Multi-Period Mean-Variance Portfolio Selection with Regime-Switching

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Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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Abstract

In this thesis, we study the multi-period mean-variance portfolio selection under regime-switching framework. This thesis consists of two aspects. The first aspect of research focuses on the portfolio selection model for a defined contribution(DC) pension scheme in the multi-period regime-switching framework under the mean-variance criterion. We assume that there are one risk-free asset and multiple risky assets in the finance market, which is followed by the finite-state observable Markov chain. And both of the asset return rates and the wage income depend on the market state. Based on the fundamental formulation, we further study three important factors. The first factor is mortality risk, considering that some members in a pension plan may die during the accumulation phase. To safeguard the right of the members who have mortality risks, we introduce the premiums return policy as the second factor in which the contribution from the deceased member in a plan will be paid back to their heirs. The last factor we study is incomplete information market which is driven by a hidden Markov chain, and the return rates of risky assets rely on both the observable market states and the unobservable market states which are driven by a hidden Markov chain, the risk-free asset return rate only depends upon the observable market state. By the dynamic programming approach with the embedding technique or the Lagrange multiplier method, we derive the results. Some special cases and numerical analysis illustrate that the factors have effect on the efficient frontier.

In the second part, we study the asset allocation with exit probability where the exit time distribution follows the market state. When the market enters the bankruptcy state, investors are assumed to get back δ part of the wealth from the bankrupt company, where δ refers to the retrieval rate. By introducing the Lagrange multiplier λ, we create an innovative expression for the wealth process and the iterative representation of the value function to obtain the result. Various special cases are investigated and our numerical analysis shows that the effect of the exit probability and bankruptcy state on the investment strategy is significant.

List of publications during PhD candidature

- Y Wang, Y Wu, X Zhang. Multi-period Mean-Variance Portfolio Selection with State-Dependent Exit Probability and Bankruptcy State, Journal of Mathematical Finance, 9(2), 2019.

  The following papers were completed during the PhD candidature and are currently under review:

- Y Wang, Y Wu, X Zhang. Portfolio selection for a DC pension plan with premiums return policy and incomplete information.

- Y Wang, Y Wu, X Zhang. Portfolio selection for a defined contribution plan with regime-switching and mortality risk.

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Contents

CHAPTER 1

Introduction

1.1 Background

Since the precursory research of Markowitz(1952) [1], the mean-variance (MV) model has become the most commonly used theoretical assumption and basis in portfolio selection. However, due to the inseparability for the objective function, the early studies focused only on the single period case. In this sense, the Markowitzs MV portfolio selection has not been fully utilized in multi-period case until 2000. By the embedding techniques, Li and Ng(2000) [2] transformed the original portfolio selection model into an auxiliary problem which can be solved by the dynamic programming approach to get the analytical solution and the corresponding efficient frontier. Since then, there has been a great development in the multi-period portfolio selection theory to incorporate different factors into the portfolio selection. For example, Zhu et al.(2004) [3] studied the multi-period MV portfolio selection problem with bankruptcy risk management. Guo and Hu(2005) [4] studied the multi-period MV asset allocation with uncertain exit time.

In reality, however, the market state includes different states. For example, when the market is at the bullish state, the market response may be enthusiastic and the market may be profitable beyond the expectation of the public. When the state is bearish, investors will have pessimistic attitude toward the market and the return rates of assets may be negative. Therefore, the regime-switching model, which is pioneered by Neftci(1984) [5], has become one of the most important extensions of the MV portfolio selection problem. Zhou and Yin(2003) [6], Yin and Zhou(2004) [7] and Çakmak and Özekici(2006) [8] investigated the MV portfolio optimization with regime-switching. Wei and Ye(2007) [9] considered the multi-period MV portfolio selection model with bankruptcy risk control under regime-switching framework.

Recently, the MV portfolio selection has been extended to the investment management of a defined contribution pension plan. Yao et al.(2013) [10], Vigna(2014) [11] and Guan and Liang(2015) [12] investigated the optimal investment management for the DC pension plan. Yao et al.(2014) [13] mainly considered factors of the stochastic income and mortality risk. Wu and Zeng(2015) [14] studied the MV equilibrium investment strategies for the DC annuity scheme.

In this thesis, we consider two kinds of factors in the portfolio selection model: uncertain time horizon and special market conditions. We establish the portfolio selection model for the defined contribution annuity scheme by taking account of mortality risk, premiums return policy and incomplete information market. In addition, we also study the factor of state-dependent exit probability and bankruptcy state in the portfolio selection problem. The computation process will be given, followed by numerical analysis.

1.2 Objectives of the Thesis

This thesis focuses on the study of three portfolio selection models under the regime-switching framework with mainly two kinds of constraints: uncertain investment time horizon and special market conditions. The main purpose in the thesis is to establish proper models to find the optimum investment policy and the corresponding efficient frontier. Besides, we attempt to study the effect of various factors on the optimal investment strategy and the efficient frontier. Specifically, the aims of the thesis are as follows.

(i) Study the portfolio selection for the DC pension scheme under regime-switching with mortality risk, establish the proper wealth process and create the corresponding iteration processes and expressions, then use the embedding technique to obtain the closed-form investment strategy, and finally examine the impact of mortality risk and regime-switching on the efficient frontier.

(ii) Under regime-switching criterion, investigate the portfolio choice for the DC annuity plan in the incomplete information market, taking account of a premiums return policy, then establish the suitable wealth and iteration processes, and use the sufficient statistics to convert the problem to the one with complete information. The embedding technique is then used to obtain the analytical solution. Finally, the effect of incomplete information market and premiums return policy on the efficient frontier is analyzed.

(iii) Explore the asset allocation with state-dependent exit probability and bankruptcy state, formulate the corresponding model and use the Lagrange method to determine the optimum investment policy, then investigate the impact of state-dependent exit probability and bankruptcy state on the efficient frontier.

All the three topics above are based on the portfolio selection where we study factors of the uncertain time horizon and regime-switching framework in different situations and try to explain their impacts on the optimal strategies. The first topic is about portfolio selection for a DC pension scheme where we mainly focus on situations of the change to the market state and the mortality risk, which also lays the fundamental for this thesis. Based on the result of the first topic, the second topic mainly studies the factor of incomplete information in a changing market and investigates a more detailed situation on the uncertain time horizon in which a premiums return policy is considered. We obtain the optimal strategy for this specific situation and explain the impact of these factors. Compared with the first two topics, the third topic further concretizes the regime-switching market from the perspective of a company: we define an extreme market state as a bankruptcy state where it is the best choice for a company to declare bankruptcy when struggling with heavy liability and unhealthy financial performance. Besides, the state-dependent exit probability is also considered in the topic as a more realistic form of uncertain time horizon.

1.3 Outline of the Thesis

The thesis consists of five chapters.

Chapter 1 presents a brief introduction and the objectives of the thesis.

Chapter 2 provides a literature review for the portfolio selection problem under the mean-variance framework. The main precursory research and methodologies for portfolio selection are also introduced in this chapter.

Chapter 3 concerns portfolio selection on the defined contribution annuity scheme under regime-switching and mortality risk. Section 3.1 gives the overview of the relevant research. In section 3.2, we formulate the finance market and establish the mean-variance asset allocation model with mortality risk. In addition, we use the embedding technique to make the model feasible in the sense of dynamic programming. Section 3.3 gives the optimal strategy for both the auxiliary and original problems. In section 3.4, we obtain the efficient frontier for the problem. Special cases are given in section 3.5 to show the relationship between the established model and the models in some existing literature. Section 3.6 provides numerical analysis to illustrate the impact of the regime-switching and mortality risk on the efficient frontier.

Chapter 4 studies the portfolio selection for the DC pension scheme taking account of premium return policy and incomplete market information. It starts the overview of the research on the portfolio selection for the DC pension plan in section 4.1, and some literature concerned with premiums return policy and incomplete information market are also given. Section 4.2 assumes that the market state consists of the unobservable and observable market states. Then we establish the finance market and construct the wealth process combined with a premiums return policy from the perspectives of the fund manager. In addition, we introduce the sufficient statistics and the embedding technique to transform the problem into the auxiliary problem. Section 4.3 provides the optimal investment policy for the problem. Some special cases are given in section 4.4 to demonstrate that the model is a generalized form of the previous works. In section 4.5, we use numerical examples to illustrate the effect of the incomplete information and premiums return policy on the efficient frontier.

Chapter 5 begins with the overview of the portfolio selection in sense of the exit probability and bankruptcy state in section 5.1. Section 5.2 establishes the finance market and wealth process and adopts the Lagrange method for the problem transformation. In section 5.3, by the dynamic programming approach, we obtain the optimal strategy for the problem. Section 5.4 provides the corresponding efficient frontier for the model. In section 5.5, we provide special cases to show the generalization of the model. Numerical analysis is given in section 5.6 to investigate the effect of some special market conditions on the efficient frontier for the problem.

CHAPTER 2

Literature Review

2.1 General Overview

Portfolio selection is to determine the optimum wealth allocation among securities in the market. It was pioneered by Markowitz(1952 [1],1959 [15]) in a single-period setting, which laid the root for the modern finance theory. Under the model, an investor makes the optimal investment strategy in order to optimize the expected terminal wealth with the constraint of investment risk quantified by the variance. The analytical solution of the efficient frontier was derived by Merton(1972) [16] with the assumption of the positive covariance matrix and short-selling that is allowed. Since then, the mean-variance portfolio selection model stimulated numerous investigations and stimulations. One of the most important developments is extending the single-period setting to the multi-period framework.

Smith(1967) proposed a transition model as a framework for selecting and revising portfolio, and the corresponding empirical test were also presented. Mossin (1968) [17] sought the optimal solution for the expected quadratic utility function in the multi-period framework with no constraint of inter-temporal consumption. Samuelson(1975) [18] generalized Phelps' model to include portfolio selection and consumption choice under a multi-period version. Hakansson(1971) [19] did multi-period mean-variance analysis with the von Neumann-Morgenstern utility function. Elton and Gruber(1974) [20] concluded the study on the multi-period consumption-investment decision to analyze the single-period model validity. Perold(1984) [21] investigated an algorithm for large-scale mean-variance asset allocation optimization with transaction limits and costs. Duffie and Richardson(1991) [22] investigated the mean-variance hedging problem and presented the corresponding hedging results and policies. For more detailed relevant studies, we refer the reader to Elton and Gruber(1975) [23], Francis(1976) [24], Dumas and Liucinao(1991) [25] and Grauer and Hakansson(1993) [26]. However, the studies under a multi-period version mentioned above, comparable to those under a single-period framework, did not obtain the corresponding analytical result. The mean-variance criterion dominated the research studies on the multi-period portfolio choice, namely maximizing the expected terminal wealth utility functions which can be mainly divided into four formats as power form, log form, exponential form and quadratic form. And the corresponding optimal strategies under these formulations, in fact, tend to be shortsighted choices, which can not strictly seen as real optimal choices. In addition, when studying the mean-variance portfolio selection optimization under multi-period framework, there are mainly two problems in the previous research. Firstly, the utility functions are hard to be elicited from the perspectives of the investors. Secondly, trade-off information between the expected return and the investment risk is obscure which significantly confuses the investors. By this measure, the mean-variance criterion is not totally applicable to the multi-period portfolio selection model. The relevant research study has been stagnant until the important work of Li and Ng(2000) [2] and Zhou and Li(2000) [27] in which they obtained the analytical solutions for optimizing the asset allocation under multi-period and continuous-time versions, respectively. Both works adopted the embedding technique to convert the original problem into a feasible structure in which the dynamic programming approach can be utilized to get the result. Then the following two decades have witnessed a great expansion of the dynamic mean-variance model and numerous constraints have been considered in the past two decades. Lim and Zhou(2002) [28] considered the mean-variance asset allocation optimization with stochastic risk-free rate, value-added ratios and variance parameters in a continuous-time setting. Zhou and Yin(2003) [6] concerned the mean-variance model under regime-switching framework. Leippold et al.(2004) [29] investigated the multi-period mean-variance asset-liability management. Zhu et al.(2004) [3] studied the dynamic portfolio selection with risk control over bankruptcy. Guo and Hu(2005) [4] studied the asset allocation with stochastic exit time under a multi-period setting. Xiong and Zhou(2007) [30] focused on the mean-variance portfolio selection with partial information. Elliott et al.(2010) [31] considered the mean-variance portfolio choice under a hidden Markov regime-switching model. Yao et al.(2014b) [32] focused on the continuous-time asset allocation with only risky assets.

In recent years, numerous studies has been carried out to apply the portfolio selection optimization to a pension plan problem. Pension plans can be divided into two major types in terms of the pension purchasing and payoff mode: defined benefit (DB) plans and defined contribution (DC) plans. As the names suggest, a defined benefit pension plan is a plan in which an employer guarantees the fixed amount of pension payments, thus contributions of the DB pension changes constantly according to the payment goal. A defined contribution plan refers to a pension plan whose contribution rates are constant during all time periods. DB pension plans dominated the capital market in the past because it is easy to be managed and risk-free for employees. In the United States, for example, most of public employees were covered by DB plans. However, an increasing number of companies have abandoned DB plans in recent years because of its open-ended nature. Employers tend to take the risk of the DB pension plan since they have the responsibility to guarantee the benefit of the plan. Contrary to a DB pension plan, a DC pension plan is much more popular among employers because members in the plan take all the investment risk. For this reason, a large number of countries have stopped providing DB plans. Unisuper, one of the Australian super funds, recently launched a plan which provided defined benefits and also required defined contributions, but the trustee have right to reduce the defined benefits when there exists a prolonged economic recession. Such a plan in which investment risks are transferred from employers to employees may be defined as an hybrid of a DB and a DC plan. Nowadays DC plans are becoming increasingly popular and influential in the world. For example, the UK personal pension scheme(PPP), the Australia Superannuation scheme, the Germany Riester plans and the New Zealand KiwiSaver plan. Individual pension plans also exist in most of the European and Asian countries such as Greece, Spain, Japan and China. Therefore, it is important and meaningful to study the investment problem for the DC plan.

As the cornerstone of the study on portfolio selection for the DC annuity scheme, Vigna and Haberman (2001) [33] derived the optimum investment policy by the variance minimization. Hjgaard et al.(2007) [34] investigated the mean-variance portfolio choice and efficient frontier for the DC pension scheme. Vigna(2009) [35] discussed the unavailability of CRRA and CARA utility functions in the asset allocation under the DC scheme. Nkeki(2012) [36] studied the inflation hedging strategy under mean-variance portfolio selection for the DC annuity plan. He and Liang(2013) [37] investigated the stochastic optimal control problem for the DC pension plan in a continuous-time setting. Yao et al.(2013) [10] used Conditional Value-at-Risk (CVaR) to measure risk and adopted the methodology of nonparametric estimation to explore the meanCVaR portfolio selection problem. Vigna(2014) [11] discussed the advantage of mean-variance criterion based on the portfolio choice with the DC scheme. Wu and Chen(2015) [38] considered the Nash equilibrium strategy for the multi-period mean-variance asset allocation under regime-switching framework.

In this thesis, we mainly study the portfolio choice for the DC scheme under regime-switching framework. Based on the fundamental formulation, we extend the work by considering two types of constraints: uncertain time horizon and special market conditions. In addition, we use different approaches such as the embedding technique and the Lagrange method to transform the original objective function to an auxiliary feasible problem in the sense of the dynamic programming. In the following sections, we will propose an overview of the relevant constraints and methodologies.

2.2 Mathematical Models and Methodologies

2.2.1 Portfolio Selection with Constraints

(A) Regime-Switching Framework

In recent years, the regime-switching model, which aims to reflect the change in the market state, has become increasingly popular in the area of the portfolio selection. Specifically, there are mainly two kinds of market states in the market: the bullish market state and the bearish market state. When the market is in the bullish state, the perspectives from investors are enthusiastic and the market may be profitable beyond the expectation of the public. When the state is bearish, investors have pessimistic attitudes toward investment and the return rates of assets are even be negative. In fact, a variety of factors such as the GDP growth rate, the government policy, the inflation rate, the exchange rate and severe epidemics, have been dominating process of the financial market state, which undoubtedly determines the asset return rates and further influences the corresponding investment strategies. Therefore, it is necessary to take the account of regime-switching model into the portfolio selection problem. One of the most important work is from Hamilton (1989) [39] in which changes in the market is modeled by a finite-state Markov process. Since then, numerous studies focused on the portfolio selection optimization under regime-switching framework. Zhou and Yin(2003) [6] considered the Markowitz's mean-variance asset allocation with regime-switching in a continuous-time setting. Yin and Zhou(2004) [7] concerned the multi-period mean-variance asset allocation model and discussed the limits from the multi-period model to the continuous-time problem. Çakmak and Özekici(2006) [8] investigated various utility functions in the portfolio choice with a regime-switching model which is driven by a Markov chain with the completely observable state. Xie(2009) [40] studied the continuous-time mean-variance asset-liability management with regime-switching. Wu and Chen(2015) [38] focused on the Nash equilibrium strategy for the multi-period mean-variance portfolio choice optimization under regime-switching framework. Chen and Yao(2018) [41] considered the no-shorting constraints for the mean-variance portfolio choice with regime-switching criterion. In addition, some studies also investigated the portfolio selection for the DC annuity scheme in a regime-switching market. Korn et al.(2011) [42] and Chen and Delong(2015) [43] studied the continuous-time model for the DC pension plan under regimes-switching environment. Yao et al.(2016) [44] discussed the portfolio selection and mortality risk in the DC pension scheme under regime-switching. Lin et al.(2019) [45] concerned the multi-period portfolio selection optimization with the DC pension plan in the decumulation phase. More works relevant to the regime-switching including Elliott(2005) [46], Chen et al.(2008) [47], Chen and Yang(2011) [48], Yu and Zhang(2014) [49] and Wu and Chen(2015) [38].

(B) Uncertain Time Horizon

An important extension for the portfolio selection model is to break the routine assumption that the investment horizon is deterministic, that is to say, the investment time horizon is assumed to be stochastic at the beginning of the investment. In fact, there exist numerous exogenous and endogenous factors that enforce investors to quit the market. Of various factors, exogenous factors include natural disaster, serious epidemics and mortality risk, and a typical example of endogenous events is that the investor exits the market as long as he/she achieves the predetermined investment return target during the investment horizon. Therefore, many studies on the portfolio choice model focused on the uncertain time horizon in the recent decades. Martellini and Urošević(2006) [50] generalized the static mean-variance analysis combined with stochastic time horizon. Yi et al.(2008) [51] studied the multi-period asset-liability management with the assumption of uncertain investment horizon. Wu and Li(2011) [52] considered the mean-variance asset allocation in a regime-switching market where the exit time is exogenously uncertain. Zhang and Li(2012) [53] studied the multi-period mean-variance portfolio choice problem with uncertain investment horizon with serially correlated return rates. Yao et al.(2013) [10] investigated mean-variance portfolio selection with endogenous liabilities and an exogenous investment exit time in a multi-period version. Zeng et al.(2013) [54] concerned an investment-consumption situation under regime-switching framework with exogenous exit time. Wu et al.(2014) [55] considered the state-dependent exit time in a multi-period mean-variance model with regime-switching. For previous studies on stochastic time horizon, one can refer to Yaari(1965) [56], Hakansson(1969) [57], Merton(1969) [58] and Richard(1975) [59]. The assumption of uncertain time horizon has also been applied to the portfolio selection optimization for the DC pension scheme and are often represented in the form of mortality risk. Yao et al.(2014) [13] concerned the portfolio selection for the DC pension scheme with stochastic income and mortality risk. Wu and Zeng(2015) [14] investigated the equilibrium investment policy for the DC pension model with mortality risk. Yao et al.(2016) [44] considered the multi-period portfolio choice for the DC plan with mortality risk.

(C) Special Market Condition

Besides the regime-switching model, which has been seen as the research basis for portfolio selection model nowadays, there are increasing number of research studying special market condition assumptions such as complete, incomplete information market and bankruptcy market state. Lim and Zhou(2002) [28] considered the continuous-time mean-variance investment management under a perfect market in which interest rate is assumed to be random. Honda(2003) [60] studied the portfolio selection optimization with the unobservable and regime-switching mean return. Bäuerle and Rieder(2005) [61] considered the optimal portfolio choice with unobservable Markovian drifted process. Cheung and Yang(2007) [62] investigated consumption model with a bankruptcy state under regime-switching framework. Bensoussan et al.(2009) [63] focused on partially observed real prices for consumption and portfolio choice. Elliott et al.(2010) [31] concerned a hidden Markovian regime-switching model for the mean-variance portfolio selection where the market can not be fully observable. Wu and Zeng(2013) [64] investigated the multi-period mean-variance model with a bankruptcy state under regime-switching framework. Zhang et al.(2018) [65] considered the optimum investment policy with the DC pension plan and incomplete information. For more research, we refer to Gennotte(1986) [66], Brendle(2006) [67], Xiong and Zhou(2007) [30] and Liesiö et al.(2008) [68].

