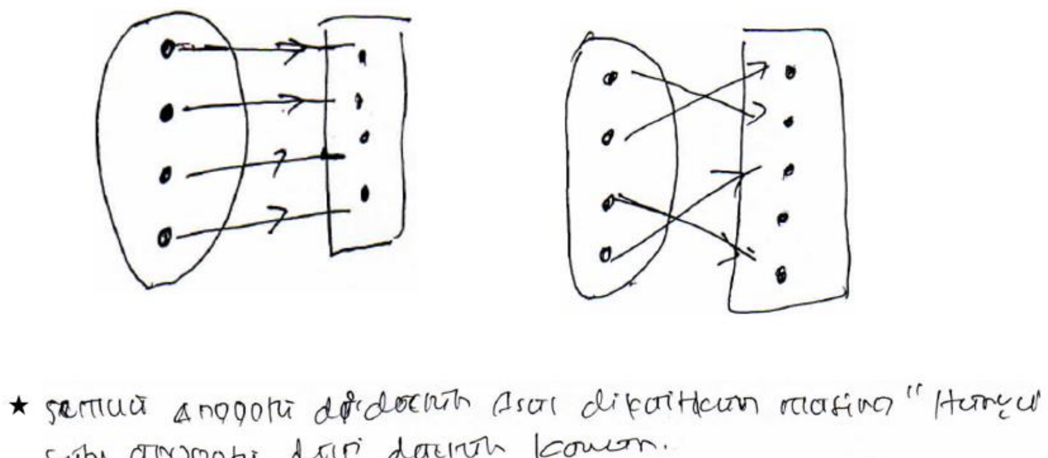
Figure 5.23 The Responses of Adel–Tanti to the Concept of Association Between Sets’ Elements

For the concepts of the Cartesian product and relation, Adel and Tanti’s responses were dominantly in formalizing-symbolism representations. In Figure 5.24, it was observed that the responses commenced with symbolic representations which were then connected to the formal representations for expressing the formal definition of the Cartesian product of two sets.



★ All elements of the set in a are elements of the set in d.
✚ The set is the results of the association between all elements of L and all elements of C.
⬆ Let $k \in L$ and $w \in C$, then the set could be written $Hp = \{(k,w)\ k \in L,\ w \in C$

Figure 5.24 The Responses of Adel–Tanti to the Concept of the Cartesian Product of Two Sets



★ All elements of the domains are associated respectively with only one element of the codomains.

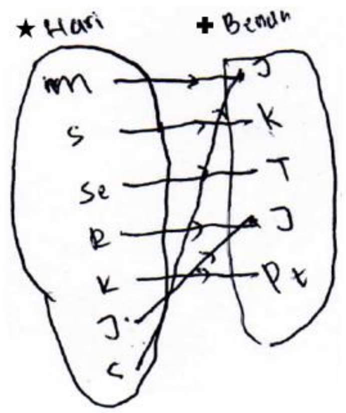
Figure 5.25 The Responses of Adel–Tanti to the Operational Formal Definition of Function

The last three concepts of the exhaustive domain property, the unique image property, and the operational formal definition of function were responded to dominantly with embodied representations. In formulating the operational formal definition of function, Adel and Tanti drew arrow diagrams to illustrate two different functions. Then, they completed their responses by writing the common property of the illustrated functions in a sentence as shown in Figure 5.25. And, Adel and Tanti did not proceed to formally define the property they had identified.

Adel and Tanti provided limited responses to the verification of relations as to whether they were legitimate functions or not. The only relation verified was the table form, which they verified in an embodied representation. They represented the relation in another embodied expression of an arrow diagram, based upon which they concluded that the relation was a function (Figure 5.26).

Days	Things
Sunday	Orange
Monday	Computer
Tuesday	Assignment
Wednesday	Jakarta
Thursday	Aeroplane
Friday	Jakarta
Saturday	Orange

Relation



Verification

★ Days ✚ Things

Figure 5.26 A Verification of Embodied Relation (Table) by Adel–Tanti

5.3 The Appropriateness of the Representations Used in Constructing the Operational Formal Definition of Function

The second aspect of the use of the representations focused on in this study was the appropriateness. This analysis was in order to further describe how the representation categories were related to the appropriateness of the representations in the students' responses.

An appropriate representation is a configuration containing correct signs, notations, diagrams, graphs, models, formulae, equations, pictures, or symbolic expressions to accurately encode, express, or symbolize mathematical objects, notions, or relationships in a proper situation or problem. The appropriateness comprises both the correctness of the configuration and the properness of the situation in which it is used. Therefore, a representation is considered inappropriate when it either uses incorrect expressions or it is in an unsuitable context. The representations were used in the whole responses of students. They were used to express the mathematical objects or ideas, to answer the questions, or to solve problems in the calculus tasks. The students may start the responses with mathematical ideas expressed appropriately in mathematical representations. They may proceed with their responses with mathematical ideas expressed in appropriate representations and finished the problems with a correct solution or answer expressed in appropriate mathematical representations. However, they may also continue the responses with inappropriate representations and finished the problems with inappropriately represented mathematical ideas. Based upon all possible patterns of the appropriateness of representations used in the whole responses from the beginning to the end, the rubric for analyzing the appropriateness of the used representations was developed. There were six patterns of the appropriateness of the used representations. These six patterns were modified from those used by Smart (2013).

The following are the six patterns with their description, respectively.

1. The response was started, continued, and finished all with *appropriate* representations (App-App).
2. The response was started with *appropriate* representations, but it was continued and finished with *inappropriate* representations (App-Inapp).
3. The response was started with *appropriate* representations, but it was left *incomplete* (App-Inco).
4. The response was started with *inappropriate* representations, but it was continued and finished with *appropriate* representations (Inapp-App).
5. The responses was started, continued, and finished all with *inappropriate* representations (Inapp-Inapp).

6. The response was started with *inappropriate* representations and was left *incomplete* (Inapp-Inco).

Similar to the analysis of representation categories, the analysis of the representation appropriateness was implemented based upon the seven knowledge elements constructed in the abstraction processes for understanding the operational formal definition of function. The results of the analysis are recorded in Table 5.2.

Table 5.2 The Appropriateness of Representations Used by Students in Constructing the Operational Formal Definition of Function

Knowledge Elements	Student Pairs				
	Iful–Vito	Dina–Yuni	Dyn–Sam	Naya–Amzi	Adel–Tanti
Association between sets' elements (ASC)	Inapp-App	App-App	App-App	App-App	Inapp-App
Ordered pairs (ORP)	Inapp-App	App-App	App-App	App-App	App-App
The Cartesian product of two sets (CAR)	App-App	App-App	App-App	App-App	App-App
Relation (REL)	App-App	App-App	App-App	App-App	App-App
The exhaustive domain property (EXH)	App-App	App-App	App-Inapp	App-App	App-Inco
The unique image property (UNI)	App-App	App-App	App-Inapp	App-Inapp	Inapp-Inco
Operational formal definition of function (FUN)	App-App	App-App	App-Inapp	App-Inapp	App-Inco

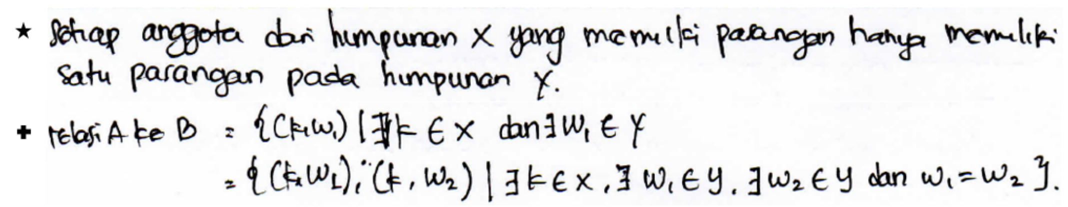
The results of the analysis of representation appropriateness showed that the representations used by the students in their responses varied. Of six models of appropriateness, five models were evident. No responses were started with inappropriate representations, continued, and finished also with inappropriate representations. From Table 5.2, it could be stated that in general, most of the representations used by the students to start the responses to the calculus tasks concerning the seven knowledge elements were appropriate representations. Out of 35 responses focused on this analysis, there were 31 responses that were started with mathematical ideas appropriately expressed in mathematical representations. In addition, it means that those appropriately represented mathematical ideas actually could lead to a correct answer if the students continued their responses with these representations.

Further, there existed only four responses that were started with inappropriately represented mathematical ideas. These four were shared equally between two pairs of students, namely, Iful–Vito and Adel–Tanti. As described in Table 5.2, Iful and Vito had errors at the beginning of their responses to the association of sets' elements and ordered pairs, while Adel and Tanti had inappropriate representations in their responses to the association of sets' elements and the unique image property of relation.

On the other hand, regarding the appropriateness of the representations used in continuing and finishing the solution of the problems in calculus tasks, most of the representations used by the students in continuing and finishing their responses to the calculus tasks were appropriate representations. There were five of 35 responses which continued and finished with mathematical ideas expressed in inappropriate representations. These five responses progressed and finished with incorrect answers to the calculus tasks, despite being started with appropriate representations. Three of the five were of Dyn and Sam's responses, namely, the responses to the exhaustive domain property, the unique image property, and the operational formal definition of function. Two were of Naya and Amzi's responses, namely, the responses the unique image property and the operational formal definition of function. Meanwhile, three of 35 responses were incomplete. These were all of Adel and Tanti's responses to the exhaustive domain property, the unique image property, and the operational formal definition of function. Two of the three incomplete responses were started with appropriate representations and one with inappropriate representations.

With respect to the performance of each pair of students, it could be observed in Table 5.2 that Dina and Yuni had their responses with appropriate mathematical representations in the starting part, and they continued and finished the responses also with appropriate representations. An example of their responses could be seen in Figure 5.12 (p. 201). The only concern in this particular response was that they wrote $x \to y$ instead of writing (x, y), to express that x was related to y. In this particular case, however, the use of an arrow to express the relation between x and y could still be approved.

Dyn and Sam also had appropriately represented mathematical ideas in the beginning part of all their responses. The responses for the first four knowledge elements of association between sets' elements, ordered pairs, the Cartesian product of two sets, and relation continued and finished with appropriate representations. However, Dyn and Sam did not continue and finish the responses for the special properties of relation and the operational formal definition of function with appropriate mathematical representations. An example of this can be seen in Figure 5.27 following, where they expressed the unique image property incorrectly. The property should not be expressed as a set. It should be expressed as a mathematical statement. In the second from the bottom line, they did not make a complete notation of a set. Besides, in the bottom line, they used two ordered pairs to represent the element of the set, namely, $(k, w1)$ and $(k, w2)$.



★ Each element of set X which has partners has only one partner in set Y.

✚ Relation A to B = {(k,wl)|∃k∈X and ∃wl∈Y
= {(k,wl), (k,w2)|∃k∈X, ∃wl∈Y, ∃w2∈Y and wl = w2}.

Figure 5.27 The Responses of Dyn–Sam to the Concept of the Exhaustive Domain Property

Naya and Amzi started start all their responses for the seven knowledge elements with mathematical ideas expressed in appropriate mathematical representations. However, they made inappropriate representations similar to those in the responses of Dyn and Sam to the exhaustive domain property. They also could not finish their responses to the operational formal definition of function with appropriate representations.

In contrast to the performance of the three pairs of students aforementioned, some of Iful and Vito's responses were not started with appropriately represented mathematical ideas. The responses to association between sets' elements and ordered pairs were started in inappropriate representations. The incorrect expressions written by Iful and Vito can be seen in Figure 5.1 (p. 185) where they expressed sets in wrong representations: $M = a, b, c$ and $K = 3,$ while they should write $M = \{a, b, c\}$ and $K = \{3\}$. The expression $K = 3$ means K is an ordinary real number, 3, which is different from $K = \{3\}$ representing a singleton (a set containing only one element).

Moreover, $M = a, b, c$ may represent a list of indices, which is traditionally written in a small alphabet and numbers, for instance, $Q_i = l + \frac{h}{f}\left(\frac{iN}{4} - c\right)$; $i = 1, 2, 3$ (this is the formula for locating the quartiles of grouped data in statistics). In Figure 5.3 (p. 188), Iful and Vito showed a different inappropriate representation of sets, where they expressed $L = (2, 3)$ and $C = (2, 4, 6)$, instead of $L = \{2, 3\}$ and $C = \{2, 4, 6\}$. The expression $L = (2, 3)$ traditionally represents an ordered pair while $C = (2, 4, 6)$ represents a triple or 3-tuple (a sequence or an ordered list of 3 elements). Iful and Vito could finish all their responses with mathematical ideas stated in appropriate representations.

Adel and Tanti used inappropriate representations in two responses, namely, the responses for the association between sets' elements and the unique image property. As could be seen in Figure 5.23 (p. 210), they had inappropriate representations of set *K*, where it was expressed as $K = 2 < x < 4$ and $K = 3$. Further, their response for the association of sets' elements could be finished in appropriate representations, and so could be the responses for the concepts of ordered pairs, the Cartesian product of two set, and relation. However, the last three responses of the special properties of relation and the operational formal definition of function were left incomplete.

5.3.1 The Comparison Between the Categorizations and the Appropriateness of Representations

As presented in the previous sections, it was found that of the nine possible categories of representations which could be used dominantly by the students, there were five representation categories used in their responses to the calculus tasks concerning the knowledge elements for constructing the understanding of the operational formal definition of function. The categories were embodied representations (ER), symbolic representations (SR), symbolizing-embodied representations (SE), formalizing-symbolic representations (FS), and formal representations (FR). It was also found that of the six possible patterns of the representation appropriateness in the students' responses, there were five patterns identified in their responses to the same calculus tasks. The patterns were: the response was started, continued, and finished all with appropriate representations (App-App); the response was started with appropriate representations, but it was continued and finished with inappropriate representations (App-Inapp); the response was started with appropriate representations, but it was left incomplete (App-Inco); the response was started with inappropriate representations, but it was continued and finished with appropriate representations (Inapp-App); the response was started with inappropriate representations, and was left incomplete (Inapp-Inco). A comparison between the categories and the appropriateness of the students' representations was analyzed and the results are depicted in Table 5.3.

Table 5.3 The Categories/Appropriateness of Representations Used by the Students in Constructing the Operational Formal Definition of Function

Knowledge Elements	**Student Pairs**				
	Iful–Vito	Dina–Yuni	Dyn–Sam	Naya–Amzi	Adel–Tanti
Association between sets' elements (ASC)	ER/ Inapp-App	ER/ App-App	ER/ App-App	SR/ App-App	ER/ Inapp-App
Ordered pairs (ORP)	SE/ Inapp-App	SR/ App-App	SE/ App-App	SR/ App-App	SE/ App-App
The Cartesian product of two sets (CAR)	FS/ App-App	FS/ App-App	FR/ App-App	FS/ App-App	FS/ App-App
Relation (REL)	FS/ App-App	FS/ App-App	FR/ App-App	FS/ App-App	FS/ App-App
The exhaustive domain property (EXH)	FR/ App-App	FR/ App-App	FR/ App-Inapp	FR/ App-App	ER/ App-Inco
The unique image property (UNI)	FR/ App-App	FR/ App-App	FR/ App-Inapp	FR/ App-Inapp	ER/ Inapp-Inco
Operational formal definition of function (FUN)	FR/ App-App	FR/ App-App	FR/ App-Inapp	FR/ App-Inapp	ER/ App-Inco

With regard to the responses with dominant embodied representations, the appropriateness of the representations the students used varied. Of seven responses in embodied representation category, less than half were started, continued, and finished all with mathematical ideas expressed in appropriate representations. These were all the responses to the association between sets' elements. There were three responses started with inappropriate representations. However, two of them were still continued and finished with correctly represented mathematical ideas, while one was left incomplete. Three responses with dominant embodied representations were left incomplete. They were the responses of Adel and Tanti to the concepts of the exhaustive domain property, the unique image property, and the operational formal definition of function. The analysis based on the pairs of students showed no particular trend of relationship between the categories and the appropriateness of the representations. However, certain trends could be asserted when the analysis was based on the concepts under study. For the concept of association between sets' elements, the responses which were dominantly expressed in embodied representations were finished with appropriate representations, disregarding the appropriateness of their beginning representations.

An observation from Table 5.3 showed a trend that the responses using formalizing-symbolic representations were started and finished with mathematical ideas expressed in appropriate representations. This trend was evident in the responses to the concepts of the Cartesian product and relation. All pairs of students, except Dyn–Sam, had this type of responses. The responses of Dyn and Sam for the two concepts were expressed in formal representations, and they also commenced and finished with appropriate representations. Another trend was also identified when the observation was expanded to cover all responses containing a symbolic representation component. Such a component could be found in the symbolically represented responses and the other responses dominantly expressed in either symbolizing-embodiment or formalizing-symbolism representations. It was evident that all responses in this particular type were finished with appropriate representations, disregarding the appropriateness of their beginning representations. And, this type of representations could be seen in the responses to the concepts of ordered pair, the Cartesian product of two sets, and relations. Naya and Amzi's responses to the concept of association between sets' elements was also included in this kind of representations.

There were 14 responses which fell into the category of formal representations, and 12 of them were the responses to the exhaustive domain property, the unique image property, and the operational formal definition of function. Of 14 responses with dominant formal representations, nine were continued and finished with appropriately represented mathematical ideas. The other five responses were also progressed and finished but with inappropriate mathematical representations. In terms of the representation appropriateness, a trend was shown in the responses to the concepts of the exhaustive domain and unique image property as well as the operational formal definition of function, namely, the formally represented responses commenced with mathematical ideas represented appropriately, disregarding the uncertainty of whether or not they were finished with appropriate representations. This trend seemed to be an opposite of the one in the embodiedly represented responses.

5.4 Discussion

This chapter was written essentially to respond to the second research question: *How do students use mathematical representations in constructing the operational formal definition of function?* The Three Worlds of Mathematics was used to frame the data analysis, and this framework provided one way of explaining the students' use of representations. The Three Worlds of Mathematics is a theory of the development of mathematical thinking, where mathematical representations are positioned as the objects of cognitive activities (Tall, 2004a, 2008). The mathematical representations hold a central position within this theory because they are the only visible expressions of what the students are thinking of and what the students understand. The research question was resolved by focusing on two aspects of the representation use, namely, the categories and the appropriateness. The discussion of the students' responses to the question above is based upon the research results presented in the previous sections.

5.4.1 Mathematical Representations Used by the Students

The first focus of the discussion is the kinds of representations used by the participant students in responding to the calculus tasks concerning the construction of the operational formal definition of function. The kinds were determined based upon the nine categories of representations. By and large, it was found the mathematical representations dominantly used by the students in their responses were limited. Focusing on the seven knowledge elements investigated in this present study, evidence presented in the previous sections (Table 5.1, p. 179) showed that the students' mathematical representations expressed in their responses were classified into five different categories, namely, *embodied representations* (ER), *symbolic representations* (SR), *formal representations* (FR), *symbolizing-embodied representations* (SE), and *formalizing-symbolic representations* (FS). According to the theory of the Three Worlds of Mathematics, the embodied, symbolic, and formal representations can be considered as basic models of representations, and the symbolizing-embodied and formalizing-symbolic representations are two models of connected representations. The dominance of the three categories does not mean that there were only five types of representations used by the students to express their thinking.

Solving the problems concerning the concept of association between sets' elements (ASC), most students provided responses that were categorized as dominantly embodied representations. Responding to the question asking the kinds of association which can be made between the elements of two different sets, all but one pair embodied their answers in the form of an arrow diagram illustrating all possible models of association they thought of. One pair of students, Naya–Amzi, presented their responses as a set of ordered pairs (refer to Figure 5.16, p. 204). All students made only one direction of the association, that is, from the first given set to the second set. The question was "What kinds of association between members of the two sets could you make?"; it was not "What kinds of association from the members of the first set to those of the second set?". Despite the open nature of the question which did not specify one particular direction of the association, all students' responses provided only one direction: a diagram with arrows pointing from the left to the right side, and did not provide the alternative opposite direction. Whereas, they could easily change the direction of the arrows in the diagram and obtained different models of sets' elements correspondence. Similarly, with the set of ordered pairs, the students could reverse the position of elements in the ordered pairs and produce different models of correspondence. Such embodied representations provide evidence of the influence of the "met-befores" or impressive prior experiences (Tall, 2008, p. 6). In this regard, Tall (2008) and Nogueira de Lima and Tall (2008) argued that an individual's current mental capacity results from their previous experiences which might be either support for or a hindrance to the development of mathematical thinking. Traditionally introduced to students since their junior secondary school, the notion of associating elements of two groups of things was presented in the form of an arrow diagram. Such a diagram is used by mathematics teachers as a way of expressing the association. It is an opening illustration for introducing the concept of relations and functions. It exemplifies a generic instantiation representing the concept of association with minimal extraneous information which is close to abstract rules (Kaminski, Sloutsky,

& Heckler, 2008). It is expected to lead the students to think of the variety of models they could make, and therefore, it is supportive to their cognitive development.

In terms of the concept of ordered pairs, the representations dominantly used by the students to express their responses were classified into symbolic representations and symbolizing-embodied representations. This type of representation was evident in the responses of three pairs of students, namely, Iful–Vito, Dyn–Sam, and Adel–Tanti. The problem in the calculus tasks instrument concerning this concept was expressed in symbolic representations: "Let A and B be two non-empty sets, $k \in A$ and $w \in M$. If we associate k to w, we can express the results (k, w) as an element of the association. … (c) Are the results in (a) the same as in (b)? Why? Explain your reason" (refer to Figure 4.1 in Chapter 4, p. 128). This question, to some extent, was intended to provide the students with an example of symbolic representations related to the concept of ordered pairs. Asking the students to explain their reasons could be answered in any kind of representations. Essentially, by this question, the students were expected to identify the defining characteristics of an ordered pair, and these characteristics could be expressed in symbolic representations. The students focused on two aspects of an ordered pair which they considered as distinguishing properties, namely, the rule of association resulting in the ordered pairs, and consequently, the position of the elements of the pairs which were not convertible. Their responses were:

> "Different, because the associations' results are different, reversed" (Iful–Vito).
>
> "They are different, right? … In Part a, it associates elements of L with elements of C …. In Part b, elements of C to elements of L …." (Dyn–Sam).
>
> "Because it is clearly said if it is from 2 to 4; if it is from 4 to 2, it means it is different" (Adel–Tanti).

All these responses only express the forms of mathematical entities based on perception of their properties which are either visible in the real world or imaginable in the mind (Tall, 2008, 2013). Thus, they are basically embodied representations. They reflect neither the characteristics of symbolic representations nor formal representations. In their whole responses, these pairs of students started with embodied diagrams and then made connections and transformations to symbolic expressions. Therefore, their responses were classified as symbolizing-embodied representations.

In terms of the targeted objective, the representations provided by the students as their responses to the concepts of association of sets' elements and ordered pairs are considered to have achieved the minimum standard based upon the prescribed operational definition of those two concepts (refer to Table 4.2 in Chapter 4, p. 130). Beyond the minimal target, the students actually could go further to provide responses that are formally represented. For the ASC, With this respect, they could express their response no more with embodied representations but with formal representations, such as, given $n(A) = a,\ n(B) = b,\ a, b \in \boldsymbol{N}$, the number of associations from A to B is 2^{ab}, thus, if $n(M) = 3 \wedge n(K) = 1,$ then the number of association models from M to K is $2^{3.1} = 8.$ This formula automatically includes the empty set as one model of association (Rosen, 2019). For ORP, the students could go further to formally define an ordered pair: $(a,b) = \{\{a\},\{a,b\}\}$ and $(a,b) \neq (b,a).$ However, the data showed no students attempting to provide such formally represented responses.

After finishing their solutions to the concepts of ASC and ORP, the students moved to answer the questions concerning the concepts of CAR and REL. The representations used by the students to state their understanding of the concepts of the Cartesian product of two sets (CAR) and the general relation (REL) were generally similar. Four pairs of students expressed their responses in connected formalizing-symbolism representations. The principal objective of the learning activities taking place in this present study was that the students formulated the formal definition of functions which was operational in nature. Accordingly, the students were facilitated to achieve the level of formal mathematical thinking, where their constructed understanding was expressed in formal representations.

It is interesting to note that there was a transformation from symbolic expressions where the students started their responses, to formal expression at the end where they expressed the definition of the Cartesian product of two sets and relation. This trend was shown by four pairs of students. In their responses, however, the proportion of the formal representations was not dominant over the symbolic ones. Therefore, they were considered as falling into the category of formalizing-symbolism representations. An exception was found in the response provided by Dyn and Sam, where the dominant formal representations were evident (see Figure 5.20, p. 208 and Figure 5.21, p. 208).

The connected representation of the formalizing-symbolic model functions to bridge the students' thought to reach the formal level of thinking of, and expressing mathematical concepts eventually. The data further showed that in the last three concepts, which were considered as the core concepts investigated in this present study, the students' responses were classified as formal representations. Four pairs of students, except Adel–Tanti, successfully reached the formal level that their responses to the concepts of the exhaustive domain property (EXH), the unique image property (UNI), and the definition of function (FUN) were represented in formal expressions. Adel and Tanti could not provide responses other than embodied representations.

According to the assumption of the Three Worlds of Mathematics theory, the cognitive development traditionally tends to move from conceptual-embodiment to proceptual-symbolism to axiomatic-formalism (Tall et al., 2001). Focusing on the whole processes of constructing an understanding of the seven knowledge elements, an emerging special path could be identified in the responses of two pairs of students, namely, Iful–Vito and Dina–Yuni. Their responses showed a development trend: embodied representations to symbolic (symbolizing-embodiment) representations to formalizing-symbolic representations to formal representations. Given the small number of participants, one might question the degree of representativeness of this trend as it was shown by two out of five pairs of participant students. This trend, to some extent, was similar to the one assumed in the Three Worlds of Mathematics (Tall et al., 2001). The formalizing-symbolism representations evident in the emerging trend found in this present study could be considered as a bridge for the development of thinking from the symbolic level to the formal one.

Further, another trend identified showed a cognitive development starting from symbolic representations to embodied representations to formal representations. This trend was evident in the activity of verifying relations as to whether they were legitimate functions. The relations which were presented symbolically were verified by firstly transforming them into embodied representations. This transformation results in a response categorized as embodying-symbolism representations. This was illustrated in the verification of a relation $x = 5$, where this symbolic relation was re-presented as a graph or an arrow diagram (see, Figure 5.11, p. 197, Figure 5.19, p. 207, and Figure 5.22, p. 209). Actually, the given relations could be disproven directly in a formal way using the counterexample approach. However, apparently, the students tried to convince themselves by the embodied representation of the relations before applying the formal counterexample verification. The result supports the claim that the formal thinking can be performed when symbolic representations were enhanced with embodied representations (Christou et al., 2005). The embodied representations increased the availability of concept representations and enriched the students' thoughts (Stewart & Thomas, 2007).

The aforementioned trend, to some extent, confirms the same trend found by Christou et al. (2005). The findings of Christou et al. (2005) showed that students transform from symbolism to embodiment to formalism. When they dealt with symbolically represented mathematical ideas, they made a detour to embodiment then move to formalism, instead of moving from symbolic representations to formal representations directly. In terms of understanding, this developmental trend of thinking of function does suggest that symbolic comprehension is itself insufficient to lead to formal comprehension (Christou et al., 2005). And, it has been suggested by researchers that students usually deal with the concept of function by first operating within the algebraic, symbolic representations and then transitioning to the graphical, embodied representations (Leinhardt et al., 1990; Yerushalmy & Schwartz, 1993).

The traditional development trend of mathematical cognition within the Three Worlds of Mathematics, from the embodied world to the symbolic world and finally to achieve the formal world, is not strictly hierarchical in nature (Smart, 2010, 2013). Tall (2008) states that the development within the worlds is not necessarily linear, and the flexible back and forth translation between two worlds normally takes place. This flexibility was another finding worth noting in this present study. Again, the notion of representational flexibility is related to the connected representations.

As mentioned earlier, there was limited evidence showing the dominant connected representations which were used in the students' responses to the seven concepts being investigated. Instead, the connected embodying-symbolism representations were found in their examination of the functionality of the relations. Dina and Yuni made a transition from the symbolic world to the embodied one when verifying the functionality of $x = 5$, as can be seen in Figure 5.28.

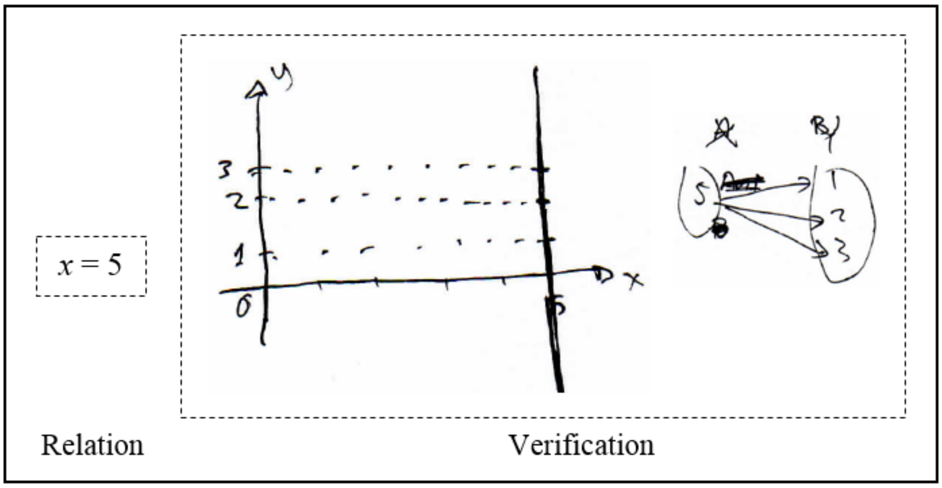


Figure 5.28 A Verification of Symbolic Relation (Algebraic Equation) by Dina–Yuni

Next, they constructed examples of functional relation where they moved from the embodied world (functions depicted in arrow diagrams) back to the symbolic world (functions formulated in algebraic expressions) (Figure 5.29). Although these two responses were for different relations, they could be considered as evidence confirming the non-linearity of the transition path from the embodied world to the symbolic world (Tall, 2008). The use of an arrow diagram as an embodied representation of function was found also in other responses (see Figure 5.9, p. 196, Figure 5.10, p. 197, Figure 5.11, p. 197, Figure 5.13, p. 202, Figure 5.19, p. 207, Figure 5.26, p. 212, and Figure 5.28, p. 226), so it could be claimed that an arrow diagram seemed to be a pivotal representation for the students. Arrow diagrams and graphs are two competing representations of functions (Stewart & Tall, 2015).

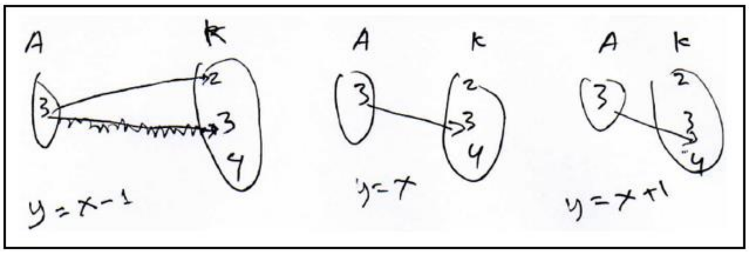


Figure 5.29 Functions from set *A* to set *K* Formulated by Dina–Yuni

Connecting among types of mathematical representations has been investigated in many studies focusing on the use of multiple representations (Bal, 2015; Cai, 2004; Cifarelli, 1998; Dreher & Kuntze, 2014; Duval, 2006; Even, 1998; Gagatsis &

Shiakalli, 2004; Hitt, 1998). The successful back and forth translations between the embodied and the symbolic representations are similar to the results of an investigation conducted by Bal (2015). Gagatsis and Shiakalli (2004) found that the ability to translate between representations of function fosters the problem-solving ability. This inter-representations translation ability is crucial for the learning of mathematical concepts and problem solving (National Council of Teachers of Mathematics, 2000). It means that transitioning dynamically between representations improves the understanding of concepts. Linking coherently and translating fluently between representations help build a meaningful version of the represented concepts (Hitt, 1998; Lesh, 2000).

The representation categories provided clear descriptions against which the students' representations can be classified. The analysis has provided evidence that not all nine categories were evident to appear as a dominant representation in the students' responses. For this case, based upon her investigation, Smart (2013) has argued that the model of the calculus tasks influences the variety of the representations used in the students' responses. How the questions are formulated affects the students' perception of what is expected from the questions. This claim is supported by the findings of this present study. In particular, the confirming data were taken from the responses to the questions concerning the seven concepts being studied. When the questions for the concepts of ordered pairs (ORP) and the Cartesian product of two sets (CAR) asked "Why?", "Explain your reason", or "What is the uniqueness of the results?", the students responded to them with symbolic, symbolizing-embodied, or formalizing-symbolic statements. These questions actually could be answered with formally represented responses. And, this was exemplified by a pair of students, Dyn and Sam (see Table 5.1, p. 179), where they formulated the formal definition of the Cartesian product of two sets. When the questions clearly stated: "How do you define the object?" or "What will be your definition?", all students attempted to provide their responses dominated by formal representations. This was evident in the students' responses to the problems concerning the exhaustive domain property (EXH), the unique image property (UNI), and the operational formal definition of function. Most of the students could devise formal explanations. Only one of five pairs of students who did not succeed in providing answers expressed in formal representations. Four pairs of students responded to the problems concerning the EXH, UNI, and FUN with dominant formal representations. However, not all the responses provided were correct. The appropriateness of the representations will be discussed in the next section.

5.4.2 The Appropriateness of the Representations Used by the Students

The second focus of the discussion is the appropriateness of representations used by the participant students in responding to the calculus tasks concerning the construction of the operational formal definition of function. The appropriateness of representations was examined on the basis of six categories which varied according to whether the representations the students used in starting, proceeding, and finishing their responses were appropriate or not.

From the students' responses to the calculus tasks concerning the seven concepts under investigation, the findings presented in Table 5.2 (p. 214) showed that generally, the students expressed their mathematical knowledge in appropriate representations. Almost all (31 out of 35) responses provided by the students were started with ideas presented in appropriate mathematical representations. The four responses which were started with inappropriate representations expressed sets in inappropriate notations, symbols, or illustrations. Rather than expressing a set, they resulted in different mathematics objects, or the expressions were meaningless statements because they did not follow the standard mathematical symbolization. This mistake apparently is the use of wrong rules, including symbols and notations, for a particular context (Hiebert & Wearne, 1985). Instead of expressing $K = \{x \mid 2 < x < 4,\ x \in \boldsymbol{N}\}$, which they could copy directly from the question, Adel and Tanti expressed $K = 2 < x < 4$ and $K = 3$ which resulted in *a fake equality* $2 < x < 4 = 3$. This equality may look appropriate, however, it is formally a meaningless expression as it equates a *falsely notated* interval (set) with a real number that should be an element of the set. The embodied representation of set *K* could have been transformed by Adel and Tanti into a degraded verbal expression: *If x is a natural number greater than* 2 *and less than* 4, *then x is* 3. This interpretation has simply discarded the idea of *K* as a set, and talks only about the properties of natural numbers. As a result, when Adel and Tanti expressed the results of their thought, they left out the characterizing symbols in the notational representation of a set. In the end, *K* is expressed merely as a real number, $K = 3$.

This phenomenon is similar to the one found by Gagatsis and Shiakalli (2004) in their study of the translation between representations of function. They claimed such a mistake as an indication of the absence of a complete and coherent cognitive structure for the concept. A coherent articulation of different representations of a concept is an implication of a complete understanding of the concept.

Among the whole responses analyzed, there were five responses that were finished with inappropriately represented mathematical ideas. Despite being started with appropriate representations, the responses were finished with inappropriate representations. Taking the case of Dyn and Sam, the mistake which they displayed in their responses fell under the domain of mathematical symbolic logic (Dawkins, 2017). Their responses to the exhaustive domain property (EXH), the unique image property (UNI), and the operational formal definition of function (FUN) were finished with inappropriate representations. The formal definitions for the special properties of function were expressed inappropriately. Despite their success in identifying the exhaustive domain property, their definition was "$\{(k,w) \mid \forall k \in A \text{ and } \exists w \in B\}$" (see Figure 5.27, p. 216). More explanation of the responses of Dyn and Sam to the two special properties could be seen in section 4.3.2.5 (p. 146) and section 4.3.2.6 (p. 149). According to the standard symbolic logic expression, the definition of an exhaustive domain relation R from A to B could be stated as $\forall k \in A,\ \exists w \in B,\ (k,w) \in R$ or $R = \{z \mid \forall k \in A, \exists w \in B, z = (k,w)\}$. Neither of these expressions is equivalent to Dyn and Sam' definition. In addition, Dyn and Sam formulated that expression on the basis of their appropriate definition of the Cartesian product of two sets: $A \times B = \{(k,w) \mid \forall k \in A \text{ and } \forall w \in B\}$ (see Figure 5.20, p. 208). Based upon their discussion, it was found that they considered the Cartesian product as exhaustive in both the domain and the codomain which they had dealt beforehand. Therefore, to define the exhaustive domain relation, they just changed the quantifier for the membership of the codomain (B), from $\forall w \in B$ (in the Cartesian product definition) to $\exists w \in B$ (in the exhaustive domain property definition). Such results were similar to those of Oppenheimer and Hunting (1999). In their investigation, they have found that the students were confused by the symbolic configuration in somewhat similar problems solved earlier and end up applying incorrect logical rules. The students did not incorrectly use the rules, but they used incorrect rules which they modified in order to make the answers that looked right for them (Hiebert & Wearne, 1985; Oppenheimer & Hunting, 1999). The similar mistakes were displayed in the responses of Naya and Amzi to the unique image property and the operational formal definition of function.

The inappropriate representations used by the students identified in this present study generally resulted from their failure to follow the interpretation standard and meanings prevailing in mathematical logic. This is problematic because in practice, mathematical knowledge and understanding are communicated through symbolic, logic statements with language precision beyond the strictures of everyday discourse (Dawkins, 2017). The use of symbolism is the power of mathematical thinking (Arcavi, 2005; Janvier, 1987b). According to Ball (2003), knowledge of topics, concepts, and procedures is fundamental to knowing mathematics, however, to learn and use mathematics successfully require the fluent use of symbolic notation as part of representational practice domain. Maharaj (2008) claimed the urgency for the students to have the established meanings of mathematical symbols before they engage in the study of mathematical ideas. His suggestion was based upon his research findings revealing that difficulties encountered by the students were due to their lack of mastery of the appropriate use of mathematical symbols. Some studies have reported on the difficulty experienced by undergraduate mathematics students in terms of mathematical language and symbols (e.g., Durand-Guerrier, 2003; Epp, 2003; Ferrari, 2004). Such a difficulty could inhibit students' engagement in mathematical learning that is formal deductive in nature (Dawkins, 2017). It might be less problematic when the students perform just a deviation from mere notational standards (Dawkins, 2017), such as the one experienced by Dina and Yuni. To define the exhaustive domain property, they wrote: Let domain = L, codomain = C, $x \in L$, $y \in C$, $\forall x \in L\ \exists y \in C,\ \ x \to y$ (Figure 5.12, p. 201). Also, verifying a formal relation, they wrote: $\forall a \in \boldsymbol{N},\ \forall b,\ c \in \boldsymbol{Z},\ a \to b \wedge a \to c \Rightarrow a+b=0$ and $a+c=0$ then $b=c$ (see Figure 5.15, p. 203). In their responses, they expressed an element of relation as $x \to y,$ instead of (x, y). This might be seen as deviating from the notation convention, however, it was still acceptable. Dina and Yuni were the only pair of students who responded to all the concepts under investigation with appropriate representations.

The last thing to discuss is the comparison between the categories and the appropriateness of the representations. In this present study, there were adequate findings to support the existence of a conclusive unique relationship between the categories and the appropriateness patterns of the representations which applied to the whole concepts. The trends of relationship were identified according to particular concepts under investigation. It was asserted that the responses which were categorized as formalizing-symbolic representations commenced and finished with appropriately represented mathematical ideas. This assertion applied to the responses the students gave in answering the problems concerning the concepts of the Cartesian product of two sets and relation. The second trend was that the responses which were categorized as embodied representations finished with correctly represented mathematical ideas, disregarding the appropriateness of the representations they commenced with. This trend applied in the students' responses to the concept of association between sets' elements. Also, it was asserted that the responses which were categorized as formal representations commenced with appropriately represented mathematical ideas, disregarding the appropriateness of the representations they finished with. This assertion applied to the students' responses to the concepts of special properties of relation and the operational formal definition of function.

Focusing on the first assertion, it is interesting to note that those responses which dominantly used connected representations, namely, formalizing-symbolism representations, seemed to correlate to more chance of the appropriate representations expressed in the response, from the commencement to the finish. When we considered those representations as expressions of the solutions to the problems given to the students, then it can be further implied that the use of connected representations correlated with a better chance of correct solutions. Some studies have found the relationship which was evident between how students make connections from mathematical worlds and their performance at problem solving (Nogueira de Lima & Tall, 2008; Stewart & Thomas, 2007). From the perspective of multiple representations, the representation use in learning mathematics concerns not only expressing ideas, concepts, or other mathematical entities in different representations which can be in isolation from one to another but, more importantly, it concerns the transition from one representation to another. Learning mathematics is a domain where possessing, using, linking, and coordinating representations are considered as requirements for developing a conceptual understanding (de Jong et al., 1998; Goldin, 2014; Wood et al., 2007).

The inappropriate representations found either in the beginning, the middle, or the end of the responses indicated that the students still faced difficulties in representing mathematical ideas. Some representation errors seemed to be less influential or less significant performed by the students in their responses to the concepts of association between sets' elements. This might be because the incorrect symbols were used or situated in the embodied world. Therefore, despite students making those errors at the beginning of their answers, they still could proceed and finish the responses with appropriate representations as claimed in the second assertion above. The case was different from the use of symbols in formal representations where an expression of mathematical statements was highly dependent upon the logical symbols and rules as well as the set theory (Tall, 2008). As stated in the third assertion above, the students' responses expressed in formal representations all commenced with appropriate representations, but not all of them could finish with appropriate representations due to their mistakes in symbolic logic statements.

5.5 Conclusion

The mathematical representations functioned as the tools to perform the abstraction process. The students used them to express their thinking and understanding of mathematical ideas. The research results showed that there were five kinds of representations dominantly used by the students in their solutions to the problems concerning the development of an understanding of the operational formal definition of function. The embodied representations were dominantly used in the students' responses to the concept of association between sets' elements. The connected formalizing-symbolism representations were dominantly used in the responses to the concept of the Cartesian product of twos sets and relation. The formal representations were dominantly used in the responses to the special properties of relation and the operational formal definition of function. Regarding the appropriateness, most of the responses commenced and finished with mathematical ideas expressed in appropriate representations.

Comparing the categories and the patterns of appropriateness of the representations used by the students revealed particular trends. The responses categorized dominantly as formalizing-symbolism representations commenced and finished with appropriately represented mathematical ideas. The students made some mistakes in representing mathematical ideas. The research revealed that the mistakes were related to the use of symbols and notations and to the meaning and interpretation of mathematical logic. In the next Chapter 6, further discussion about the results of the research is presented. It deals with the relationship between the abstraction and the mathematical representation in developing the understanding of the definition of the concept of function.

CHAPTER 6
THE RELATIONSHIP BETWEEN MATHEMATICAL ABSTRACTION AND REPRESENTATIONS

"A powerful means of abstraction, I should like to claim, is representation."
(Damerow, 1996, p. 373)

6.1 Introduction

This chapter presents a general discussion tying together the two research foci whose related findings and discussion were discussed separately in Chapters 4 and 5. The discussion deals with the synthesis between the aspects of abstraction and the representations in developing an understanding of the operational formal definition of function. This synthesis deals with how the abstraction is related to the representations in the students' attempt to develop their understanding of the definition of function.

6.2 Abstraction and Representations in Constructing an Understanding of the Formal Operational Definition of Function

The study of introductory calculus in undergraduate programs of mathematics education and pure mathematics is approached with the formal conceptual model in which developing a deductive argument is considered as a core competence. Using the rules of logical inference, the argument is developed on the bases of formal definitions, axioms, postulates, and their interrelationship to verify the truth of mathematical statements. Therefore, a comprehension of formal definitions plays a very crucial role in formal mathematics (Vinner, 2002). At the same time, the mathematical logic which is considered as the language of the formal mathematics should be comprehended by the students (Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012; Ganesalingam, 2013; Waner & Costenoble, 2011; Woleński, 2012). Students encounter conceptual problems because they know the formal definitions but fail to use them appropriately to devise valid arguments for examining, verifying, or proving the truth of mathematical texts, conjectures, or statements (Leinhardt et al., 1990; Vinner &

Dreyfus, 1989). The students face challenges caused by their problematic understanding of the definitions and inability to use or work with the formal definitions.

Many formal definitions are expressed in a statement that is not operational for the students. To make a formal definition operable requires a reconstruction of the formal definition or a formulation of the operational version of the formal definition. Constructing or formulating definitions of mathematical concepts is considered as an advanced mathematics learning activity (Ouvrier-Buffet, 2011; Zandieh & Rasmussen, 2010). One of the formal definitions which is not operational for the students is the definition of the concept of function. The students know the definition but cannot use it to deductively verify the legitimation of relations as a valid function (Hoyles & Healy, 2007). The students neither understand the operational version of the formal definition nor the formulation of that operational version itself. The formal definition of function is not operable for the students (Bills & Tall, 1998).

One learning strategy to develop an understanding of a mathematical object/idea is by learning to (re)invent or (re)create the object/idea (Freudenthal, 1973, 2002b; Ouvrier-Buffet, 2004). The comprehension of mathematical definition and its role can be developed through defining activities (Edwards & Ward, 2008), and "[t]he best way to learn is to do – to ask, and to do" (Halmos, 1975, p. 469). Therefore, an understanding of the formal definition of function which is operational in nature can be developed through formulating the operational version of the formal definition of function. The students in this research were engaged in defining activities in which they attempted to define the concept of function following a set of instructional activities. In this present study, the students were facilitated in learning activities of defining. During the learning process, the students engaged within the abstraction process through which they developed their understanding. Abstraction takes place in problem solving activities where the students perform the action of constructing knowledge (Hershkowitz et al., 2001). Working in pairs, the students discussed and solved the calculus tasks designed specifically to encourage them to perform abstraction processes through which they developed their understanding of the operational formal definition of function.

As discussed in Chapter 4, it was found that the student pairs could perform the theoretical abstraction processes in their endeavor to develop an understanding of the operational formal definition of function. Following the learning trajectory which was embodied in a sequence of activities of solving the problems set in the calculus tasks, the student pairs showed the epistemic actions of recognizing, building-with, and constructing (Dreyfus et al., 2015; Hershkowitz et al., 2001; Schwarz et al., 2009). While forming their knowledge of the concept of general relation, the results of the analysis of a pair of students, Dyn and Sam, showed that they recognized the concepts of membership of a set and the arbitrariness, and then built-with them to construct their understanding of the association between the elements of two sets. Next, they recognized and built-with the concepts of association between sets' elements, the membership of sets, and the elements' order to construct their understanding of the concept ordered pair. The epistemic actions were evident in their abstraction processes of the other higher level concepts of the Cartesian product of two sets, relation, the exhaustive domain property, the unique image property, until the culminating point of formulating the operational formal definition of function. All processes of abstraction were observable in the students' discussion and solution to the calculus tasks. The product of abstraction reflected in the level of understanding developed by the students varied. They were able to develop their understanding of the operational formal definition of function. Two pairs of students, Iful–Vito and Dina–Yuni, constructed the complete level of their understanding of all concepts under investigation. Moreover, these students also showed their ability to appropriately use the formulated definition to verify the functionality of relations with deductive arguments (see Figure 4.6, p. 160 and Figure 5.15, p. 203). The arguments could be considered as evidence that the formal definitions formulated by them were really operational or operable for them (Bills & Tall, 1998).

To perform all abstraction processes, the students utilized mathematical representations. According to Damerow (1996), quoted at the beginning of this chapter, the mathematical representation is a powerful tool for abstraction. The use of the mathematical representations was analyzed in terms of the representation categories and appropriateness. The findings revealed that among the nine possible categories of mathematical representations, five categories were found to be used by the students in expressing their mathematical knowledge and understanding. Their responses to the calculus tasks were dominated by any one of embodied representations, symbolic representations, symbolizing-embodiment representations, formalizing-symbolism representations, and formal representations. All formalizing-symbolism representations used in expressing their responses to the concepts of the Cartesian product of two sets and relation were correct. All of these responses started, proceeded, and finished with appropriate representations. Most of the responses to the concept of association between sets' elements were dominated by the embodied representations. These responses finished with appropriate representations, although some commenced with inappropriate representations. In contrast, all responses to the concepts of special properties of relation and the operational formal definition function that were formally represented commenced with appropriate representations, but some of them finished with inappropriate representations.

As an abstraction tool, mathematical representations play a major role in the process of mathematical abstraction (Ferrari, 2003). If the representation is a tool, it means that its versatility determines both the process and the result of the abstraction. Being versatile with representations means being competent to utilize representations conceptually and procedurally and to transform between them fluently (Thomas, 2008b). Representations are symbols or symbol systems to express mathematical ideas or structures. The symbols reflect the level of thinking involved in abstraction and guide the processes of abstraction (Damerow, 1996). The data in this present study confirmed this role of representations in abstraction.

In the abstraction processes to construct the first two concepts investigated, namely, association between sets' elements and ordered pairs, the data showed the current level of the students' thoughts. For these two concepts, the level of cognitive activities expected to take place could be either the embodied or symbolic level. These two concepts were considered as a starting point for the construction of the concept of relation. Following the development path of mathematical cognition in the Three Worlds of Mathematics (Tall, 2004a, 2004b, 2006, 2008, 2013), the calculus tasks concerning the two concepts were intentionally set to the level of embodied or symbolic world of mathematics. Therefore, it was adequate for the students to respond to the problems with embodied or symbolic representations. All participating students performed well in constructing their understanding of these concepts.

Most students used arrow diagrams as an embodied representation to express their responses to the concept of association between sets' elements. An example of the students' answers was displayed in Figure 4.3 (p. 134). The results of the data analysis showed that the selected representations functioned as a tool prompting the students to think about all possible models of the association between sets' elements. The arrow diagram inspired the students to think about the notion of arbitrariness of the association because they could easily modify the model of association between the elements of the two given sets. Further, representing the results of the association of set A to set B as a pair of elements (a,b), where $a \in A$ and $b \in B$, functioned to encourage the students to think about further concepts. In the view of Mason (1992), the students engaged in abstraction processes by firstly looking at the expression and then shifted to looking through the expression. At first sight, students may look at and just focus on the property of the representations, and next, they shifted their focus to the property of the concept which was represented (Mason, 1992). When Dyn and Sam attempted to construct the concept of ordered pairs, they succeeded in enlisting the results of two associations where the results of the first association were the inversion of those of the second association. A more detailed explanation of the abstraction process for constructing the concept of ordered pairs can be found in section 4.3.2.2 (p. 136).