2.2.2 Methodology

In the multi-period Markowitz's mean-variance portfolio selection model, the asset is allocated at each point of time in a given investment time horizon T. The investor can adjust his portfolio choice at the beginning of each time period. Moreover, instead of representing the mean and variance of the portfolio as two isolated investment targets, Markowitz's model enables the investor to maximize the expectation of final wealth with an acceptable risk level valued by the variance, or minimize the investment risk for the predetermined expected final wealth. In the following section, two typical methods are introduced for solving the mean-variance portfolio choice optimization under a multi-period version.

(A) Embedding Technique

To derive the optimum investment strategy in the sense of dynamic programming, we need to use relevant methodology to convert the original problem due to the inseparability of the variance in the objective function. Specifically, denote I^t an information set at time t, the expectation operator $E(\cdot)$ satisfies the smoothing property: for any $m > n$, $E(E(\cdot|I^m)|T_n) = E(\cdot|I^n)$. However, the variance does not hold this property. Because of the optimization principle no longer applicable, we can not directly use dynamic programming approach in such cases. Li and Ng(2000) [2] firstly proposed the embedding technique to transform the original optimization problem into an auxiliary problem which can be solved by dynamic programming under a multi-period version. They suppose that there are n risky assets and one risk-free asset in the finance market. An investor possesses initial wealth x_0 and enters the market at time 0 to allocate his asset in the finance market. And at the beginning of each of the following $T-1$ time periods, the investor can adjust the investment policy. Denote $\boldsymbol{e_t} = [e_t^1, e_t^2, ..., e_t^n]'$ the return of risky assets in the market at time period t, where e_t^i is the i-th asset return at time period t while letting e_t^0 the risk-free asset return at period t, $i = 1, 2, ..., n$. Denote x_i the investor's wealth at time period t and u_t^i the amount invested in the i-th risky asset at period t. Then the model is shown as follows:

$$E(\omega) \quad \begin{cases} max\ E(x_T) - \omega Var(x_T) \\ s.t.\ x_{t+1} = \sum_{i=1}^n e_t^i u_t^i + \left(x_t - \sum_{i=1}^n u_t^i\right) e_t^0 = e_t^0 x_t + \boldsymbol{P_t'} \boldsymbol{u_t} \\ t = 0, 1, ..., T-1, \end{cases} \tag{2.1}$$

where $\boldsymbol{P_t} = [P_t^1, P_t^2, ..., P_t^n]' = [e_t^1 - e_t^0, e_t^2 - e_t^0, ..., e_t^n - e_t^0]'$, $\boldsymbol{u_t} = [u_t^1, u_t^2, ..., u_t^n]'$, $E[x_T]$ and $Var[x_T]$ denote the expected terminal wealth and the corresponding variance respectively, $\omega \geq 0$ denotes the risk aversion coefficient. Then the auxiliary problem is represented as follows:

$$A(\lambda,\ \omega)\quad \begin{cases} max\ E(-\omega x_T^2 + \lambda x_T) \\ s.t.\ x_{t+1} = e_t^0 x_t + \boldsymbol{P_t' u_t},\ t = 0, 1, ..., T-1, \end{cases} \tag{2.2}$$

where $\lambda \in (-\infty, +\infty)$. The problem $A(\lambda,\ \omega)$ provides a separable structure for the dynamic programming, and it has a quadratic form of objective function with a linear constraints. We can derive the optimal solution of problem $E(\omega)$ by problem $A(\lambda,\ \omega)$ with $\lambda^* = 1 + 2\omega E(x_T)|u^*$ where u^* is the corresponding optimal strategy.

The embedding technique has been investigated in numerous studies on the multi-period mean-variance portfolio choice. For instance, by the embedding technique, Yi et al.(2008) [51] derived the analytical solution for the asset-liability management with stochastic time horizon. Wu and Li(2011) [52] solved the problem with stochastic cash flow by the embedding technique. Wu et al.(2014) [55] investigated the model with stochastic time frame under regime-switching criterion, and obtained the closed form of the solution by adopting the embedding technique.

(B) Lagrange Multiplier Method

Another commonly adopted method in multi-period mean-variance portfolio choice is named the Lagrange multiplier method, which is first proposed by Li et al.(2002) [69]. We formulate the following model to demonstrate the method:

$$\begin{cases} min\ Var(x_T) = E(x_T - d)^2 \\ s.t.\ E(x_T) = d \\ x_{t+1} = e_t^0 x_t + \boldsymbol{P_t' u_t},\ t = 0, 1, ..., T-1. \end{cases} \tag{2.3}$$

Due to the convexity of the above problem, we introduce a Lagrange multiplier $\lambda \in R$ to deal with the constraint $E(x_T = d)$, then we have the transformed objective function as follows:

$$minE(u_T - d)^2 + 2\lambda(E(u_T) - d) = E(x_T - (d - \lambda))^2 - \lambda^2. \tag{2.4}$$

Since λ^2 has no effect on the choice of the optimal strategy, we neglect the term to obtain the following problem:

$$\begin{cases} min\ E(x_T - (d-\lambda))^2 \\ s.t.\ x_{t+1} = e_t^0 x_t + \boldsymbol{P_t'} \boldsymbol{u_t},\ t = 0, 1, ..., T-1. \end{cases} \tag{2.5}$$

Compared with the embedding technique, the Lagrange multiplier method does not need complexed computation and transformation for the objective function. Lim and Zhou(2002) [28] used Lagrange duality theory to solve the continuous-time portfolio choice problem. Wei and Ye(2007) [9] studied the mean-variance portfolio choice with risk control and derived the corresponding analytical solution by Lagrange multiplier method. Ma et al.(2015) [70] considered the continuous-time asset-liability management and got the result by Lagrange multiplier method.

2.3 Concluding Remarks

This section reviews the studies of mean-variance portfolio selection and discusses significant extensions with various constraints such as regime-switching, uncertain time horizon and some special market conditions. The transformation methods, the embedding technique and the Lagrange multiplier method are also introduced, which simplify the model formulation and offer good ways to derive the corresponding optimal solution.

This thesis mainly focuses on the multi-period mean-variance portfolio selection optimization with constraints including regime-switching, mortality risk, state-dependent exit probability, bankruptcy state and incomplete information market. The embedding technique and the Lagrange method will be used to derive the analytical solution and efficient fronter in our study.

CHAPTER 3

Mean-Variance Portfolio Selection for a Defined Contribution Plan with Regime-Switching and Mortality Risk

3.1 General Overview

In this chapter, we study the mean-variance portfolio selection for a defined contribution pension plan with mortality risk under regime-switching framework. Investors, especially senior ones, have mortality risks during the accumulation phase of the DC pension plan. That is, some investors may die due to some unexpected reasons such as a deadly disease and a car accident. Therefore, mortality risk has been studied recently by an increasing number of researchers in portfolio selection optimization problem. For example, Yao et al.(2014) [13] considered the asset allocation problem for the DC pension plan with mortality risk and stochastic income, and they used the Lagrange method to obtain the optimal strategy. Wu and Zeng(2015) [14] studied the time-consistent investment policy for the defined contribution scheme with mortality risk.

As far as we know, for mean-variance asset allocation for the DC pension plan in a multi-period setting, no study has examined the feasibility of the embedding technique in the model due to the consideration of the stochastic income in the wealth process which makes the computation even more complex. Therefore, this chapter presents the first attempt of this method. Specifically, we assume that both the asset returns and wage income depend on the stochastic market which is dominated by a discrete time-homogeneous Markov chain. By the embedding technique and innovative iteration process we obtain the closed form of the solution and the corresponding efficient frontier with the dynamic programming.

In addition, some special cases and numerical analysis are presented to show the impact of regime-switching and mortality risk on the efficient frontier.

3.2 Model Formulation

We first assume that an investor joins a DC pension plan at time 0 and plans to retire after T time periods. The employee has probability to pass away at time τ ($\tau = 0, 1, 2, ...T$) due to deadly illness or any other accidents, which is shown as follows:

$$P_t = P(\tau = t), t = 0, 1, .., T-1; \quad P_T = P(\tau \geq T). \tag{3.1}$$

We construct a multi-period financial market whose state is dominated by a time-homogeneous Markov chain $\{X_t, X_t = 1, 2, .., S; t = 0, 1, .., T-1\}$ with transition matrix Q in a complete probability space (Ω, F, P). For example, if the market mode at time 0 is i then it would be j at time 1 with probability $Q(i, j)$ which could be considered as the (i, j)th entry of the transition matrix Q. We assume there are n risky asset represented as asset $1, 2, ..., n$ and one risk-free asset noted as asset 0 which are all dependent on the market state. The risky asset returns at time t with market state $X_t = i$ are denoted as $e_t(i) = (e_t^1(i), e_t^2(i), .., e_t^n(i))'$ where $e_t^k(i)$ ($t = 0, 1, .., T-1; k = 1, 2, .., n$) stands for the return of the kth asset at time t with market state i and $'$ refers to the transpose of the vector or matrix, and denote $r_f(i)$ the deterministic interest rate with market state i. We then define the investor receives the payoff $y_t(i)$ which also depends on the market state i, denote c_t ($0 \leq c_t \leq 1$) the deterministic contribution rate at time $t = 0, 1, .., T-1$. The investor pays $c_t y_t(i)$ as defined contribution into his DC pension fund account from which he can receive pension on his retirement. The DC pension plan has to be stopped at time τ if the investor passes away due to diseases or any other accidents.

Throughout the paper, the following assumptions have been made:

A3.1 We assume that $E(e_t(i)e_t'(i))$ ($t = 0, 1, .., T-1; i = 1, 2, .., S$) is positive definite, $\sigma_{kl}(i) = Cov(e_t^k, e_t^l)$; and $\sigma(i) = (\sigma_{kl}(i))_{k,l=1,2,..,n}$, we can then easily get the conclusion that $\sigma(i)$ is also positive definite.

A3.2 Note that the transition property of the Markov market state X_t exists, and the vectors $e_t(X_t), t = 0, 1, 2, ..., T-1$ are not independent. However, when the market state is fixed, i.e., the fixed market mode is i, then vectors $e_t(i), for\ t = 0, 1, ..., T-1$ are considered to be independent.

A3.3 For any $i = 1, 2.., S$ and $t = 0, 1, .., T-1$, the wage income $y_t(i)$ and return rates of the stocks $e_t(i)$ are independent.

A3.4 For any $i = 1, 2.., S$ and $t = 0, 1, .., T-1$, The interest rate $r_f(i)$ is independent of return rates of the stocks $e_t(i)$ and the wage income $y_t(i)$.

A3.5 We assume that $P_T \neq 0$, if $P_T = 0$ it means that the stopped time $\tau \leq T$, which suggests the formulation of T time period is redundant.

A3.6 short sales are permitted and transaction costs are neglected. Credit and loan are allowed. Capital raising and withdrawal are prohibited.

Remark 3.1. A3.1 is derived From Li and Ng (2000), which means there exists a perfect market.

Note the property of Markov transition matrix Q, denote $Q(i, j)$ the transition probability which is the (i, j)th entry of Q if the market state switches from i to j within one time period. If the switches take n time periods, the corresponding probability is $Q^n(i, j)$ which is the (i, j)th entry of Q^n, the n powers of Q.

D3.1 For any S-dimension column vector q, define that $\bar{Q}_q$ is $Qdiag(q)$, Q_q^n means n powers of Q_q, and the S-dimension vector $\bar{Q}_q^n$ is equivalent to $Q_q^n\mathbb{I}$ where $\mathbb{I} = (1, 1, .., 1)'$ is a S-dimension column vector. Particularly, we define $\bar{Q}_q^0 = \mathbb{I}$, $\bar{Q}_q = \bar{Q}_q^1 = Q_q\mathbb{I} = Qq$ and $\bar{Q}_q(i) = \sum_{j=1}^{S} Q(i, j)q(j)$. For $k = 0$, $\bar{Q}_q^k = \mathbb{I}$; for $k < 0$, $\bar{Q}_q^k = \mathbf{0}$. Furthermore, for any time-dependent matrix $A_t(S \times S)$ and any vector $a_t(S \times 1)$, define $\sum_{t=1}^{0} a_t = \mathbf{0}$ where $\mathbf{0}$ is the $S \times 1$ zero vector and $\prod_{t=1}^{0} A_t = \mathbb{I}$ where $\mathbb{I}$ is the $S \times S$ unit matrix.

D3.2 For any S-dimension vector a we define $\widetilde{Q}_q[a(j)](i) = \sum_{j=1}^{S} Q(i, j)a(j)q(i)$; For $k = 0$, $\widetilde{Q}_q^k = \mathbb{I}$, for $k \leq 0$, $\widetilde{Q}_q^k = \mathbf{0}$.

D3.3 For any S-dimension vectors a, b and c, $a * b/c$ stands S-dimension vector whose ith element is $a(i) * b(i)/c(i)$.

D3.4 $r_k(i) = E(e_t^k)$; $r_k^e(i) = r_k(i) - r_f(i)$; $r^e(i) = (r_1^e(i), r_2^e(i), .., r_n^e(i))$.

D3.5 $R_t(i) = [e_t^1(i) - r_f(i), e_t^2(i) - r_f(i), \ldots, e_t^n(i) - r_f(i)]'$; We define S-dimension vector h, $q_k (k = 1, 2, ..., 5)$ whose ith entry is $h(i) = r^e(i)'E^{-1}[R_t(i)R_t(i)']r^e(i)$; $q_1(i) = r_f^2(i)(1 - h(i))$; $q_2(i) = r_f(i)(1 - h(i))$; $q_3 = r_f(i)y_t(i)(1 - h(i))$; $q_4(i) = r_f^2(i)y_t(i)(1 - h(i))$; $q_5(i) = r_f^2(i)y_t^2(i)(1 - h(i))$ respectively.

Remark 3.2. Note that in **D**3.1-**D**3.3, the meaning of i is different from that in **A**3.1-**A**3.6 and **D**3.4-**D**3.5, i refers to the ith vector element or the ith matrix row in **D**3.1-**D**3.3.

Now assume that the investor in the pension plan can make investment choices in the finance market. We denote x_t the wealth at time t $(t = 0, 1, .., T)$ and u_t^k $(k = 1, 2, .., n; t = 0, 1, .., T - 1)$ the investment amount in the kth risky asset at time t. We then have the wealth dynamics as follows:

$$\begin{aligned} x_{t+1} &= \left(x_t + c_t y_t(X_t) - \sum_{k=1}^{n} u_t^k \right) r_f(X_t) + \sum_{k=1}^{n} e_t^k(X_t) u_t^k \\ &= r_f(X_t) \left(x_t + c_t y_t(X_t) \right) + R_t'(X_t) u_t, \ t = 0, 1, .., T - 1. \end{aligned} \tag{3.2}$$

The investor adjusts his investment strategy at each time point t, we formulate the following portfolio selection model under the MV criterion with mortality risk at each time period during the time horizon:

$$E(\omega) \begin{cases} max \ E_i(x_{T\wedge\tau}) - \omega Var_i(x_{T\wedge\tau}) \\ s.t. \ x_{t+1} = r_f(X_t) \left(x_t + c_t y_t(X_t) \right) + R_t'(X_t) u_t, \\ t = 0, 1, .., T - 1; \ \omega > 0, \end{cases} \tag{3.3}$$

where $E_i(x_{T\wedge\tau})$ means $E(x_{T\wedge\tau} | X_1 = i)$ and $Var_i(x_{T\wedge\tau})$ stands for $Var(x_{T\wedge\tau} | X_1 = i)$ respectively.

By Li and Ng (2000) [2], we use the embedding technique to convert problem $E(\omega)$ into the following auxiliary formulation:

$$A(\lambda, \ \omega) \begin{cases} max \ E_i(-\omega x_{T\wedge\tau}^2 + \lambda x_{T\wedge\tau}) \\ s.t. \ x_{t+1} = r_f(X_t) \left(x_t + c_t y_t(X_t) \right) + R_t'(X_t) u_t, \\ t = 0, 1, .., T - 1; \ \omega > 0. \end{cases} \tag{3.4}$$

Note that $\lambda^* = 1 + 2\omega E_i(x_{T\wedge\tau})|_{u^*}$. From Li and Ng (2000) [2], We could easily clarify the relationship between problem $E(\omega)$ and problem $A(\lambda,\ \omega)$: If u^* is an optimal policy of $E(\omega)$, then u^* is also an optimal strategy of $A(\lambda^*,\ \omega)$; if u^* is an optimal strategy for $A(\lambda,\ \omega)$, a necessary condition for u^* optimizing $E(\omega)$ is $\lambda = \lambda^*$.
We could use dynamic programming approach to solve $A(\lambda^*,\ \omega)$ because of its inseparability. When we obtain the optimal strategy u^* for $E(\omega)$, we could also get a set of coordinate $(E_i(x_{T\wedge\tau})|_{u^*}, Var(x_{T\wedge\tau})|_{u^*})$ which is called the efficient frontier.

3.3 Analytical Solution for $A(\lambda,\ \omega)$

Note that $E_i(x_{T\wedge\tau}) = E_i(\sum_{t=0}^{T} P_t x_t)$, so we could rewrite problem $A(\lambda,\ \omega)$ as follows:

$$A(\lambda,\ \omega)\ \begin{cases} max\ E_i\Big(\sum_{t=0}^{T} P_t(-\omega x_t^2 + \lambda x_t)\Big) \\ s.t.\ x_{t+1} = r_f(X_t)\Bigg(x_t + c_t y_t(X_t)\Bigg) + R_t'(X_t)u_t, \\ t = 0, 1, .., T-1;\ \omega > 0. \end{cases} \tag{3.5}$$

Lemma 3.1. For $k = 1, 2, ..., 5;\ i = 1, 2, .., S,\ \ 0 < h(i) < 1,\ \ q_k(i) > 0.$

Proof. We first know

$$E(R_t(i)R_t'(i)) = \sigma(i) + r^e(i)r^e(i)'. \tag{3.6}$$

According to the Woodbury formula in Max [71], we inverse the matrixes on both sides of the $Equation$(3.6) and have

$$\begin{aligned} E^{-1}(R_t(i)R_t'(i)) &= [\sigma(i) + r^e(i)r^e(i)']^{-1} \\ &= \sigma^{-1}(i) - \frac{\sigma^{-1}(i)r^e(i)r^e(i)'\sigma^{-1}(i)}{1 + r^e(i)'\sigma^{-1}(i)r^e(i)}. \end{aligned} \tag{3.7}$$

Then premultiply $r^e(i)'$ and postmultiply $r^e(i)$ on both sides of the $Equation$ (3.7), we get

$$h(i) = \frac{r^e(i)'\sigma^{-1}(i)r^e(i)}{1 + r^e(i)'\sigma^{-1}(i)r^e(i)}. \tag{3.8}$$

Obviously, $r^e(i) > 0$ and $\sigma(i)$ is positive definite, and we get $0 < h(i) < 1$. From

D3.5 we have $q_k(i) > 0,\ \ k = 1, 2, ..., 5.$ □

Lemma 3.2. For S-dimension vector a, we have

$$E_i\Big(\prod_{l=1}^{t} a(X_l)\Big) = \bar{Q}_a^t(i),\ \ t = 1, 2, .., T. \tag{3.9}$$

Proof. When $t = 1$, *Equation* (3.9) reduces to

$$E_i(a(X_1)) = \sum_{j=1}^{S} Q(i,j)a(j) = \bar{Q}_a(i).$$

Therefore, *Equation* (3.9) holds true when $t = 1$, we then assume it holds true when $t = m$. Then for $t = m + 1$ we have

$$\begin{aligned} E_i\Big(\prod_{l=1}^{m+1} a(X_l)\Big) &= \sum_{j=1}^{S} Q(i,j)a(j)E\Big(\prod_{l=2}^{m+1} a(X_l)|S_1 = j\Big) \\ &= \sum_{j=1}^{S} Q(i,j)a(j)E_j\Big(\prod_{l=2}^{m} a(X_l)\Big) \\ &= \sum_{j=1}^{S} Q_a(i,j)\bar{Q}_a^{m-1}(j) = \Big(Q_a\bar{Q}_a^{m-1}\mathbb{I}\Big)(i) = \bar{Q}_a^m(i). \end{aligned}$$

Hence, *Equation* (3.9) holds true for $t = m + 1$, By the mathematical induction we could prove *Lemma* 3.2 holds for $t = 1, 2, .., T$. □

Now we focus on the analytical solution for $A(\lambda,\ \omega)$. First, we define the value functions for $A(\lambda,\ \omega)$ as follows:

$$f_t(i, x_t) = max_{u_t,..,u_T} E\Big\{\sum_{k=t}^{T} P_k(-\omega x_k^2 + \lambda x_k|X_t = i, x_t)\Big\}. \tag{3.10}$$

By the value functions above, we can obtain the following Bellman's equation:

$$\begin{aligned} f_t(i, x_t) &= max_{u_t} E\Big\{P_t(-\omega x_t^2 + \lambda x_t) + f_{t+1}(X_{t+1}, x_{t+1})\Big\} \\ &= P_t(-\omega x_t^2 + \lambda x_t) \\ &\quad + max_{u_t} E\Big\{\sum_{j=1}^{S} Q(i,j)f_{t+1}(j, r_f(i)(x_t + c_t y_t(i)) + R_t'(i)u_t)\Big\} \\ f_T &= P_T(-\omega x_T^2 + \lambda x_T). \end{aligned} \tag{3.11}$$

Where $t = 0, 1, .., T-1$.