[0262]	Dyn	To me, they are different.
[0263]	Sam	They are different, right? Different because of the condition of its associatio In Part a, it associates elements of L with elements of C whose values a greater than those of L. In Part b, elements of C to elements of L whose valu are less than those of C. It means they are different.
[0264]	Dyn	Different. The types of associations are different.
[0265]	Sam	Because the types of associations, the types of associations from set L to C and C to L are different.

In this excerpt [0262–0265], it could be observed that the students just looked at the representation (the list of the associations' results) and identified the difference between the results on the basis of the association producing them. They were different “[b]ecause the types of associations, the types of associations from set L to C and C to L are different” [0265].

[0280]	Dyn	If it is written like this, 4 comma 2 and 2 comma 4, they are different, right?
[0281]	Sam	Different, different. For this [(2,4)], its origin point is 2; for this [(4,2)], its origin point is 4. The origin points are different.
. . .		
[0323]	Sam	To me, 2 comma 4 couldn’t be written as 4 comma 2 because this writing shows which one is the origin, which one is the result. So, we couldn’t write 4 comma 2 as 2 comma 4, because if we do it … orderly, the position shows which one is the origin, which one is the result.

Then, Dyn and Sam seemed to shift away from the notion of the different associations to the position of the elements of the pairs by comparing two pairs, one was the inverse of the other. This shift was considered as looking through the lists of the associations’ results which after all, led Sam to conclude that the order of the elements mattered [0323]. This conclusion distinguished an ordered pair (a,b) from a set $\{a,b\}$.

More examples were found in the data pertaining to the change in the way the students dealt with the representations in the abstraction processes. Such changes could be observed in the abstraction process of constructing the other concepts under investigation in this present study. The following excerpt was part of the analysis result in section 5.2.2.5 (p. 191).

[0410]	Vito	Oh, there is an element in *B* which is not in *D*, meaning that *B* is not a subset of the Cartesian product.
[0411]	Iful	Yes, *A* is its subset, while *B* is the reverse of *A*, meaning that *B* is not a subset of *D*.
[0412]	Lec	Here, you have to see *A* and *D*, and find what their unique property is; it is not about the relationship between *A* and *D*.
[0413]	Vito	Oh, yes, *B* is not a subset of *D*.
. . .		
[0425]	Iful	Yes, *A* is *L* to *C*, and *A* is the same as *D*.
[0426]	Vito	*B* … *C* to *L*.
[0427]	Lec	Is it unique?
[0428]	Vito	No, let’s see the sets. This is *A*, this is *B*, this *D*, *A*, *B*, and *D*. They are common in … *A* and *D*, 2 and 3 have partners.
. . .		
[0432]	Iful	Yes, the first group … their elements must have partners.
[0433]	Lec	Yes, what about here [*B*]?
[0434]	Vito	Not, it is not.

In the above excerpt, it can be seen how Iful and Vito shifted their focus from the relationship between *A*, *B*, and *D* to the unique property belonging to both *A* and *D* but not to B. First, when they looked at the expression of sets A, B, and D, they identified the relationship between A and D, namely, $A \subseteq D$. They also found that B was not a subset of D. Further, they shifted their attention to look through the representations, and they succeeded in identifying the uniqueness of A and D which made them different from B, namely, all elements of the domains have partners [0432–0434].

All the changes from merely looking at the expression to looking through it confirmed the notion of abstraction theorized by Mason (1989) that abstraction was "a delicate shift of attention from seeing an expression *as* an expression of generality, to seeing the expression *as* an object or properties" (p. 2). From the viewpoint of Abstraction in Context (Dreyfus et al., 2015; Hershkowitz et al., 2001; Schwarz et al., 2009), the change of the attention focus takes place when the students move from the action of recognizing to the action of building-with. Recognizing is realizing, knowing, or making an association with previous relevant mathematical entities in the given situation, problem, or context. Then, building-with is using, combining, or assembling the mathematical entities to attain particular goals which can be a strategy of problem solving, a justification of statements, or an answer to questions. The action of recognizing is parallel to the process of looking at the problems represented and the building-with is parallel to looking through the problems. Thomas and Hong (2001) stated that the looking at was a kind of observing the representation of the problems at a surface level, while the looking through was a deeper level of observation targeting the structure or properties of the representation.

Supporting the claim of Damerow (1996) that the mathematical representations are abstraction tools, Thomas and Hong (2001) argue that the mathematical representations become a conceptual tool in students' mathematical learning (Thomas & Hong, 2001). For this notion, Thomas and Hong (2001) take a spade as a metaphor where the spade becomes a tool not because of its shape or properties, but because of the action of using it. A conceptual interaction with representations enables the students to investigate and build a meaningful understanding of mathematical concepts represented (Thomas, 2008a). The ability to interact with the representations conceptually is part of the representational versatility (Thomas, 2008a, 2008b). And, because conceptual ideas can be constructed from multiple representations, the students should be exposed to a number of representations to facilitate the development of their understanding of the ideas with an emphasis on explicitly linking of ideas across representations (Thomas, 2008a).

The role of representations as conceptual tools was shown in the response of Dina–Yuni to the problem concerning relations with the unique image property. They were given two different relations illustrated in arrow diagrams based on which they identified and then formally defined the concept of the unique image property (see Item 2(h) in Figure 4.2, p. 128).

[0896]	Yuni	Yes, yes, each element is linked. These [the elements of the domain that are not linked] are not included, because they are not linked.
[0897]	Lec	That's it. Dina, do you get it? Each, but not all each element of the domain. There is an extra condition.
[0898]	Dina	Yes, each element of the domains that is linked ….
[0899]	Yuni	It is linked to only one element in the codomain. So, each element that is linked ….
[0900]	Dina	Each element of the domain that is linked to the codomain ….
[0901]	Yuni	Each element of the domain that is linked to the codomain ….
[0902]	Lec	Yes, you stop there, so how?
[0903]	Dina/ Yuni	Each element of the domain that is linked … is linked to only one element of the codomain.

Dina and Yuni used the representations of the relations given in the question item conceptually (Thomas & Hong, 2001). They did not need to perform any process, and, after identifying the unique image property common to the two relations, they transformed the statement into formal form of representation, namely, the symbolic logic statement formally defining the unique image property $\forall a \in X, \forall b, c \in Y, a \to b \wedge a \to c \Rightarrow b = c$ (Figure 6.1). Their first attempt resulted in the statement: $\forall a \in X, \exists b, c \in Y, a \to b, a \to c, b = c,$ which then be improved to the statement: $\forall a \in X, \forall b, c \in Y, a \to b \wedge a \to c \Rightarrow b = c.$ It could be seen in their definition that Dina and Yuni did not specify the name of the relation to be defined, for instance, R is a relation from X to Y. They just mentioned that the relation had X as its domain and Y as its codomain. An arrow diagram which they drew was to clarify that if a is associated with both b and c and $b \neq c$ $(a \to b \wedge a \to c \Rightarrow b \neq c),$ then a would not have only one partner in the codomain. Dina and Yuni symbolized "a is related to b" as $a \to b.$ The expression $a \to b$ is not the standard representation as in mathematical practices we traditionally write $(a,b) \in R$ to express a is related to b

under the relation *R*. Dina and Yuni might be influenced by the notion of relation as some element goes to some other element, so they used the arrow →. However, such symbolization is an expression of students' mathematics signifying the same meaning as that of standard expression, and since the mathematical ground could be identified, it is acceptable (Steffe & Thompson, 2000). The students might use less accurate representations resulting from the correct inference of the situation (Cox & Brna, 1995). Creating representations could lead to a better comprehension of the encountered situation (Ainsworth, 2006).

★ – Setiap anggota domain yang dikaitkan ~~hanya satu~~ ke kodomain hanya dikaitkan satu di kodomain.

✚ Domain = X , misal $a \in X$

Kodomain: Y , misal $b, c \in Y$

$\forall a \in X \, \exists b,c \in Y, \; a \to b, \; a \to c, \; b = c$

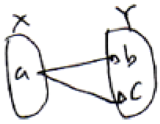
$\forall a \in X, \; \forall b, c \in Y$

$a \to b$

$a \to c$

⬆ berarti $b = c$ — Ketika $b \neq c$

$a \to b \wedge a \to c \Rightarrow b = c$



★ Each element of the domain that is linked to the codomain is linked to only one in the codomain.

✚ Domain = X, let $a \in X$

Codomain = Y, let $b, c \in Y$

⬆ It means $b = c$ — When $b \neq c$

Figure 6.1 The Responses of Dina–Yuni to the Concept of the Unique Image Property

The role of representations as tools of abstraction could also be viewed in terms of the representation difficulty. The difficulty was embodied in the use of inappropriate representations. As a tool, when the representations are inappropriate, the students should experience problems in progressing their abstraction. Dyn and Sam's construction of the definition of the unique image property of a relation showed that their abstraction progress was impeded by their incorrect formal representations. While they succeeded in identifying the exhaustive domain property from the given relations: "For *A* and *D*, each element of their domain has partners in codomain … while for *B*, not all elements of its domain have partners" [Sam, 0580]. However, they failed to formulate the formal definition of the property because the representations they used in the logical statement were incorrect: "A special relation *A* to *B* = $\{(k,w)|\forall k\in A$ and $\exists w\in B\}$" [Sam, 0590], which should be $\forall k \in A, \exists w \in B, (k,w) \in R,$ where *R* is a relation from *A* to *B*. In this case, they used inappropriate representations in a particular context, not using particular representations in an unsuitable context (Hiebert & Wearne, 1985; Oppenheimer & Hunting, 1999). Adel and Tanti, also experienced the problem of abstraction progress. They correctly identified the exhaustive domain property of a relation, but they could not see the unique image property in the given relations. Further, they did not make a formal definition of the properties because they had no ideas about expressing statements in a symbolic-logic sentence.

The theoretical abstraction in the framework of AiC is based upon the idea of Davydov (1972/1990). He theorized that abstraction begins with an initial, vague, undeveloped form, which is not necessarily consistent. Abstraction ends with a more elaborated, developed, and consistent form. In the sense of Davydov (1972/1990), formulating a formal definition of mathematical concepts is a process of abstraction. In this present study, constructing the operational formal definition of function started from an initial vague, undeveloped statement, and ended with an elaborated, developed, and consistent expression. The abstraction process, which is dialectic in nature (Davydov, 1972/1990), transforms from undeveloped to a developed statement of the definition where the features of the concept of function are emphasized (Gilboa, Kidron, & Dreyfus, 2019). Within AiC, abstraction is a vertical reorganization of the constructed mathematical knowledge within the existing mathematical structure using mathematical means in order to lead to established mathematical knowledge (Dreyfus et al., 2015). In this respect, the process of constructing and developing the definition of mathematical concepts into the accepted formulation is facilitated by the mathematical representations as an abstraction means.

This dialectic process of abstraction to formulate the operational formal definition of function, of course, involved mathematical logic which functions as the language of formal mathematics (Ganesalingam, 2013; Woleński, 2012). In this present study, it was found that the students, basically, could identify and express the concepts under investigation. However, when they moved to formally define the concepts requiring symbolic logic, some students encountered difficulties. Those students who were able to progress with the symbolic logic statement of the definition did not directly achieve the final accepted form of the definition statements. Iful and Vito's definition of the unique image property of a relation showed that they had several attempts of refining to obtain the final statement of the definition (see Figure 5.7, p. 194). They started with $\forall k \in X, \exists m \in Y, n \in Y \mid (k,m) = (k,n)$, to $\forall k \in X, \forall m \in Y, \forall n \in Y \mid (k,m) = (k,n)$, and then $\forall k \in X, \forall m, n \in Y \mid (k,m) \wedge (k,n) \Rightarrow m = n$. This logical refinement is parallel to the finding of the studies carried out by Gilboa et al. (2019) and Kidron (2008, 2011). Gilboa et al. (2019) found that the constructed definition was refined by increasing the language precision of the definition statement.

Overall, the performance in abstraction was determined by the representational ability. This relationship between mathematical abstraction and representations confirmed the finding of the study conducted by Kato, Kamii, Ozaki, and Nagahiro (2002) revealing that the higher the level of abstraction performed, the higher the complexities of representations required.

6.3 Conclusion

Mathematical abstraction requires mathematical representations as conceptual tools. The success of the abstraction processes to construct mathematical knowledge is, to some extent, determined by the versatility of mathematical representations, namely, the ability to use, interact, and interpret representations procedurally and conceptually as well as to translate and transform between representations. In this study, the abstraction process to construct various concepts leading to formulating the operational formal definition of function was facilitated by the mathematical representations. The success of the students to construct all the concepts being investigated was supported by their representational ability. As the targeted definition was the operational formal definition of function, the mathematical logic as the language of formal mathematics played a determining role in terms of representing the concept definitions considered as acceptable in mathematics. The conclusions, recommendations, and limitations of the present study are presented in the next and final Chapter 7.

CHAPTER 7
CONCLUSIONS, RECOMMENDATIONS, AND LIMITATIONS

"… education is not an affair of "telling" and being told, but an active and constructive process, …."
(Dewey, 1916, p. 46)

7.1 Overview

The concept of function is an important, core concept in the Introductory Calculus course studied in undergraduate programs for mathematics education in Indonesian universities. In this research, learning the function concept was dealt with by the abstract formal theoretical approach characterized by precise mathematical definitions, upon which the theorems, properties, and propositions are logically deduced. In general, the students understood the formal definition of function but encountered difficulties in providing a definition-based argument to verify or prove whether a relation is a legitimate function of not. These difficulties are because the definition of function that the students had was not formally operational. The operational definition could be formulated through learning activities of constructing or formulating mathematical definitions. This study was conducted with the main purpose of investigating the students' construction of an understanding of the operational formal definition of function. Two research questions were addressed. The first question pertained to how students constructed the operational formal definition of function through theoretical abstraction and the second question concerned how students used mathematical representations in constructing the operational formal definition of function?

This present study was designed as a multiple case study employing an explorative qualitative approach. It was implemented as the teaching interview which was an adaptation of the teaching experiment strategy. The study involved first year students undertaking an introductory calculus unit in the mathematics education undergraduate program in one large state university located in the eastern part of Indonesia. The calculus tasks were designed to encourage the students to perform the theoretical abstraction process. The qualitative data were gathered through interviewing the students while they were working in pairs to solve the calculus tasks by thinking aloud. The students' worksheets were collected as written documents. The data were analyzed within the framework of the Abstraction in Context to obtain the answer to the first research question. The analysis resulted in the description of the processes and the results of the abstraction underwent by the students in constructing their understanding of the operational formal definition of function. The data were also analyzed within the framework of the Three Worlds of Mathematics to answer the second research question concerning the students' use of mathematical representations in abstraction.

This present chapter presents the conclusions of the thesis based on the research findings and discussion. The conclusions are followed by the discussion of the contribution of the study to the existing literature. The recommendations are explained in the penultimate section. This chapter concludes with a section addressing the limitations of the study.

7.2 Conclusions

Based on the previous chapters, the following sections are presented to summarize the major findings and discussions for each research question.

7.2.1 How Do Students Construct the Operational Formal Definition of Function?

Based on the research results, it was found that the students succeeded in constructing an understanding of the operational formal definition of function through the process of abstraction. Based on the *a priori* analysis, there were seven knowledge elements or concepts hierarchically structured in the learning trajectory for constructing the intended definition of function. The trajectory covered the concepts of association between sets' elements, ordered pairs, the Cartesian product of two sets, relation, the exhaustive domain property, the unique image property, and operational formal definition of function. Several prerequisite concepts were assumed to be constructed previously by the students, namely, the membership of set, subset of set, arbitrariness of association, order of elements, domain of relation, and codomain of relation. The abstraction performed by the students showed that the epistemic actions of recognizing, building-with, and constructing were evident in the process of abstraction. In the process, the students started recognizing the previously constructed concepts as relevant in the problems encountered, building-with them, and constructing the new concepts following the prescribed learning trajectory. The actions were performed in a nested way. The construction of the concept of association between sets' elements required several previously constructed concepts. In the abstraction process, the students started recognizing the concepts of the membership of set and arbitrariness of association. The action of recognizing these two concepts was followed by the action of building-with the concepts which led to the action of constructing the association of the elements between two given sets. A similar process of abstraction took place in the construction of the concepts. In the end, the students constructed the operational formal definition of function based upon the previously constructed concepts of the exhaustive domain property and the unique image property. All the processes showed that Abstraction in Context as the model of the theoretical mathematical abstraction was performed by the students in their attempt to construct an understanding of the operational formal definition of function.

Regarding the product of abstraction, the findings revealed that all students successfully constructed their complete-level understanding of the concepts association between sets' elements, ordered pairs, and relation. However, only two pairs of students, Iful–Vito and Dina–Yuni, successfully constructed their complete level understanding of the concepts of the exhaustive domain property, the unique image property, and the operational formal definition of function (see Table 4.2, p. 130). Further, they could implement the formulated definition of function to deductively verify the functional status of a relation with a formal argument. This definition-based deductive verification proved that the definition which they constructed was formally operational. Naya and Amzi successfully attained their complete understanding of the exhaustive domain property only, while the other two pairs of students, Dyn–Sam and Adel–Tanti, did not succeed in constructing their understanding of the three concepts to the required/complete level (see Table 4.2, p. 130).

Those students who did not develop their complete level understanding of several concepts encountered difficulties in the abstraction process (see Chapter 4, section 4.3.3.3, p. 161 and section 4.3.3.4, p. 164). Naya–Amzi and Adel–Tanti faced difficulties in identifying the unique image property correctly, and therefore, they did not further attempt to formulate the definition of the property. The difficulties were partly caused by their lack of understanding of mathematical logic. In particular, the students encountered difficulties in identifying and defining the unique image property because they could not see that the additional condition should be assigned to the elements of the domain to allow the use of the universal quantifier in the formal statement. Dyn and Sam could identify formulate the correct statement of the definitions of the special properties of relation. However, they encountered difficulties in expressing the definition in a statement using the appropriate symbolic logic. The logic-related difficulties encountered by Dyn and Sam concerned the interpretation and the symbolic expression. They experienced problems in terms of using the wrong symbolic expressions in logical statements, not in terms of using the correct symbolic expression in an inappropriate context (see Chapter 4, section 4.3.2.5, p. 146 and section 4.3.2.6, p. 149).

7.2.2 How Do Students Use Mathematical Representations in Constructing the Operational Formal Definition of Function?

The students used mathematical representations to express their mathematical ideas during the abstraction process of constructing an understanding of the operational formal definition of function. Their representations were observed in their responses to the concepts investigated as well as in their verification of relations as to whether they were legitimate functions or not. The research findings showed that there were five categories of mathematical representations dominantly used by the students to express their mathematical ideas in the solutions to the problems concerning the development of an understanding of the operational formal definition of function. With respect to the concepts of association between the elements of two sets, the students' responses were dominantly expressed in embodied representations. The formalizing-symbolism representations were also used by the students in their responses. This representation category was found dominant in the responses to the concepts of the

Cartesian product of two sets and relation. The formal representations were used dominantly to express the students' responses to the concepts of the exhaustive domain property, the unique image property, and the operational formal definition of function (see Table 5.1, p. 179).

In justifying relations which were given in various representations, it was found that the students used various representations covering embodied representations, symbolic representations, formal representations, embodying-symbolism representations, and embodying-formalism representations, and symbolizing-formalism representations (see the summary in section 5.2.3, p. 199). The research findings also revealed that to verify the functional status of relations, some students had a preference to use embodied representations including those connected to embodied representations such as embodying-symbolism and embodying-formalism representations.

Regarding the appropriateness of the representations, most of the students' responses commenced and finished with mathematical ideas expressed in appropriate representations (refer to Table 5.2, p. 214). As described in section 5.3.1 (p. 217) and summarized in Table 5.3 (p. 218), the responses categorized as formalizing-symbolism representations commenced and finished with correctly represented mathematical ideas. This trend was evident in the students' responses to the concept of the Cartesian product of two sets and relation. When the students expressed their responses to the concept of association between sets' elements in embodied representations, it was found that although not all responses began with appropriate representations, all responses proceeded and finished with mathematical ideas stated in appropriate representations. In contrast, all responses dominated by formal representations started with mathematical ideas expressed in appropriate representations, however, some of them proceeded and finished with inappropriate representations. This trend was shown in the responses to the concepts of the exhaustive domain property and the unique image property as well as the operational formal definition of function. The students made several representation errors in expressing their ideas which were evident in their responses. The mistakes were related to the use of notations and symbols as well as to the meaning and interpretation of logic in mathematical statements.

7.2.3 The Relationship Between Abstraction and the Mathematical Representations

Mathematical representations are the conceptual tools for mathematical abstraction. According to the research findings, the success of the abstraction processes to construct mathematical knowledge was, to some extent, determined by the representation versatility. The ability to interpret and utilize representations conceptually and procedurally and to change and translate from one representation model to another affected the students' progress in abstraction. The students' construction of various concepts leading to the formulation of the operational formal definition of function was facilitated by the representations. The construction of the operational formal definition of function was the main focus of this present study. And, from the viewpoint of formal mathematics, the mathematical logic as the language of formal mathematics played a significant role in terms of representing the concept definitions considered as acceptable in the world of mathematics.

7.2.4 Contribution of the Research

The extant research investigating the concept of function focuses more on the students' general understanding of function. The study on the concept of function in the context of formal, axiomatic, deductive mathematics seems to be under-represented in the existing research literature. With respect to the research on undergraduate mathematics education, the investigation of the concept of function tends to deal with the concept as an independent topic. It means that the concept of function is not treated as part of a particular course unit, although it is considered as a prerequisite, unifying concept for advanced mathematical studies. In terms of the definition of function, some studies involving first year university students, such as those conducted by Bardini et al. (2014), Christou et al. (2005), and Vinner and Dreyfus (1989), have asked the students to provide the definition of the concept function. However, these authors did not focus on the formal definition of function. The students' responses displayed various definitions which to some extent, were the concept image of function (Tall & Vinner, 1981). As part of these studies, the students were asked to justify the functional status of relations and, as a result, the students determined the legitimacy of the function based on their concept images, not on the concept definition, let alone providing a formal definition-based verification (Bardini et al., 2014; Christou et al., 2005; Vinner & Dreyfus, 1989). They relied more on the inductive inference rather than any logical deductive argument to justify the validity of the function (Hoyles & Healy, 2007).

This present study was conducted in the context of studying the concept of function set as a topic in the unit course of Introductory Calculus for mathematics education undergraduate students. The unit itself was studied through the formal deductive approach. Thus this study contributes to enrich the extant literature regarding the students' concept definition of function. The findings showed that the students could formulate the formal definition of function which was operational in nature. Further, it was found that they could verify the legitimacy of a function with their formulated definition which proved that the definition was indeed operational.

The study on defining mathematical concepts is a developing research field (see, e.g., Edwards & Ward, 2008; Gilboa et al., 2019; Kobiela & Lehrer, 2015; Ouvrier-Buffet, 2011; Rasmussen et al., 2005; Zandieh & Rasmussen, 2010; Zaslavsky & Shir, 2005). The investigation of defining as a mathematics learning activity currently gains its moment. The investigation focused on developing formal definitions of a mathematical concept that were more advanced than merely stating the concept images of a mathematical concept. This present study went further in order to investigate the formulation of the operational version of the formal definition of function.

Another contribution of this present study concerns the research frameworks. In the Indonesian university context, the Abstraction in Context framework has been implemented by Nurhasanah (2018) in her doctoral research studying mathematical abstraction of pre-service mathematics teachers in learning non-conventional mathematics concepts. This present study appears to be the second study in the same context, but with a different focus, namely, constructing the operational formal definition of function, a conventional concept in mathematics. The learning trajectory resulted from the a priori analysis of this study could be considered as enriching the notion of genetic decomposition of function (see Arnon et al., 2014).

Regarding the study of mathematical representations, a plethora of research has been conducted focusing on the development of understanding of mathematical concepts, mathematical reasoning, and problem solving (see, e.g., Serrazina & Rodrigues, 2017;

Smart, 2013; Stylianou & Silver, 2004; Thomas, 2003, 2008a, 2008b; Thompson & Chappell, 2007; Tripathi, 2008). The findings of this present study, to some extent, contribute to the literature on the relationship between mathematical representations and abstraction.

7.3 Recommendations

Based on the conclusions and the contribution explained in the previous section, some recommendations from the current investigation are proposed. Some students in this present research succeeded in constructing the operational formal definition of function in the context of introductory calculus. This outcome shows that defining concepts as a mathematics learning activity, which is part of the formal approach to studying introductory calculus, is worth implementing. The students should benefit more from the formal approach to introductory calculus learning especially in terms of developing established mathematical thinking to prepare them for the more advanced mathematical study.

The implementation of a learning approach necessitates a detailed preparation. The students' success in this present study was allowed and supported by the calculus tasks particularly designed to encourage the performance of the abstraction processes. Therefore, the further implementation of the defining mathematical concept activity in particular, and the theoretical abstraction model of Abstraction in Context in general needs mathematical tasks specifically designed based on the *a priori* analysis for the concept to be investigated.

This present study has also examined some part of the relationship between abstraction and mathematical representations from a qualitative perspective. Research on learning with representations has found that the students' ability to interact and use appropriate representations enhanced their learning achievement (Ainsworth, 2006). A quantitative study has been carried out by Kato et al. (2002) finding the positive relationship between the levels of abstraction ability and the representations skill. A further qualitative study should be conducted to investigate the interaction between abstraction performance and mathematical representation versatility.

7.4 Limitations

During the investigation, several limitations were identified that may affect the legitimacy of the findings of this present study. These limitations include the number of students participating in this study, the voluntary nature of the students' involvement, and the role as a lecturer-researcher.

7.4.1 The Numbers of Participant Students

The first limitation regards the number of participants. The data of this present study were gathered from 10 volunteer students. In terms of representativeness, this number of participants did not represent the population of first year students in the undergraduate program for mathematics education in the university in which the research took place. Due to the number of students involved, the notion of generalizing the findings could become an issue. Therefore, detailed research descriptions were provided which allow the readers to judge the extent to which the findings could be generalized analytically in their situation. The thick and rich report of the research is expected to allow the readers to develop a proper understanding of the phenomenon. In this way, the authority is in the hands of the readers to decide for themselves whether the findings apply to their context.

7.4.2 The Voluntary Nature of the Participants' Involvement

The recruitment of volunteer students to be research participants emerged as an issue in this investigation. As described in section 3.4.2 (p. 99), all participants in this present study were volunteers. They had been informed about the possible risks and benefits of their involvement in the research and they consciously understood that their participation had no impact whatsoever on their final marks in the unit of Introductory Calculus. Despite complying with the ethics of the research, this situation potentially restricted the participant students to perform maximally when they solved the problems given in the calculus tasks instrument. Their solution to the given calculus problems might not represent their best mathematical ability. Therefore, during the interviews, some probing questions were posed to clarify their answers and to get more detailed and accurate statements.

7.4.3 My Role as a Lecturer-Researcher

My position as a lecturer-researcher created a limitation in this study. Although I endeavored to position myself as an external researcher, the participants who were my students in the class of Introductory Calculus, to some extent, kept themselves in the position as students rather than as interviewees. This situation, more or less, constrained them in terms of the freedom to comfortably express all of their ideas and understanding during the interviews.

In this research investigation, I provided the students with guidance to build their concepts. In the interviews, I posed questions to encourage them to reflect on their constructed understanding. Hints were offered to help the students break the barrier when they could not progress in problem-solving. Such interventions were given limitedly. However, they might have impacted the credibility of the data.

To conclude, it is hoped that the outcomes of this study will provide the lecturers of prospective mathematics education teachers some understanding of how the first-year students construct their understanding of the operational formal definition of function through the process of abstraction and how they use the mathematical representations in the process. Also, it is expected that the outcomes of this research will give the lecturers insights into the difficulties the students face in learning the concept of function.

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APPENDICES

Appendix A1 Calculus Tasks

Calculus Worksheets

FUNCTIONS: THE OPERATIONAL FORMAL DEFINITION

Direction:

Please, read and answer all questions as completely as possible. In answering each question, you are expected to express the process and the thought you employ freely and clearly. You could use all your apparatuses and tools. Be advised that this is part of your calculus learning. However, the answers will not be judged, and your work has no relations whatsoever to the final mark you will get in this subject.

Name: ____

Partner's Name: ____

Date: ____

Operational Formal Definition OFD-002-ENG 10/09/2016

FUNCTIONS: THE OPERATIONAL FORMAL DEFINITION

1. The followings are pairs of two sets. What kinds of association between members of the two sets could you make? (*Think about all possible associations and write your answer within the box provided.*)

(a) $M = \{a, b, c\}$ and $K = \{x \mid 2 < x < 4, x \in \boldsymbol{N}\}$.

(b) $P = \{z \mid z^2 + 2z + 2 = 0, z \in \boldsymbol{R}\}$ and $R = \{6, 7\}$.

2. Let A and M be two non-empty sets, $k \in A$, and $w \in M$. If we associate k to w, we can express the result as (k, w). Further, (k, w) is considered as an element of the association. Given two sets: $L = \{2, 3\}$ and $C = \{2, 4, 6\}$.

(a) Write all the results of the association if each element of L is associated with C's elements whose values are greater than that of the element of L. (*Write your answer within the box provided.*)

(b) Write all the results of the association if each element of C is associated with L's elements whose values are less than that of the element of C. (*Write your answer within the box provided.*)

(c) Are the results in (a) the same as in (b)? Why? Explain your reason. (W*rite your answer within the box provided.*)

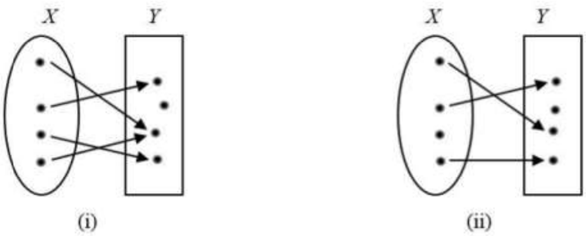
(d) Write all the results of the association if each element of L is associated with C's elements which are even numbers. (W*rite your answer within the box provided.*)

(e) Compare the results in (a), (b), and (d). What is the uniqueness of the results in (d) compared to those both in (a) and (b)? (W*rite your answer within the box provided.*)

(f) Let us group the results in (a) and (d), respectively, into different sets. Take into account the two initial sets involved, that is, *L* and *C*. What relationships could be identified between the two sets? What are these two sets? If you can define this special mathematical object, what will be your definition? (*Write your answer within the box provided.*)

(g) Look at the results in (a) and (d). Group them into different sets, respectively. In (f), as a special mathematical object you have named these two set, and have defined the object as well. What commonality do they have which does not belong to the set of results in (b)? Express this special property. How do you define the object which has this particular property? (*Write your answer within the box provided.*)

(h) Look at and compare the pictures beside. What differences could you identify? If you focus on (ii), what is its special property? Express this special property. How do you define the object which has this particular property? (*Write your answer within the box provided.*)

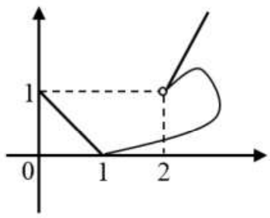


(i) Consider the two special properties, each you have discussed in (h) and (g). Construct a relation which fulfils both of the properties. (*Write your answer within the box provided.*)

(j) How do you define the special relation you have constructed in (i)? (*Write your answer within the box provided.*)

3. The special relation you have defined in the previous question is called a *function*. Now, look at the following list of expressions. Examine the following expressions. Check the box beside [F] if the expression is a function or the one beside [N] if the expression is not a function. Give an explanation to justify your answer. (*Write your idea within the box provided.*)

(a)[1] [F] ☐ [N] ☐



Because:

(b)[2] [F] ☐ [N] ☐

Days	Things
Sunday	Orange
Monday	Computer
Tuesday	Assignment
Wednesday	Jakarta
Thursday	Aeroplane
Friday	Jakarta
Saturday	Orange

Because:

(c) $x = 5$ [F] ☐ [N] ☐

Because:

(d) $x(y) = \begin{cases} y & \text{if} \quad y \text{ is rational} \\ -y & \text{if} \quad y \text{ is irrational} \end{cases}$ [F] ☐ [N] ☐

Because:

(e) ***R*** is the set of real numbers. Relation β is from ***R*** to ***R***; [F] ☐ [N] ☐
$\beta(y) = \log(2 - y)$, for each $y \in \boldsymbol{R}$.

Because:

(g) Let $\boldsymbol{N}$ be the set of natural numbers and $\boldsymbol{Z}$ be the set of integers. Relation L is from $\boldsymbol{N}$ to $\boldsymbol{Z}$ where $L = \{(a, b) \mid a+b=0,\ a\in \boldsymbol{N},\ b\in \boldsymbol{Z}\}$.

[F] ☐ [N] ☐

Because:

4. Give your best example of function. Using the definition you have built, justify why it is a function. (*Write your answer within the box provided.*)

5. If we have two sets $A = \{3\}$ and $K = \{2, 3, 4\}$, can we create functions between the two sets? If yes, please list the functions which you make. (*Write your answer within the box below.*)

6. How do you define a relation which is not a function? (*Write your answer within the box provided.*)

7. Give an example of a non-function and explain why it is not a function. (*Write your answer within the box below.*)

Notes:

1. Adapted from: Christou, C., Pitta-Pantazi, D., Souyoul, A., & Zachariades, T. (2005). The embodied, proceptual, and formal worlds in the context of functions. *Canadian Journal of Science, Mathematics and Technology Education, 5*(2), 241-252. doi: 10.1080/14926150509556656
2. Adapted from: Feeley, S. J. (2013). *Assessing understanding of the concept of function: A study comparing prospective secondary mathematics teachers' responses to multiple-choice and constructed-response items* (Doctoral Dissertation). Available from ProQuest Dissertations & Theses Global database. (UMI No. 3576452)
3. Adapted from: Nicholson, W. K. (2012). *Introduction to abstract algebra* (4 ed.). Hoboken, NJ: John Wiley & Sons.

Appendix A2 Calculus Tasks (Indonesian Version)

Lembar Kerja Kalkulus

FUNGSI: DEFINISI FORMAL OPERASIONAL

Petunjuk:

Mohon dibaca dan dikerjakan semua pertanyaan secara lengkap. Dalam menjawab setiap pertanyaan, Anda diharapkan mengungkapkan proses dan pemikiran yang digunakan secara bebas dan jelas. Silahkan menggunakan semua peralatan yang Anda miliki. Ingat bahwa ini adalah bagian dari proses belajar Kalkulus 1. Hasil kerja atau jawaban tidak akan diberi nilai dan tidak ada kaitannya dengan nilai akhir yang Anda peroleh pada mata kuliah ini.

Nama: ____

Nama Pasangan: ____

Tanggal: ____

FUNGSI: DEFINISI FORMAL OPERASIONAL

1. Berikut ini terdapat beberapa pasangan himpunan. Tentukan hubungan antar anggota kedua himpunan tersebut yang dapat Anda buat. (*Pikirkan semua hubungan yang mungkin dibuat, kemudian tuliskan jawaban Anda pada kotak yang telah disediakan.*)

(a) $M = \{a, b, c\}$ dan $K = \{x \mid 2 < x < 4, x \in \boldsymbol{N}\}$.

(b) $P = \{z \mid z^2 + 2z + 2 = 0, z \in \boldsymbol{R}\}$ dan $R = \{6, 7\}$.

2. Misalkan A dan M dua himpunan tidak kosong, $k \in A$, dan $w \in M$. Jika kita mengaitkan k ke w, maka hasil pengaitan dituliskan dalam bentuk (k, w). Selanjutnya, (k, w) dinyatakan sebagai anggota pengaitan tersebut. Diberikan dua himpunan: $L = \{2, 3\}$ dan $C = \{2, 4, 6\}$.

(a) Tuliskan semua hasil pengaitan jika setiap anggota L dikaitkan dengan anggota-anggota C yang nilainya lebih dari anggota L. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(b) Tuliskan semua hasil pengaitan jika setiap anggota C dikaitkan dengan anggota-anggota L yang nilainya kurang dari anggota C. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(c) Apakah hasil di bagian (a) sama dengan hasil di bagian (b)? Mengapa? Jelaskan alasan Anda. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

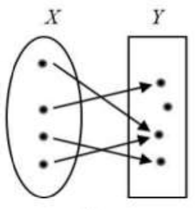
(d) Tuliskan semua hasil pengaitan jika setiap anggota L dikaitkan dengan anggota-anggota C yang merupakan bilangan genap. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(e) Bandingkan hasil yang Anda peroleh pada bagian (a), (b), dan (d). Apa yang unik dari hasil pengaitan di (d) dibanding hasil di (a) dan (b). (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

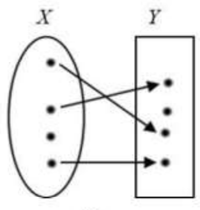
(f) Coba kita kelompokkan hasil pengaitan di (a) dan (d) masing-masing menjadi dua himpunan berbeda. Perhatikan kedua himpunan yang kita gunakan di awal, yakni, L dan C. Hubungan apa yang dapat ditemukan dari kedua himpunan tersebut? Himpunan apakah itu? Bagaimana Anda mendefinisikan obyek matematika tersebut? (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(g) Pada bagian (f), Anda telah menamai dan mendefinisikan himpunan yang memuat hasil pengaitan yang diperoleh pada bagian (a) dan (d). Dapatkah Anda menemukan sifat yang sama dimiliki kedua himpunan hasil pengaitan ini, tetapi tidak dimiliki oleh himpunan hasil pengaitan pada bagian (b)? Bagaimana Anda mendefinisikan obyek matematika yang memiliki sifat yang dimaksud? (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(h) Perhatikan dan bandingkan kedua gambar di samping. Tulislah perbedaan yang Anda temukan dari kedua gambar. Perhatikan gambar (ii), pikirkan sifat khusus dari gambar tersebut. Tuliskan. Bagaimana Anda mendefinisikan obyek yang memiliki ciri-ciri seperti itu? (*Tuliskan jawaban Anda pada kotak yang tersedia.*)



(i)



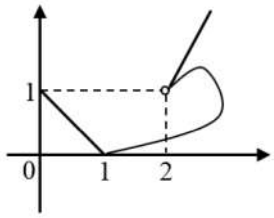
(ii)

(i) Perhatikan kedua ciri khusus sebagaimana yang Anda temukan di (h) dan (g). Buatlah satu relasi yang memenuhi kedua ciri tersebut. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(j) Bagaimana mendefinisikan relasi khusus yang telah Anda buat di bagian (i) (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

3. Relasi khusus yang telah Anda definisikan pada pertanyaan sebelumnya disebut *fungsi*. Sekarang, perhatikan ekspresi-ekspresi berikut. Beri tanda centang (√) pada kotak di sebelah [F] jika ekspresi tersebut merupakan fungsi, atau kotak di sebelah [N] jika bukan fungsi. Beri penjelasan untuk jawaban Anda. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(a)[1] [F] ☐ [N] ☐



Karena:

(b)[2] [F] ☐ [N] ☐

Hari	Benda
Minggu	Jeruk
Senin	Komputer
Selasa	Tugas
Rabu	Jakarta
Kamis	Pesawat terbang
Jumat	Jakarta
Sabtu	Jeruk

Karena:

(c) $x = 5$ [F] ☐ [N] ☐

Karena:

(d) $$x(y)=\begin{cases} y & \text{jika } y \text{ rasional} \\ -y & \text{jika } y \text{ irrasional} \end{cases}$$ [F] ☐ [N] ☐

Karena:

(e) $\boldsymbol{R}$ adalah himpunan bilangan riil. Relasi β dari $\boldsymbol{R}$ ke $\boldsymbol{R}$; [F] ☐ [N] ☐
$\beta(y)=\log(2-y)$, untuk setiap $y\in\boldsymbol{R}$.

Karena:

(f) Misalkan $\boldsymbol{N}$ adalah himpunan bilangan asli dan $\boldsymbol{Z}$ adalah himpunan bilangan bulat. Relasi L menghubungkan $\boldsymbol{N}$ ke $\boldsymbol{Z}$ dimana $L = \{(a, b) \mid a+b=0,\ a \in \boldsymbol{N}, b \in \boldsymbol{Z}\}$. [F] ☐ [N] ☐

Karena:

4. Berikan satu contoh fungsi. Dengan menggunakan definisi yang Anda telah buat, jelaskan mengapa contoh tersebut merupakan fungsi.

5. Jika terdapat dua himpunan $A = \{3\}$ dan $K = \{2, 3, 4\}$, dapatkah kita membuat fungsi yang menghubungkan kedua himpunan tersebut? Jika ya, tuliskan fungsi-fungsi yang dapat Anda buat. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

6. Bagaimana Anda mendefinisikan relasi yang bukan merupakan fungsi? (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

7. Berikan satu contoh relasi yang bukan fungsi, dan jelaskan mengapa relasi tersebut bukan fungsi. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

Catatan:

1. Diadaptasi dari: Christou, C., Pitta-Pantazi, D., Souyoul, A., & Zachariades, T. (2005). The embodied, proceptual, and formal worlds in the context of functions. *Canadian Journal of Science, Mathematics and Technology Education, 5*(2), 241-252. doi: 10.1080/14926150509556656
2. Diadaptasi dari: Feeley, S. J. (2013). *Assessing understanding of the concept of function: A study comparing prospective secondary mathematics teachers' responses to multiple-choice and constructed-response items* (Doctoral Dissertation). Available from ProQuest Dissertations & Theses Global database. (UMI No. 3576452)
3. Diadaptasi dari: Nicholson, W. K. (2012). *Introduction to abstract algebra* (4 ed.). Hoboken, NJ: John Wiley & Sons.

Appendix B1 Interview Guidance

Some Possible Questions in Interview

This interview, or I prefer to call it discussion, is to understand your thought processes while solving these calculus problems. Please feel free to express your thoughts or ideas, but please do not hesitate to let me know if you feel uncomfortable with the interview.

After examining the relation in question 3d, you decided that it is a function. Could you verify it with more formal proof? You could refer back to the definition of function you have constructed.

If I may suggest, you could try to focus on the entities between the braces: $\{(a, b) \mid a+b=0, a\in\mathbf{N}, b\in\mathbf{Z}\}$.

You might think about some indicators when you examine certain expression whether it was a set or not. Could you please list them?

[Here, the interviewer makes a suggestion. This is not always the case in the interviews. It is done when it is found necessary to provide a prompt, otherwise the participants will be in the situation of being stuck.]

Now, we have completed the first part. Do you mind if we proceed to the next part?

How would you explain the interrelation between those concepts in the context of the formal definition of function?

I am interested in the way you define a function in the formal style.

Now, if you have a chance to express the definition of a function not in formal way, how would you like to say it? You might think of a situation where your friend probably asks you to explain to them how you understand a function, but avoiding the use of formal terms.

Appendix B2 Interview Guidance (Indonesian Version)

Beberapa Pertanyaan yang Mungkin dalam Wawancara

Wawancara ini dilaksanakan guna memahami proses berpikir kalian pada saat menyelesaikan masalah kalkulus. Mohon ungkapkan pikiran atau gagasan kalian secara bebas, dan jangan segan memberitahu saya manakala kalian merasa tidak nyaman dalam rangkaian wawancara ini.

Setelah memeriksa relasi pada soal 3d, kalian telah memutuskan bahwa relasi tersebut adalah fungsi. Bisakah kalian memberi bukti yang lebih formal? Kalian bisa mengacu kembali pada definisi fungsi yang telah dibuat.

Jika saya boleh menyarankan, kalian bisa mencoba berfokus pada unsur-unsur di antara kurung kurawal: $\{(a, b) \mid a+b=0,\ a \in \mathbf{N},\ b \in \mathbf{Z}\}$.

Kalian bisa memikirkan beberapa indikator yang diperhatikan pada saat kita memeriksa ekspresi tertentu apakah ia merupakan himpunan atau bukan. Bisakah kalian menyebutkan indikator-indikator tersebut?

[Di sini, pewawancara memberikan petunjuk. Namun, ini tidak selalu dilakukan. Hal ini hanya ditempuh manakala diperlukan untuk menghindarkan mahasiswa dari situasi kebuntuan akibat ketiadaan gagasan.]

Sekarang, kita telah menyelesaikan bagian pertama. Bisa kita lanjut ke bagian berikutnya?

Bagaimana kalian menjelaskan hubungan atau keterkaitan antara konsep-konsep tersebut dalam konteks definisi formal fungsi?

Saya tertarik dengan cara kalian mendefinisikan fungsi secara formal.

Selanjutnya, jika kalian diminta menyatakan definisi fungsi secara tidak formal, bagaimana kalian akan menjelaskannya? Kalian bisa saja membayangkan seorang teman yang meminta penjelasan bagaimana kalian memahami konsep fungsi. Tetapi, temanmu tidak menginginkan penjelasan secara formal.

Appendix C1 Student Information Sheet

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Student Information Sheet

Dear Student,

My name is Sabri. I am a PhD student at Curtin University under the supervision of Professor David F. Treagust. I am currently working on a research project investigating university students' learning of calculus concepts and their use of mathematical representations. I would like to invite you to take part in this study. The research procedures are as following:

- This Introductory Calculus class will be taught by the researcher.
- All students will solve a set of calculus problems in groups.
- The researcher will walk around the class and support students' learning.
- Those students who are willing to participate in the study will be asked to 'think aloud' when they solve the problems.
- The lecturer of the unit will be present at all time to support the students and answer questions.

If you decide to participate in this study, I will ask you several questions to help you solve the calculus problems. You will need to say what you are thinking about the calculus problems so that I can ask appropriate questions. Your answers will be audio-recorded and transcribed later for analysis.

The mathematics department and its staff members have agreed to participate in this project. However, your involvement in this project is completely voluntary. Participation in this study does not affect your unit assessment marks. If you don't want to be a part of this project, you can stop at any time without any consequences to you. When you sign the consent form, please return it to me so that I know you have agreed to participate in the research and allow me to use your data for this project.

The information you provide for this project will be kept confidential and private. This means I won't talk to other lecturers in the department or your peers about any individual students (unless I am legally required to disclose the information). This project won't go on your university record or count toward your unit marks. The interview transcript won't have your name or any other identifying information on it. In adherence to the university policy, the interview records and transcripts will be kept in a locked cabinet for seven years. If you wish to learn about the research results, please email me and I will provide a summary of research findings.

This research has been reviewed and given approval by Curtin University Human Research Ethics Committee (Approval Number RDSE-55-15). If you have any ethical concerns, you can contact the Human Research Ethics Committee at Curtin University (+ 61 8 9266 9223 or hrec@curtin.edu.au). If you would like further information about the study, please feel free to contact me at the university or my doctoral supervisor Professor David F. Treagust at +61 8 9266 7924 (D.Treagust@curtin.edu.au).

Thank you very much for your support for this research.

Kind regards,

Sabri

Appendix C2 Student Information Sheet (Indonesian Version)

Curtin University

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Lembar Informasi Mahasiswa

Untuk mahasiswa,

Saya Sabri, mahasiswa PhD Curtin University of Technology di bawah supervisi Professor David F. Treagust. Saya sedang melaksanakan proyek penelitian yang mengkaji pembelajaran mahasiswa tentang konsep kalkulus dan penggunaan representasi matematika. Saya bermaksud mengajak Anda untuk mengikuti penelitian ini. Proses penelitian adalah sebegai berikut:

- Mata kuliah Kalkulus 1 akan diampu oleh peneliti.
- Semua mahasiswa akan menyelesaikan serangkaian soal kalkulus secara berkelompok.
- Peneliti akan berada di ruang kelas dan membantu mahasiswa dalam belajar.
- Mahasiswa yang bersedia mengikuti penelitian ini akan diminta berpikir dan berbicara saat mereka menyelesaikan soal.
- Dosen pengampu mata kuliah akan selalu berada di tempat untuk membantu mahasiswa dan menjawab pertanyaan.

Apabila Anda bersedia mengikuti penelitian ini, saya akan menanyakan beberapa pertanyaan untuk membantu Anda menyelesaikan soal kalkulus. Anda akan diminta mengungkapkan pemikiran tentang soal kalkulus tersebut supaya saya bisa mengajukan pertanyaan yang tepat. Jawaban Anda akan direkam dan kemudian ditranskripsi untuk kepentingan analisis data.

Jurusan Matematika dan stafnya telah menyetujui untuk ikut serta dalam penelitian ini. Akan tetapi, keterlibatan Anda dalam penelitian ini sepenuhnya bersifat sukarela. Keikutsertaan dalam penelitian ini tidak mempengaruhi nilai akhir Anda dalam mata kuliah ini. Jika Anda tidak ingin ikut serta dalam proyek ini, Anda bisa berhenti kapan saja dan tidak akan berakibat apapun untuk Anda. Setelah Anda menandatangani formulir persetujuan, mohon Anda kembalikan ke saya supaya saya mengetahui bahwa Anda setuju untuk mengikuti penelitian dan mengizinkan saya menggunakan data Anda untuk kepentingan penelitian ini.

Informasi yang Anda berikan kepada proyek ini akan bersifat pribadi dan dijaga kerahasiaanya. Hal ini berarti bahwa saya tidak akan membicarakan tentang Anda dengan dosen lain di Jurusan Matematika atau dengan teman kelas Anda (kecuali saya secara hukum diminta mengungkap informasi tersebut). Proyek penelitian ini tidak akan menjadi bagian laporan kemajuan perkuliahan Anda dan juga tidak akan diperhitungkan dalam penentuan nilai akhir Anda. Transkripsi wawancara tidak akan mencantumkan nama Anda atau informasi apapun yang bisa digunakan untuk mengenali Anda. Sesuai dengan kebijakan Curtin University of Technology, rekaman dan transkripsi wawancara akan disimpan dalam lemari terkunci selama tujuh tahun. Jika Anda ingin mempelajari hasil penelitian ini, mohon kirimkan e-mail ke saya dan saya akan memberikan ringkasan hasil penelitian.

Penelitian ini telah ditinjau dan disetujui oleh Curtin University Human Research Ethics Committee (Komisi Etik Penelitian Manusia Curtin University) dengan Nomor Persetujuan RDSE-55-15. Jika Anda mempunyai pertanyaan tentang masalah etik, Anda dapat menghubungi Komisi Etik Penelitian Manusia di Curtin University (Telepon: +61 8 9266 9223 atau hrec@curtin.edu.au). Jika Anda membutuhkan informasi lebih lanjut tentang penelitian ini, silahkan menghubungi saya atau pembimbing saya, Professor David F. Treagust di nomor telepon: +61 8 9266 7924 (D.Treagust@curtin.edu.au).

Saya mengucapkan terima kasih yang sebesar-besarnya untuk semua bantuan Anda dalam penelitian ini.

Salam hormat,

Sabri

Appendix D1 Student's Consent Form

Curtin University

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Student's Consent Form

- I understand the purpose and procedures of the study.
- I have been provided with the participation information sheet.
- I have been given the opportunity to ask questions about this research.
- I understand that the procedure itself may not benefit me.
- I understand that my involvement is voluntary and I can withdraw at any time without problem.
- I understand that no personal identifying information like my name and address will be used in any published materials.
- I understand that all information will be securely stored for at least 7 years before a decision is made as to whether it should be destroyed.
- I agree to participate in the study outlined to me.