Theorem 3.1. The optimal investment strategy for $A(\lambda,\ \omega)$ is of the following form:

$$
\begin{aligned}
u_t^*(i,x_t) &= \Big\{\frac{\lambda\sum_{k=t+1}^{T}P_k\bar{Q}_{q_2}^{k-(t+1)}(i)}{2\omega\sum_{k=t+1}^{T}P_k\bar{Q}_{q_1}^{\;k-(t+1)}(i)} - r_f(i)x_t - cr_f(i)y(i) \\
&\quad -\frac{cQ\{\sum_{l=1}^{T-(t+2)}[\widetilde{Q}_{q_2}^{l-1}(\sum_{k=t+1+l}^{T}P_k\bar{Q}_{q_1}^{k-(t+1+l)}q_4)]\}(i)}{\sum_{k=t+1}^{T}P_k\bar{Q}_{q_1}^{k-(t+1)}(i)} \\
&\quad -\frac{c^2[Q_{q_2}^{T-(t+2)}(\bar{Q}_{q_5})](i)}{2\sum_{k=t+1}^{T}P_k\bar{Q}_{q_1}^{k-(t+1)}(i)}\Big\}E^{-1}(R_t(i)R_t(i)')r^e(i), \\
t &= 0,1,..,T-1;\ i=1,2,..,S. \qquad (3.12)
\end{aligned}
$$

And the corresponding value functions are as follows:

$$
\begin{aligned}
f_t(i,x_t) &= -[\omega q_1(i)\sum_{k=t+1}^{T}P_k\bar{Q}_{q_1}^{k-(t+1)}(i)+\omega P_t]x_t^2 \\
&\quad +[\lambda q_2(i)\sum_{k=t+1}^{T}P_k\bar{Q}_{q_2}^{k-(t+1)}(i)+\lambda P_t]x_t \\
&\quad -2\omega c\Big\{\sum_{j=1}^{T-(t+1)}[\widetilde{Q}_{q_2}^{j-1}(\sum_{k=t+j}^{T}P_k\bar{Q}_{q_1}^{k-(t+j)}q_4)](i)\Big\}x_t \\
&\quad -\omega c^2\Big\{\{[Q_{q_2}^{T-(t+2)}(\bar{Q}_{q_5})]q_2\}(i)\Big\}x_t+M_t(i), \\
t &= 0,1,..,T-1;\ i=1,2,..,S, \\
f_T(i,x_T) &= -\omega P_Tx_T^2+\lambda P_Tx_T, \qquad (3.13)
\end{aligned}
$$

where $M_t(i)$ is a constant term which dose not include the wealth x_t.

Proof. When $t=T$, by definition of the value function we have

$$f_T(i,x_T) = -\omega P_T x_T^2 + \lambda P_T x_T.$$

Thus *Equation* (3.13) holds true when $t = T$. We then consider when $t = T - 1$

$$\begin{aligned}
f_{T-1}(i, x_{T-1}) &= max_{u_{T-1}} E\Big\{P_{T-1}(\lambda x_{T-1} - \omega x_{T-1}^2) + \sum_{j=1}^{S} Q(i,j) P_T \\
&\quad \cdot\{\lambda[r_f(i)(x_{T-1} + cy(i)) + R'_{T-1}(i)u_{T-1}] - \omega[r_f^2(i)x_{T-1}^2 \\
&\quad + c^2 r_f^2(i) y^2(i) + u'_{T-1} R_{T-1}(i) R'_{T-1}(i) u_{T-1} + 2cr_f^2(i)y(i)x_{T-1} \\
&\quad + 2r_f(i)x_{T-1}R'_{T-1}(i)u_{T-1} + 2cr_f(i)y(i)R'_{T-1}(i)u_{T-1}]\}\Big\} \\
&= max_{u_{T-1}} \Big\{P_{T-1}(\lambda x_{T-1} - \omega x_{T-1}^2) + P_T \\
&\quad + \{\lambda r_f(i)x_{T-1} + c\lambda r_f(i)y(i) + \lambda r^e(i)'(i)u_{T-1} - \omega r_f^2(i)x_{T-1}^2 \\
&\quad + c^2\omega r_f^2(i)y^2(i) + \omega u'_{T-1}E(R_{T-1}(i)R'_{T-1}(i))u_{T-1} \\
&\quad + 2c\omega r_f^2(i)y(i)x_{T-1} 2\omega r_f(i)x_{T-1}r^e(i)'(i)u_{T-1} \\
&\quad + 2c\omega r_f(i)y(i)r^e(i)'(i)u_{T-1}\}\Big\}.
\end{aligned}$$

Since $E^{-1}(R_{T-1}(i)R'_{T-1}(i)) > 0$, we know the optimal u_{T-1}^* exists and it can be derived by solving $\frac{\partial f_{T-1}}{\partial u_{T-1}} = 0$, then we have

$$u_{T-1}^* = \Big(\frac{\lambda}{2\omega} - r_f(i)x_{T-1} - cr_f(i)y(i)\Big)E^{-1}(R_{T-1}(i)R_{T-1}(i)')r^e(i).$$

Thus the *Equation* (3.12) holds true for $t = T - 1$. Substituting u_{T-1}^* back to f_{T-1}, we have the following equation:

$$\begin{aligned}
f_{T-1}(i, x_{T-1}) &= -\omega\Big(P_T q_1(i) + P_{T-1}\Big)x_{T-1}^2 + \lambda\Big(P_T q_2(i) + P_{T-1}\Big)x_{T-1} \\
&\quad -2c\omega q_4(i)x_{T-1} + P_T\Big(\frac{\lambda^2}{4\omega}h(i) + c\lambda q_3(i) - c^2\omega q_5(i)\Big).
\end{aligned}$$

Hence, *Equation* (3.13) holds true when $t = T - 1$. Based on the mathematical induction process, now we suppose *Equation* (3.12) and (3.13) hold true when

$t = m+1$ where $m \leq T-2$, then for $t = m$ we have

$$
\begin{aligned}
& f_m(i, x_m) \\
= \;& P_m(\lambda x_m + \omega x_m^2) + max_{u_m}\Big\{ \sum_{j=1}^{S} Q(i,j) \\
& \cdot f_{m+1}(j, r_f(i)(x_m + cy(i)) + R(i)' u_m) \Big\} \\
= \;& P_m(\lambda x_m + \omega x_m^2) \\
& + max_{u_m}\Big\{ -\sum_{j=1}^{S} Q(i,j)[\omega q_1(j) \sum_{k=m+2}^{T} P_k \bar{Q}_{q_1}^{k-(m+2)}(j) + \omega P_{m+1}] x_{m+1}^2 \\
& + \sum_{j=1}^{S} Q(i,j)[\lambda q_2(j) \sum_{k=m+2}^{T} P_k \bar{Q}_{q_2}^{k-(m+2)}(j) + \lambda P_{m+1}] x_{m+1} \\
& - \sum_{j=1}^{S} Q(i,j)\{2c\omega \sum_{l=1}^{T-(m+2)} [\widetilde{Q}_{q_2}^{l-1}(\sum_{k=m+1+l}^{T} P_k \bar{Q}_{q_1}^{k-(m+1+l)} q_4)](j)\} x_{m+1} \\
& - \sum_{j=1}^{S} Q(i,j)\{c^2 \omega [Q_{q_2}^{T-(m+3)} \bar{Q}_{q_5} q_2](j)\} x_{m+1} \Big\},
\end{aligned}
$$

then we extend the equation above and then have

$$
\begin{aligned}
& f_m(i, x_m) \\
= \;& \Big\{ P_m(\lambda x_m + \omega x_m^2) \\
& - \sum_{j=1}^{S} Q(i,j)[\omega q_1(j) \sum_{k=m+2}^{T} P_k \bar{Q}_{q_1}^{k-(m+2)}(j) + \omega P_{m+1}] \\
& \cdot [r_f^2(i) x_m^2 + c^2 r_f^2(i) y^2(i) + u_m' E(R_m(i) R_m(i)') u_m \\
& + 2c r_f^2(i) y(i) x_m + 2 r_f(i) x_m r^e(i)' u_m + 2c r_f(i) y(i) r^e(i)'] \\
& + \sum_{j=1}^{S} Q(i,j) \Big\{ [\lambda q_2(j) \sum_{k=m+2}^{T} P_k \bar{Q}_{q_2}^{k-(m+2)}(j) + \lambda P_{m+1}] \\
& - \{2c\omega \sum_{l=1}^{T-(m+2)} [\widetilde{Q}_{q_2}^{l-1}(\sum_{k=m+1+l}^{T} P_k \bar{Q}_{q_1}^{k-(m+1+l)} q_4)](j)\} \\
& - \{c^2 \omega [Q_{q_2}^{T-(m+3)} \bar{Q}_{q_5} q_2](j)\} \Big\} [r_f(i) x_m + c r_f(i) y(i) + r^e(i)' u_m].
\end{aligned}
\tag{3.14}
$$

Similarly, we obtain the following equation by solving $\frac{\partial f_m}{\partial u_m} = 0$:

$$
\begin{aligned}
&-\sum_{j=1}^{S} Q(i,j)[\omega q_1(j) \sum_{k=m+2}^{T} P_k \bar{Q}_{q_1}^{k-(m+2)}(j) + \omega P_{m+1}] \\
&\cdot[2E(R_m(i)R_m(i)')u_m + 2r_f(i)r^e(i)x_m + 2cr_f(i)y(i)r^e(i)] \\
&+\sum_{j=1}^{S} Q(i,j)\Big\{[\lambda q_2(j) \sum_{k=m+2}^{T} P_k \bar{Q}_{q_2}^{k-(m+2)}(j) + \lambda P_{m+1}] \\
&-\{2c\omega \sum_{l=1}^{T-(m+2)} [\widetilde{Q}_{q_2}^{l-1}(\sum_{k=m+1+l}^{T} P_k \bar{Q}_{q_1}^{k-(m+1+l)} q_4)](j)\} \\
&-\{c^2\omega[Q_{q_2}^{T-(m+3)} \bar{Q}_{q_5} q_2](j)\}\Big\} r^e(i) = 0.
\end{aligned}
$$

Consider $E^{-1}(R_m(i)R'_m(i)) > 0$, we know that the optimal strategy u_m^* exists, and by *Lemma* 3.2 we obtain u_m^* as follows:

$$
\begin{aligned}
u_m^* \;=\;& \Big\{\frac{\lambda \sum_{k=m+1}^{T} P_k \bar{Q}_{q_2}^{k-(m+1)}(i)}{2\omega \sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(i)} - r_f(i)x_m - cr_f(i)y(i) \\
&-\frac{\sum_{j=1}^{S} Q(i,j)\{c \sum_{l=1}^{T-(m+2)} [\widetilde{Q}_{q_2}^{l-1}(\sum_{k=m+1+l}^{T} P_k \bar{Q}_{q_1}^{k-(m+1+l)} q_4)](j)\}}{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(i)} \\
&-\frac{c^2[Q_{q_2}^{T-(m+2)}(\bar{Q}_{q_5})](i)}{2\sum_{k=t+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(i)}\Big\} E^{-1}(R_m(i)R_m(i)')r^e(i)\Big\}.
\end{aligned}
\tag{3.15}
$$

Rewrite *Equation* (3.15), we have

$$
\begin{aligned}
u_m^* \;=\;& [\frac{B_m(i)}{2A_m(i)} - r_f(i) - cr_f(i)y(i) - \frac{U_m(i)}{2A_m(i)} \\
&-\frac{V_m(i)}{2A_m(i)}]E^{-1}(R_m(i)R_m(i)')r^e(i),
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
A_m(i) &= \omega \sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(i), \\
B_m(i) &= \lambda \sum_{k=m+1}^{T} P_k \bar{Q}_{q_2}^{k-(m+1)}(i), \\
U_m(i) &= \sum_{j=1}^{S} Q(i,j)\{2c\omega \sum_{l=1}^{T-(m+2)} [\widetilde{Q}_{q_2}^{l-1}(\sum_{k=m+1+l}^{T} P_k \bar{Q}_{q_1}^{k-(m+1+l)} q_4)](j)\}, \\
V_m(i) &= c^2\omega[Q_{q_2}^{T-(m+2)} \bar{Q}_{q_5}](i).
\end{aligned}
$$

We then substitute *Equation* (3.16) into *Equation* (3.14) and obtain the following equation:

$$\begin{aligned}
f_m(i,x_m) &= P_m(\lambda x_m - \omega x_m^2) \\
&\quad -A_m(i)\Big\{r_f^2(i)x_m^2 + r_f^2(i)h(i)x_m^2 - \frac{B_m(i)}{A_m(i)}r_f(i)h(i)x_m \\
&\quad +\frac{U_m(i)+V_m(i)}{A_m(i)}r_f(i)h(i)x_m + 2cr_f^2(i)y(i)h(i)x_m + 2cr_f^2(i)y(i)x_m \\
&\quad +2r_f(i)h(i)x_m[\frac{B_m(i)}{2A_m(i)} - \frac{U_m(i)+V_m(i)}{2A_m(i)} - r_f(i)x_m - cr_f(i)y(i)] \\
&\quad +2cr_f(i)y(i)h(i)[\frac{B_m(i)}{2A_m(i)} - \frac{U_m(i)+V_m(i)}{2A_m(i)} - r_f(i)x_m - cr_f(i)y(i)]\Big\} \\
&\quad +[B_m(i) - (U_m(i)+V_m(i)]\Big\{r_f(i)x_m + cr_f(i)y(i) \\
&\quad +[\frac{B_m(i)}{2A_m(i)} - r_f(i) - cr_f(i)y(i) - \frac{U_m(i)}{2A_m(i)} - \frac{V_m(i)}{2A_m(i)}]h(i)\Big\} \\
&= [-\omega P_m - A_m(i)q_1(i)]x_m^2 \\
&\quad +[\lambda P_m - 2cA_m(i)q_4(i) \\
&\quad +B_m(i)q_2(i) - (U_m(i)+V_m(i))q_2(i)]x_m + M_m(i), \qquad (3.17)
\end{aligned}$$

where $M_m(i)$ is the constant term which excludes x_m. Rewriting *Equation* (3.17), we get

$$\begin{aligned}
f_m(i,x_m) &= -[\omega P_m + \omega q_1(i)\sum_{k=m+1}^{T} P_k\bar{Q}_{q_1}^{k-(m+1)}(i)]x_m^2 \\
&\quad +[\lambda P_m + \lambda q_2(i)\sum_{k=m+1}^{T} P_k\bar{Q}_{q_2}^{k-(m+1)}(i)]x_m \\
&\quad -2c\omega\Big\{\sum_{l=1}^{T-(m+1)}[\widetilde{Q}_{q_2}^{l-1}(\sum_{k=m+l}^{T} P_k\bar{Q}_{q_1}^{k-(m+l)}q_4)](i)\Big\}x_m \\
&\quad -c^2\omega\Big\{\{[Q_{q_2}^{T-(m+2)}(\bar{Q}_{q_5})]q_2\}(i)\Big\}x_m + M_m(i).
\end{aligned}$$

Hence, according to *Equation* (3.12) and *Equation* (3.13), *Equation* (3.16) and *Equation* (3.17) hold true when $t = m$ where $m \leq T-2$. By mathematical induction, the theorem is true when $t = 0, 1, ..., T-1$. □

3.4 Analytical Solution for $E(\omega)$

Given the optimal strategy for $A(\lambda,\ \omega)$ and the dynamics of wealth process, in order to derive the solution for $E(\omega)$, we first substitute *Equation* (3.12) into *Equation* (3.2), then we obtain the following equation about the wealth x_{m+1}:

$$\begin{aligned}
x_{m+1} &= r_f(X_m)x_m + cr_f(X_m)y(X_m) + R_m'(X_m)u_m^* \\
&= r_f(X_m)x_m + cr_f(X_m)y(X_m) \\
&\quad +[\frac{B_m(X_m)}{2A_m(X_m)} - r_f(X_m) - cr_f(X_m)y(X_m) - \frac{U_m(X_m)}{2A_m(X_m)} - \frac{V_m(X_m)}{2A_m(X_m)}] \\
&\quad \cdot R_m'(X_m)E^{-1}(R_m(X_m)R_m(X_m)')r^e(X_m) \\
&= [1 - R_m'(X_m)E^{-1}(R_m(X_m)R_m(X_m)')r^e(X_m)][r_f(X_m)x_m \\
&\quad +cr_f(X_m)y(X_m)] + [\frac{B_m(X_m)}{2A_m(X_m)} - \frac{U_m(X_m)}{2A_m(X_m)} - \frac{V_m(X_m)}{2A_m(X_m)}] \\
&\quad \cdot R_m'(X_m)E^{-1}(R_m(X_m)R_m(X_m)')r^e(X_m), \\
m &= 0, 1, .., T-1.
\end{aligned} \tag{3.18}$$

Then square both sides of *Equation* (3.18), we get the following formulation of the square of wealth x_{m+1}^2:

$$\begin{aligned}
x_{m+1}^2 &= [1 - R_m'(X_m)E^{-1}(R_m(X_m)R_m(X_m)')r^e(X_m)]^2 \\
&\quad \cdot [r_f(X_m)x_m + cr_f(X_m)y(X_m)]^2 \\
&\quad +[\frac{B_m(X_m)}{2A_m(X_m)} - \frac{U_m(X_m)}{2A_m(X_m)} - \frac{V_m(X_m)}{2A_m(X_m)}]^2 \\
&\quad \cdot r^e(X_m)'E^{-1}(R_m(X_m)R_m(X_m)') \\
&\quad \cdot R_m(X_m)R_m(X_m)'E^{-1}(R_m(X_m)R_m(X_m)')r^e(X_m) \\
&\quad +[\frac{B_m(X_m)}{2A_m(X_m)} - \frac{U_m(X_m)}{2A_m(X_m)} - \frac{V_m(X_m)}{2A_m(X_m)}] \\
&\quad \cdot R_m(X_m)'E^{-1}(R_m(X_m)R_m(X_m)')r^e(X_m) \\
&\quad \cdot [1 - R_m'(X_m)E^{-1}(R_m(X_m)R_m(X_m)')r^e(X_m)] \\
&\quad \cdot [r_f(X_m)x_m + cr_f(X_m)y(X_m)], \\
m &= 0, 1, .., T-1.
\end{aligned} \tag{3.19}$$

Taking conditional expectation on both sides of *Equation* (3.18) and (3.19), we get

$$\begin{aligned} E(x_{m+1}|X_0,X_1,..,X_m) &= q_2(X_m)E(x_m|X_0,X_1,..,X_m)+cq_3(X_m) \\ &+[\frac{B_m(X_m)}{2A_m(X_m)}-\frac{U_m(X_m)}{2A_m(X_m)}-\frac{V_m(X_m)}{2A_m(X_m)}]h(X_m), \end{aligned} \tag{3.20}$$

$$\begin{aligned} E(x^2_{m+1}|X_0,X_1,..,X_m) &= q_1(X_m)E(x^2_m|X_0,X_1,..,X_m)+c^2q_5(X_m) \\ &+2cq_4(X_m)E(x_m|X_0,X_1,..,X_m) \\ &+[\frac{B^2_m(X_m)}{4A^2_m(X_m)}+\frac{U^2_m(X_m)}{4A^2_m(X_m)}+\frac{V^2_m(X_m)}{4A^2_m(X_m)} \\ &-\frac{B_m(X_m)U_m(X_m)}{2A^2_m(X_m)}-\frac{B_m(X_m)V_m(X_m)}{2A^2_m(X_m)} \\ &+\frac{U_m(X_m)V_m(X_m)}{2A^2_m(X_m)}]h(X_m), \end{aligned} \tag{3.21}$$

where $h(X_m),\ q_1(X_m),\ q_2(X_m),\ q_3(X_m),\ q_4(X_m)$ and $q_5(X_m)$ are shown in **D**3.5. Noting that x_m is known given the market state X_{m-1}, and rewriting *Equation* (3.20) and (3.21), we obtain

$$\begin{aligned} & E(x_{m+1}|X_0,X_1,..,X_m) \\ = \; & q_2(X_m)E(x_m|X_0,X_1,..,X_m)+\frac{\lambda}{2\omega}b^1_m(X_m)+b^2_m(X_m) \\ = \; & q_2(X_m)E(x_m|X_0,X_1,..,X_{m-1})+\frac{\lambda}{2\omega}b^1_m(X_m)+b^2_m(X_m), \end{aligned} \tag{3.22}$$

$$\begin{aligned} & E(x^2_{m+1}|X_0,X_1,..,X_m) \\ = \; & q_1(X_m)E(x^2_m|X_0,X_1,..,X_m)+2cq_4(X_m)E(x_m|X_0,X_1,..,X_m) \\ & +\frac{\lambda^2}{4\omega^2}d^1_m(X_m)-c\frac{\lambda}{\omega}d^2_m(X_m)-\frac{c^2}{2}\frac{\lambda}{\omega}d^3_m(X_m)+d^4_m(X_m) \\ = \; & q_1(X_m)E(x^2_m|X_0,X_1,..,X_{m-1})+2cq_4(X_m)E(x_m|X_0,X_1,..,X_{m-1}) \\ & +\frac{\lambda^2}{4\omega^2}d^1_m(X_m)-c\frac{\lambda}{\omega}d^2_m(X_m)-\frac{c^2}{2}\frac{\lambda}{\omega}d^3_m(X_m)+d^4_m(X_m), \end{aligned} \tag{3.23}$$

where

$$
\begin{aligned}
b_m^1(X_m) &= \frac{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_2}^{k-(m+1)}(X_m)}{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(X_m)} h(X_m), \\
b_m^2(X_m) &= cq_3(X_m) + [-\frac{U_m(X_m)}{2A_m(X_m)} - \frac{V_m(X_m)}{2A_m(X_m)}] h(X_m), \\
d_m^1(X_m) &= \Big[\frac{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_2}^{k-(m+1)}(X_m)}{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(X_m)}\Big]^2 h(X_m), \\
d_m^2(X_m) &= \frac{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_2}^{k-(m+1)}(X_m)}{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(X_m)} h(X_m) \\
&\quad \cdot \frac{\sum_{j=1}^{S} Q(X_m, j)\{\sum_{l=1}^{T-(m+2)} [\widetilde{Q}_{q_2}^{l-1}(\sum_{k=m+1+l}^{T} P_k \bar{Q}_{q_1}^{k-(m+1+l)} q_4)](j)\}}{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(X_m)}, \\
d_m^3(X_m) &= \frac{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_2}^{k-(m+1)}(X_m)}{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(X_m)} h(X_m) \\
&\quad \cdot \frac{\{Q_{q_2}^{T-(m+2)} \bar{Q}_{q_5}\}(X_m)}{\sum_{k=m+1}^{T} P_k \bar{Q}_{q_1}^{k-(m+1)}(X_m)}, \\
d_m^4(X_m) &= c^2 q_5(X_m) + [\frac{U_m^2(X_m)}{4A_m^2(X_m)} + \frac{V_m^2(X_m)}{4A_m^2(X_m)} + \frac{U_m(X_m)V_m(X_m)}{2A_m^2(X_m)}] h(X_m).
\end{aligned}
$$

Noting that *Equation* (3.22) and (3.23) are recursive, By using mathematical induction, it is easy to obtain the following equations:

$$\begin{aligned} E(x_{m+1}) &= x_0 \prod_{k=0}^{m} q_2(X_k) + \frac{\lambda}{2\omega} \sum_{k=0}^{m} b_k^1(X_k) \prod_{l=k}^{m-1} q_2(X_{l+1}) \\ &\quad + \sum_{k=0}^{m} b_k^2(X_k) \prod_{l=k}^{m-1} q_2(X_{l+1}), \end{aligned} \tag{3.24}$$

$$\begin{aligned} E(x_{m+1}^2) &= x_0^2 \prod_{k=0}^{m} q_1(X_k) + 2c \sum_{k=0}^{m} q_4(X_k) \prod_{j=0}^{k-1} q_2(X_j) \prod_{l=k}^{m-1} q_1(X_{l+1}) x_0 \\ &\quad + c\frac{\lambda}{\omega} \sum_{k=1}^{m} q_4(X_k) \sum_{j=0}^{k-1} b_j^1(X_j) \prod_{h=j}^{k-2} q_2(X_{h+1}) \prod_{l=k}^{m-1} q_1(X_{l+1}) \\ &\quad + 2c \sum_{k=1}^{m} q_4(X_k) \sum_{j=0}^{k-1} b_j^2(X_j) \prod_{h=j}^{k-2} q_2(X_{h+1}) \prod_{l=k}^{m-1} q_1(X_{l+1}) \\ &\quad + \frac{\lambda^2}{4\omega^2} \sum_{k=0}^{m} d_k^1(X_k) \prod_{l=k}^{m-1} q_1(X_{l+1}) - c\frac{\lambda}{\omega} \sum_{k=0}^{m} d_k^2(X_k) \prod_{l=k}^{m-1} q_1(X_{l+1}) \\ &\quad - \frac{c^2}{2} \frac{\lambda}{\omega} \sum_{k=0}^{m} d_k^3(X_k) \prod_{l=k}^{m-1} q_1(X_{l+1}) + \sum_{k=0}^{m} d_k^4(X_k) \prod_{l=k}^{m-1} q_1(X_{l+1}). \end{aligned} \tag{3.25}$$

According to the law of total probability, we know the following equation can be established:

$$E_i(x_m) = E_i\Big(E(x_m | X_0 = i, X_1, .., X_{m-1})\Big), \tag{3.26}$$

by *Equation* (3.26) and *Lemma* 3.3.2, we can transform *Equation* (3.24) and (3.25) into the following equations:

$$\begin{aligned}
E_i(x_{m+1}) &= x_0 q_2(i)\bar{Q}_{q_2}^{m}(i) + \frac{\lambda}{2\omega}\sum_{k=0}^{m}[Q^k(b_k^1\bar{Q}_{q_2}^{m-k})](i) \\
&\quad + \sum_{k=0}^{m}[Q^k(b_k^2\bar{Q}_{q_2}^{m-k})](i), \\
E_i(x_{m+1}^2) &= x_0^2 q_1(i)\bar{Q}_{q_1}^{m}(i) + 2cx_0q_2(i)\sum_{k=0}^{m}\{Q_{q_2}^{k-1}[Q_{q_4}\bar{Q}_{q_1}^{m-k}]\}(i) \\
&\quad + c\frac{\lambda}{\omega}\sum_{k=1}^{m}\sum_{j=0}^{k-1}\left\{Q^j\{b_j^1 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{m-k})\}\right\}(i) \\
&\quad + 2c\sum_{k=1}^{m}\sum_{j=0}^{k-1}\left\{Q^j\{b_j^2 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{m-k})\}\right\}(i) \\
&\quad + \frac{\lambda^2}{4\omega^2}\sum_{k=0}^{m}[Q^k(d_k^1\bar{Q}_{q_1}^{m-k})](i) + c\frac{\lambda}{\omega}\sum_{k=0}^{m}[Q^k(d_k^2\bar{Q}_{q_1}^{m-k})](i) \\
&\quad + \frac{c^2}{2}\frac{\lambda}{\omega}\sum_{k=0}^{m}[Q^k(d_k^3\bar{Q}_{q_1}^{m-k})](i) + \sum_{k=0}^{m}[Q^k(d_k^4\bar{Q}_{q_1}^{m-k})](i).
\end{aligned}$$

Rewriting the two equations above into the generalized formats, we then have

$$\begin{aligned}
E_i(x_m) &= x_0 q_2(i)\bar{Q}_{q_2}^{m-1}(i) + \frac{\lambda}{2\omega}\sum_{k=0}^{m-1}[Q^k(b_k^1\bar{Q}_{q_2}^{m-k-1})](i) \\
&\quad + \sum_{k=0}^{m-1}[Q^k(b_k^2\bar{Q}_{q_2}^{m-k-1})](i),\ m = 1, 2, .., T, \qquad (3.27) \\
E_i(x_m^2) &= x_0^2 q_1(i)\bar{Q}_{q_1}^{m-1}(i) + 2cx_0q_2(i)\sum_{k=1}^{m-1}\{Q_{q_2}^{k-1}[Q_{q_4}\bar{Q}_{q_1}^{m-k-1}]\}(i) \\
&\quad + c\frac{\lambda}{\omega}\sum_{k=1}^{m-1}\sum_{j=0}^{k-1}\left\{Q^j\{b_j^1 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{m-k-1})\}\right\}(i) \\
&\quad + 2c\sum_{k=1}^{m-1}\sum_{j=0}^{k-1}\left\{Q^j\{b_j^2 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{m-k-1})\}\right\}(i) \\
&\quad + \frac{\lambda^2}{4\omega^2}\sum_{k=0}^{m-1}[Q^k(d_k^1\bar{Q}_{q_1}^{m-k-1})](i) + c\frac{\lambda}{\omega}\sum_{k=0}^{m-1}[Q^k(d_k^2\bar{Q}_{q_1}^{m-k-1})](i) \\
&\quad + \frac{c^2}{2}\frac{\lambda}{\omega}\sum_{k=0}^{m-1}[Q^k(d_k^3\bar{Q}_{q_1}^{m-k-1})](i) + \sum_{k=0}^{m-1}[Q^k(d_k^4\bar{Q}_{q_1}^{m-k-1})](i), \\
m &= 1, 2, .., T. \qquad (3.28)
\end{aligned}$$

Noting that

$$\begin{aligned} E_i(x_{T\wedge\tau}) &= E_i[\sum_{t=0}^{T} P_t x_t] = \sum_{t=0}^{T} P_t E_i(x_t), & (3.29)\\ E_i(x^2_{T\wedge\tau}) &= E_i[\sum_{t=0}^{T} P_t x^2_t] = \sum_{t=0}^{T} P_t E_i(x^2_t), & (3.30)\end{aligned}$$

by substituting *Equation* (3.27) and (3.28) into *Equation* (3.29) and (3.30) respectively, we have the equations as follows:

$$\begin{aligned} E_i(x_{T\wedge\tau}) &= P_0 x_0 + \sum_{t=1}^{T} P_t x_0 q_2(i)\bar{Q}^{t-1}_{q_2}(i) + \frac{\lambda}{2\omega}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^{k-1}(b^1_k \bar{Q}^{t-k-1}_{q_2})](i)\\ &\quad + \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^{k-1}(b^2_k \bar{Q}^{t-k-1}_{q_2})](i), \qquad (3.31)\\ E_i(x^2_{T\wedge\tau}) &= P_0 x^2_0 + \sum_{t=1}^{T} P_t x^2_1 q_1(i)\bar{Q}^{t-1}_{q_1}(i)\\ &\quad + 2c x_0 q_2(i)\sum_{t=1}^{T} P_t \sum_{k=1}^{t-1}\{Q^{k-1}_{q_2}[Q_{q_4}\bar{Q}^{t-k-1}_{q_1}]\}(i)\\ &\quad + c\frac{\lambda}{\omega}\sum_{t=1}^{T} P_t \sum_{k=1}^{t-1}\sum_{j=0}^{k-1}\big\{Q^j\{b^1_j Q^{k-j-1}_{q_2}(Q_{q_4}\bar{Q}^{t-k-1}_{q_1})\}\big\}(i)\\ &\quad + 2c\sum_{t=1}^{T} P_t \sum_{k=1}^{t-1}\sum_{j=0}^{k-1}\big\{Q^j\{b^2_j Q^{k-j-1}_{q_2}(Q_{q_4}\bar{Q}^{t-k-1}_{q_1})\}\big\}(i)\\ &\quad + \frac{\lambda^2}{4\omega^2}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d^1_k \bar{Q}^{t-k-1}_{q_1})](i) + c\frac{\lambda}{\omega}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d^2_k \bar{Q}^{t-k-1}_{q_1})](i)\\ &\quad + \frac{c^2}{2}\frac{\lambda}{\omega}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d^3_k \bar{Q}^{t-k-1}_{q_1})](i) + \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d^4_k \bar{Q}^{t-k-1}_{q_1})](i). \qquad (3.32)\end{aligned}$$

By Li and Ng (2000) [2], we have

$$\lambda^* = 1 + 2\omega E_i(x_{T\wedge\tau})|_{u^*}. \qquad (3.33)$$

By substituting *Equation* (3.31) into (3.33), we have

$$\lambda^* = \frac{1 + 2\omega\Big\{ \sum_{t=1}^{T} P_t x_0 q_2(i)\bar{Q}_{q_2}^{t-1}(i) + \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^2\bar{Q}_{q_2}^{t-k-1})](i) + P_0 x_0 \Big\}}{1 - \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^1\bar{Q}_{q_2}^{t-k-1})](i)}. \tag{3.34}$$

Therefore, we obtain the analytical solution for $P1(\omega)$ by the following theorem:

Theorem 3.2. The optimal investment strategy for $P1(\omega)$ is of the following form:

$$\begin{aligned} u_t^*(i, x_t) &= \Big\{ \frac{\lambda^* \sum_{k=t+1}^{T} P_k \bar{Q}_{q_2}^{k-(t+1)}(i)}{2\omega \sum_{k=t+1}^{T} P_k \bar{Q}_{q_1}^{\ k-(t+1)}(i)} - r_f(i)x_t - cr_f(i)y(i) \\ &- \frac{cQ\{\sum_{l=1}^{T-(t+2)}[\widetilde{Q}_{q_2}^{l-1}(\sum_{k=t+1+l}^{T} P_k \bar{Q}_{q_1}^{k-(t+1+l)} q_4)]\}(i)}{\sum_{k=t+1}^{T} P_k \bar{Q}_{q_1}^{k-(t+1)}(i)} \\ &- \frac{c^2[Q_{q_2}^{T-(t+2)}(\bar{Q}_{q_5})](i)}{2\sum_{k=t+1}^{T} P_k \bar{Q}_{q_1}^{k-(t+1)}(i)} \Big\} E^{-1}(R_t(i)R_t(i)')r^e(i), \\ t &= 0, 1, .., T-1;\ i = 1, 2, .., S, \end{aligned} \tag{3.35}$$

where λ^* is given by *Equation* (3.34).

3.5 The Efficient Frontier for $P1(\omega)$

By *Equation* (3.31) and (3.32), we could use the equation $Var_i(x_{T\wedge\tau}) = E_i(x^2_{T\wedge\tau}) - [E_i(x_{T\wedge\tau})]^2$ to obtain the following equation:

$$\begin{aligned}
& Var_i(x_{T\wedge\tau}) = E_i(x^2_{T\wedge\tau}) - [E_i(x_{T\wedge\tau})]^2 \\
= \; & P_0 x_0^2 + \sum_{t=1}^{T} P_t x_1^2 q_1(i)\bar{Q}_{q_1}^{t-1}(i) + 2cx_0 q_2(i)\sum_{t=1}^{T} P_t \sum_{k=1}^{t-1}\{Q_{q_2}^{k-1}[Q_{q_4}\bar{Q}_{q_1}^{t-k-1}]\}(i) \\
& + c\frac{\lambda}{\omega}\sum_{t=1}^{T} P_t \sum_{k=1}^{t-1}\sum_{j=0}^{k-1}\left\{Q^j\{b_j^1 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{t-k-1})\}\right\}(i) \\
& + 2c\sum_{t=1}^{T} P_t \sum_{k=1}^{t-1}\sum_{j=0}^{k-1}\left\{Q^j\{b_j^2 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{t-k-1})\}\right\}(i) \\
& + \frac{\lambda^2}{4\omega^2}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d_k^1\bar{Q}_{q_1}^{t-k-1})](i) + c\frac{\lambda}{\omega}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d_k^2\bar{Q}_{q_1}^{t-k-1})](i) \\
& + \frac{c^2}{2}\frac{\lambda}{\omega}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d_k^3\bar{Q}_{q_1}^{t-k-1})](i) + \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d_k^4\bar{Q}_{q_1}^{t-k-1})](i) \\
& - \Big\{P_0 x_0 + \sum_{t=1}^{T} P_t x_0 q_2(i)\bar{Q}_{q_2}^{t-1}(i) + \frac{\lambda}{2\omega}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^{k-1}(b_k^1\bar{Q}_{q_2}^{t-k-1})](i)\Big\}^2.
\end{aligned} \tag{3.36}$$

Referring to *Equation* (3.31) we have

$$\begin{aligned}
\frac{\lambda}{\omega} \; = \; & \frac{2[E_i(x_{T\wedge\tau}) - \sum_{t=1}^{T} P_t x_0 q_2(i)\bar{Q}_{q_2}^{t-1}(i)]}{\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^1\bar{Q}_{q_2}^{t-k-1})](i)} \\
& - \frac{2[\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^2\bar{Q}_{q_2}^{t-k-1})](i) - P_0 x_0]}{\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^1\bar{Q}_{q_2}^{t-k-1})](i)}.
\end{aligned} \tag{3.37}$$

In order to obtain the efficient frontier we have the lemma as follow:

Lemma 3.3.

$$0 < \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^1\bar{Q}_{q_2}^{t-k-1})](i) = \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d_k^1\bar{Q}_{q_1}^{t-k-1})](i) < 1. \tag{3.38}$$

Proof. First, we have

$$b_k^1 = \frac{\sum_{l=k+1}^{T} P_l \bar{Q}_{q_2}^{l-(k+1)}}{\sum_{l=k+1}^{T} P_l \bar{Q}_{q_1}^{l-(k+1)}} h; \;\; d_k^1 = \Big[\frac{\sum_{l=k+1}^{T} P_k \bar{Q}_{q_2}^{l-(k+1)}}{\sum_{l=k+1}^{T} P_l \bar{Q}_{q_1}^{l-(k+1)}}\Big]^2 h.$$

Substituting the above two equations into both sides of *Equation* (3.38), we then have

$$\begin{aligned}
&\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1} [Q^k(b_k^1 \bar{Q}_{q_2}^{t-k-1})](i) = \sum_{k=0}^{T-1} Q^k [b_k^1 \sum_{t=k+1}^{T} P_t(\bar{Q}_{q_2}^{t-k-1})](i) \\
&= \sum_{k=0}^{T-1} Q^k \{ \frac{\sum_{l=k+1}^{T} P_l \bar{Q}_{q_2}^{l-(k+1)}}{\sum_{l=k+1}^{T} P_l \bar{Q}_{q_1}^{l-(k+1)}} h \sum_{t=k+1}^{T} P_t(\bar{Q}_{q_2}^{t-(k+1)}) \}(i) \\
&= \sum_{k=0}^{T-1} Q^k \{ \frac{[\sum_{l=k+1}^{T} P_l \bar{Q}_{q_2}^{l-(k+1)}]^2}{\sum_{l=k+1}^{T} P_l \bar{Q}_{q_1}^{l-(k+1)}} h \}(i) > 0, \\
&\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1} [Q^k(d_k^1 \bar{Q}_{q_1}^{t-k-1})](i) = \sum_{k=0}^{T-1} Q^k [d_k^1 \sum_{t=k+1}^{T} P_t(\bar{Q}_{q_1}^{t-(k+1)})](i) \\
&= \sum_{k=0}^{T-1} Q^k \{ \Big[\frac{\sum_{l=k+1}^{T} P_k \bar{Q}_{q_2}^{l-(k+1)}}{\sum_{l=k+1}^{T} P_l \bar{Q}_{q_1}^{l-(k+1)}} \Big]^2 h \sum_{t=k+1}^{T} P_t(\bar{Q}_{q_1}^{t-(k+1)}) \}(i) \\
&= \sum_{k=0}^{T-1} Q^k \{ \frac{[\sum_{l=k+1}^{T} P_l \bar{Q}_{q_2}^{l-(k+1)}]^2}{\sum_{l=k+1}^{T} P_l \bar{Q}_{q_1}^{l-(k+1)}} h \}(i).
\end{aligned}$$

Hence, the first part of *Equation* (3.38) holds true. We then prove its second part, when $T = 1$ we have

$$\begin{aligned}
\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1} [Q^k(b_k^1 \bar{Q}_{q_2}^{t-k-1})](i) &= P_1 b_0^1(i) = P_1 \frac{\sum_{l=1}^{1} P_l \bar{Q}_{q_2}^{l-1}}{\sum_{l=1}^{1} P_l \bar{Q}_{q_1}^{l-1}} h(i) \\
&= P_1 h(i) \leq h(i).
\end{aligned}$$