Name: ______________________________

Signature: ______________________________

Date: ______________________________

Appendix D2 Student's Consent Form (Indonesian Version)

Curtin University

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Formulir Persetujuan Mahasiswa

- Saya memahami tujuan dan prosedur penelitian ini.
- Saya telah diberikan lembar informasi keikutsertaan.
- Saya telah diberikan kesempatan untuk mengajukan pertanyaan tentang penelitian ini.
- Saya memahami bahwa prosedur penelitian ini mungkin saja tidak memberi saya manfaat apapun.
- Saya memahami bahwa keikutsertaan saya bersifat sukarela dan saya bisa berhenti kapan saya tanpa ada akibat apapun.
- Saya memahami bahwa tidak ada informasi untuk mengenali pribadi saya, seperti nama dan alamat, yang akan digunakan dalam bahan publikasi dari penelitian ini.
- Saya memahami bahwa semua informasi akan disimpan dengan aman selama paling kurang 7 tahun sebelum diputuskan untuk memusnahkannya.
- Saya setuju untuk ikut serta dalam penelitian yang telah dijelaskan kepada saya.

Nama: ______________________________

Tandatangan: ______________________________

Tanggal: ______________________________

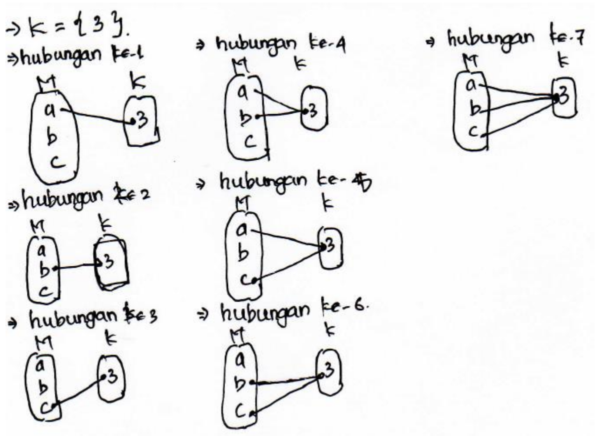
Appendix E The Example of Interview Transcripts

Interview: Dyn–Sam

No.	Name	Utterance
[0001]	Lec	Well, Dyn and Sam. Thank you for your readiness to participate in this interview. As we have agreed on, your participation is voluntary. This interview will gather data of the cognitive processes in your minds. I will never ask about your personal matter. So, if you find any question you think concerning your privacy and you don't need to answer it. I make sure that I will not ask about it, but just in case I pose such question unintentionally. This interview will be purely about how you think about something. OK. We could start by working on the items of problems.
[0002]	Sam	Number one. The followings are pairs of two sets. Find associations between the members of the two sets you could make. So ….
[0003]	Dyn	First, we determine the elements of each set, respectively, because the elements of set *M* are there already, that is, *a*, *b*, and *c* … and that set *K* … its element is clear, but we need to write it first.
[0004]	Sam	So, the element of set *K* is 3. So … the association … make an association between elements …. So, we just make an association, should we use a diagram?
[0005]	Dyn	An arrow diagram.
[0006]	Sam	An arrow diagram. So the first relation is ….
[0007]	Dyn	Set *M*.
[0008]	Sam	The first set, where *M* is …. Set *M* is … *a*, *b*, and *c* … like this, right? And set *K*, *K* is 3.
[0009]	Dyn	Yes.
[0010]	Sam	The first association is only *a* and *K*, isn't it?
[0011]	Dyn	It could be only *a* with 3, *b* has no association with 3, and *c* has no association with 3.
[0012]	Sam	So, the first association is *a* with 3
[0013]	Dyn	It could be the second.
[0014]	Sam	The second relation … maybe it is quicker if … the second one.
[0015]	Dyn	Only *a* and *b*.
[0016]	Sam	The second is ….
[0017]	Dyn	Only *b* having an association with the set … having an association with the set *K*, that is, 3.
[0018]	Sam	Yes … an association of *M*. Amm … associating *a*, *b*, and *c* … *b* …. Amm … the third association ….
[0019]	Dyn	Only *c* ….
[0020]	Sam	*c* only, isn't it? c only.
[0021]	Dyn	It could also be the fourth.
[0022]	Sam	The fourth association has *a* and *b* only, and not *c*. The fifth association could *b* and *c* only, not *a*, or could *a* and *c*, *b* has no association. OK.
[0023]	Dyn	*a* and *c* only, *b* has no association. OK.
[0024]	Sam	The fifth is this … the fifth is … *a* and *c* only, *b* is not. The sixth … *b* and *c* only, *a* is not.
[0025]	Dyn	*b* and *c* only.
[0026]	Sam	Yes, *b* and *c* only ….
[0027]	Dyn	The seventh is ….
[0028]	Sam	All.

[0029] Dyn All could be. All could have a relation with 3.

[0030] Sam The seventh … is like this ….
(Drawing)



So, the relations we could make are seven. So, are they all?

[0031] Dyn To me, it is all, except you still have something to add.

[0032] Sam Hang on. We have got *a* and *b* already, *a* and *c*, *b* and *c*, yes. All them, right? OK.

[0033] Sam Amm … for Part b, that is …, first we have to find … amm … the elements of *P*, *P* right? The first.

[0034] Dyn We solve the equation first.

[0035] Sam We solve the equation using the *ABC* formula, don't we?

[0036] Dyn Yes, we can, if we want to use the formula. Factorization … we can also use it.

[0037] Sam We use the *ABC* formula, right?

[0038] Dyn Yes, *ABC*.

[0039] Sam If set *P* equals *z*, where *z* squared plus 2*z* plus 2 equals 0, *z* is an element of real numbers. So, we find the factors first … for *z* squared plus 2*z* plus 2 equals 0. We use the *ABC* formula where ….

[0040] Dyn To find the value of *z* ….

[0041] Sam *x*1, *x*2 equal negative *b* plus minus the square root of *b* squared minus 4ac per 2*a*.
Is it like this? Oh, yes.
Negative b … negative 2 plus minus the square root of 2 squared minus 4 times 1 times 2 per 2 times 1.
Negative 2 …. The result?

[0042] Dyn Your equal [sign] ….

[0043] Sam Where? Why?

[0044] Dyn Your equal [sign] isn't parallel to your fraction sign.

[0045] Sam Is equal to ….
So, *x*1 is equal to negative 2 plus the square root of negative 4 per 2.
And *x*2 is equal to negative 2 minus the square root of negative 4 per 2.
So, the elements of set *P* are … amm … negative 2 plus the square root of negative 4 per 2.

[0046] Dyn It couldn't be solved further. This is it. The point is … its form is just like that.

[0047] Sam But, how is its solution?
Are they an element of the set [*x*1, *x*2] because they are imaginary, while *z* is an element of *R*, real … these are imaginary, are these included, are

these the elements of set *P*, because these are imaginary? Could we put them in?

[0048] Dyn While the *z* is elements of real numbers. What we got are, for *z*, imaginary, the square root of negative 4.

[0049] Sam Could we work on them or how?

[0050] Dyn These numbers, if we do some operations on them, won't we get real numbers?

[0051] Sam But this is the square root of negative 4, can't be processed.
We couldn't … how … yes, the square root of negative 4 ….

[0052] Dyn You want to see them as real numbers if you want to process them further.

[0053] Sam So, how is the solution or we just put it in this form?

[0054] Dyn Just like that ….

[0055] Sam The set, set … for the elements of set *P*, how about that?
If we just make them like this, meaning that the existing relation is amm … negative 2 plus the square root of negative 4 per 2 goes to 6 and negative 2 minus the square root of negative 4 per 2 gets … 7, and negative 2 minus the square root of negative 4 … it means that the relation possible is that only ….
Or, we could just write the existing relation like this. I mean … the form of … the form of the elements of set *P*.

[0056] Lec What is the element of *P*?

[0057] Dyn The elements of *P* are *z* where *z* squared plus 2*z* plus 2 equals 0, where *z* is a real number.

[0058] Sam To find out the value of *z*, we solve this equation, look for its factors.

[0059] Lec And then?

[0060] Sam To get its factors ….
For *x*1 or *z*, negative 2 plus the square root of negative 4 per 2 and negative 2 minus the square root of negative 4 per 2.

[0061] Lec So, how about them?

[0062] Sam So … this form is ….

[0063] Dyn They are not suitable … here its *z* is a real number and what we get is an imaginary number.

[0064] Lec So, how is it?

[0065] Sam Amm … how?

[0066] Dyn Nothing, there are no elements of set *P* … because ….

[0067] Lec What did you write there?

[0068] Sam Here is negative 2 plus the square root of negative 4 per 2. That is ….
Maybe this square root of negative 4 couldn't be solved because it is imaginary.

[0069] Dyn But, that one could be ….

[0070] Sam But, the values of the elements of set, *z* is amm …, that is, negative 2 plus the root of negative 4 per 2. This is the *z* value.

[0071] Lec The *z* of what?

[0072] Dyn *z* is a set ….

[0073] Sam The set of number, *P*.

[0074] Dyn The set of elements of *P*.

[0075] Lec What is *z*?

[0076] Dyn *z* is real numbers.

[0077] Lec Then, what are your real numbers there?

[0078] Dyn Negative 2 plus the square root of negative 4 per 2 ….
Is that a real number?

[0079] Lec *P* contains of *z*, what kind of number is *z*?

[0080] Sam Real numbers.

[0081] Lec Now, what did you write there?
[0082] Sam It is not a real number, because a real number is *a* over *b*, isn't it, Sir, *a* is an element of ….
[0083] Dyn That is a rational number.
[0084] Sam Oh, is that rational? Rational or real. Yes, rational.
Oh, yes, yes, so, is this a rational number?
[0085] Dyn If it is an imaginary, is it real or not?
[0086] Sam Imaginary? No, no.
[0087] Dyn Is it?
[0088] Sam No, because it is imaginary, rational. Real numbers are rational numbers.
[0089] Dyn No, rationals are reals. Rationals are reals, irrational as well are reals, integers are also reals.
[0090] Sam Irrationals are reals, oh, yes. But, this is imaginary, right?
[0091] Dyn Amm … that is, the square root of negative 4 is imaginary, but, if it is operated along with ….
[0092] Sam Could imaginary numbers be operated?
Is the square root of negative 4 imaginary, imaginary or rational?
[0093] Dyn Imaginary.
[0094] Sam Could imaginary be operated? Amm … the square root of negative 1 ….
It can't be. Amm … it is clear for set *R* elements, they are 6 and 7.
[0095] Lec Yes, that is clear. Now, the questions is what is set *P*?
[0096] Sam *P* is a set whose element is *z*, *z* is an element of real number set. While, if it is processed, we got negative 2 plus the square root of negative 4 per 2, it is getting more imaginary.
[0097] Dyn It means that it is real, because it is *z*, while, *z* is a real number.
[0098] Sam So, it means that these [the roots] are real number.
[0099] Lec How could this be a real number?
[0100] Dyn Because this is *z* … *z* is obtained from the factors of *z* squared plus 2*z* plus 2 equals 0.
[0101] Lec This one [the equation in *z*], what does it mean, actually? How is *z*?
[0102] Sam The quadrat of *z* ….
[0103] Lec That *z*, it satisfies *z* squared plus 2*z* plus 2 equals 0 and it is a real number. Now, you look for the member of *P* like that.
[0104] Dyn This [the root of the equation] is, when we plug it into this [the equation], we get zero, right? But, not real.
[0105] Lec Here, there is an additional condition that *z* must also be real. So, not only from here [the equation] do you get the value of *z*. It is not that *z* fulfils this [the equation] only, but, *z* satisfies this [the equation] and is also the real numbers. Now, You should find *z*.
[0106] Dyn Amm … *z* which satisfies is … 1 no, 2 no ….
[0107] Lec What is your way to find the correct *z*? Is it like that?
Your way to find the value from the equation *z* squared plus 2*z* plus 2 equals 0?
[0108] Dyn We use the *ABC* formula.
[0109] Sam Factoring.
[0110] Lec You have used this, right? What did you get?
[0111] Sam The value of *z* ….
[0112] Dyn The value of *z*, *z* is negative 2 plus the square root of negative 4 per 2 and negative 2 minus the square root of negative 4 per 2.
[0113] Lec Then, what is the condition of *z*?
[0114] Dyn *z* must be real.
[0115] Sam Must be real.

[0116]	Lec	So, what then? Here, you got this [the root of the equation] from the formula, what did you get from the formula?
[0117]	Sam	The result of factoring this, z squared plus $2z$ plus 2 equals 0.
[0118]	Lec	Why did you say factoring? What did you get from that formula?
[0119]	Dyn	The value of z.
[0120]	Sam	The value of z.
[0121]	Lec	The value of z, what is it?
[0122]	Sam	An element ….
[0123]	Lec	How about the z value there?
[0124]	Dyn	Its' value is imaginary.
[0125]	Lec	Then, what is its condition?
[0126]	Dyn	z is a real number.
[0127]	Sam	Real number.
[0128]	Lec	What is requested in P?
[0129]	Sam	z is a real number.
[0130]	Dyn	z is a real number satisfying z squared plus $2z$ plus 2 equals 0.
[0131]	Lec	So, what satisfies that [the condition of P's element]?
[0132]	Sam	It means P is … the empty set.
[0133]	Dyn	It means, these [the roots of the equation] are not its members. These [the roots of the equation] are not P's members.
[0134]	Sam	No, not the members of P.
[0135]	Dyn	Because they are not real numbers.
[0136]	Sam	Is there satisfying this [equation]? Try to substitute them into …. Oh, yes, yes.
[0137]	Lec	What are you going to do, what for?
[0138]	Sam	It means, because for this condition, z squared plus $2z$ plus 2 equals 0, the roots satisfy it. But they don't fulfil this [z is a real number]. So, these [the roots of the equation] are not P's elements.
[0139]	Dyn	Not those [the roots of the equation].
[0140]	Sam	Because the numbers other than these [the root of the equation] are not the element of this [P], meaning P is an empty set, nothing.
[0141]	Dyn	So, there is not set.
[0142]	Sam	There is not set.
[0143]	Lec	P doesn't exist, what do you mean?
[0144]	Sam	Because ….
[0145]	Dyn	It has no elements. P has no elements.
[0146]	Sam	Set P has no element … because for the condition z squared plus $2z$ plus 2 equals 0, z values satisfying it are these [the roots of the equation] and other than these don't satisfy it.
[0147]	Lec	Yes ….
[0148]	Sam	But these [the roots of the equation] are not real numbers.
[0149]	Dyn	Not real numbers.
[0150]	Sam	If so ….
[0151]	Dyn	P is a null set.
[0152]	Lec	Go ahead, you discuss it.
[0153]	Sam	So, the conclusion, it has no set, right?
[0154]	Dyn	Amm … because ….
[0155]	Lec	What do you mean it has no set?
[0156]	Dyn	P has no elements.
[0157]	Sam	Element, element. Set P has no elements.
[0158]	Dyn	Amm … set P has no elements.
[0159]	Sam	So … oh, yes, because ….

[0160]	Dyn	Then, *P* is a null set.
[0161]	Sam	Are you sure these [the roots of the equation] are imaginary?
[0162]	Dyn	For me, I am sure, because there is the square root of negative 4.
[0163]	Sam	The square root of negative 4 is imaginary, it couldn't be operated.
[0164]	Dyn	No.
[0165]	Sam	Oh, so, because these [the roots of the equation] are imaginary, then *P* ….
[0166]	Dyn	*P* has no elements.
[0167]	Sam	Oh, yes.
[0168]	Dyn	Set *P* has no elements.
[0169]	Sam	Yes. Because … because negative 2 plus the square root of negative 4 per 2 and negative 2 minus the square root of negative 4 per 2 are real numbers ….
[0170]	Dyn	Imaginary numbers.
[0171]	Sam	Imaginary numbers. Are these numbers?
[0172]	Dyn	Amm … these are ….
[0173]	Sam	These are not numbers.
[0174]	Dyn	The results are imaginary numbers.
[0175]	Lec	If it has results, what do you mean it has results?
[0176]	Sam	It means there exists results.
[0177]	Dyn	The results of the operation are imaginary numbers.
[0178]	Lec	So, what are they, actually?
[0179]	Sam	Not a number … they are, amm … the operation in which there are ….
[0180]	Dyn	Imaginary numbers.
[0181]	Sam	One of … yes. One of … an operation which has ….
[0182]	Lec	If 2 plus 3 is equal to 5 … what is 2 plus 3?
[0183]	Dyn	2 plus 3 is ….
[0184]	Sam	An operation ….
[0185]	Dyn	Amm ….
[0186]	Sam	An addition operation.
[0187]	Lec	It is said, it is equal to 5 … what is 5?
[0188]	Dyn	5 is the result.
[0189]	Sam	A number.
[0190]	Lec	5 is a number. So, what 2 plus 3?
[0191]	Dyn	2 plus 3 is … what is it?
[0192]	Sam	An addition operation.
[0193]	Dyn	A product, a product.
[0194]	Sam	A sum of 5, amm … a sum 5.
[0195]	Lec	But, what is a sum?
[0196]	Sam	The sum is the result 5.
[0197]	Dyn	A sum is ….
[0198]	Lec	What is a sum, actually? What is it?
[0199]	Dyn	It is a number.
[0200]	Lec	So, that 2 plus 3 is?
[0201]	Dyn	It is also a number.
[0202]	Lec	A number. So, what are they [the roots of the equation]?
[0203]	Dyn	A number.
[0204]	Sam	A number.
[0205]	Dyn	Yes.
[0206]	Sam	2 plus 3?
[0207]	Dyn	It is also a number because it equals 5 and 5 is a number.
[0208]	Sam	So, these [the roots of the equation] are numbers. In these numbers … there are processes.
[0209]	Dyn	The square root of negative 4 is a number or not. Is the square root of negative 2 a number or not?

[0210]	Sam	The square root of negative 2. A number. An imaginary number.
[0211]	Dyn	It means this is also a number, because the square root of negative 4 because, amm … is the square root of negative 4 a number or not?
[0212]	Sam	It is not … amm … it is a number because it is imaginary number, right? But, I am in doubt because there are processes in these [the roots of the equation].
[0213]	Dyn	The square root of 4 is also a process … 4 … the square root of 4 is also a number.
[0214]	Sam	An imaginary number.
[0215]	Dyn	The square root of 2 is also a number and the process is finding the square root.
[0216]	Sam	It means it can be called a number. An imaginary number.
[0217]	Dyn	Yes.
[0218]	Sam	An imaginary number … imaginary … or z is ….
[0219]	Dyn	z is an imaginary number, then the value of z doesn't satisfy the condition. z value doesn't satisfy this [z is a real number].
[0220]	Sam	Then, the z we obtain ….
[0221]	Dyn	The z value satisfying the first condition, doesn't fulfill the second condition, that is, z is a real number.
[0222]	Sam	So, the set of z, namely, ….
[0223]	Dyn	Not them [the roots of the equation.]
[0224]	Sam	Not these [the roots of the equation], and neither are others.
[0225]	Dyn	Because the values of z satisfying this condition [the equation] are these [the roots] only. But, these [the roots] don't satisfy this [z is a real number]. So, there is not any number as an element of *P*, because no numbers could satisfy both conditions. Any number could only satisfy either of them.
[0226]	Sam	So, *P* has no set ….
[0227]	Dyn	Has no elements.
[0228]	Sam	So, set *P* has no elements or *P* is equal to … like this [{ }, a null set].
[0229]	Dyn	Its relation with this [set *R*]?
[0230]	Sam	How could it be associated with if it has no elements?
[0231]	Dyn	It means no association, because what we can associate with are its elements. If it has no element, meaning it can't be related, right?
[0232]	Sam	Is that so? No association?
[0233]	Dyn	So, the association between set *P* and set *R* ….
[0234]	Sam	So, the association between set *P* and set *R*, they have no association.
[0235]	Dyn	Because it [*P*] has no elements which could be associated with set *R*.
[0236]	Sam	If we describe the elements of set *R*, they are 6, 7, and the null set, what do you think?
[0237]	Dyn	You mean, the null set is a subset of set *R*?
[0238]	Sam	Isn't' the null set a subset of all set?
[0239]	Dyn	Yes, but the null set has no elements. Then, if you want to write a null set, we can't also express it as 0.
[0240]	Sam	It means *P* is a subset of *R*, because the null set is a subset of each set.
[0241]	Dyn	The problem, the questions is the association between the elements of two sets. It means the elements only.
[0242]	Sam	It means *P* is a member of *R*.
[0243]	Dyn	No. This is about the association between the elements of the two sets. So, if this [*P*] has no elements, this [*R*] has elements, how do we relate them?
[0244]	Sam	Oh, association of their elements, right? It means the association of the elements of the two … two sets doesn't exist.
[0245]	Dyn	Yes.

[0246] Lec OK. Just write it.

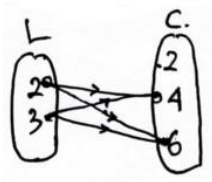
[0247] Sam So, the association between *P*'s elements and *R*'s elements doesn't exist because *P* …. Like that?

[0248] Dyn Yes, done.

[0249] Sam Let *A* and *B* be two non-empty sets, *k* is an element of *A*, and *w* is an element of M. If we associate *k* with *w*, then the result is written in the form *k* comma *w*, and it is considered as the member of the association.

[0250] Dyn The elements of *L* are 2 and 3. The element, that is, 2, has relation with ….

[0251] Sam It means … yes, right … it means 2 sets, right? We relate each … set *L* to set *C* which is greater, which the elements of *L* are greater than those of *C*. It means 2 goes to 4 and 6, 3 … gets 4 and 6. Right?
(Drawing an arrow diagram)



[0252] Dyn Could we use other than an arrow diagram?

[0253] Sam Is the relation like that?

[0254] Dyn The direction of the arrows.

[0255] Sam The relation is like this, or could be written in ordered diagram, something ….

[0256] Dyn Could be written in the form of *L* comma *C* amm … *k* comma *w*.

[0257] Sam Then, set *C*, that is, 2 comma 4, 2 comma 6, 3 comma 4, and 3 comma 6.
(Writing)
If it is written in the form (k,w), then the relation between set L and set C results in (2,4), (2,6), (3,4), (3,6).
Like this?
Write all results of the mapping, if each element of *C* is associated with elements of *L* which are less than those of *C*.

[0258] Dyn It means 4 comma 2, 4 comma 3, 6 comma 2, 6 comma 3.

[0259] Sam Oh, yes. Just go with a diagram like this.
(Writing a list of ordered pairs)
The relation between set C and set L, that is, (4,2), (4,3), (6,2), (6,3).

[0260] Dyn Are the results in Part a the same as those in Part b.

[0261] Sam What? Explain your reason!

[0262] Dyn To me, they are different.

[0263] Sam They are different, right? Different, because of the condition of its association. In Part a, it associates elements of *L* with elements of *C* whose values are greater than those of *L*. In Part b, elements of *C* to elements of *L* whose values are less than those of *C*. It means they are different.

[0264] Dyn Different. The types of associations are different.

[0265] Sam Because the types of associations, the types of associations from set *L* to *C* and *C* to *L* are different.

[0266] Lec OK. Done.

[0267] Sam (Reading the question) Write all results of the association where each element of *L* ….
Set *L* is 2, 3. For *C*, it is 2, 4, and 6. It means, the elements of *L* to all, right? So, 2 to 2, 2, to 4, 2 to 6, then 3 to 2, because all are even, right?

[0268] Dyn The elements of *C* are all even.

[0269] Sam If each element of *L* is associated with elements of *C* which are even numbers … so, it could be written … the association of the elements of *L* with the elements of *C*, that is, 2 comma 2, 2 comma 4, 2 comma 6, 3 comma 2, 3 comma 4, and 3 comma 6.

[0270]	Dyn	Compare the results you got from Parts a, b, and d. What is unique in the results in Part d? The unique, that is, because the association in d ….
[0271]	Sam	What is unique?
[0272]	Dyn	It [d] covers all.
[0273]	Sam	All elements of *A* have partners …. How? What is unique?
[0274]	Dyn	What is unique in d is all elements of the sets have a partner either in *L* or in *C*. All elements of *L* and *C* have a partner.
[0275]	Sam	Oh, yes, yes. In the results of association in d, each element of *L*, what?
[0276]	Dyn	Elements of set *L* and elements of set *C* have a partner.
[0277]	Sam	While, in Parts a and b ….
[0278]	Dyn	Not all elements have a partner.
[0279]	Sam	Let's group the results of association in a and d.
[0280]	Dyn	If it is written like this, 4 comma 2 and 2 comma 4, they are different, right?
[0281]	Sam	Different, different. For this [(2,4)], its origin point is 2; for this [(4,2)], its origin point is 4. The origin points are different.
[0282]	Sam	Different, different. For this [(2,4)], its origin point is 2; for this [(4,2)], its origin point is 4. The origin points are different.
[0283]	Lec	You could write it here. I would like to ask you something. Set *A* has its elements 1 and 2. Set *B* has 2 and 4. Now, the first association, the elements of *A* are factors of the elements of *B*. What are the results?
[0284]	Dyn	It means, for 2 in *B*, it gets 1 and 2, for 4, it gets 1 and 2 also.
[0285]	Sam	Are they factors?
[0286]	Dyn	Yes. The factors of 2 are 1 and 2.
[0287]	Sam	Is it? 2, 1 and 2?
[0288]	Lec	The results of the association, what are they? You just write them down here.
[0289]	Sam	1 comma 2.
[0290]	Dyn	2 comma 2.
[0291]	Sam	1 comma 4. Isn't it 1 comma 4? For 4, 2 only?
[0292]	Dyn	Just write 2 comma 4 first.
[0293]	Sam	2 comma 2?
[0294]	Dyn	2 comma 2 also.
[0295]	Sam	All them? All them?
[0296]	Dyn	The elements of *A* are factors of the elements of *B*. Which one is the origin, which one is the end?
[0297]	Sam	The origin is *A*. It is clear that *A* is the origin. The end is *B*.
[0298]	Lec	*A* is factor of *B*, so *A* is connected to *B* in the form of the elements of *A* are factors of the elements of *B*, it is related to its factors.
[0299]	Dyn	1 comma 2, 1 comma 4, 2 comma 2, 2 comma 4. OK. That is all, Sir.
[0300]	Lec	Now, the second association. The elements of *A* are less than the elements of *B*.
[0301]	Dyn	1 is less than 2, 1 is less than 4, 2 is less than 2, 2 is less than 4.
[0302]	Sam	2 is less than 4, there is not 2 less than 4 because they are the same.
[0303]	Dyn	Oh, yes, yes,
[0304]	Lec	Done. The third association. Less than or equal to.
[0305]	Sam	Like this. 1 comma 2, 1 comma 4, 2 comma 2, 2 comma 4.
[0306]	Lec	Is that all? Yes. Now, compare the results of association 1 and 3. Are they equal?

[0307] Dyn The results are the same, but the types of association are different.
[0308] Lec OK, what is the question here, number 2 Part c?
[0309] Dyn The result in Part a and the result in Part b, are they the same?
[0310] Lec Are they the same or not?
[0311] Sam Oh, the results, not the types.
[0312] Dyn If so, they are different, because the origin point ….
[0313] Sam The origin set and the end set are different.
[0314] Dyn The origin set in the association a is 2 amm … in *L*, the origin are the elements of set *L*, while for b, the origins are the elements of set *C*. So, the origin points are different, their end points ….
[0315] Sam The origin set and the partner set different. Is it like that?
[0316] Lec It means you still look at these again, *L* and *C*.
Now, you just look at the results.
[0317] Dyn 4 comma 2, 2 comma 4 ….
[0318] Lec Here is it asking about the results. It is not about whether the origin sets are the same or different, how the rule of association is, it just asks you to compare the results.
[0319] Dyn If they could be inverted, the same.
[0320] Sam Of course they are different. 2 comma 4, 4 comma 2.
[0321] Dyn But the relationship just involves 2 and 4. I can also invert them.
[0322] Lec So, could this [(4,2)] be written here as 2 comma 4?
[0323] Sam To me, 2 comma 4 couldn't be written as 4 comma 2 because this writing shows which one is the origin, which one is the result. So, we couldn't write 4 comma 2 as 2 comma 4, because if we do it … orderly, the position shows which one is the origin, which one is the result.
[0324] Dyn But, in the end, they are the same. The result, they are just the same.
[0325] Lec Which ones are the same?
[0326] Dyn The association between 2 and 4 … if it is an association between 4 and 2 …, oh, different.
[0327] Sam Different, because, for example, like this, for example, *C* to *L* where the association is less than … the elements of *L* which are less than the elements of *C*. It means, 2 is less than 4. If it is inverted, 4 is less than 2, it is wrong.
[0328] Dyn They are not the same.
[0329] Sam Not the same, right? So?
[0330] Lec So, why?
[0331] Dyn They remain different.
[0332] Lec OK.
[0333] Dyn For these [relation 1 and 3], they are the same, because the results are the same, the expressions are the same.
[0334] Lec But the association is different. What are you asked to see?
[0335] Dyn The results.
[0336] Lec Compare the results.
[0337] Dyn Here [relation 1 and 3], the results are exactly the same, 1 comma 2 … 1 comma 2, here [relation 2], they are inverted.
[0338] Lec These results are exactly the same. You see the results, right? So, when you are asked to see the results, just see the results. Now, you look at these all results [relation1], and these all [relation2]. Are the results the same?
[0339] Sam Different.
[0340] Lec Then, just now, you suddenly referred back to the sets involved, the rule of the association. Here, it is only the results needed. For the associations, I don't' ask whether they are different. What is interesting is these [the results]. Because some of them look being inverted. The problem is, do

you consider 2 comma 4 the same as its inversion 4 comma 2? This is the question, are they the same, if 2 comma 4 is inverted into 4 comma 2?

[0341] Sam 2 comma 4 is different from 4 comma 2.
[0342] Lec How is it Dyn?
[0343] Dyn Hahaha … 2 comma 4 … 4 comma 2. The results? For the results, the same.
[0344] Lec What are the same?
[0345] Dyn The relation of 2 and 4, it means between 2 and 4. This [(2,4)] is 2 and 4 … this [(4,2)] is the same, 2 and 4.
[0346] Lec Now, what about the association?
[0347] Dyn For this [(2,4)], the association is from 2 to 4, and for this [(4,2)] it is from 4 to 2.
[0348] Lec Yes.
[0349] Sam 2 to 4 and 4 to 2 are different.
[0350] Dyn Oh, yes. Different … hahaha.
[0351] Sam 2 goes to 4, 4 goes to 2.
[0352] Dyn 2 to 4, it is 2 less than 4 … 2 to 4, it is 2 plus 2, for 4 goes to 2, it is not 4 plus 2 but 4 negative 2. They are different.
[0353] Sam So, they are different.
[0354] Dyn If so, they are different.
[0355] Sam Are they different? To me, they are.
[0356] Lec Why?
[0357] Sam Because 2 to 4, 4 to 2, different.
[0358] Lec What is different?
[0359] Sam The written expression.
[0360] Lec Besides that?
[0361] Sam The relations for 2 to 4 and 4 to 2 are also different. If we associate 2 to 4, it is different from associating 4 to 2. It could be that 2 is less than 4, but 4 is not less than 2. So, they are different.
[0362] Lec You finish, right? Now, move to the next item.
[0363] Dyn Compare the results you get in Parts a, b, and d. What is unique in the results in Part d compared to the results in Parts a and b.
[0364] Sam What is unique in the results?
[0365] Dyn In Part d, each element of set *C* and set *L* all them have partners. If Part a, set *C* has an element that has no partner; all elements of *L* have a partner. In b, for set *C*, not all elements have a partner. So, only in Part d, all elements have a partner.
[0366] Sam The results in d show each element of *L* ….
[0367] Dyn Let's group the results in a and b into 2 different sets, respectively. Consider the two set we use earlier, *L* and *C*. What relation could we identify between the two sets?
Write all elements of the set in Part a.
[0368] Sam Let's group the results in a and b into 2 different sets, respectively.
[0369] Dyn Capital letter.
[0370] Sam So, set *A* ….
2 comma 4, 2 comma 6, 3 comma 4, 3 comma 6.
And, *D*, 2 comma 2, 2 comma 4, 2 comma 6, 3 comma 2, 3 comma 4, 3 comma 6.
(Writing sets)
$A=\{(2,4), (2,6), (3,4), (3,6)\}$
$D=\{(2,2), (2,4), (2,6), (3,2), (3,4), (3,6)\}$
What relation could we identify between the two sets?
[0371] Dyn What I see is all elements of *A* are in *D*.
[0372] Sam So, *A* is a subset of *D* … right?

[0373]	Dyn	And for their relation. The relation is between elements of *A* which are the same as elements of *D*.
[0374]	Sam	What? The relation of elements of *A* which are the same as elements of *D*?
[0375]	Dyn	Yes, elements of *A* which are the same as elements of *D*.
[0376]	Lec	What is asked there?
[0377]	Sam	The relation. What relation we could identify between the two sets.
[0378]	Dyn	It means, *A* is a subset of *D*. Each element of set *A* is an element of set *D*.
[0379]	Sam	So, *A* is a subset of *D*.
[0380]	Dyn	*A* is a subset of *D*. It means set *A* is a subset of *D*.
[0381]	Sam	So … the relation is *A* is a subset?
[0382]	Dyn	*A* is a subset of *D*.
[0383]	Sam	The relation of *A* to *D*, that is, *A* is a subset of *D*.
[0384]	Dyn	If *A* is a subset of *D*, then, certainly, its set is *D* itself.
[0385]	Lec	How?
[0386]	Dyn	If *A* is a subset of *D*, each element of *A* is in set *D*, then the set, certainly, is set *D* itself. All elements are in *A* and *D*, it is set *D*.
[0387]	Lec	What do you mean?
[0388]	Dyn	The set in *A* is in *D*, and the set in *D* is in *D*, then set is a set of *A* and *D*, *A* union *D*.
[0389]	Sam	Oh, so, what set is it? For this set, the set is *A*, that is, *A* union *D*. Like this? (Writing) The set is *A* union *D* equals *D*
[0390]	Lec	What is *A* union *D*?
[0391]	Sam	*D*.
[0392]	Dyn	*A* union *D* results in elements, all elements are in *D*.
[0393]	Lec	Yes. Now, what is unique in *D*?
[0394]	Dyn	In *D*, all elements of set *C* and *L* have a partner.
[0395]	Lec	OK. The mathematical term of *D* is the Cartesian product. You could write it.
[0396]	Sam	Sorry, Sir?
[0397]	Lec	*D*. *D* is a set, and it is called the Cartesian product.
[0398]	Sam	*D* is the Cartesian product.
[0399]	Lec	Yes. It could be written *D* equals *L* times *C*. That is the symbol of the Cartesian product. So, for the Cartesian product of *L* and *C*, *L* times *C* and *C* times *L*, they are different. In this case, because if it is from *L* to *C*, then we write *L* times *C*, which results in this [*D*]. It pairs all off ….
[0400]	Sam	All elements of *L* with all elements of *C*.
[0401]	Lec	It is called the Cartesian product. Now, what is the Cartesian product?
[0402]	Sam	The results relating ….
[0403]	Dyn	The results relating all elements ….
[0404]	Sam	Relating each element in ….
[0405]	Dyn	Each element of *L* ….
[0406]	Sam	To each ….
[0407]	Dyn	To each element of *C*. Each element of a set to another set.
[0408]	Lec	So, what is the Cartesian product?
[0409]	Sam	The results of relating each element of set *A* to each element of set *B*.
[0410]	Lec	About the results of relating *L* to *C*, what are they?
[0411]	Dyn	They are not a set yet.
[0412]	Lec	So, what is the Cartesian product?
[0413]	Dyn	A set covering all … all elements … amm … all elements of a set amm ….
[0414]	Sam	All elements of set *A* to each element of set *B*.
[0415]	Dyn	Relating … it is a set of pairs of each element of *A* to each element of *B*.
[0416]	Lec	The Cartesian product, what is it?

[0417]	Dyn	The Cartesian product … amm … *A* times *B* … what was it?
[0418]	Sam	The product between set *A* and set *B*.
[0419]	Dyn	The Cartesian product is a set of pairs of each element of a set to each element of another set.
[0420]	Sam	The Cartesian product is a set … a set of pairs of each element of set *A* ….
[0421]	Dyn	Each element of a set to elements of another set.
[0422]	Sam	To each ….
[0423]	Dyn	Each element of another set.
		Why did you use *A* and *B* again? When you use them, it means this is not the Cartesian product in general, but the Cartesian product *A* times *B*.
[0424]	Sam	Elements of another set.
[0425]	Lec	Next, if you want to define it, what is the Cartesian product? You define it symbolically.
[0426]	Dyn	Can we use letters as symbols?
[0427]	Lec	The request is just a symbolic definition.
[0428]	Dyn	Just write it down. The Cartesian product is equal to *x* where *x* is a pair of each element of *A* and *B* amm …
[0429]	Sam	We should use set *X*, shouldn't we?
[0430]	Dyn	You got me wrong. What I mean is the Cartesian product is equal to ….
[0431]	Sam	You write it.
[0432]	Dyn	(Writing)
		$A \times B = \{x \mid x$
[0433]	Sam	In the form of a sentence, right?
[0434]	Lec	Symbolically.
[0435]	Sam	The sentence is in?
[0436]	Dyn	Because that couldn't be symbolized. Hahaha ….
[0437]	Lec	Which one couldn't be symbolized?
[0438]	Dyn	If *x* is a real number, for this, the symbol is *R*. But, if *x* is a pair of each element, how to symbolize it?
[0439]	Lec	If you look at the introduction of the question, what information could you find in it? Do you understand it?
[0440]	Dyn	Oh, yes, yes. *k* comma *w*.
		How could we symbolize them all? For each *k* comma *w* is a pair in *A* and *B* ….
[0441]	Sam	*x* is equal to … where *x* is *k* comma *w*, for each *k* comma *w* is a pair of each element ….
[0442]	Lec	Then, what is the Cartesian product?
[0443]	Dyn	A set of pairs of each element.
[0444]	Dyn	It means this [*x* in the expression $A \times B = \{x \mid x$ …] is *k* comma *w* where that is … or do you want to use other symbols?
[0445]	Sam	Just *k* comma *w*.
[0446]	Lec	What is it? What?
[0447]	Dyn	This is a pair of ….
		Oh, yes … correct, I got it.
		Where *k* is … for each element *k* of *A* … for each element *k* of *A* ….
[0448]	Lec	OK. Go on.
[0449]	Dyn	*k* comma *w* where for each *k* element of *A* and for each ….
[0450]	Sam	For each element *k* of *A*?
[0451]	Dyn	Yes … and for each *w* element of *B*.
[0452]	Sam	It is for all, right? For each, right? If for each element *k* of *A* ….
[0453]	Dyn	*k* is an element of *A*, meaning all elements of *A*.
		Then, *w* is also for all elements of *B*.

[0454] Sam (Writing)
The Cartesian product $A \times B = \{(k,w) | \forall k \in A$ and $\forall w \in B\}$
Each element *k* … element *k* of … each element of *A*. How is it?

[0455] Dyn For each element *k* of *A*, correct.

[0456] Sam Hang on. We need to question this.

[0457] Lec What are you questioning?

[0458] Dyn And for each element *w* of *B*.

[0459] Sam If for each element *w* of *B*, it means there are still elements of *B* which are not include here?

[0460] Dyn No.

[0461] Lec What do mean? What is the meaning of for each?

[0462] Dyn For each means all *B* included … it could replace amm … all *B*, it could be *w*. Hahaha.

[0463] Lec What happens with you, Sam?

[0464] Sam Sorry. If for each element *w* of *B* meaning all *B* is represented by this *w*.

[0465] Lec What do you thing the meaning of for each?

[0466] Dyn They could not be all included … I mean they could not be included at once, but they just could be included, not at once.

[0467] Lec What do you mean?

[0468] Dyn All them … sorry, confused. Hahaha.

[0469] Sam Because this Cartesian product is a set of the pairs of each element of a set off with each element. It means all elements of *A* are paired off with all elements of *B*.

[0470] Dyn So, if *k* is an element of *A* and *A* has 2, 4, 6, meaning *k* could be 2, 4, or 6, right?

[0471] Sam Yes, OK, yes.

[0472] Dyn It is similar with *w*. If *w* is an element of *B*, and *B* has 1, 2, 3, meaning *w* could be 2, could be 2, or 3. Get it? Yes?

[0473] Sam Yes, yes, I understand … I understand.

[0474] Dyn Finished.

[0475] Lec So, what is the relationship between *A* and *D*?

[0476] Dyn For *A* and *D*, each element of *A* is in *D*.

[0477] Lec Then, what is the relationship?

[0478] Sam *A* is a subset of *D*.

[0479] Dyn *D* is the Cartesian product *L* times *C*.

[0480] Lec *D* is the Cartesian product. According to mathematics concept, because *A* is a subset of *D* while *D* is the Cartesian product of two sets, that is, *L* and *C*, so *D* = *L* times *C*, then *A* is called a relation from *L* to *C*.

[0481] Sam (Writing)
Because $A \subseteq D$ and D is the Cartesian product $L \times C$, then A is a relation from L to C.

[0482] Lec Then, what is a relation? What is its definition?

[0483] Sam A relation from *L* to *C*.
Because *A* is a subset of *D* and *D* is the Cartesian product *L* times *C*, then *A* is a relation from *L* to *C*.

[0484] Dyn A relation is Part of ….

[0485] Lec Maybe, my question is not clear enough. What is actually a relation from *A* to *C*?

[0486] Dyn A relation of *L* to *C*.
A relation is a Part of the Cartesian product *L* times *C* or a relation is some of pairs in the Cartesian product *L* times *C*.

[0487] Sam So, a relation is just some pairs?

[0488] Dyn Because *A* is a subset of *D*, right?

[0489]	Sam	Some pairs relating *L* to *C*.
[0490]	Lec	If you just say pairs, you miss the essence … a relation is a set.
[0491]	Dyn	A relation from *L* to *C* is a set of some pairs of the Cartesian product *L* times *C*.
[0492]	Sam	(Writing) A relation from L to C is a set of some pairs of the Cartesian product.
[0493]	Lec	Now, what is the definition of a relation from *L* to *C*?
[0494]	Sam	A set of some pairs ….
[0495]	Dyn	Could we write it symbolically, Sir?
[0496]	Lec	Yes.
[0497]	Dyn	A relation is …. A relation of *L* to *C* is equal to ….. If you want to use *A* times *B*, it would be just the same.
[0498]	Sam	Relation *A* from *L* to *C* … like this? It is equal to … is equal to?
[0499]	Dyn	*k* comma *w* where … some ….
[0500]	Sam	Where some ….
[0501]	Dyn	Some elements *k* of *A* or ….
[0502]	Sam	Should we say 'or'?
[0503]	Dyn	Yes.
[0504]	Sam	Or some elements *w* of *B*?
[0505]	Lec	Why do you use 'or'?
[0506]	Dyn	Because, to me, it could be that here [*A*] some of its elements have no partner or it could be some of the elements of here [*B*] have no partner as well.
[0507]	Lec	How could you pair them off if there is nothing in there [*B*]?
[0508]	Sam	'And' ... because there are only some, it could be 1, could be 2.
[0509]	Dyn	Oh, yes. It is right, 'and' ….
[0510]	Sam	If we use 'or', it means … it means it could be only this [$k \in A$] ….
[0511]	Dyn	Yes, could be this [$k \in A$] only, could be this [$w \in B$] only
[0512]	Lec	If it is for that [$k \in A$] only, what would be the result?
[0513]	Dyn	There would be no partner. It means it doesn't result in a pair. It means, we have to use 'and'.
[0514]	Sam	(Writing) Relation A from C to L $= \{(k,w) \mid \exists k \in A \text{ and } \exists w \in B\}$.
[0515]	Lec	OK.
[0516]	Sam	You have named and define sets containing the results of the association on Parts a and b. Could you identify the commonalities of the two sets? Their common properties.
[0517]	Dyn	The common properties of a and d.
[0518]	Sam	It is different if we say relation.
[0519]	Dyn	It is just the same. The common properties … it means both of them are sets.
[0520]	Sam	It could also be that *D* is a relation from *L* to *C*.
[0521]	Lec	How could it be?
[0522]	Sam	Because *D* relates *L* to *C*.
[0523]	Lec	If it relates, must it be a relation? What was the definition of a relation?
[0524]	Sam	A relation is some ….
[0525]	Dyn	Just some.
[0526]	Sam	If you say some, I could also mean for all. It won't be wrong if you take all.
[0527]	Lec	If you look back to some of these sentences, what is a relation, actually?
[0528]	Dyn	A set of pairs of Cartesian product. Because *A* is a subset of *D*, and *D* is the Cartesian product, then *A* is a relation from *L* to *C*.
[0529]	Lec	If, so, what is a relation from *L* to *C*, actually?

[0530] Dyn A set of some pairs.

[0531] Lec Just look at here. *A* is a subset of *D*, *D* is the Cartesian product. Therefore, *A* is called a relation from *L* to *C*. Because *A* is a subset of *D* which is the Cartesian product *L* times *C*. Now, what is a relation?

[0532] Sam A set connecting ….

[0533] Lec You have said that.

[0534] Sam (Writing)

A relation from L to C, that is, $\subseteq$ of the Cartesian product $L \times C$.

[0535] Lec OK, now, is *D* also a relation?

[0536] Dyn Not, because it is not a subset … amm ….

[0537] Lec Think about it again.

[0538] Dyn It is also a relation.

[0539] Lec Because?

[0540] Dyn Because *D* is also a subset of *D*.

[0541] Lec Now. You move to the question.

[0542] Dyn It means that the common property is *A* and *D*.

[0543] Sam You have name and define the sets of the results of the association of *A* with *D*. Could you find the common property of the two? The common property is ….

[0544] Dyn *A* and *D* are relations from *L* to *C*.

[0545] Sam *B* is a relation from *C* to *L*.

[0546] Lec Now, is *B* also a relation?

[0547] Dyn *B* is also a relation, but *B* is a relation from *C* to *L*, not from *L* to *C*.

[0548] Lec You go back to their origins. What is the essence, if they are relations,
Sam what is the difference between the two relations? Relations *A* and *D*, what makes them different from *B*?

(Writing)

A is a relation

$A = \{(2,4), (2,6), (3,4), (3,6)\}$

$D = \{(2,2), (2,4), (2,6), (3,2), (3,4), (3,6)\}$

$B = \{(4,2), (4,3), (6,2), (6,3)\}$

[0549] Lec Yes. For *A*, it a relation from what to what?

[0550] Sam *A* is a relation from *L* to *C*.

[0551] Lec You may write it at the end of each *A*. And, for the other sets.

[0552] Dyn *L* to *C* … that is … *L* to *C* also.

[0553] Lec The below one?

[0554] Sam *C* to *L*

(Writing)

$L = \{2, 3\}$

$A = \{(2,4), (2,6), (3,4), (3,6)\}$ (L to C) $C = \{2, 4, 6\}$

$D = \{(2,2), (2,4), (2,6), (3,2), (3,4), (3,6)\}$ (L to C)

$B = \{(4,2), (4,3), (6,2), (6,3)\}$ (C to L)

[0555] Lec OK, now look at them. What is unique? What is the difference between both *A* and *D* and *B*?

[0556] Dyn *A* and *D* is a relation from *L* to *C* and *B* is a relation from *C* to *L*.

[0557] Lec Is that the difference?

[0558] Dyn *B* is a subset of *D* or not?

[0559] Sam Is *B* a subset of *D*? Not.

[0560] Dyn No, it is not. *A* and *D* are subsets of *D*, but *B* is not a subset of *D*.

[0561] Lec Besides that, if you think again … what is the unique property in *A* and *D*, which can’t be found in *B*, *B* doesn’t have that uniqueness? By taking into account that sets involved in relation *A*, that is, *L* as the first set and *C* as the second set. It similar to *D*. For *B*, the first set is *C* and the second set is *L*. And, you got the relations. What is unique in *A* and *D*, which is not in *B*?

[0562] Sam The form is ….

[0563] Dyn For relation *A*, it is *k* comma *w*, but for *B*, it is not *k* comma *w*, it is *w* comma *k* instead.

[0564] Lec Yes, it is because for *B*, it is from its domain *C* to *L*, so you look at these [*A* and *D*] now together, and then look at this [*B*] which is from *C* to *L*, what is unique in them [*A* and *D*]? It belongs to this [*A*] and this [*D*], but it is not here [*B*]. The one that you mention is only an order of the elements. You have to focus on the properties of the relations by taking into account their domain and codomain, respectively.

[0565] Sam Oh … the domain and codomain.

[0566] Lec One again, don't look at both *A* and *D* with their domain *L* and range *C*, while *B* has the opposite. No. It is not the one. Find a specific, a holistic property.

[0567] Sam *A* is a subset of the Cartesian product, and *B* is not a subset of the Cartesian product.

[0568] Lec If you take the Cartesian product *C* times *L*?

[0569] Dyn Then, you got that.

[0570] Sam Oh, yes, I got it … got it.

[0571] Dyn For *A* and *D*, all elements of the domain have partners; for *B*, not all elements have partners.

[0572] Lec Not all what?

[0573] Dyn The elements of the domain. For *A* and *D*, we focus on the elements of the domain, that is, *L*, they have partners. For *B*, not all them … there is 6 which has no partner.

[0574] Sam Not 6.
It is 2 … 2 which has no partner in *B*, it is from *C* to *L*.

[0575] Dyn For *A* and *D*, each element of ….
Each element of the domain has a partner; for *B*, only some of elements of the domain have partners.

[0576] Sam Yes, all them. 2 and 3 have partners. But their partners ….

[0577] Lec What do you mean?

[0578] Sam Each element of the domain *L* has a partner in *A* and *D*. For *B*, the domain *C*, not all elements have partners.

[0579] Lec OK. So you have seen the essence, something beyond the order of *C* and *L* in the sets.

[0580] Sam For *A* and *D*, each element of their domain has partners in codomain … while for *B*, not all elements of its domain have partners.

[0581] Lec Now. That is unique property, then it is called a special relation. The special relation is an exhaustive domain relation.

[0582] Dyn Are those *A* and *D*, sir?

[0583] Lec Yes, they are unique, exhaustive relations, their domains are exhaustive, right? Now, how do you define such a property?

[0584] Dyn It seems similar to the Cartesian product, because each element of *A* … amm … oh only the domain. It means, it could be some, some elements *w* of *B*.

[0585] Sam And for the other … for *k* … for each *k*.

[0586] Dyn For each element *k* of *A*. Yes.

[0587] Sam *A* and *D* are special relations.

[0588] Dyn A special relation …..

[0589] Lec You don't need to say *A* and *D* anymore, but now for a special relation like *A* and *D*, how do you define it?

[0590] Sam A special relation from *A* to *B*, *k* comma *w* where for each element *k* of *A* and there exist an element *w* of *B*.
(Writing)
A special relation A to B $= \{(k,w) | \forall k \in A \text{ and } \exists w \in B\}$.

[0591] Dyn OK.