By *Lemma* 3.1 we have $h(i) < 1$, then the second part of *Equation* (3.38) holds true when $T = 1$. When $T > 1$, note that $Var_i(x_{T\wedge\tau}) > 0$ and assume that $x_0 = 0,\ c = 0$, then we have

$$\begin{aligned}
Var_i(x_{T\wedge\tau}) &= \frac{\lambda^2}{4\omega^2} \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1} [Q^k(d_k^1 \bar{Q}_{q_1}^{t-k-1})](i) \\
&\quad - \Big\{ \frac{\lambda}{2\omega} \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1} [Q^k(b_k^1 \bar{Q}_{q_2}^{t-k-1})](i) \Big\}^2 > 0,
\end{aligned}$$

that is

$$\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1} [Q^k(d_k^1 \bar{Q}_{q_1}^{t-k-1})](i) > \Big\{ \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1} [Q^k(b_k^1 \bar{Q}_{q_2}^{t-k-1})](i) \Big\}^2.$$

The equation above means $\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1} [Q^k(d_k^1 \bar{Q}_{q_1}^{t-k-1})](i) < 1$. Therefore, we have proven *Lemma* 3.3. □

Substituting *Equation* (3.37) into *Equation* (3.36) and by *Lemma* 3.3, we then obtain the following theorem

Theorem 3.3. The efficient frontier of the original problem is given by

$$\begin{aligned} Var_i(x_{T\wedge\tau}) &= \frac{1-\Theta_2}{\Theta_2}[E_i(x_{T\wedge\tau}) - \frac{\Theta_1-\Theta_5}{1-\Theta_2}]^2 + [\Theta_3 + P_0 - \frac{(\Theta_8+P_0)^2}{1-\Theta_2}]x_0^2 \\ &+ \frac{2(\Theta_8+P_0)(\Theta_5-\Theta_9)}{1-\Theta_2}x_0 + \frac{(2\Theta_5-\Theta_9)\Theta_2\Theta_9-\Theta_5^2}{\Theta_2(1-\Theta_2)} \\ &+ \Theta_4 + \Theta_6 + \Theta_7, \end{aligned} \tag{3.39}$$

where

$$\begin{aligned} \Theta_1 &= \sum_{t=1}^{T} P_t x_0 q_2(i)\bar{Q}_{q_2}^{t-1}(i) + \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^2\bar{Q}_{q_2}^{t-k-1})](i) + P_0 x_0, \\ \Theta_2 &= \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^1\bar{Q}_{q_2}^{t-k-1})](i), \\ \Theta_3 &= \sum_{t=1}^{T} P_t q_1(i)\bar{Q}_{q_1}^{t-1}(i), \\ \Theta_4 &= 2cx_0q_2(i)\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}\{Q_{q_2}^{k-1}[Q_{q_4}\bar{Q}_{q_1}^{t-k-1}]\}(i), \\ \Theta_5 &= c\sum_{t=1}^{T} P_t \sum_{k=1}^{t-1}\sum_{j=0}^{k-1}\left\{Q^j\{b_j^1 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{t-k-1})\}\right\}(i) \\ &+ c\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d_k^2\bar{Q}_{q_1}^{t-k-1})](i) \\ &+ \frac{c^2}{2}\sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d_k^3\bar{Q}_{q_1}^{t-k-1})](i), \\ \Theta_6 &= 2c\sum_{t=1}^{T} P_t \sum_{k=1}^{t-1}\sum_{j=0}^{k-1}\left\{Q^j\{b_j^2 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{t-k-1})\}\right\}(i), \\ \Theta_7 &= \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(d_k^4\bar{Q}_{q_1}^{t-k-1})](i), \\ \Theta_8 &= \sum_{t=1}^{T} P_t q_2(i)\bar{Q}_{q_2}^{t-1}(i), \\ \Theta_9 &= \sum_{t=1}^{T} P_t \sum_{k=0}^{t-1}[Q^k(b_k^2\bar{Q}_{q_2}^{t-k-1})](i). \end{aligned}$$

Note that the efficient frontier above is not of a form of perfect square, which means that senior investors in the DC pension plan can not hedge the total investment risks by putting all the contributions into the bank account. The reason is that the dynamics of both the wage income and the return rate of risk-free asset depend on the the market state. Therefore, the wage income and the interest rate are stochastic and can not be determined before the investment, this result is rather different from those in existing literature.

3.6 Special Cases

3.6.1 The Case with No Mortality Risk

We assume that there is no mortality risk for investors, we define $P_0 = P_1 = .. = P_{T-1} = 0,\ P_T = 1$, then we have

$$\lambda^* = \frac{1 + 2\omega\Big\{x_0 q_2(i)\bar{Q}_{q_2}^{T-2}(i) + \sum_{k=0}^{T-1}[Q^k(b_k^2\bar{Q}_{q_2}^{T-k-1})](i)\}}{1 - \sum_{k=0}^{T-1}[Q^k(b_k^1\bar{Q}_{q_2}^{T-k-1})](i)}, \tag{3.40}$$

the optimal investment strategy is

$$\begin{aligned} u_t^*(i, x_t) &= \Big\{\frac{\lambda^*\bar{Q}_{q_2}^{T-t-1}(i)}{2\omega\bar{Q}_{q_1}^{T-t-1}(i)} - r_f(i)x_t - cr_f(i)y(i) \\ &- \frac{cQ\{\sum_{l=0}^{T-t-2}[\widetilde{Q}_{q_2}^l(\bar{Q}_{q_1}^{T-(t+l)}q_4)]\}(i)}{\bar{Q}_{q_1}^{T-t}(i)} \\ &- \frac{c^2[Q_{q_2}^{T-t-2}(\bar{Q}_{q_5})](i)}{2\bar{Q}_{q_1}^{T-t-1}(i)}\Big\}E^{-1}(R_t(i)R_t(i)')r^e(i), \\ t &= 0, 1, .., T-1;\ i = 1, 2, .., S, \end{aligned} \tag{3.41}$$

where λ^* is given by *Equation* (3.34), and the corresponding efficient frontier is

$$\begin{aligned} Var_i(x_{T\wedge\tau}) &= \frac{1-\Theta_2}{\Theta_2}[E_i(x_{T\wedge\tau}) - \frac{\Theta_1 - \Theta_5}{1-\Theta_2}]^2 + [\Theta_3 - \frac{\Theta_8^2}{1-\Theta_2}]x_0^2 \\ &+ \frac{2\Theta_8(\Theta_5 - \Theta_9)}{1-\Theta_2}x_0 + \frac{(2\Theta_5 - \Theta_9)\Theta_2\Theta_9 - \Theta_5^2}{\Theta_2(1-\Theta_2)} + \Theta_4 + \Theta_6 + \Theta_7, \end{aligned} \tag{3.42}$$

where

$$\begin{aligned}
\Theta_1 &= x_0 q_2(i)\bar{Q}_{q_2}^{T-1}(i) + \sum_{k=0}^{T-1}[Q^k(b_k^2\bar{Q}_{q_2}^{T-k-1})](i),\\
\Theta_2 &= \sum_{k=0}^{T-1}[Q^k(b_k^1\bar{Q}_{q_2}^{T-k-1})](i),\\
\Theta_3 &= q_1(i)\bar{Q}_{q_1}^{T-1}(i),\\
\Theta_4 &= 2cx_0 q_2(i)\sum_{k=0}^{T-1}\{Q_{q_2}^{k-1}[Q_{q_4}\bar{Q}_{q_1}^{T-k-1}]\}(i),\\
\Theta_5 &= c\sum_{k=1}^{T-1}\sum_{j=0}^{k-1}\big\{Q^j\{b_j^1 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{T-k-1})\}\big\}(i)\\
&\quad +c\sum_{k=0}^{T-1}[Q^k(d_k^2\bar{Q}_{q_1}^{T-k-1})](i)\\
&\quad +\frac{c^2}{2}\sum_{k=0}^{T-1}[Q^k(d_k^3\bar{Q}_{q_1}^{T-k-1})](i),\\
\Theta_6 &= 2c\sum_{k=1}^{T-1}\sum_{j=0}^{k-1}\big\{Q^j\{b_j^2 Q_{q_2}^{k-j-1}(Q_{q_4}\bar{Q}_{q_1}^{T-k-1})\}\big\}(i),\\
\Theta_7 &= \sum_{k=0}^{T-1}[Q^k(d_k^4\bar{Q}_{q_1}^{T-k-1})](i),\\
\Theta_8 &= q_2(i)\bar{Q}_{q_2}^{T-1}(i),\\
\Theta_9 &= \sum_{k=0}^{T-1}[Q^k(b_k^2\bar{Q}_{q_2}^{T-k-1})](i),\\
b_m^1 &= \frac{\bar{Q}_{q_2}^{T-m-1}}{\bar{Q}_{q_1}^{T-m-1}}h,\\
b_m^2 &= cq_3 + [-\frac{U_m}{2A_m} - \frac{V_m}{2A_m}]h,\\
d_m^1 &= \Big[\frac{\bar{Q}_{q_2}^{T-m-1}}{\bar{Q}_{q_1}^{T-m-1}}\Big]^2 h,\\
d_m^2 &= \frac{\bar{Q}_{q_2}^{T-m-1}}{\bar{Q}_{q_1}^{T-m-1}}h\frac{\sum_{j=1}^{S}Q(X_m,j)\{\sum_{l=1}^{T-m-2)}[\widetilde{Q}_{q_2}^{l-1}(\bar{Q}_{q_1}^{T-(m+l)-1}q_4)](j)\}}{\bar{Q}_{q_1}^{T-m-1}},\\
d_m^3 &= \frac{\bar{Q}_{q_2}^{T-m-1}}{\bar{Q}_{q_1}^{T-m-1}}h\frac{\{Q_{q_2}^{T-m-1}\bar{Q}_{q_5}\}}{\bar{Q}_{q_1}^{T-m-1}},\\
d_m^4 &= c^2 q_5 + [\frac{U_m^2}{4A_m^2} + \frac{V_m^2}{4A_m^2} + \frac{U_m V_m}{2A_m^2}]h,
\end{aligned}$$

$$
\begin{aligned}
A_m(i) &= \omega \bar{Q}_{q_1}^{T-m-1}(i), \\
B_m(i) &= \lambda \bar{Q}_{q_2}^{T-m-1}(i), \\
U_m(i) &= \sum_{j=1}^{S} Q(i,j)\{2c\omega \sum_{l=1}^{T-m-2} [\widetilde{Q}_{q_2}^{l-1}(\bar{Q}_{q_1}^{T-(m+l)-1} q_4)](j)\}, \\
V_m(i) &= c^2 \omega [Q_{q_2}^{T-m-2} \bar{Q}_{q_5}](i).
\end{aligned}
$$

We find when there is no mortality risk, the model reduces to the one which only contains regime-switching. That is, the investor will retire at time T doubtlessly while the market changes constantly, and for the case, our result is similar to the one in Chen and Yang (2011) [48].

3.6.2 The Case with No Regime-Switching

Based on 3.6.1, we assume that there exists a market without regime-switching, thus we have

$$
\lambda^* = \frac{1 + 2\omega(x_0 q_2^{T-1} + \sum_{k=0}^{T-1} b_k^2 q_2^{T-k-1})}{(1-h)^{T-1}}, \tag{3.43}
$$

and the corresponding optimal strategy is

$$
\begin{aligned}
& u_t^*(i, x_t) \\
= \ & \Big\{ \frac{\lambda^*}{2\omega} \frac{1}{r_f^{T-t-1}} - r_f x_t - c r_f y + c \frac{r_f y}{1 - r_f} (1 - \frac{1}{r_f^{T-t-2}}) \\
& - \frac{c^2 y^2}{2} \frac{1}{r_f^{T-t-2}} \Big\} E^{-1}(R_t R_t') r^e, \ t = 0, 1, .., T-1,
\end{aligned} \tag{3.44}
$$

where λ^* is given by *Equation* (3.43), and the efficient frontier is as follow:

$$
\begin{aligned}
Var_i(x_{T\wedge\tau}) = \ & \frac{1-\Theta_2}{\Theta_2}[E_i(x_{T\wedge\tau}) - \frac{\Theta_1 - \Theta_5}{1-\Theta_2}]^2 + [\Theta_3 - \frac{\Theta_8^2}{1-\Theta_2}]x_0^2 \\
& + \frac{2\Theta_8(\Theta_5 - \Theta_9)}{1-\Theta_2} x_0 + \frac{(2\Theta_5 - \Theta_9)\Theta_2\Theta_9 - \Theta_5^2}{\Theta_2(1-\Theta_2)} + \Theta_4 + \Theta_6 + \Theta_7,
\end{aligned} \tag{3.45}
$$

where

$$
\begin{aligned}
\Theta_1 &= x_0 q_2^T + \sum_{k=0}^{T-1} b_k^2 q_2^{T-k-1}, \\
\Theta_2 &= \frac{h}{r_f}\frac{q_2^T - 1}{q_2 - 1}, \\
\Theta_3 &= q_1^T, \\
\Theta_4 &= 2cx_1 q_4 \sum_{k=0}^{T-1} q_1^{T-k-1} q_2^{k-1}, \\
\Theta_5 &= cq_4 \frac{h}{r_f} \sum_{k=1}^{T-1}\sum_{j=0}^{k-1} q_1^{T-k-1} q_2^{k-j} + c\sum_{k=0}^{T-1} d_k^2 q_1^{T-k-1} + \frac{c^2}{2}\sum_{k=0}^{T-1} d_k^3 q_1^{T-k-1}, \\
\Theta_6 &= 2cq_4 \sum_{k=1}^{T-1}\sum_{j=0}^{k-1} b_j^2 q_1^{T-k-1} q_2^{k-j}, \\
\Theta_7 &= \sum_{k=0}^{T-1} d_k^4 q_1^{T-k-1}, \\
\Theta_8 &= q_2^T, \\
\Theta_9 &= \sum_{k=0}^{T-1} b_k^2 q_2^{T-k-1}, \\
b_m^1 &= \frac{h}{r_f^{T-m-1}}, \\
b_m^2 &= cr_f y(1-h) + \{c\frac{r_f y}{1-r_f}(1 - \frac{1}{r_f^{T-m-2}}) - \frac{c^2 y^2}{2}\frac{1}{r_f^{T-m-2}}\}h, \\
d_m^1 &= \frac{h}{r_f^{2(T-m-1)}}, \\
d_m^2 &= \frac{y}{r_f^{T-m-2}}\frac{1}{r_f - 1}(1 - \frac{1}{r_f^{T-m-2}})h, \\
d_m^3 &= \frac{y^2}{r_f^{T-m-1}}h, \\
d_m^4 &= c^2 r_f^2 y^2 (1-h) + [\frac{U_m^2}{4A_m^2} + \frac{V_m^2}{4A_m^2} + \frac{U_m V_m}{2A_m^2}]h, \\
A_m &= \omega q_1^{T-m-1}, \\
B_m &= \lambda q_2^{T-m-1}, \\
U_m &= 2c\omega q_4 \sum_{l=1}^{T-m-2} q_1^{T-(m+l)-1} q_2^{l-1}, \\
V_m &= c^2 \omega q_2^{T-m-2} q_5.
\end{aligned}
$$

In this case, we find that our results are similar to the ones in Li and Ng (2000) [2].

3.6.3 The Case with Zero Contribution Rate

Based on 3.6.1 and 3.6.2, we assume that the contribution rate $c = 0$, thus we have

$$\lambda^* = \frac{1 + 2\omega x_1 r_f^T (1-h)^T}{(1-h)^T}, \tag{3.46}$$

and the optimal strategy is shown as

$$\begin{aligned} u_t^*(i, x_t) &= \Big\{ \frac{\lambda^*}{2\omega} \frac{1}{r_f^{T-t-1}} - r_f x_t \Big\} E^{-1}(R_t R_t') r^e, \\ t &= 0, 1, .., T-1, \end{aligned} \tag{3.47}$$

where λ^* is given by *Equation* (3.37), and we can get the efficient frontier as follow:

$$Var_i(x_{T\wedge\tau}) = \frac{1-\Theta_2}{\Theta_2}[E_i(x_{T\wedge\tau}) - \frac{\Theta_1}{1-\Theta_2}]^2 + [\Theta_3 - \frac{\Theta_8^2}{1-\Theta_2}]x_0^2, \tag{3.48}$$

where

$$\begin{aligned} \Theta_1 &= x_0 q_2^T + \sum_{k=0}^{T-1} b_k^2 q_2^{T-k-1}, \\ \Theta_2 &= \frac{h}{r_f} \frac{q_2^T - 1}{q_2 - 1}, \\ \Theta_3 &= q_1^T, \\ \Theta_8 &= q_2^T. \end{aligned}$$

We conclude that the result above is consistent with the work of Wu and Li (2011) when the contribution rate $c = 0$.

3.7 Numerical Examples

In this section, numerical analysis is given to reflect impacts of different factors on the corresponding efficient frontiers.

First, we suppose that there are two market states during three time periods

$T = 3$. When $X_t = 1$, the market is in the bearish mode and when $X_t = 2$, it is bearish. Assume that an investor has higher wage $y_t(1) = 1.05$ in the bullish market state and lower income $y_t(2) = 0.95$ in the bearish market state. In addition, the investor has initial wealth $x_0 = 1$ and fixed contribution rate $c = 0.2$ for his DC pension plan. Parameters regarding asset return rates in different market modes are shown as follows:

$$r_f(1) = 1.03;\ r_f(2) = 1.162;$$

$$r_1(1) = 1.14;\ r_1(2) = 1.246;$$

$$r_2(1) = 1.19;\ r_2(2) = 1.228;$$

$$cov(e_t(1)) = \begin{pmatrix} 0.0312 & 0.0215 \\ 0.0215 & 0.0179 \end{pmatrix}, cov(e_t(2)) = \begin{pmatrix} 0.0154 & 0.0104 \\ 0.0104 & 0.0089 \end{pmatrix} for\ t = 0, 1, 2.$$

3.7.1 Initial Market State

In this example, we show the impact of initial market state on the corresponding efficient frontier. First, we suppose the transition matrix $Q = (0.5, 0.5; 0.5, 0.5)$, and then we set the mortality probabilities in each time point as follow:

$$P_0 = 0.1,\ P_1 = 0.2,\ P_2 = 0.3,\ P_3 = 0.4.$$

The setting above means the mortality risk increases when the investor gets older, which is realistic and reasonable. Then we obtain the efficient frontiers by $Theorem$ 3.3. Figure 3.1 shows the corresponding efficient frontiers, which suggests that when the variance is fixed, the investor has higher expected final wealth given a bullish initial market compared with a bearish initial market state. This result is consistent with the fact, compared with the result when being in the bearish market state, the bullish initial market mode makes positive and profitable investment impression to investors, which stimulates investment enthusiasm and increase the expectation of the terminal wealth. Regardless of the effects of wage income and mortality risks, this result is also consistent with the one in Chen et al.(2008) [47].

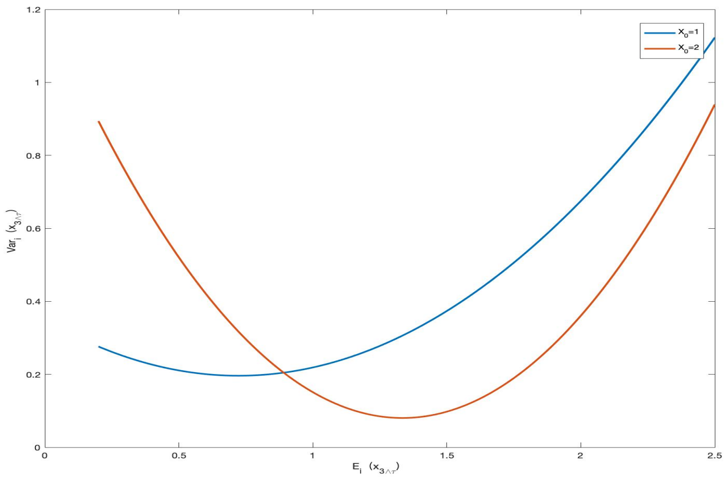

Figure 3.1: Mean-variance efficient frontiers for 3.7.1

3.7.2 Regime-Switching

This example focuses on the market state transition probability and we assume that we always start our model under the bearish market mode $X_0 = 1$. By increasing the market mode transition probability p_{11} from 0 to 1, we get their corresponding efficient frontiers in Figure 3.2.