[0592] Lec OK. You move to the next item. To be more focused, what is common, want is unique again, which you could see in both i and ii. You might just ignore all the accessories, like the ones that you firstly identify in the previous problem, when you examined the uniqueness in *A* and *D*, but not in *B*. Focus on the essence. Now, what is the uniqueness belonging to both i and ii?

[0593] Sam Together?

[0594] Lec To be more focused, you just directly answer the question. What is the uniqueness belonging to both i and ii?

[0595] Dyn In both, only several elements …..

[0596] Lec What do you mean?

[0597] Sam So, we work on both pictures, Sir? Not Picture i, and then Picture ii?

[0598] Lec Yes. So, it is the same when you tried to identify the uniqueness of *A* and *D* which distinguished them from *B*. Here, you are to find the uniqueness which belongs to both. Just now, you did the same approach to *A* and *D*, which both had a common property that … all elements of the domain had partners.
Now, how about pictures i and ii? What unique property is in both pictures i and ii?

[0599] Dyn Only several elements of the codomain which have partners, because for this [codomain in Illustration (i)], it is for each element, for this [domain of in Illustration (ii)] some elements. Then, for this [codomain in Illustration (i)] some elements, for this [codomain in Illustration (ii)] also some elements. It means, in the codomains, only some elements having partners.

[0600] Sam The special property?

[0601] Dyn Do you see other things? Yes, only that.

[0602] Lec Is there any other, besides that property?

[0603] Sam That is 4 [the number of elements of the domain in Illustration (i) which have partners], that is 3 [the number of elements of the domain in Illustration (ii) which have partners].

[0604] Lec That could be considered just as accessories, if you turn to focus on that. Could you find an essential property in the relations?

[0605] Dyn The elements of *X* that have partners, have only 1 partner.

[0606] Sam 1 … 1 ... 2. There are two in this [domain].

[0607] Dyn I mean, these [elements of the domain of relation (i)] have only one partner.

[0608] Sam Each element of *X* which has partners, has only 1 partner.

[0609] Dyn Has only 1 partner.

[0610] Sam Each element of *X* which has partners, has only 1 partner in *Y*.

[0611] Lec Now, how is its definition? This is also a special relation.

[0612] Dyn Amm … OK. Amm …. Definition of special relation … what is its name … its name?

[0613] Lec For the previous one in Part g, its specialty is the all elements of domain are paired off. In Part h, you have seen in Part h that each element of set *X* which has partners, has only one partner in set *Y*.

[0614] Dyn How to make it like this [definition in Part g]?

[0615] Sam It is almost similar to this.

[0616] Dyn No, no.

[0617]	Sam	It is different, different. Because there is also … some elements of *A*. Oh, yes, yes, I got it … a relation ….
[0618]	Dyn	What is the symbol for only 1?
[0619]	Lec	Only one.
[0620]	Dyn	Only one element b.
[0621]	Lec	What are thinking about that?
[0622]	Dyn	Its partner.
[0623]	Lec	The partner is only 1. Yes, it is interesting. How to show or write it when its partner is only 1?
[0624]	Sam	It is only *k*, right? Because *w* could be ….
[0625]	Dyn	Amm … *k* is only 1 … *k* is only 1.
[0626]	Lec	Remember, if *k* is only 1, how to show that *k* is only 1?
[0627]	Dyn	It means, we don't use for each element *k* of ….
[0628]	Sam	There exists, there is, there is.
[0629]	Lec	There exists, it doesn't necessarily mean only 1.
[0630]	Dyn	*k* is only 1.
[0631]	Sam	*k* is only 1 … *k* is only 1.
[0632]	Lec	Do you remember my story of something existing which is only one?
[0633]	Sam	We have to take another *k*. We have to take another same thing ….
[0634]	Dyn	Take another thing, but the result is just the same.
[0635]	Lec	Yes, go on with that. How to say it?
[0636]	Sam	*k* is an element of ….
[0637]	Dyn	It means *k* only, *k* only.
[0638]	Sam	What do you mean by *k* only?
[0639]	Lec	Only 1 *k*, this is the one we want to symbolize. But, it is quite difficult to write directly that there is only 1 *k*, because that 'only' is still unclear. How to express something that it is only 1? You have said that, how?
[0640]	Dyn	Haha… I got it. I got it.
[0641]	Sam	What? What?
[0642]	Dyn	*k* is equal to … *k* is equal to … what was that?
[0643]	Sam	*k* equals *k*?
[0644]	Dyn	No.
[0645]	Sam	*k* is not more than 1?
[0646]	Dyn	*k* … *k* is equal to … where was it … *k* is only 1 in *A*.
[0647]	Sam	*k* doesn't appear twice.
[0648]	Dyn	No, how to write it amm ….
[0649]	Sam	What? In a sentence? Amm ….
[0650]	Lec	In a sentence. What is *k* actually?
[0651]	Dyn	*k* is an element of *A*, but *k* must be only 1 amm ….
[0652]	Lec	What is *A*? The first set or the second set?
[0653]	Sam	The first.
[0654]	Lec	Which one is only one?
[0655]	Dyn	An element of *A*. One of the elements of *A*.
[0656]	Lec	What do you mean? Which one is only 1?
[0657]	Sam	*k*.
[0658]	Lec	*k* … what is *k*?
[0659]	Dyn	*k* is an element of *A* amm ….
[0660]	Lec	Here. Is it only 1 here? What is *k*, actually? You should look at this first, your previous answers.
[0661]	Dyn	*k* is a partner of *w*.
[0662]	Lec	*k* is a partner of *w* ... where is *w*?
[0663]	Dyn	An element of *Y*.
[0664]	Lec	An element of *Y*. The second set. Then, *k* is only 1? How about *w*?

[0665]	Dyn	There are some *w*'s.
[0666]	Sam	It could be all them. Yes, it could be several.
[0667]	Lec	What is it which could be some?
[0668]	Sam	Some of them have partners in *X*. How is it?
[0669]	Dyn	*w* is an element … an element of set *Y*.
[0670]	Lec	Yes, and then? What about *k* as an element of *X*? In this case, which one is only one, *k* or *w*?
[0671]	Sam	*k*. *k* is the one which is the only one. Set *X* has only one partner in set *Y*.
[0672]	Lec	So, where is that which is only one?
[0673]	Sam	In *Y*.
[0674]	Dyn	In *X*. It is an element of *X*.
[0675]	Lec	You say that there is only one partner. Think about it again.
[0676]	Sam	Set *X* has only one partner in set *Y*. It means the partner is in *Y*.
[0677]	Dyn	It is *y* as the only one, it means the *y* ….
[0678]	Lec	If it is only one, what is its symbol, *k* or *w*?
[0679]	Sam	Has only one partner in set *Y*. It means each element of set *X* which has a partner, has only one partner in set *Y*.
[0680]	Lec	If so, where is the one considered as the only one? Is it in *X* or in *Y*?
[0681]	Dyn	In *Y*.
[0682]	Sam	In *Y*, its partner.
[0683]	Dyn	In *w*. Yes, in *w*.
[0684]	Sam	Why it is in *w*?
[0685]	Dyn	Because here … for each element *k* of *X*, meaning that the *k* could be 1, 2, 3, 4, right? For the *w*, there is only one here. It means *k* comma *w* … if here is 1, 2, 3, 4, it means that here is 1 comma *w*, 2 comma *w*, 3 comma *w*, 4 comma *w*. How to define it?
[0686]	Lec	Here, could you clarify, which is one only one?
[0687]	Dyn	*Y*.
[0688]	Lec	What is *Y*?
[0689]	Dyn	*Y* is a partner of …. *Y* is an element of set of … amm ….
[0690]	Lec	What is only one?
[0691]	Dyn	The partner of *X*.
[0692]	Lec	What did you symbolize there?
[0693]	Dyn	*w*.
[0694]	Lec	It means it is an element of *Y*. So, it is not *Y* which is only one. No. But, its element which is only one. Now, how to show that the partner of *k*, that is, *w*, which is in *Y*, which is in *Y*. Right. The partner of *k* is *w*. So, what is the partner of *k*?
[0695]	Dyn	*w* is in *Y*.
[0696]	Lec	*w* is in *Y* … *w* is only one.
[0697]	Dyn	It means *w* equals *Y*. It means, only for each element *k* of *A*. That is it, *w* equals *Y*.
[0698]	Lec	Why could *w* equals *Y*, whereas *w* is an element of *Y*? How to show, that *w* is only 1.
[0699]	Sam	We have to bring something up ….
[0700]	Lec	We have to bring something up … show another element of *Y*, which is not *w*, and in fact ….
[0701]	Sam	It is the same.
[0702]	Lec	It is equal to *w*, the one that you show.
[0703]	Sam	Where *w*1 equals *w*2.
[0704]	Lec	So, you could symbolize the ones that you take as *w*1 and *w*2, but in fact ….
[0705]	Dyn	*w*1 and *w*2 is *w*.
[0706]	Lec	Are the same. Not equal to *w*.

[0707] Sam *w*1 equals *w*2.

[0708] Lec Yes.

[0709] Dyn So, some ….

[0710] Sam Some *k*’s ….

[0711] Lec Some *k*’s, what do you mean?

[0712] Dyn Because it is not all.

[0713] Lec Oh, yes, yes.

[0714] Sam Some *k*’s are elements of *X* and *w*’s are elements of ….

[0715] Dyn *w*, *w* equals *w*1 equals *w*2 equals *wn*.

[0716] Lec Why did you write them as the same?

[0717] Sam *w* is an element of *Y*, where *w*1 equals *w*2.

[0718] Lec Why are *w*1, *w*2 the same?

[0719] Dyn Because *w* is also an element of *Y*. *w*1 is also an element of *Y*. It means *w*1, all them are elements of *Y*.

[0720] Lec *w*1 is an element of *Y* … if they are all elements of *Y*, must they be the same?

[0721] Dyn No, *w*1, *w*2, *w*3 ….

[0722] Sam What? *w* equals *w*1 and *w* equals *w*2, where *w*1, *w*2.

[0723] Lec But, what are *w*1 and *w*2?

[0724] Dyn *w*1 and *w*2 are elements of *Y*.

[0725] Lec Just because they are elements of the same set, are they the same?

[0726] Dyn No.

[0727] Lec What is the purpose of taking *w*1 and *w*2?

[0728] Sam To prove that *w* is only 1.

[0729] Lec But, what is *w*?

[0730] Sam An element of *Y*.

[0731] Lec What is its function there?

[0732] Dyn The partner of *k*.

[0733] Lec The partner of *k*. So, it has to be paired off with *k*, both of them, and at the end, you will find that the two partners of *k* are just the same. So, you couldn’t conclude in a sudden that *w*1 equals *w*2, without knowing what *w*1 and *w*2 are, except that they are elements of *Y*.

[0734] Sam It must be *k* comma *w*1 ….

[0735] Dyn And then, *k* comma *w*2 ….

[0736] Lec Yes, *k* comma *w*2. So, they both are paired off with *k*, and at the end you will find ….

[0737] Dyn *w* is also an element of *Y*.

[0738] Lec No. You will find that *w*1 equals *w*2. So, you don’t need *w* anymore, because you have taken other symbols. Do you get it?

[0739] Dyn OK.

[0740] Sam (Writing)

Relation from A to B $= \{(k,w1) \mid \exists k \in X$ and $\exists w1 \in Y$

$= \{(k,w1),(k,w2) \mid \exists k \in X, \exists w1 \in Y, \exists w2 \in Y$ and $w1 = w2\}$.

[0741] Lec OK. You have just discussed the first property of a special relation. Here, what is being asked is ….

[0742] Sam Look at the two characteristics as you have identify in Parts h and g.

[0743] Lec Make a relation satisfying the two properties. That is, … the exhaustive domain, the partner of each element of the domain is only one.

[0744] Sam It satisfies this [the first property], and also this [the second property]

[0745] Lec Yes, it satisfies the two properties.

All you have to do is to make a relation. This is an example. You are asked to provide an example of relation.

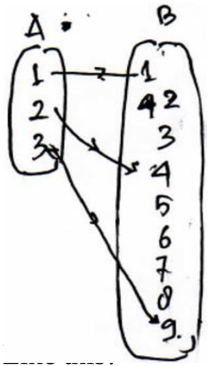
[0746] Sam For example, *A* and *B* … relation *A*. The first is the set … set … I just make it, for instance, relation … 1, 2 … set *A* is 1, 2, 3, set *B* contains 1, 2, 3, 4, 5, 6, 7, 8, 9. Just like that.

Next … where the relation is … relation from *A* to *B* is equal to 1 … 1, 2 … 4, 3 … 9. Relation of *A* as the square root of *B*.

[0747] Dyn The square root of *B*.

[0748] Sam It is the same. Is it like that?

(Writing)



⇒ relation from A to B : $\{(1,1), (2,4), (3,9)\}$

Does it satisfy the properties?

Where for each element l of *A*, and some *w* elements of *B*, where there is only 1 such an element of *B*, right?

Amm … where *A* has only 1 partner in *B*, it satisfies this special property [the exhaustive domain] and this [the unique image]. Amm ….

Is it like this? Just check it first.

How to define a special relation which you have made in Part i?

[0749] Dyn How to define?

[0750] Sam Define.

[0751] Dyn It means a relation from *A* to *B* is equal to ….

[0752] Sam Relation from *A* to *B* … *k* comma *w* again?

[0753] Dyn Yes. For each … amm … *k* comma *w*1, *k* comma *w*2 … for each *k* ….

[0754] Sam For each element *k* of *A*, there exist *w*1 and *w*2 elements of *B*, and *w*1 equals *w*2. Is it correct? Because for each *k*, so, all *k* in *A* has only 1 partner … the domain has only one partner.

(Writing)

Relation A to B = $\{(k,w1), (k,w2) \mid \forall k \in A, \exists w2, w2 \in B \text{ and } w1=w2\}$

[0755] Dyn What is left?

[0756] Lec Done?

[0757] Sam Yes, Sir.

[0758] Dyn Done, Sir.

[0759] Lec Now, move to number 3.

[0760] Dyn The specific relation you have defined in the previous question is called a function.

[0761] Sam Examine the following expression. Check the box beside F if the expression is a function or the one beside N if the expression is not a function. Give an explanation to justify your answer.

[0762] Dyn Isn't it not the Cartesian diagram? *x*-axis and *y*-axis

[0763] Lec It is up to you. It depends on you.

[0764] Sam The Cartesian diagram where this [horizontal axis] is its domain, this [vertical axis] is its range.

[0765] Dyn This [horizontal axis] is set *X*, this [vertical axis] is set *Y*. Right?

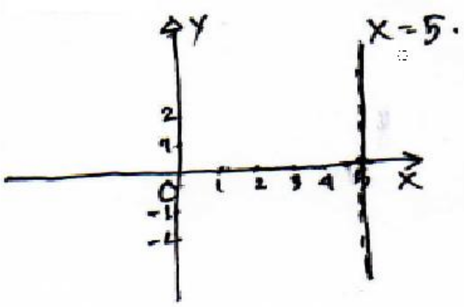
[0766] Sam Yes, yes. This is set *S*, this is set *Y*. Domain, range.

[0767] Dyn They are not, I reckon.

[0768]	Sam	Why?
[0769]	Dyn	Because of the definition of a function, a special relation, right? That is a function. Function is a special relation, the definition is each element of *X* which has a partner, has only 1 partner in *Y*, right? If you look at this, this 2 is paired off with 1, but it has another one here, 2 is paired off with 0 point something.
[0770]	Sam	So, if we see it from this view …. We could say that based upon the specific properties, where *k* has 2 partners or its domain has 2 partners, is that what you think? So, the answer could be written symbolically, or any other idea?
[0771]	Dyn	There is an element in the domain ….
[0772]	Sam	There is an element in the domain which has 2 partners in the elements of range.
[0773]	Lec	Codomain.
[0774]	Dyn	That is 2 ….
[0775]	Lec	Look at the graph carefully. Look at it all entirely. Find the details.
[0776]	Sam	The graph is like this, so …. There are some points in the graph ….
[0777]	Lec	If you see the graph, this is a corner, it is a hole. Do you see a hole there?
[0778]	Dyn	Where?
[0779]	Lec	At the tip of the meeting point of the two dashed lines.
[0780]	Sam	Yes, it is hollow, there is a hole.
[0781]	Lec	It is hollow there, right? It is to indicate that the graph is discontinuous there, and it then goes up.
[0782]	Sam	It means …. A point is still there.
[0783]	Lec	Which one?
[0784]	Dyn	There is 2 point something here, it has two partners, 0 point something and 1 point something.
[0785]	Sam	So, there is a graph passing, amm … how to say it?
[0786]	Dyn	There is an element of the domain which has 2 partners, it has more than 1 partner.
[0787]	Sam	Or, the graph passes … what … the line *x* equals 2. … (2 point something) … passes 3 points. Is it like that?
[0788]	Dyn	Write its relation first.
[0789]	Sam	Relation?
[0790]	Dyn	Amm … how? Set *Y* is real numbers, then, set *X* is also real numbers. Then ….
[0791]	Sam	Which one is *X* and *Y*? But the graph is ….
[0792]	Dyn	So, next is its relation … a relation of *x* of *y*, it has 0 comma 1, 1 comma 0, 2 comma 0, 2 comma 1.
[0793]	Sam	It is not 2 comma 1, right? It is discontinuous here.
[0794]	Dyn	Oh … it means 2 comma 1 is not. It is 2 point something, it is more than 2.
[0795]	Lec	What do you mean?
[0796]	Dyn	Its *x* is 2, the *y* is 0 point something.
[0797]	Lec	*x* is 2, *y*?
[0798]	Dyn	0 point something.
[0799]	Lec	Which are you talking about?
[0800]	Dyn	This one [a point in the right side of 2].
[0801]	Lec	Then?
[0802]	Dyn	The *x* is 2 point something.
[0803]	Lec	What is something?
[0804]	Sam	2 point 1 or it could be 2 point 2.
[0805]	Dyn	*x* is 2, *y* is 0 point something.
[0806]	Lec	What for do you take them?

[0807] Dyn Looking for the partner of the domain that has more than 2 partners … amm more than 1 partner. There are elements of *X* that has more than 1 partner, that is, 2 point something.
[0808] Sam 2, what? It could be 2 point 1, but it is not precise.
[0809] Dyn Just 2 point something.
[0810] Lec What is something?
[0811] Sam Or, couldn't it be like this, we say. For the domain, the codomain where there is a point in the codomain which is passed by … the graph passes 2 common points in *X* region. How to say it?
Could we represent this [a point on the graph] *k* comma *w* here [a point on the graph] *k* comma *w*, where *w*1 and *w*2 are not the same?
[0812] Dyn What? Say it again.
[0813] Sam Couldn't it be represented here, *k* comma *w* here *k* comma *w*1 here, *k* comma *w*2 here where *w*1 is not equal to *w*2, where for the special property, it says *w*1 equals *w*2. So, if we prove it here, there exists *w*1 not equal to *w*2, which means it doesn't fulfill the special property. Is it like that?
[0814] Dyn Yes.
[0815] Sam Where the special property, that is, function, it is ….
[0816] Dyn A function is ….
[0817] Sam A function is a relation from *A* to *B* where ….
[0818] Lec Why do you call it function?
[0819] Sam Because the special relation we have define previously is a function.
[0820] Lec How?
[0821] Dyn A function has this definition, and then this graph doesn't fulfill this definition.
[0822] Lec Because?
[0823] Dyn Because this graph has one point in the domain … that has elements of ….
[0824] Sam Like this … because there exists a point, that is, *k* comma *w*1 and *k* comma *w*2, where *w*1 and *w*2 are not the same.
[0825] Lec What is *k* there?
[0826] Dyn An element of set *X*.
[0827] Lec Yes, so what is *k*?
[0828] Sam A set of domain.
[0829] Lec I think I should ask, how many is *k*? Not, what *k*?
[0830] Dyn *k* is 2 … could be 2, amm …. k is 2 plus *x*.
[0831] Sam Couldn't we represent the point on *x*-axis here? *k* is ….
[0832] Lec What is the value of *k* you choose?
[0833] Sam It is not exact, 2 point something more.
[0834] Dyn We measure it. If we could take a point here [*x*-axis] *x*, it means 2 plus *x*.
[0835] Sam *k* is a value of *x*.
[0836] Lec Yes, but what value do you take?
[0837] Dyn 2 point 3 … 2 point 1, could be? 2 point 1 is in this side, then it gets, 2 point 1 is not in the graph. Hang on … but the graph gets 2 point 1.
[0838] Sam It could be. But, the problem is the value is not precise.
[0839] Lec What is the importance of finding the exact point which is touched?
[0840] Dyn The distance of this [0] to this [1] is 1, right? Then, this [1] to this [2] is exactly 1. It means if I take here [in the right side of 2], it means it is a halfway, it is 2 point 5 right? It means 2 point 5 more to this [the point he takes].
[0841] Lec So, what happens then?
[0842] Dyn It means 2 point 5 has more than one partner in the codomain.
[0843] Sam Amm … that is?
[0844] Dyn Thai is, here and here. Here is *y*, where 0 is less than *y* is less than 1 … for the one here, *y* is greater than 1.

[0845] Sam Greater than 1?
[0846] Dyn Yes.
[0847] Sam It is not a function because … because a function is a relation ….
[0848] Lec You may just directly say that it is not a function because of what.
[0849] Sam Because there exists domain which ….
[0850] Lec An element ….
[0851] Sam An element of the domain that has more than 1 partner in the codomain. Right?
[0852] Dyn Yes.
[0853] Dyn For this item [item 3b] it is a function.
[0854] Sam Yes, because it fulfils the special relation.
[0855] Dyn Amm … days … a set of elements of a set of days, Sunday until Saturday, right? For the things, they are orange, computer, assignment, Jakarta, and ….
[0856] Sam Because it is clear that each day has only one.
[0857] Dyn But, actually Jakarta and orange could be excluded.
[0858] Sam They could be the same, right? Because it still fulfills the special properties.
[0859] Dyn Because it has fulfilled the properties or conditions, that us, each element of the set of days has a partner and its partner is only one … that is ….
[0860] Sam Each element of the set of days ….
[0861] Dyn Has a partner ….
[0862] Sam Has a partner.
[0863] Dyn And ….
[0864] Sam And its partner is only one in the set of things.
[0865] Sam For this [x = 5], it should be clarified first.
[0866] Dyn The equation x equals 5, if we draw its Cartesian diagram, the graph is ….
[0867] Sam (Drawing a graph)



It is not … because the domain, 5, doesn't have a unique partner in the codomain.

[0868] Dyn It is a partner of each element in *Y*. It is certain that it doesn't fulfill the conditions, satisfying this condition [exhaustive domain] but not satisfying this condition [the unique image].
[0869] Sam Because the domain has more than one partner in the codomain.
[0870] Lec What has a partner, actually?
[0871] Dyn The domain.
[0872] Lec What has a partner? Domain? How could the domain have a partner? Why could it have a partner?
[0873] Dyn Each domain which has a partner.
[0874] Lec How many domains are there, if you say each?
[0875] Dyn Oh … each element of the domain … element … element … because an element in the domain ….
[0876] Sam Because elements of the domain have more than one partner in elements of set of the codomain.
[0877] Lec OK. Next.

[0878] Sam For that item [question number 3d], it is a function. How could we draw the graph of this? *y* is positive if *y* is rational, meaning ….
[0879] Dyn It is *x* times *y*. Not? Or, function *x* of *y*.
[0880] Sam Function *xy*? If *y* is rational then *y* is positive, if *y* is irrational then *y* is minus, it means it is a function because … the domain is rational, then the range is positive, if the domain is irrational, the range is minus.
[0881] Dyn Is it like this? Write it.
[0882] Sam Because if *y* is rational, then *y* is greater than 0, and if *y* is irrational, then *y* is less than 0.
[0883] Lec What is *y* less than 0?
[0884] Sam The result, the codomain.
[0885] Dyn If *y* is rational, then *xy* equals *y*.
You may erase that *y* is greater than 0.
It means if this is the domain, this [$x(y) = y$ and $x(y) = -y$] is its range.
[0886] Sam It means it satisfies the special properties.
[0887] Lec Two properties or one of them? There are two special relations.
[0888] Dyn The two properties of special relation.
[0889] Lec When you finished, just go ahead.
[0890] Dyn *R* is the set of real numbers.
[0891] Sam A function, because if we take *y* as its domain ….
[0892] Dyn No, no ….
[0893] Lec How could *y* be as its domain?
[0894] Sam Oh, *R*.
[0895] Lec The domain *R*, what is the codomain?
[0896] Sam *R*. *R* to *R*.
[0897] Lec *R* to *R* means the domain and the codomain are both *R*, is it a function?
[0898] Sam A function. Amm … where β is a relation corresponding ….
Is a function?
It is clear that it is from *R* to *R*, right? But, the name of its relation is ….
[0899] Dyn Relation β.
[0900] Sam log 2 minus *y*).
[0901] Lec So, β*y* equals log 2 minus *y*.
[0902] Sam This [(2 – *y*)] will become the power.
(Writing)
$^{a}\log b = c \Rightarrow a^{c} = b$
If this [(2 – *y*)] is minus, it means it will also be minus here [the square root of b in the written statement], so it is real. It is imaginary, right?
[0903] Lec If it is not real, then?
[0904] Sam It means the condition … *y* must be less than ….
[0905] Dyn It means not all ….
[0906] Sam For this [(2 – *y*)] not to be minus, *y* must be less than 2. Because *y* must be real, it means that there is a non-real partner for any *y*. It means it is not a function.
[0907] Lec OK.
[0908] Sam What is to write here?
[0909] Dyn It is not a function, because it doesn't satisfy this condition.
[0910] Lec What condition?
[0911] Sam It must be verified. How to verify it?
[0912] Dyn Just write it.
[0913] Sam To make β*y* equals 2 log 2 minus *y* non-imaginary ….
[0914] Lec What do you mean by imaginary?
[0915] Sam Real, real, it results in a real number ….
Then, *y* must be less than 2.

[0916]	Dyn	Less than 2 and greater than 0, or only less than 2.
[0917]	Sam	Less than 2. Because, if you take negative 1, 2 minus negative 1 equals 3, it is not negative.
[0918]	Dyn	Oh, yes, yes.
[0919]	Sam	Because *y* is an element of *R*, than *R* is less than 2. Is it like that?
[0920]	Dyn	Yes, it is. It is enough. Then, *y* is an element of *R*, *y* is less than 2, *y* is an element of *R*.
[0921]	Lec	So, what is the conclusion?
[0922]	Sam	It means not all.
[0923]	Dyn	Correct. There exists.
[0924]	Sam	There exists an element of the domain which has no partner in the codomain.
[0925]	Dyn	So, it is not a function.
[0926]	Sam	So that, relation β from *R* to *R* is not a function.
[0927]	Dyn	Let *N* be a set of natural numbers and *Z* is a set of integers.
[0928]	Sam	Let *N* be a set of natural numbers and *Z* is a set of integers. Relation *L* is from *N* to *Z*. It means, to get 0, and *b* must be ….
[0929]	Dyn	*b* must be negative *a*.
[0930]	Sam	Yes. Or it is like this, *a* is greater than 0 and *b* is less than 0 or *a* is less than 0 …. (Writing) $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$
[0931]	Lec	What is it for?
[0932]	Dyn	Amm …, no, no, it is not the case.
[0933]	Lec	Just let it be there. But, if you do not agree, what is that for?
[0934]	Sam	So that *a* plus *b* equals 0.
[0935]	Lec	How could it be? This is an addition.
[0936]	Sam	Oh, yes, yes.
[0937]	Dyn	To get *a* plus *b* equals 0, then *b* equals negative *a*.
[0938]	Sam	*a* plus *b* equals 0, then *b* equals negative *a*.
[0939]	Lec	So, is it function or not?
[0940]	Dyn	*a* comma *b*. A function.
[0941]	Sam	Oh … relation *L* is from *N* to *Z*, so automatically ….
[0942]	Dyn	The relation is *a* comma *b*, right? It means *a* comma negative *a* because *b* equals negative *a*, meaning that *a* comma negative *a*.
[0943]	Sam	*a* comma negative *a*?
[0944]	Dyn	Yes. If *a* is 1, the partner is negative 1.
[0945]	Sam	Oh, yes, I got it. How to write it?
[0946]	Dyn	Then, *a* comma *b* equals *a* comma negative *a*.
[0947]	Lec	What are you going to next?
[0948]	Dyn	It satisfies the conditions, right?
[0949]	Sam	Hang on, you didn't say that relation *L* is from *N* to *Z*.
[0950]	Dyn	This is still its relation.
[0951]	Sam	And *a* comma *b* equals *N* comma *Z*.
[0952]	Lec	How could they be the same?
[0953]	Sam	Because *a* is an element of *N*, *b* is an element of *Z*. Am I right?
[0954]	Dyn	But, *a* and *b* are elements, while this [*N* and *Z*] are sets.
[0955]	Sam	They are set. OK. So ….
[0956]	Dyn	Each element *a*, each element of *N* has partners in *N* which is a negative of *a*.
[0957]	Sam	It is like this? *a* is an element of *N* and *b* is an element of *Z*, then, there should be ….
[0958]	Dyn	Hold on, *b* equals negative *a* element of *Z*

[0959] Sam Oh, yes. Because it a relation.

[0960] Dyn It must be a function.

[0961] Sam It is clear.

[0962] Lec So, why did you say that it is clearly a function?

[0963] Dyn Because each element of N has a partner in set Z, namely, b, where b is negative a.

[0964] Lec Oh, OK. You just repeat the previous reason.

[0965] Dyn The conditions are fulfilled. Just write it.

[0966] Sam What?

[0967] Dyn Each element of N has a partner in set Z and each element having a partner has only one partner.

[0968] Sam Each element of N has a partner … has a partner in the elements of Z, and ….

[0969] Dyn Each element which has a partner, has only one partner.

[0970] Sam Each element which has a partner, only has one partner.

[0971] Lec Each element which has a partner, only has one partner.

[0972] Dyn Then it is a function. Relation L ….

[0973] Lec It is very long. Done. Just tick here … so, it is a function. OK.

[0974] Dyn Give an example of a function.

[0975] Lec Give a function.

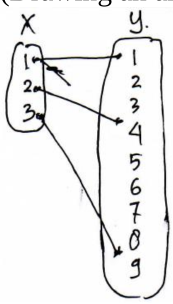
[0976] Dyn Do you want to use a graph or an arrow diagram?

[0977] Sam Just a diagram.

[0978] Dyn Just take y equals the square root of x.

[0979] Sam x equals the square root of y. But it is long.

(Drawing an arrow diagram)



[0980] Dyn If we have two sets A equals 3 and K equals 2, 3, 4, can we create functions between the two sets?

[0981] Sam (Writing)

First function

A to K = {3,2}

Is it like this?

[0982] Lec What is that [{3, 2}]?

[0983] Dyn A relation from A to K.

[0984] Lec Is a function like that?

[0985] Sam No, this is … this is its domain.

[0986] Lec This is a relation from A to K. Is it a function? Why is it a function?

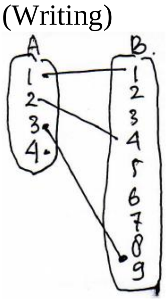
[0987] Dyn Because all elements of A have a partner in K.

[0988] Lec Which ones are in a pair?

[0989] Dyn 3 comma 2.

[0990] Lec Where is it?

[0991] Dyn It isn't written.

[0992]	Lec	Which Part says that 3 is paired off with 2 there?
[0993]	Sam	The first function ….
[0994]	Lec	Look at this [{3, 2}] first, I am asking you which one here says that 3 and 2 are in a pair?
[0995]	Dyn	In the bracket … in the bracket.
[0996]	Lec	What do you to write?
[0997]	Dyn	No, I mean here … in the bracket 3 comma 2.
[0998]	Sam	What?
[0999]	Lec	Just write the answer.
[1000]	Sam	You write it.
[1001]	Dyn	3 comma 2, right? (Writing) A to K $= \{(3,2)\}$
[1002]	Lec	Any others?
[1003]	Sam	The second function.
[1004]	Dyn	Use *A* to *K* or? (Writing) ii A to K $= \{(3,3)\}$ iii A to K $= \{(3,4)\}$ Finished.
[1005]	Dyn	Its definition, a relation which is not a function, the negation of a function. A relation which is not a function is equal to a set … what was the definition?
[1006]	Sam	A function is *k* comma *w* … *A* to *B* which is equal to … just make a negation, right?
[1007]	Dyn	Just write it. Just make it like that. You just change 'for each' and 'there exists'.
[1008]	Sam	Oh, just write it directly. (Writing) A relation which is not a function, that is, A to B $= \{(k,w1), (k,w2) \mid \exists k \in A,$ and $\forall w1, w2 \in B$ and $w1 \neq w2\}$ Is it like that? Think about it again.
[1009]	Lec	An example.
[1010]	Dyn	Just make a domain which has no partner in a range.
[1011]	Sam	(Writing)  Is it a function?
[1012]	Lec	Why?
[1013]	Sam	Because not all ….
[1014]	Dyn	Or, it could be written there exists or some, they are just the same.
[1015]	Sam	Yes, not all elements of set *A* have a partner in the elements of set *B*.
[1016]	Lec	Finish? OK. Thank you.

Appendix F The Example of Students' Worksheets

Lembar Kerja Kalkulus

FUNGSI: DEFINISI FORMAL OPERASIONAL

Petunjuk:

Mohon dibaca dan dikerjakan semua pertanyaan secara lengkap. Dalam menjawab setiap pertanyaan, Anda diharapkan mengungkapkan proses dan pemikiran yang digunakan secara bebas dan jelas. Silahkan menggunakan semua peralatan yang Anda miliki. Ingat bahwa ini adalah bagian dari proses belajar Kalkulus 1. Hasil kerja atau jawaban tidak akan diberi nilai dan tidak ada kaitannya dengan nilai akhir yang Anda peroleh pada mata kuliah ini.

Nama: IFUL - VITO

Nama Pasangan:

Tanggal:

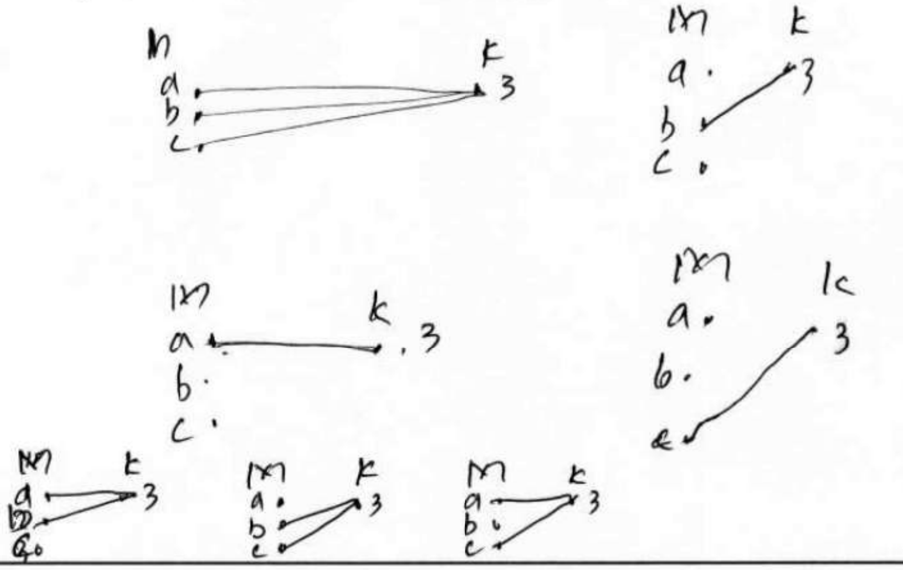
Definisi Formal Operasional DFO-002-IND 10/09/2016

FUNGSI: DEFINISI FORMAL OPERASIONAL

1. Berikut ini terdapat beberapa pasangan himpunan. Tentukan hubungan antar anggota kedua himpunan tersebut yang dapat Anda buat. (*Pikirkan semua hubungan yang mungkin dibuat, kemudian tuliskan jawaban Anda pada kotak yang telah disediakan.*)

(a) $M = \{a, b, c\}$ dan $K = \{x \mid 2 < x < 4, x \in \boldsymbol{N}\}$.

$M = a, b, c \qquad K = 3$



(b) $P = \{z \mid z^2 + 2z + 2 = 0, z \in \boldsymbol{R}\}$ dan $R = \{6, 7\}$.

$P = a = 1 \quad b = 2 \quad c = 2$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_1 = \frac{-2 + \sqrt{4 - 8}}{2}$$

$$= \frac{-2 + \sqrt{-4}}{2}$$

$$= -1 + \sqrt{-1}$$

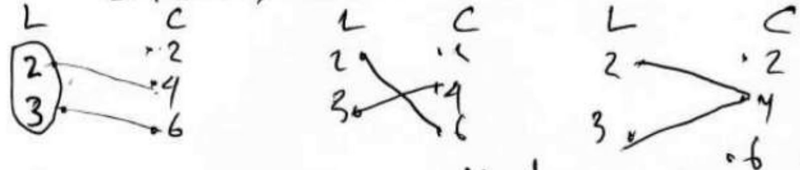
$$x_2 = -1 - \sqrt{-1}$$

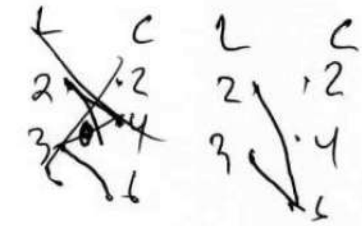
2. Misalkan A dan M dua himpunan tidak kosong, $k \in A$, dan $w \in M$. Jika kita mengaitkan k ke w, maka hasil pengaitan dituliskan dalam bentuk (k, w). Selanjutnya, (k, w) dinyatakan sebagai anggota pengaitan tersebut. Diberikan dua himpunan: $L = \{2, 3\}$ dan $C = \{2, 4, 6\}$.

(a) **Tuliskan semua hasil pengaitan jika setiap anggota L dikaitkan dengan anggota-anggota C yang nilainya lebih dari anggota L.** (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

L = (2,3)   C = (2,4,6)

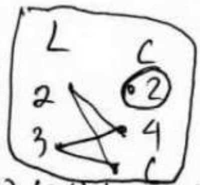
L ke C, c > L





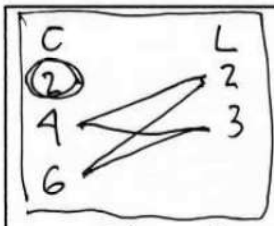
Hasil pengaitan

1. {(2,4), (3,6)}
2. {(2,6), (3,4)}
3. {(2,4), (3,4)}
4. {(2,6), (3,6)}



1. {(2,4), (2,6), (3,4), (3,6)}

(b) **Tuliskan semua hasil pengaitan jika setiap anggota C dikaitkan dengan anggota-anggota L yang nilainya kurang dari anggota C.** (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

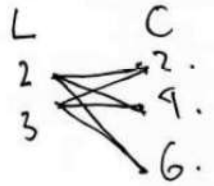


HP = {(4,2), (4,3), (6,2), (6,3)}

(c) Apakah hasil di bagian (a) sama dengan hasil di bagian (b)? Mengapa? Jelaskan alasan Anda. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

Beda . karena hasil pengaitannya berbeda (berkebalikan)
Misal, pada bag. (a) = Notasinya (2,4)
——→ bag (b) = Notasinya (4,2)

(d) Tuliskan semua hasil pengaitan jika setiap anggota *L* dikaitkan dengan anggota-anggota *C* yang merupakan bilangan genap. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)



{(2,2), (2,4), (2,6), (3,2), (3,4) (3,6)}

(e) Bandingkan hasil yang Anda peroleh pada bagian (a), (b), dan (d). Apa yang unik dari hasil pengaitan di (d) dibanding hasil di (a) dan (b). (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(a) (2,4), (2,6), (3,4), (3,6) x,2 – tidak ada

(b) (4,2), (4,3), (6,2), (6,3) 2,1 tidak ada

(d) (2,2), (2,4), (2,6), (3,2), (3,4), (3,6) Lengkap

{2} {3}
{2,3}
L = 2 & 3
{2}, {3} {2,3}

~~a~~ C
b C . 6

(2), (4), (
2, 4, 6, (2,4), (4,6), (2,6)

(f) Coba kita kelompokkan hasil pengaitan di (a) dan (d) masing-masing menjadi dua himpunan berbeda. Perhatikan kedua himpunan yang kita gunakan di awal, yakni, L dan C. Hubungan apa yang dapat ditemukan dari kedua himpunan tersebut? Himpunan apakah itu? Bagaimana Anda mendefinisikan obyek matematika tersebut? (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

Hasil kali kartesius 2 himpunan adalah pasangan setiap elemen himpunan yg. semua anggotanya harus berkaitan.

A B
k n
x ∈ R

A B a c
k n b d
k ∈ A n ∈ B e
A

Hasil kali kartesius ~~adalah~~

$A \times B = \{(k, n)\}$

$A = \{(2,4), (2,6), (3,4), (3,6)\}$ L ke C $L = \{2, 3\}$ $C = \{2, 4, 6\}$ = {2} {4} {6} {2,4} {2,6}

$D = \{(2,2), (2,4), (2,6), (3,2), (3,4), (3,6)\}$

$A \subseteq D$

(k, n)

Relasi : ~~Bagian~~ $\subseteq$ HKK

Relasi adalah bagian dari Hasil kali Kartesius 2 Himpunan.

(g) Pada bagian (f), Anda telah menamai dan mendefinisikan himpunan yang memuat hasil pengaitan yang diperoleh pada bagian (a) dan (d). Dapatkah Anda menemukan sifat yang sama dimiliki kedua himpunan hasil pengaitan ini, tetapi tidak dimiliki oleh himpunan hasil pengaitan pada bagian (b)? Bagaimana Anda mendefinisikan obyek matematika yang memiliki sifat yang dimaksud? (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

~~B ∉ D~~

Setiap ~~Himp~~ Elemen Himp. Pertama pada bagian a dan d pasti memiliki pasangan

Pada bagian b, ada satu elemen Himp. Pertama, tidak memiliki pasangan.

Relasi khusus adalah relasi antara kedua anggota himp. dimana setiap anggota himp. pertama mempunyai pasangan di himp. 2

(h) Perhatikan dan bandingkan kedua gambar di samping. Tulislah perbedaan yang Anda temukan dari kedua gambar. Perhatikan gambar (ii), pikirkan sifat khusus dari gambar tersebut. Tuliskan. Bagaimana Anda mendefinisikan obyek yang memiliki ciri-ciri seperti itu? (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(i) (ii)

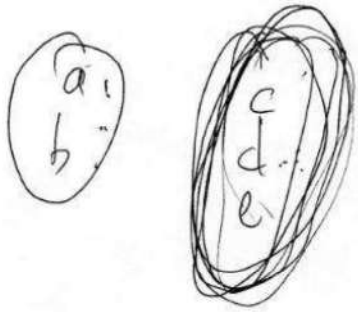
$A = \forall \in L$ memiliki pasangan di himpunan C

~~A = himpunan t~~ Relasi antara L & C

$\forall \in L$ & C

$\forall k \in L$ me

$\forall k \in L \; \exists m \in C$ ~~dimana~~ $[(k,m)]$



Setiap anggota x yang memiliki pasangan di y hanya mengikat 1 anggota

$\forall k \in X, \exists m \in Y, n \in Y \mid (k,m) = (k,n)$

(k,m)

|k,n

$\forall (k,\theta) \cap (k,\theta), \exists m,n \in Y$

$\forall (k,m) \cap (k,n), m,n \in Y \mid \cancel{(k,m) = (k,n)}\, m = n$

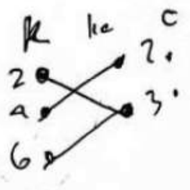
$\forall k \in X, \forall m \in Y, \forall n \in Y \mid (k,m) = (k,n)$

$\forall m,n \in Y$

$\forall k \in X, \forall m,n \in Y \mid (k,m) \wedge (k,n) \Rightarrow m = n$

(i) Perhatikan kedua ciri khusus sebagaimana yang Anda temukan di (h) dan (g). Buatlah satu relasi yang memenuhi kedua ciri tersebut. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

Setiap anggota X harus memiliki



(j) Bagaimana mendefinisikan relasi khusus yang telah Anda buat di bagian (i) (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

Setiap anggota X harus memiliki tepat 1 anggota di Y

Definisi fungsi

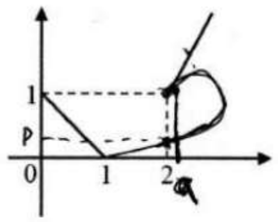
Domain X, kodomain Y

Relasi f:

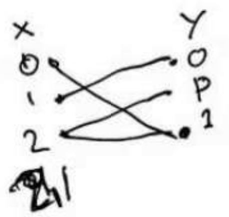
* $\forall k \in X \; \exists m \in Y \mid (k,m)$
* $\forall k \in X, \; \forall m,n \in Y \mid (k,m) \wedge (k,n) \Rightarrow m = n$

3. Relasi khusus yang telah Anda definisikan pada pertanyaan sebelumnya disebut *fungsi*. Sekarang, perhatikan ekspresi-ekspresi berikut. Beri tanda centang (√) pada kotak di sebelah [F] jika ekspresi tersebut merupakan fungsi, atau kotak di sebelah [N] jika bukan fungsi. Beri penjelasan untuk jawaban Anda. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

(a)[1] [F] ☐ [N] ☑



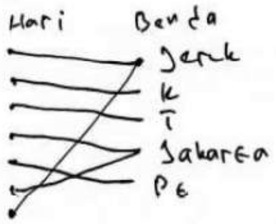
Karena:



(b)[2] [F] ☑ [N] ☐

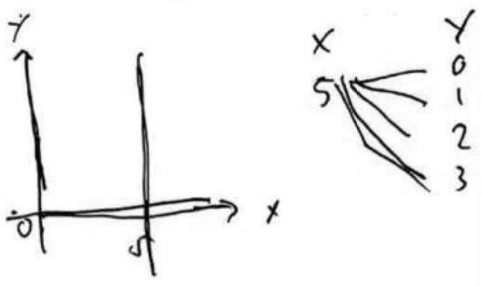
Hari	Benda
Minggu	Jeruk
Senin	Komputer
Selasa	Tugas
Rabu	Jakarta
Kamis	Pesawat terbang
Jumat	Jakarta
Sabtu	Jeruk

Karena:



(c) $x = 5$ [F] ☐ [N] ☑

Karena:



5, 0
5, 1
5, 2
. . .

(d) $x(y) = \begin{cases} y & \text{jika } y \text{ rasional} \\ -y & \text{jika } y \text{ irrasional} \end{cases}$ [F] ☑ [N] ☐

Karena:

$x(2) = 2$

$x(\sqrt{2}) = -\sqrt{2}$

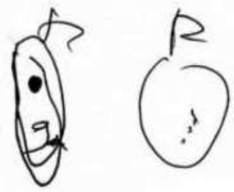
(e) $\boldsymbol{R}$ adalah himpunan bilangan riil. Relasi β dari $\boldsymbol{R}$ ke $\boldsymbol{R}$; $\beta(y) = \log(2-y)$, untuk setiap $y \in \boldsymbol{R}$. [F] ☐ [N] ☑

Karena:

$\beta(-8) = \log(2-(-8))$

$= \log 10$

$= 1$



$\beta(-98) = \log 100$

$= 2$

$-y \neq 0$ x

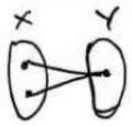
(f) Misalkan N adalah himpunan bilangan asli dan Z adalah himpunan bilangan bulat. Relasi L menghubungkan N ke Z dimana $L = \{(a, b) \mid a + b = 0,\ a \in N, b \in Z\}$. [F] ☑ [N] ☐

Karena:

N ke Z

karena himpunan pertama adalah bilangan + dan mempunyai pasangan di Z yang menyebabkan $a+b=0$ dimana $a \in N$ dan $b \in Z$

4. Berikan satu contoh fungsi. Dengan menggunakan definisi yang Anda telah buat, jelaskan mengapa contoh tersebut merupakan fungsi.



karena setiap elemen x punya tepat 1 pasangan di Y

$\underline{(x = y+1)}$, $x, y \in R$

$1 = Y + 1$
$Y = 0$

$L: a+b=0$

$a \in N$; $a+b=0$

$b \in Z$ $\quad b=-a$

* Ambil sebarang $a \in N$ maka terdapat $b=-a$

sehingga ~~$a+b=0$~~ $a+b=a+(-a)=0$

* Ambil sebarang $a \in N$

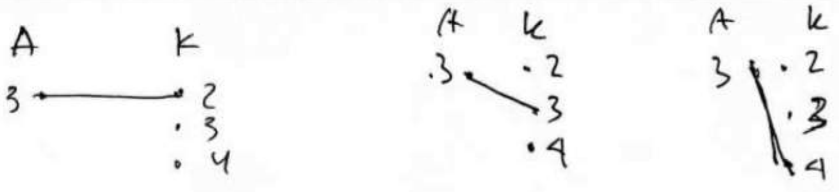
Ambil sebarang $m, n \in Z$

$(a,m) \wedge (a,n)$

- $\Rightarrow a+m=0 \Rightarrow m=-a$
- $a+n=0 \Rightarrow n=-a$

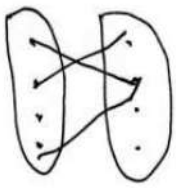
~~Karena~~ $(m=-a$ dan $n=-a) \Rightarrow m=n=-a$

5. Jika terdapat dua himpunan $A = \{3\}$ dan $K = \{2, 3, 4\}$, dapatkah kita membuat fungsi yang menghubungkan kedua himpunan tersebut? Jika ya, tuliskan fungsi-fungsi yang dapat Anda buat. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)



6. Bagaimana Anda mendefinsikan relasi yang bukan merupakan fungsi? (*Tuliskan jawaban Anda pada kotak yang tersedia.*)

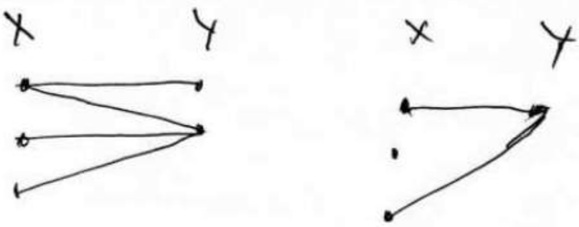
$\exists\, k \in X$



Ada $k \in X$ yang memiliki anggota di Y -

Ada $k \in X$ memiliki >1 anggota di Y

7. Berikan satu contoh relasi yang bukan fungsi, dan jelaskan mengapa relasi tersebut bukan fungsi. (*Tuliskan jawaban Anda pada kotak yang tersedia.*)



Catatan:

1. Diadaptasi dari: Christou, C., Pitta-Pantazi, D., Souyoul, A., & Zachariades, T. (2005). The embodied, proceptual, and formal worlds in the context of functions. *Canadian Journal of Science, Mathematics and Technology Education*, *5*(2), 241-252. doi: 10.1080/14926150509556656
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3. Diadaptasi dari: Nicholson, W. K. (2012). *Introduction to abstract algebra* (4 ed.). Hoboken, NJ: John Wiley & Sons.