Figure 3.2 suggests that when the transition probability from the bearish market to the bullish market state p_{11} increases, namely, it is 0, 0.2, 0.4, 0.6, 0.8 and 1 respectively, the corresponding efficient frontier moves upward, which means that we will bear more investment risk with the high transition probability p_{11} given the same expectation in the market with the low transition probability. This result also reflects that investors will be more optimistic if the bullish market state is less likely to transform into the bearish market.

3.7.3 Mortality Risk

This example considers the element of different mortality risk probabilities. For the sake of simplicity, we set $P_1 = P_2 = 0$ and make P_0 represent the sole mortality probability in the working period and P_3 stands for the probability after retirement. Other parameters are the same as ones in 3.7.1. Figure 3.3 suggests that the efficient frontier generally rises up when P_0 increases, which means that when the variance is fixed, the investor with lower mortality risk in the working period has higher expected final wealth, which is obviously consistent with the fact. Because if the investor has high mortality risk before his retirement,

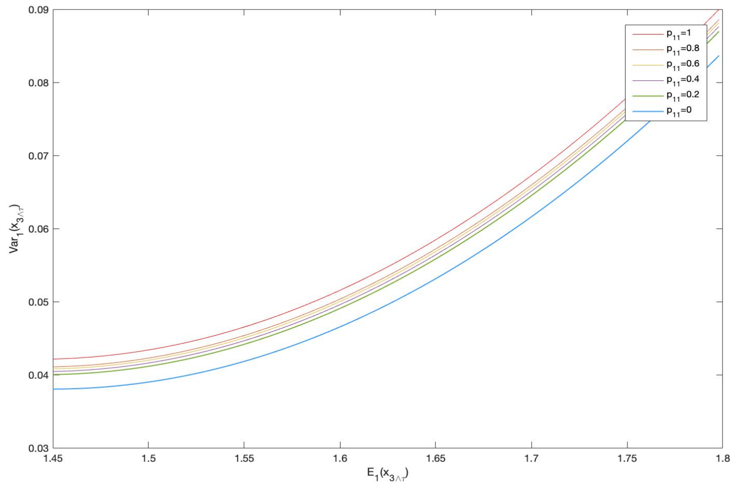

Figure 3.2: Mean-variance efficient frontiers for 3.7.2

his corresponding DC pension plan will be more likely to be terminated, which further has negatively impact on the investment return and the expected terminal wealth. Therefore, we have the conclusion that compared with the case in which there is low working mortality risk, an investor with high mortality probability in the working period has lower expected final wealth.

3.8 Concluding Remarks

In this chapter, we investigate the multi-period mean-variance portfolio selection for the defined contribution pension scheme under regime-switching framework with mortality risk. We assume that both the asset returns and the income wage depend on the market state and investors have mortality risk during the investment time horizon. Unlike existing literature in which the mainly method to solve the portfolio model is the Lagrange problem, this thesis firstly adopts the embedding technique and solves the computation complexity for the dynamic programming by using innovative expressions of the value function and relevant iteration process. The closed form of the optimal investment strategy and the corresponding efficient frontier are obtained. Special cases and numerical analysis are also used at the end of the chapter. This chapter considers the mortality risk based on the portfolio selection model and generalizes some of the existing work such as Chen and Yang(2011) [48], Wu and Li(2011) [52] and Li and Ng(2000) [2]. And we obtain three interesting results. (i) Under the same variance, the expected terminal wealth with a bullish initial market state is higher than the one for a

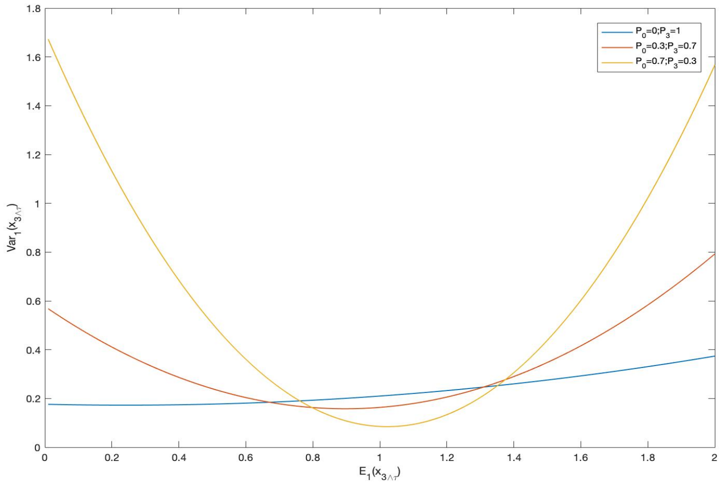

Figure 3.3: Mean-variance efficient frontiers for 3.7.3

bearish initial market state. (ii) With the increase of the transition probability from the bearish market state to the bullish market state, the investor bears more investment risks with the same level of the expected terminal wealth. (iii) The investor who has low mortality probability will get a higher expected terminal wealth.

CHAPTER 4

Portfolio Selection for a DC Pension Scheme with Premiums Return Policy and Incomplete Information

4.1 General Overview

This chapter considers the mean-variance asset allocation for the DC pension scheme in an incomplete information market. In addition, we consider the impact of mortality risk on the terminal investment wealth. An increasing number of companies nowadays adopt a premiums return policy where the mortality risk is entrusted to the fund manager who is in charge of the DC pension plan and the corresponding investment. In such a premiums return policy, heirs of the dead members in the pension plan can withdraw the premiums the member have contributed before. He and Liang(2013) [37] and Bian et al.(2018) [72] firstly incorporated the premiums return policy into the portfolio selection for the DC pension plan under the continuous-time and multi-period framework respectively. Other studies in the field include Sun et al.(2016) [73] and Li et al.(2017) [74]. Accordingly, the purpose of this chapter is to consider both the incomplete information market and the premiums return policy in the multi-period portfolio selection for the DC pension plan. By sufficient statistics, we convert the incomplete information problem into the one with complete information, and then we adopt the embedding technique and dynamic programming approach to derive the analytical solution and the corresponding efficient frontier. Besides, we investigate numerically the impact of the incomplete information condition and the regime-switching market, as well as the premiums return policy, on the corresponding efficient frontier, from which some interesting conclusion has been found.

4.2 Model Formulation

In this chapter, we study a defined contribution annuity scheme in the multi-period finance market including both the observable and unobservable market states. A pension member enters the plan at time 0 with initial wealth W_0 and plans to retire at time T. During the accumulation phase of the plan, the member contributes a defined sum of money C_t into the member's superannuation account at the beginning of the $(t+1)$-th period, where the $(t+1)$-th period means the time interval $[t, t+1)$ $(t = 0, 1, ..., T-1)$.

To protect the rights of the member who has mortality risk, a plan introduces the premiums return policy in which all the contribution the member has paid before will be paid to his/her heir if the member passes away due to disease or the accidents. That is, the heir can obtain the money worth of $\sum_{m=0}^{t} C_m$ if the member dies in time interval $[t, t+1)$ $(t = 0, 1, ..., T-1)$. However, the corresponding increment of the contribution in the pension plan can not be withdrawn but be equally portioned to the surviving members. Let q_t $(0 \leq q_t < 1)$ be the member mortality probability during the $(t+1)$-th time period.

4.2.1 Market State

The market state shows drastic fluctuations in response of a variety of factors including economic factors such as the GDP growth rate, financial crisis and non-economic events such as the government policies and natural disasters. Existing literature also suggests that the investment policy is highly dependent on the market state. In this sense, the assumption that the return rates of assets are the functions of market states is fairly sensible in this thesis.

Under a complete information market state, investors can observe all the market information with which they make their investment strategies. In a real world, however, the finance market consists of both the observable and unobservable state, and investors can only accumulate and update the observed market information. Denote $F = \{1, 2, ..., f\}$ the unobservable state set and U_t the unobservable market state at time t $(t = 0, 1, ..., T-1)$. And suppose that the unobservable state process $\{U_t, t = 0, 1, ..., T-1\}$ is driven by a discrete-time Markov chain with transition matrix $Q_t = Q(p_t(i_t, i_{t+1}))_{f \times f}$, where Q_t is assumed to be dependent of the market state, $p_t(i_t, i_{t+1}) = P\{U_{t+1} = i_{t+1} | U_t = i_t\}$ is the probability of the state transition from $U_t = i_t$ to $U_{t+1} = i_{t+1}$ at time $t+1$ $(i_t, i_{t+1} \in F)$.

Since the state U_t is unobservable, the state process$\{U_t, t = 0, 1, ..., T-1\}$ is a hidden Markov chain.

Let $G = \{1, 2, ..., g\}$ represent the observable state set and denote O_t the observable state at time t $(t = 0, 1, ..., T-1)$. Then we get the observable market state process $\{O_t, t = 0, 1, ...T-1\}$. In this chapter, we assume that at time t the change from the actual market only depends on the unobservable market state U_t which sends out information O_t to investors. The information O_t can be observed by the investors who, in turn, use it to update the investment strategies. Suppose the information O_t is independent of the previous information including unobservable information U_m $(m = 0, 1, .., t-1)$ and observable information $O_m(m = 0, 1, ..., t-1)$. Denote $P(O_t = j_t|U_t = i_t) = \sigma_t(i_t, j_t)$ the probability of the observable market state $O_t = j_t$ based on the unobservable market state $U_t = i_t$ at time t and $\Sigma_t = (\sigma_t(i, j))_{f\times g}$ the corresponding emission matrix.

As discussed above, the unobservable state process U, which is objective existence, is the only factor that changes the finance market. Meanwhile the process O can be totally observed by the investors. Therefore, the hidden Markov model in this chapter includes both the unobservable process U and the observable process O. For example, consider that at the beginning of time t, the unobservable market state $U_t = bearish$ usually emits negative information $O_t = no$ which correctly implies the true bearish market state. However, the bearish market may send out the positive information $O_t = yes$ that misleads investors to believe that the current market state is being bullish. The market releases the positive or negative information with different σ_t. At time $t+1$, the market state continues being bearish or transforms to the bullish state with different probability p_t, emitting new information to the market, and investors accumulate the observed information and update their investment strategies, which means that σ_t changes to σ_{t+1}. In each investment time horizon, this process repeats over and over again. Contrary to the existing literature where the transition probability is constant, in this chapter we assume that the transition matrix Q_t and the emission matrix Σ_t are both time-dependent, which makes our model more flexible and realistic.

4.2.2 Finance Market

Suppose that the market consists of one risk-free asset and n risky assets. We assume that the return of the risk-free asset only relies on the observable market state, which means that it can be observed directly in the market. While the unobservable and observable market state collectively determine the return rates of the risky assets. For $t = 0, 1, ..., T-1$, denote $r_t^0(O_t)$ the return of the risk-free asset during the $(t+1)$-th time period under the observable state O_t, and let $R_t^l(U_t, O_t)$ be the difference in returns between the risky asset l ($l = 1, 2, ..., n$) and the risk-free asset in the $(t+1)$-th time period given the observable market state O_t and the unobservable market state U_t. Denote $R_t(U_t, O_t) = (R_t^1(U_t, O_t), R_t^2(U_t, O_t), ..., R_t^n(U_t, O_t))'$ the excess return vector, $r_t(U_t, O_t) = E(R_t(U_t, O_t))$ the expected excess return vector, and $\mathbf{cov}_t(U_t, O_t) = \left(\sigma_t^{l,h}\right)$ the covariance matrix, where $\sigma_t^{l,h} = cov(R_t^l(U_t, O_t), R_t^h(U_t, O_t))$ for $h = 1, 2, ..., n$. In addition, we define $\gamma_t(U_t, O_t) = E(R_t(U_t, O_t)R_t'(U_t, O_t)) = \mathbf{cov}_t(U_t, O_t) + r_t(U_t, O_t)r_t'(U_t, O_t)$. Throughout this chapter, the short selling, transaction costs and tax are not considered.

A4.1 For $k, l = 1, 2, ..., T-1$, the excess return vectors $R_k(i_k, j_k)$ and $R_l(i_l, j_l)$ are statistically independent for any $i_k, i_l \in F$ and any $j_k, j_l \in G$ where $k \neq l$.

A4.2 $\mathbf{cov}_t(i, j)$ is positive definite for $t = 0, 1, ..., T-1$ and any $i \in F, j \in G$. Because $\gamma_t(U_t, O_t) = \mathbf{cov}_t(U_t, O_t) + r_t(U_t, O_t)r_t'(U_t, O_t)$, we can conclude that $\gamma_t(i, j)$ is also positive definite.

A4.3 $r_t(i, j) \neq \mathbf{0}$ for any $t = 0, 1, ..., T-1$ and $i \in F$, where $\mathbf{0}$ is the n-dimensional zero vector.

4.2.3 Wealth Process and Relevant Optimization

Denote W_t the wealth at time t, and C_t the amount of money that a member in the plan contributes as the premiums at time t ($t = 0, 1, ..., T-1$). The fund manager then make portfolio selection to maximize the terminal value of the fund. Denote $\pi_t^l(O_t)$ the amount invested in the l-th asset ($l = 1, 2, ..., n$) at time t ($t = 0, 1, ..., T-1$) under the observable market state O_t ($O_t = 1, 2, ..., g$) and $\pi_t(O_t) = (\pi_t^1(O_t), \pi_t^2(O_t), ..., \pi_t^n(O_t))'$ the vector of investment strategy at time t ($t = 0, 1, ..., T-1$). Then we know that the amount invested in the risk-free asset at time t can be represented by $W_t + C_t - \sum_{l=1}^n \pi_t^l(O_t)$. Based on the premiums return policy, if a member dies in the $(t+1)$-th time period, then all of the contributions $\sum_{l=0}^t C_l$ will be returned to his/her heir. The difference between his/her contributions and the investment return will not be returned but distributed to the members who are still alive in the plan. The expected refund in the fund can be represented by $aq_t \sum_{l=0}^{t} C_l$, where a is a parameter with value 0 and 1. When $a = 1$, the heir of the died member in the plan can get all the contributions he/she previously made. When $a = 0$, there is no premiums return policy in the plan, which is the identical case in existing literature. Thus, For $U_t = i_t, i_t \in F$ and $O_t = j_t, j_t \in G$, after equal distribution, the wealth W_{t+1} of each surviving member in the plan is

$$
\begin{aligned}
W_{t+1} =& \frac{\left(W_t + C_t - \sum_{l=1}^{n} \pi_t^l(j_t)\right) r_t^0(j_t)}{1 - q_t} \\
&+ \frac{\sum_{l=1}^{n} \pi_t^l(j_t)\left(r_t^0(j_t) + R_t^l(i_t, j_t)\right) - aq_t \sum_{l=0}^{t} C_l}{1 - q_t} \\
=& A_t(j_t)W_t + B_t(j_t) + \frac{R_t'(i_t, j_t)\pi_t(j_t)}{1 - q_t},
\end{aligned}
\tag{4.1}
$$

where $A_t(j_t) = \frac{r_t^0(j_t)}{1-q_t}$, $B_t(j_t) = \frac{C_t r_t^0(j_t) - aq_t \sum_{l=0}^{t} C_l}{1-q_t}$.

The fund manager needs to make an optimal investment policy to achieve the investment target. Therefore, based on the mean-variance framework, the objective function is formulated as follow:

$$
\begin{cases}
max_\pi \ E(W_T|U_0, O_0, w_0) - \omega Var(W_T|U_0, O_0, w_0) \\
s.t. \ W_{t+1} = A_t(O_t)W_t + B_t(O_t) + \frac{R_t'(U_t, O_t)\pi_t(O_t)}{1-q_t}, \ t = 0, 1, ..., T-1,
\end{cases}
\tag{4.2}
$$

where $\omega > 0$ represents the risk averse coefficient.

Since $Equation(4.2)$ contains the unobservable market state U_t, we can not directly use the dynamic programming approach to derive the optimal policy. However, according to Monahan(1982), by introducing an equivalent statistic method, the original is transformed into an auxiliary structure with perfect information. In the next section, the sufficient statistic method is introduced to adopt an equivalent statistics that can replace the unobservable process U.

4.2.4 Complete Information

At the beginning of each time point, the investor updates his observed information and then estimates the unobservable state distribution. Denote $\overline{O}_t = (O_0, O_1, ..., O_t)$ the accumulated information up to time t and $\phi_t(i) = P(U_t = i_t|\overline{O}_t)$ the conditional probability of $U_t = i, i \in F$ given the accumulated observable information at time point t. Let $\Phi_t = \{\phi_t(1), \phi_t(2), ..., \phi_t(f)\}$ the conditional probability distribution of U_t based on O_t at time t. At the next time point $t+1$, the market state switches from U_t to U_{t+1} and the investor updates the observed information. Then now the investor possesses the observable information as $\overline{O}_{t+1} = (\overline{O}_t, O_{t+1})$. For example, based on the observable information $\overline{O}_{t+1} = (\overline{O}_t, O_{t+1} = j_{t+1}), j_{t+1} \in G$, the conditional probability of $U_{t+1} = i_{t+1}, i_{t+1} \in F$ at time $t+1$ can be shown to be as follows:

$$\begin{aligned}
&\phi_{t+1}^{j_{t+1}}(i_{t+1}) = P(U_{t+1} = i_{t+1}|\overline{O}_{t+1}) = P(U_{t+1} = i_{t+1}|\overline{O}_t, O_{t+1} = j_{t+1}) \\
&= \frac{P(O_{t+1} = j_{t+1}|U_{t+1} = i_{t+1}, \overline{O}_t)P(U_{t+1} = i_{t+1}|\overline{O}_t)}{\sum_{i_{t+1}=1}^{f} P(O_{t+1} = j_{t+1}|U_{t+1} = i_{t+1}, \overline{O}_t)P(U_{t+1} = i_{t+1}|\overline{O}_t)} \\
&= \frac{P(O_{t+1} = j_{t+1}|U_{t+1} = i_{t+1}) \sum_{i_t=1}^{f} P(U_{t+1} = i_{t+1}|U_t = i_t)P(U_t = i_t|\overline{O}_t)}{\sum_{i_{t+1}=1}^{f} P(O_{t+1} = j|U_{t+1} = i_{t+1}) \sum_{i_t=1}^{f} P(U_{t+1} = i_{t+1}|U_t = i_t)P(U_t = i_t|\overline{O}_t)} \\
&= \frac{\delta_{t+1}(i_{t+1}, j_{t+1}) \sum_{i_t=1}^{f} \phi_t(i_t)p_t(i_t, i_{t+1})}{\sum_{i_{t+1}=1}^{f} \delta_{t+1}(i_{t+1}, j_{t+1}) \sum_{i_t=1}^{f} \phi_t(i_t)p_t(i_t, i_{t+1})}.
\end{aligned} \tag{4.3}$$

According to *Equation* (4.3), we know that the conditional probability distribution of the unobservable state Φ_{t+1} can be inferred by the information Φ_t and $O_t = j_t$, which means that we can use Φ_{t+1} to replace the unobservable state U_{t+1}. Denote $\Phi_{t+1}^{j_{t+1}} = \{\phi_{t+1}^{j_{t+1}}(1), \phi_{t+1}^{j_{t+1}}(2), ..., \phi_{t+1}^{j_{t+1}}(f)\}$ the conditional distribution of U_{t+1} given $O_{t+1} = j_{t+1}, j_{t+1} \in G$. Note that when $O_0 = j_0$ we have

$$\phi_0^{j_0}(i_0) = P(U_0 = i_0|O_0 = j_0) = \frac{P(U_0 = i_0)\delta_0(i_0, j_0)}{\sum_{i_0=1}^{f} P(U_0 = i_0)\delta_0(i_0, j_0)}. \tag{4.4}$$

In practice, the investor often preliminarily analyzes the unobservable market state and has an estimation of $P(U_0 = i_0)$.

According to the work of Monahan(1982) [75] in which the author proved that $\{\Phi_t^{j_t}, t \geq 0, j_t \in G\}$ is a Markov chain. Since $\{\Phi_t^{j_t}, j_t \in G\}$ is equivalent to the unobservable state U_t, we can replace U_t by $\Phi_t^{j_t}$ to convert the original problem into the one with only complete information as follows:

$$P(\omega) \begin{cases} max_\pi \ E(W_T|\Phi_0^{j_0}, j_0, w_0) - \omega Var(W_T|\Phi_0^{j_0}, j_0, w_0) \\ s.t. \ W_{t+1} = A_t(O_t)W_t + B_t(O_t) + \frac{R_t'(\Phi_t^{O_t}, O_t)\pi_t(O_t)}{1-q_t} \\ O_0 = j_0, \ O_t \in G for \ t = 0, 1, ..., T-1. \end{cases} \tag{4.5}$$

For $t = 0, 1, ..., T-1$, any time-dependent matrix $G_t(l \times l)$ and vector $g_t(l \times 1)$, we define $\sum_{m=k}^{t} G_k = \mathbb{I}$ for $k > t$ where $\mathbb{I}$ is the $l \times l$ unit matrix and $\prod_{m=k}^{t} g_k = \mathbf{0}$ where $\mathbf{0}$ is the $l \times 1$ zero vector. When $l = 1$, then we have $\sum_{m=k}^{t} G_k = 1$, $\prod_{m=k}^{t} g_k = 0$.

4.3 Auxiliary Problem

Due to the non-separable variance term $Var(W_T|\Phi_0^{j_0}, j_0, w_0)$ in $P(\omega)$, the dynamic programming approach can not be directly used to derive the optimal strategy. However, Li and Ng(2000) [2] came up with the embedding technique by which the original problem can be converted into the one with separability under the dynamic programming approach. Therefore, $P(\omega)$ can be transformed to the auxiliary problem as follows:

$$A(\lambda, \omega) \begin{cases} max_\pi \ E(\lambda W_T - \omega W_T^2 | \Phi_0^{j_0}, j_0, w_0) \\ s.t. \ W_{t+1} = A_t(O_t) W_t + B_t(O_t) + \frac{R_t'(\Phi_t^{O_t}, O_t)\pi_t(O_t)}{1-q_t} \\ O_0 = j_0, \ O_t \in G for \ t = 0, 1, ..., T-1, \end{cases} \tag{4.6}$$

for any $\lambda \in \mathbb{R}$. According to Li and Ng(2000) [2], the optimal policy for $P(\omega)$ is the one with special $\lambda^* = 1 + 2\omega E(W_T^{\pi^A} | \Phi_0^{j_0}, j_0, w_0)$ for the auxiliary problem if it exists, where $W_T^{\pi^A}$ is the terminal wealth under the optimal policy π^A in problem $A(\lambda, \omega)$. Hence, we just need to solve the problem $A(\lambda, \omega)$ and get the special λ^*, then the optimum investment policy and the corresponding efficient frontier for $P(\omega)$ can be obtained.

For any $t = 0, 1, ..., T-1$, $j_t \in G$ and the process $\Phi_t^{j_t}$, define the value function

$$v_t(\Phi_t^{j_t}, j_t, w_t) = max_{\pi_t(j_t), \pi_{t+1}(O_{t+1}), ..., \pi_{T-1}(O_{T-1})} E(\lambda W_T - \omega W_T^2 | \Phi_t^{j_t}, j_t, w_t). \tag{4.7}$$

Then we have the Bellman equation

$$
\begin{aligned}
& v_t(\Phi_t^{j_t}, j_t, w_t) = max_{\pi_t(j_t)} E\Big(v_{t+1}(\Phi_{t+1}, O_{t+1}, W_{t+1}) | \Phi_t^{j_t}, j_t, w_t\Big) \\
& = max_{\pi_t(j_t)} \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) \sum_{i_{t+1}=1}^{f} p_t(i_t, i_{t+1}) \sum_{j_{t+1}=1}^{g} \delta_{t+1}(i_{t+1}, j_{t+1}) \\
& \cdot E\Big(v_{t+1}\Big(\Phi_{t+1}^{j_{t+1}}, j_{t+1}, A_t(j_t)w_t + B_t(j_t) + \frac{R_t'(\Phi_t^{j_t}, j_t)\pi_t(j_t)}{1-q_t}\Big)\Big) \\
& = max_{\pi_t(j)} \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) \\
& \cdot E\Big(v_{t+1}\Big(\Phi_{t+1}^{j_{t+1}}, j_{t+1}, A_t(j_t)w_t + B_t(j_t) + \frac{R_t'(\Phi_t^{j_t}, j_t)\pi_t(j_t)}{1-q_t}\Big)\Big),
\end{aligned}
$$

with terminal condition

$$
v_T(\Phi_T^{j_T}, j_T, w_T) = \lambda w_T - \omega w_T^2, \tag{4.8}
$$

where $\varphi_t(i_t, j_{t+1}) = \sum_{i_{t+1}=1}^{f} p_t(i_t, i_{t+1})\delta_{t+1}(i_{t+1}, j_{t+1})$.
In order to obtain the expressions for the optimal strategy and the corresponding

value function, we introduce some backward time series as follows:

$$\hat{A}_t^2(j_t) = \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) \hat{A}_{t+1}^2(j_{t+1}) A_{t+1}^2(j_{t+1}) - r_t^{\prime\lambda}(j_t) V_t^{-1}(j_t) r_t^{\lambda}(j_t), \tag{4.9}$$

$$M_t(j_t) = (\hat{M}_t(j_t) - 2\omega \hat{A}_t^2(j_t) B_t(j_t)) A_t(j_t), \tag{4.10}$$

$$\hat{M}_t(j_t) = \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) M_{t+1}(j_{t+1}) - r_t^{\prime\lambda}(j_t) V_t^{-1}(j_t) r_t^{M}(j_t), \tag{4.11}$$

$$D_t(j_t) = \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) D_{t+1}(j_{t+1}) \tag{4.12}$$

$$+ (\hat{M}_t(j_t) - \omega \hat{A}_t^2(j_t) B_t(j_t)) B_(j_t) + \frac{1}{4\omega} r_t^{\prime M}(j_t) V_t^{-1}(j_t) r_t^{M}(j_t), \tag{4.13}$$

$$V_t(j_t) = \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) \gamma_t(i_t, j_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) \hat{A}_{t+1}^2(j_{t+1}) A_{t+1}^2(j_{t+1}), \tag{4.14}$$

$$r_t^{\lambda}(j_t) = \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) r_t(i_t, j_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) \hat{A}_{t+1}^2(j_{t+1}) A_{t+1}^2(j_{t+1}), \tag{4.15}$$

$$r_t^{M}(j_t) = \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) r_t(i_t, j_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) M_{t+1}(j_{t+1}), \tag{4.16}$$

with terminal condition

$$\hat{A}_T^2(j_T) = \frac{1}{A_T^2(j_T)}, \quad M_T(j_T) = \lambda, \quad \hat{M}_T(j_T) = \frac{\lambda}{A_T(j_T)} + 2\omega \frac{B_T(j_T)}{A_T^2(j_T)}, \quad D_T(j_T) = 0. \tag{4.17}$$

Lemma 4.1. *For all $t = 0, 1, ..., T-1$ and $i_t \in F,\ j_t \in G$, we have $\hat{A}_t^2(j_t) > 0$.*

Proof. According to **A**4.2, for $t = T-1$ and $i_{T-1} \in F,\ j_{T-1} \in G$, we have

$$\gamma_t(i_t, j_t) > 0,\ \forall t = 0, 1, ..., T-1,\ i_t \in F,\ j_t \in G.$$

Because $\phi_{T-1}^{j_{T-1}}(i_{T-1}) > 0$ for $i \in F,\ j_t \in G$.

$$\sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) \gamma_{T-1}(i_{T-1}, j_{T-1}) > 0,$$

and we create the following matrix:

$$\begin{pmatrix} 1 & r'_{T-1}(i_{T-1},j_{T-1}) \\ r_{T-1}(i_{T-1},j_{T-1}) & \gamma_{T-1}(i_{T-1},j_{T-1}) \end{pmatrix},$$

which can be easily proved that the matrix above is positive definite, and

$$\begin{aligned}
&\sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) \begin{pmatrix} 1 & r'_{T-1}(i_{T-1},j_{T-1}) \\ r_{T-1}(i_{T-1},j_{T-1}) & \gamma_{T-1}(i_{T-1},j_{T-1}) \end{pmatrix} \\
=&\begin{pmatrix} 1 & \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) r'_{T-1}(i_{T-1},j_{T-1}) \\ \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) r_{T-1}(i_{T-1},j_{T-1}) & \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) \gamma_{T-1}(i_{T-1},j_{T-1}) \end{pmatrix} \\
&> 0,
\end{aligned}$$

which means that

$$\begin{aligned}
&1 - \Big(\sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) r'_{T-1}(i_{T-1},j_{T-1})\Big) \cdot \Big(\sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) \gamma_{T-1}(i_{T-1},j_{T-1})\Big)^{-1} \\
&\cdot \Big(\sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) r_{T-1}(i_{T-1},j_{T-1})\Big) > 0,
\end{aligned}$$

i.e.

$$\hat{A}_{T-1}^2(j_{T-1}) = 1 - r_{T-1}^{\prime\lambda}(j_{T-1}) V_{T-1}^{-1}(j_{T-1}) r_{T-1}^{\lambda}(j_{T-1}) > 0.$$

Suppose that at time $t+1$ we have $\hat{A}_{t+1}^2(j_{t+1}) > 0$, based on $A_{t+1}^2(j_{t+1}) > 0$, we have

$$\zeta_t^{j_t}(i_t) = \phi_t^{j_t}(i_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t,j_{t+1}) \hat{A}_{t+1}^2(j_{t+1}) A_{t+1}^2(j_{t+1}) > 0,$$

and we have

$$\sum_{i_t}^{f} \zeta_t^{j_t}(i_t) \gamma_t(i_t,j_t) > 0,$$

and

$$\sum_{i_t}^{f} \zeta_t^{j_t}(i_t) \begin{pmatrix} 1 & r'_t(i_t,j_t) \\ r_t(i_t,j_t) & \gamma_t(i_t,j_t) \end{pmatrix} = \begin{pmatrix} \sum_{i_t}^{f} \zeta_t^{j_t}(i_t) & \sum_{i_t}^{f} \zeta_t^{j_t}(i_t) r'_t(i_t,j_t) \\ \sum_{i_t}^{f} \zeta_t^{j_t}(i_t) r_t(i_t,j_t) & \sum_{i_t}^{f} \zeta_t^{j_t}(i_t) \gamma_t(i_t,j_t) \end{pmatrix} > 0,$$

which implies that

$$\sum_{i_t}^{f}\zeta_t^{j_t}(i_t)-\Big(\sum_{i_t}^{f}\zeta_t^{j_t}(i_t)r_t'(i_t,j_t)\Big)\Big(\sum_{i_t}^{f}\zeta_t^{j_t}(i_t)\gamma_t(i_t,j_t)\Big)^{-1}\Big(\sum_{i_t}^{f}\zeta_t^{j_t}(i_t)r_t(i_t,j_t)\Big)>0,$$

i.e.

$$\begin{aligned}\hat{A}_t^2(j_t)=&\sum_{i_t=1}^{f}\phi_t^{j_t}(i_t)\sum_{j_{t+1}=1}^{g}\varphi_t(i_t,j_{t+1})\hat{A}_{t+1}^2(j_{t+1})A_{t+1}^2(j_{t+1})\\&-r_t^{'\lambda}(j_t)V_t^{-1}(j_t)r_t^{\lambda}(j_t)>0.\end{aligned}$$

For $t=0,1,...,T-1$ and $i_t\in F,\ j_t\in G,\ \hat{A}_t^2(j_t)>0$ holds true by the mathematical induction. □

The optimal strategy and the corresponding value function for the problem $A(\lambda,\omega)$ can be obtained by solving *Equation*(4.8), which is shown as the following theorem:

Theorem 4.1. *For $t=0,1,...,T-1,\ i_t\in F,\ j_t\in G$, the value function of the auxiliary problem $A(\lambda,\omega)$ is*

$$v_k(\Phi_k^{j_k},j_k,w_k)=-\omega\hat{A}_k^2(j_k)A_k^2(j_k)w_k^2+M_k(j_k)w_k+D_k(j_k),\tag{4.18}$$

and the corresponding optimal investment policy is

$$\pi_k^A(j_k)=(1-q_k)V_k^{-1}(j_k)\Big(\frac{1}{2\omega}r_k^M(j_k)-r_k^{\lambda}(j_k)B_k(j_k)-r_k^{\lambda}(j_k)A_k(j_k)w_k\Big).\tag{4.19}$$

Proof. For $t = T-1$ and $i_{T-1} \in F,\ j_{T-1} \in G$, by $Equation$(4.8) we have

$$
\begin{aligned}
& v_{T-1}(\Phi_{T-1}^{j_{T-1}}, j_{T-1}, w_{T-1}) \\
&= max_{\pi_{T-1}^A(j_{T-1})} E\Big\{\lambda w_T - \omega w_T^2 | \Phi_{T-1}^{j_{T-1}}, j_{T-1}, w_{T-1}\Big\} \\
&= max_{\pi_{T-1}^A(j_{T-1})} \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) \sum_{j_T=1}^{g} \varphi_t(i_{T-1}, j_T) \\
&\cdot E\Big\{\lambda\Big(A_{T-1}(j_{T-1})w_{T-1} + B_{T-1}(j_{T-1}) + \frac{R'_{T-1}(i_{T-1}, j_{T-1})\pi_{T-1}^A(j_{T-1})}{1-q_{T-1}}\Big) \\
&- \omega\Big(A_{T-1}(j_{T-1})w_{T-1} + B_{T-1}(j_{T-1}) + \frac{R'_{T-1}(i_{t-1}, j_{T-1})\pi_{T-1}^A(j_{T-1})}{1-q_{T-1}}\Big)^2\Big\} \\
&= max_{\pi_{T-1}^A} \lambda A_{T-1}(j_{T-1})w_{T-1} + \lambda B_{T-1}(j_{T-1}) \qquad (4.20) \\
&+ \frac{\lambda \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}} r'_{T-1}(i_{T-1}, j_{T-1})\pi_{T-1}^A(j_{T-1})}{1-q_{T-1}} \\
&- \omega A_{T-1}^2(j_{T-1})w_{T-1}^2 - \omega B_{T-1}^2(j_{T-1}) \\
&- \frac{\omega \pi'^A_{T-1}(j_{T-1}) \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}} \gamma_{T-1}(i_{T-1}, j_{T-1})\pi_{T-1}^A(j_{T-1})}{(1-q_{T-1})^2} \\
&- 2\omega A_{T-1}(j_{T-1})B_{T-1}(j_{T-1})w_{T-1} \\
&- \frac{2\omega A_{T-1}(j_{T-1}) \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}} r'_{T-1}(i_{T-1}, j_{T-1})\pi_{T-1}^A(j_{T-1})}{1-q_{T-1}}.
\end{aligned}
$$

Since $\gamma_{T-1}(i_{T-1}, j_{T-1})$ is positive definite for any $t = T-1$ and $i_{T-1} \in F,\ j_{T-1} \in G$, we have $\sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}} \gamma_{T-1}(i_{T-1}, j_{T-1}) > 0$ and $\omega > 0$, then we get the optimal solution using the first-order condition, which is shown as follows:

$$
\begin{aligned}
& \pi_{T-1}^A(j_{T-1}) = (1-q_{T-1})V_{T-1}^{-1}(j_{T-1}) \\
& \cdot \Big(\frac{1}{2\omega} r_{T-1}^M(j_{T-1}) - r_{T-1}^\lambda(j_{T-1})B_{T-1}(j_{T-1}) - r_{T-1}^\lambda(j_{T-1})A_{T-1}w_{T-1}\Big), \qquad (4.21)
\end{aligned}
$$

where

$$
r_{T-1}^M(j_{T-1}) = \lambda \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}} r_{T-1}(i_{T-1}, j_{T-1}), \qquad (4.22)
$$

$$
r_{T-1}^\lambda(j_{T-1}) = \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}} r_{T-1}(i_{T-1}, j_{T-1}). \qquad (4.23)
$$

In order to obtain the corresponding value functions, we substitute $\pi_{T-1}^A(j_{T-1})$

into $Equation(4.20)$, then we have

$$
\begin{aligned}
& v_{T-1}(\Phi_{T-1}^{j_{T-1}}, j_{T-1}, w_{T-1}) \\
= & -\omega A_{T-1}^2(j_{T-1})\Big(1 - r_{T-1}^{\prime\lambda}(j_{T-1})V_{T-1}^{-1}(i_{T-1,j_{T-1}})r_{T-1}^{\lambda}(j_{T-1})\Big)w_{T-1}^2 \\
& + \Big((\lambda - r_{T-1}^{\prime\lambda}(j_{T-1})V_{T-1}^{-1}(i_{T-1,j_{T-1}})r_{T-1}^{M}(j_{T-1})) \\
& - 2\omega(1 - r_{T-1}^{\prime\lambda}(j_{T-1})V_{T-1}^{-1}(i_{T-1,j_{T-1}})r_{T-1}^{\lambda}(j_{T-1}))B_{T-1}(j_{T-1})\Big)A_{T-1}(j_{T-1}) \\
& + \Big((\lambda - r_{T-1}^{\prime\lambda}(j_{T-1})V_{T-1}^{-1}(i_{T-1,j_{T-1}})r_{T-1}^{M}(j_{T-1})) \\
& - \omega(1 - r_{T-1}^{\prime\lambda}(j_{T-1})V_{T-1}^{-1}(i_{T-1,j_{T-1}})r_{T-1}^{\lambda}(j_{T-1}))B_{T-1}(j_{T-1}))B_{T-1}(j_{T-1})\Big) \\
& + \frac{1}{4\omega}r_{T-1}^{\prime M}(j_{T-1})V_{T-1}^{-1}(i_{T-1,j_{T-1}})r_{T-1}^{M}(j_{T-1}) \\
= & -\omega\hat{A}_{T-1}^2(j_{T-1})A_{T-1}^2(j_{T-1})w_{T-1}^2 + M_{T-1}(j_{T-1})w_{T-1} + D_{T-1}(j_{T-1}).
\end{aligned}
\tag{4.24}
$$

$Equation(4.19)$ and (4.24) show that $Equation(4.18)$ and (4.19) hold true for $t = T-1$.

Based on the mathematical induction process, we suppose that $Equation(4.18)$ and (4.19) hold true for $t = k+1$ and $i_t \in F,\ j_t \in G$ and substitute $Equation(4.18)$

into $Equation(4.8)$ to get the equations as follows:

$$
\begin{aligned}
& v_k(\Phi_k^{j_k}, j_k, w_k) \\
=& max_{\pi_k(j_k)} \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \\
& \cdot E\Big(v_{k+1}\Big(\Phi_{k+1}^{j_{k+1}}, j_{k+1}, A_k(j_k)w_k + B_k(j_k) + \frac{R_k'(\Phi_k^{j_k}, j_k)\pi_k^A(j_k)}{1-q_k}\Big)\Big) \\
=& max_{\pi_k(j_k)} \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \\
& \cdot E\Big(- \omega \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1})(A_k(j_k)w_k + B_k(j_k) + \frac{R_k'(i_k, j_k)\pi_k(j_k)}{1-q_k})^2 \\
& + M_{k+1}(j_{k+1})(A_k(j_k)w_k + B_k(j_k) + \frac{R_k'(i_k, j_k)\pi_k^A(j_k)}{1-q_k}) + D_{k+1}(j_{k+1})\Big) \\
=& - \omega \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) A_k^2(j_k) w_k^2 \\
& - \omega \sum_{i_k=1}^{f} \phi_t^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_t(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) B_k^2(j_k) \\
& - \frac{\omega}{(1-q_k)^2} \pi_k'^A(j_k) \sum_{i_k=1}^{f} \gamma_k(i_k, j_k) \phi_k^{j_k}(i_k) \\
& \cdot \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) \pi_k^A(j_k) \\
& - 2\omega \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) A_k(j_k) B_k(j_k) w_k \\
& - \frac{2\omega}{1-q_k} \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) r_k'(i_k, j_k) \\
& \cdot \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) A_k(j_k) \pi_k^A(j_k) w_k \\
& - \frac{2\omega}{1-q_k} \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) r_k'(i_k, j_k) \\
& \cdot \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) B_k(j_k) \pi_k^A(j_k)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})M_{k+1}(j_{k+1})A_k(j_k)w_k\\
&+\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})M_{k+1}(j_{k+1})B_k(j_k)\\
&+\frac{1}{1-q_k}\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)r_k'(i_k,j_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})M_{k+1}(j_{k+1})\pi_k^A(j_k)\\
&+\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})D_{k+1}(j_{k+1}).
\end{aligned}
\tag{4.25}
$$

Since $\omega>0,\ 0\le q_k<1,\ \frac{\omega}{(1-q_k)^2}>0$ and $\gamma_k(i_k,j_k)$ is positive definite. Based on *Lemma* 4.1, we have $\hat{A}^2_{k+1}(j_{k+1})>0$ and the first order condition of *Equation*(4.25) is satisfied, then

$$
\begin{aligned}
&\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\gamma_k(i_k,j_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})\hat{A}^2_{k+1}(j_{k+1})A^2_{k+1}(j_{k+1})\pi_k^A(j_k)\\
&=(1-q_k)\Big(\frac{1}{2\omega}\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)r_k(i_k,j_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})M_{k+1}(j_{k+1})\\
&-\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)r_k(i_k,j_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})\hat{A}^2_{k+1}(j_{k+1})A^2_{k+1}(j_{k+1})B_k(j_k)\\
&-\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)r_k(i_k,j_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})\hat{A}^2_{k+1}(j_{k+1})A^2_{k+1}(j_{k+1})A_k(j_k)w_k\Big),
\end{aligned}
\tag{4.26}
$$

from which we obtain the optimal strategy $\pi_k^A(j_k)$ as follow:

$$
\pi_k^A(j_k)=(1-q_k)V_k^{-1}(j_k)\Big(\frac{1}{2\omega}r_k^M(j_k)-r_k^\lambda(j_k)B_k(j_k)-r_k^\lambda(j_k)A_k(j_k)w_k\Big). \tag{4.27}
$$

In order to get the expression of $v_k(\Phi_k^{j_k},j_k,w_k)$, we substitute *Equation*(4.27) into $Equation(4.25)$ and have

$$
\begin{aligned}
& v_k(\Phi_k^{j_k}, j_k, w_k) \\
= & -\omega \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) A_k^2(j_k) w_k^2 \\
& -\omega \sum_{i_k=1}^{f} \phi_t^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_t(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) B_k^2(j_k) \\
& -\frac{1}{4\omega} r_k^{'M}(j_k) V_k^{-1}(j_k) r_k^M(j_k) + \frac{1}{2} r_k^{'M}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) \\
& +\frac{1}{2} r_k^{'M}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k(j_k) w_k + \frac{1}{2} r_k^{'M}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) B_k(j_k) \\
& -\omega r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) B_k^2(j_k) - \omega r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k(j_k) B_k(j_k) w_k \\
& +\frac{1}{2} r_k^{'M}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k(j_k) w_k - \omega r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k(j_k) B_k(j_k) w_k \\
& -\omega r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k^2(j_k) w_k^2 \\
& -2\omega \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \hat{A}_{k+1}^2(j_{k+1}) A_{k+1}^2(j_{k+1}) A_k(j_k) B_k(j_k) w_k \\
& -r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^M(j_k) A_k(j_k) w_k + 2\omega r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k(j_k) B_k(j_k) w_k \\
& +2\omega r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k^2(j_k) w_k^2 - r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^M(j_k) B_k(j_k) \\
& +2\omega r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) B_k^2(j_k) + 2\omega r_k^{'\lambda}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k(j_k) B_k(j_k) w_k \\
& +\sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) M_{k+1}(j_{k+1}) A_k(j_k) w_k \\
& +\sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) M_{k+1}(j_{k+1}) B_k(j_k) \\
& +\frac{1}{2\omega} r_k^{'M}(j_k) V_k^{-1}(j_k) r_k^M(j_k) - r_k^{'M}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) B_k(j_k) \\
& -r_k^{'M}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k) A_k(j_k) w_k + \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) D_{k+1}(j_{k+1})
\end{aligned}
$$

$$
\begin{aligned}
= & -\omega\Big(\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})\hat{A}_{k+1}^2(j_{k+1})A_{k+1}^2(j_{k+1}) \\
& - r_k^{\prime\lambda}(j_k)V_k^{-1}(j_k)r_k^{\lambda}(j_k)\Big)A_k^2(j_k)w_k^2 \\
& + \Big((\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})M_{k+1}(j_{k+1}) - r_k^{\prime M}(j_k)V_k^{-1}(j_k)r_k^{\lambda}(j_k))A_k(j_k) \\
& - 2\omega(\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})\hat{A}_{k+1}^2(j_{k+1})A_{k+1}^2(j_{k+1}) \\
& - r_k^{\prime M}(j_k)V_k^{-1}(j_k)r_k^{\lambda}(j_k))A_k(j_k)B_k(j_k)\Big)w_k \\
& + \Big(\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})M_{k+1}(j_{k+1}) - r_k^{\prime M}(j_k)V_k^{-1}(j_k)r_k^{\lambda}(j_k)\Big)B_k(j_k) \\
& - \omega\Big(\sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})\hat{A}_{k+1}^2(j_{k+1})A_{k+1}^2(j_{k+1}) \\
& - r_k^{\prime\lambda}(j_k)V_k^{-1}(j_k)r_k^{\lambda}(j_k)\Big)B_k^2(j_k) \\
& + \sum_{i_k=1}^{f}\phi_k^{j_k}(i_k)\sum_{j_{k+1}=1}^{g}\varphi_k(i_k,j_{k+1})D_{k+1}(j_{k+1}) \\
= & -\omega\hat{A}_k^2(j_k)A_k^2(j_k)w_k^2 + M_k(j_k)w_k + D_k(j_k).
\end{aligned}
\tag{4.28}
$$

Equation(4.27) and (4.28) show that *Equation*(4.18) and (4.19) hold for $t = k$. By the principle of mathematical induction, *Equation*(4.18) and (4.19) hold true for $t = 0, 1, ..., T-1$. $\square$

4.4 Solution for Problem $P(\omega)$

First, we define some notations. For $t = 0, 1, ..., T-1,\ k \le t,\ i_k, i_t \in F,\ j_k, j_t \in G$, we have

$$S_t(j_t) = \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) A_{t+1}(j_{t+1}) S_{t+1}(j_{t+1}) - r_t^{\prime\lambda}(j_t) V_t^{-1}(j_t) r_t^S(j_t), \tag{4.29}$$

$$r_t^S(j_t) = \sum_{i_t=1}^{f} \phi_t^{j_t}(i_t) r_t(i_t, j_t) \sum_{j_{t+1}=1}^{g} \varphi_t(i_t, j_{t+1}) A_{t+1}(j_{t+1}) S_{t+1}(j_{t+1}), \tag{4.30}$$

$$\begin{aligned}\Delta_k^t(j_k) = &\sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) A_{k+1}(j_{k+1}) \Delta_{k+1}^t(j_{k+1}) \\ &- r_k^{\Delta,t}(j_k) V_k^{-1}(j_k) r_k^{\lambda}(j_k),\end{aligned} \tag{4.31}$$

$$r_k^{\Delta,t}(j_k) = \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) r_k(i_k, j_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) A_{k+1}(j_{k+1}) \Delta_{k+1}^t(j_{k+1}), \tag{4.32}$$

with terminal condition

$$S_T(j_T) = \frac{1}{A_T(j_T)},\ \Delta_t^t(j_t) = B_t(j_t) \hat{A}_t^2(j_t). \tag{4.33}$$

Lemma 4.2. *For all* $t = 0, 1, ..., T-1$ *and* $i_t \in F,\ j_t \in G$,

$$M_k(j_k) = \lambda A_k(j_k) S_k(j_k) - 2\omega A_k(j_k) \sum_{t=k}^{T-1} \Delta_k^t(j_k), \tag{4.34}$$

$$r_k^M(j_k) = \lambda r_k^S(j_k) - 2\omega \sum_{t=k+1}^{T-1} r_k^{\Delta,t}(j_k). \tag{4.35}$$

Proof. For $t = T-1$ and $i_{T-1} \in F,\ j_{T-1} \in G$, by *Equation*(4.10) and (4.16), we have

$$\begin{aligned}M_{T-1}(j_{T-1}) = &\lambda A_{T-1}(j_{T-1})(1 - r_{T-1}^{\prime\lambda}(j_{T-1}) V_{T-1}^{-1}(j_{T-1}) r_{T-1}^{\lambda}(j_{T-1})) \\ &- 2\omega \hat{A}_{T-1}^2(j_{T-1}) A_{T-1}(j_{T-1}) B_{T-1}(j_{T-1}) \\ = &\lambda A_{T-1}(j_{T-1}) S_{T-1} - 2\omega A_{T-1}(j_{T-1}) \Delta_{T-1}^{T-1}(j_{T-1}),\end{aligned} \tag{4.36}$$

$$r_{T-1}^M(j_{T-1}) = \lambda \sum_{i_{T-1}=1}^{f} \phi_{T-1}^{j_{T-1}}(i_{T-1}) r_{T-1}(i_{T-1}, j_{T-1}) = \lambda r_{T-1}^S(j_{T-1}). \tag{4.37}$$

Equation(4.36) and (4.37) show that *Equation*(4.34) and (4.35) hold true when $t = T-1$. We then assume *Equation*(4.34) and (4.35) hold true when $t = k+1$. By (4.16) for $t = k$, we have

$$\begin{aligned}
r_k^M(j_k) =& \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) r_k(i_k, j_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) M_{k+1}(j_{k+1}) \\
=& \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) r_k(i_k, j_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \\
& \cdot \Big(\lambda A_{k+1}(j_{k+1}) S_{k+1}(j_{k+1}) - 2\omega A_{k+1}(j_{k+1}) \sum_{t=k+1}^{T-1} \Delta_{k+1}^t(j_{k+1})\Big) \\
=& \lambda \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) r_k(i_k, j_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) A_{k+1}(j_{k+1}) S_{k+1}(j_{k+1}) \\
& - 2\omega \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) r_k(i_k, j_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) A_{k+1}(j_{k+1}) \sum_{t=k+1}^{T-1} \Delta_{k+1}^t(j_{k+1}) \\
=& \lambda r_k^S(j_k) - 2\omega \sum_{t=k+1}^{T-1} r_k^{\Delta,t}(j_k),
\end{aligned} \tag{4.38}$$

and by *Equation*(4.11), (4.31) and (4.38), we have

$$\begin{aligned}
\hat{M}_k(j_k) =& \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) \Big(\lambda A_{k+1}(j_{k+1}) S_{k+1}(j_{k+1}) \\
& - 2\omega A_{k+1}(j_{k+1}) \sum_{t=k+1}^{T-1} \Delta_{k+1}^t(j_{k+1})\Big) \\
& - r_k^\lambda(j_k) V_k^{-1}(j_k) \Big(\lambda r_k^S(j_k) - 2\omega \sum_{t=k+1}^{T-1} r_k^{\Delta,t}(j_k)\Big) \\
=& \lambda \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) A_{k+1}(j_{k+1}) S_{k+1}(j_{k+1}) \\
& - 2\omega \sum_{i_k=1}^{f} \phi_k^{j_k}(i_k) \sum_{j_{k+1}=1}^{g} \varphi_k(i_k, j_{k+1}) A_{k+1}(j_{k+1}) \sum_{t=k+1}^{T-1} \Delta_{k+1}^t(j_{k+1}) \\
& - \lambda r_k^\lambda(j_k) V_k^{-1}(j_k) r_k^S(j_k) + 2\omega r_k^\lambda(j_k) V_k^{-1}(j_k) \sum_{t=k+1}^{T-1} r_k^{\Delta,t}(j_k) \\
=& \lambda S_k(j_k) - 2\omega \sum_{t=k+1}^{T-1} \Delta_k^t(j_k),
\end{aligned} \tag{4.39}$$

then by $Equation$(4.10) and (4.39), we have

$$M_k(j_k) = \lambda A_k(j_k)S_k(j_k) - 2\omega A_k(j_k)\sum_{t=k}^{T-1}\Delta_k^t(j_k). \tag{4.40}$$

$Equation$(4.38) and (4.40) shows that $Equation$(4.34) and (4.35) hold true for $t = 0, 1, ..., T-1$. □

Theorem 4.2. *Suppose the initial observable market state $O_0 = j_0 \in G$, the corresponding conditional distribution of the unobservable market U_0 at time 0 given $O_0 = j_0$ is $\Phi_0^{j_0}$ and the initial wealth $W_0 = w_0$, we have*

$$\begin{aligned}E(W_T|\Phi_0^{j_0}, j_0, w_0) =& A_0(j_0)S_0(j_0)w_0 + \sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))\\ &+ \frac{\lambda}{2\omega}\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{'S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))\\ &- \sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{'S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)), \end{aligned}\tag{4.41}$$

$$\begin{aligned}E(W_T^2|\Phi_0^{j_0}, j_0, w_0) =& A_0^2(j_0)\hat{A}_0^2(j_0)w_0^2 + 2A_0(j_0)\sum_{t=0}^{T-1}\Delta_0^t(j_0)w_0\\ &+ \sum_{t=0}^{T-1}(\phi\varphi)^t(B_t^2(j_t)\hat{A}_t^2(j_t))\\ &+ \frac{1}{4\omega^2}\sum_{t=1}^{T-1}(\phi\varphi)^t(r_t^{'M}(j_t)V_t^{-1}(j_t)r_t^M(j_t))\\ &+ 2\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(B_k(j_k)\Delta_k^l(j_k))\\ &+ \frac{1}{\omega}\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^M(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)), \end{aligned}\tag{4.42}$$

where

$$(\phi\varphi)^k = \sum_{i_0=1}^{f}\phi_0^{j_0}(i_0)\sum_{j_1=1}^{g}\varphi_0(i_0,j_1)\sum_{i_1=1}^{f}\phi_1^{j_1}(i_1)\sum_{j_2=1}^{g}\varphi_1(i_1,j_2)...\sum_{i_{k-1}=1}^{f}\sum_{j_{k-1}=1}^{g}\varphi_{k-1}(i_{k-1},j_k).$$

Proof. In order to prove $Equation$(4.41), we just need to prove the following equation holds true when $h = 0$:

$$\begin{aligned}
&E(W_T|\Phi_h^{j_h}, j_h, W_h)\\
=&A_h(j_h)S_h(j_h)W_h + \sum_{t=h}^{T-1}(\phi\varphi)^{t-h}B_t(j_t)S_t(j_t)\\
&+\frac{\lambda}{2\omega}\sum_{t=h}^{T-1}(\phi\varphi)^{t-h}r_t'^S(j_t)V_t^{-1}(i_t, j_t)r_t^S(j_t)\\
&-\sum_{l=h+1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k'^S(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)).
\end{aligned} \tag{4.43}$$

First, by *Equation*(4.1) and (4.19), we have the following equation when $t = T-1$:

$$\begin{aligned}
W_T =&A_{T-1}(j_{T-1})W_{T-1} + B_{T-1}(j_{T-1}) + \frac{R'_{T-1}(i_{T-1}, j_{T-1})\pi_{T-1}(j_{T-1})}{1-q_{T-1}}\\
=&A_{T-1}(j_{T-1})\Big(1 - R'_{T-1}(i_{T-1}, j_{T-1})V_{T-1}^{-1}(i_{T_1}, j_{T-1})r_{T-1}^{\lambda}(j_{T-1})\Big)W_{T-1}\\
&+B_{T-1}(j_{T-1})\Big(1 - R'_{T-1}(i_{T-1}, j_{T-1})V_{T-1}^{-1}(i_{T_1}, j_{T-1})r_{T-1}^{\lambda}(j_{T-1})\Big)\\
&+\frac{1}{2\omega}R'_{T-1}(i_{T-1}, j_{T-1})V_{T-1}^{-1}(i_{T_1}, j_{T-1})r_{T-1}^{M}(j_{T-1}),
\end{aligned} \tag{4.44}$$

taking the conditional expectation on both sides of *Equation*(4.44) given the observable market state $O_{T-1} = j_{T-1}$, the conditional distribution of the unobservable market state $\Phi_{T-1}^{j_{T-1}}$ and the wealth W_{T-1}, we then obtain

$$\begin{aligned}
&E(W_T|\Phi_{T-1}^{j_{T-1}}, j_{T-1}, W_{T-1})\\
=&A_{T-1}(j_{T-1})W_{T-1}\\
&\cdot\Big(1 - \sum_{i_{T-1}=1}^{f}\phi_{T-1}^{j_{T-1}}(i_{T-1})r'_{T-1}(i_{T-1}, j_{T-1})V_{T-1}^{-1}(i_{T_1}, j_{T-1})r_{T-1}^{\lambda}(j_{T-1})\Big)\\
&+B_{T-1}(j_{T-1})\\
&\cdot\Big(1 - \sum_{i_{T-1}=1}^{f}\phi_{T-1}^{j_{T-1}}(i_{T-1})r'_{T-1}(i_{T-1}, j_{T-1})V_{T-1}^{-1}(i_{T_1}, j_{T-1})r_{T-1}^{\lambda}(j_{T-1})\Big)\\
&+\frac{1}{2\omega}\sum_{i_{T-1}=1}^{f}\phi_{T-1}^{j_{T-1}}(i_{T-1})r'_{T-1}(i_{T-1}, j_{T-1})V_{T-1}^{-1}(i_{T_1}, j_{T-1})r_{T-1}^{M}(j_{T-1})\\
=&A_{T-1}(j_{T-1})S_{T-1}(j_{T-1})W_{T-1} + B_{T-1}(j_{T-1})S_{T-1}(j_{T-1})\\
&\cdot+\frac{1}{2\omega}r_{T-1}'^S(j_{T-1})V_{T-1}^{-1}(i_{T_1}, j_{T-1})r_{T-1}^{M}(j_{T-1}),
\end{aligned} \tag{4.45}$$

by *Lemma* 4.1 we have

$$
\begin{aligned}
& E(W_T|\Phi_{T-1}^{j_{T-1}}, j_{T-1}, W_{T-1}) \\
=& A_{T-1}(j_{T-1})S_{T-1}(j_{T-1})W_{T-1} + B_{T-1}(j_{T-1})S_{T-1}(j_{T-1}) \\
& + \frac{\lambda}{2\omega} r'^{S}_{T-1}(j_{T-1})V^{-1}_{T-1}(i_{T_1}, j_{T-1})r^{S}_{T-1}(j_{T-1}) \\
=& A_{T-1}(j_{T-1})S_{T-1}(j_{T-1})W_{T-1} + \sum_{t=T-1}^{T-1}(\phi\varphi)^{t-(T-1)}B_t(j_t)S_t(j_t) \\
& + \frac{\lambda}{2\omega}\sum_{t=T-1}^{T-1}(\phi\varphi)^{t-(T-1)}r'^{S}_t(j_t)V^{-1}_t(i_t, j_t)r^{S}_t(j_t) \\
& - \sum_{l=(T-1)+1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r'^{S}_k(j_k)V^{-1}_k(j_k)r^{\Delta,l}_k(j_k)).
\end{aligned}
\tag{4.46}
$$

Therefore, we prove that *Equation*(4.43) holds true when $h = T-1$. We assume that *Equation*(4.43) is true when $h = m+1$, that is

$$
\begin{aligned}
& E(W_T|\Phi_{m+1}^{j_{m+1}}, j_{m+1}, W_{m+1}) \\
=& A_{m+1}(j_{m+1})S_{m+1}(j_{m+1})W_{m+1} + \sum_{t=m+1}^{T-1}(\phi\varphi)^{t-(m+1)}B_t(j_t)S_t(j_t) \\
& + \frac{\lambda}{2\omega}\sum_{t=m+1}^{T-1}(\phi\varphi)^{t-(m+1)}r'^{S}_t(j_t)V^{-1}_t(i_t, j_t)r^{S}_t(j_t) \\
& - \sum_{l=m+2}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r'^{S}_k(j_k)V^{-1}_k(j_k)r^{\Delta,l}_k(j_k)).
\end{aligned}
\tag{4.47}
$$

When $h = m$, by $Equation$(4.1) and (4.19), we have

$$
\begin{aligned}
&E(W_T|\Phi_{m+1}^{j_{m+1}}, j_{m+1}, W_{m+1}) \\
=&A_{m+1}(j_{m+1})S_{m+1}(j_{m+1})\Big(A_m(j_m)W_m + B_m(j_m) + \frac{R'_m(i_m, j_m)\pi_m(j_m)}{1-q_m}\Big) \\
&+ \sum_{t=m+1}^{T-1} (\phi\varphi)^{t-(m+1)} B_t(j_t)S_t(j_t) \\
&+ \frac{\lambda}{2\omega} \sum_{t=m+1}^{T-1} (\phi\varphi)^{t-(m+1)} r_t'^S(j_t)V_t^{-1}(i_t, j_t)r_t^S(j_t) \\
&- \sum_{l=m+2}^{T-1} \sum_{k=0}^{l-1} (\phi\varphi)^k (r_k'^S(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)) \\
=&A_{m+1}(j_{m+1})S_{m+1}(j_{m+1})\Big(A_m(j_m)(1 - R'_m(i_m, j_m)V_m^{-1}(i_m, j_m)r_m^\lambda(j_m))W_m \\
&+ B_m(j_m)(1 - R'_m(i_m, j_m)V_m^{-1}(i_m, j_m)r_m^\lambda(j_m)) \\
&+ \frac{1}{2\omega}R'_m(i_m, j_m)V_m^{-1}(i_m, j_m)r_m^M(j_m)\Big) \\
&+ \sum_{t=m+1}^{T-1} (\phi\varphi)^{t-(m+1)} B_t(j_t)S_t(j_t) + \frac{\lambda}{2\omega} \sum_{t=m+1}^{T-1} (\phi\varphi)^{t-(m+1)} r_t'^S(j_t)V_t^{-1}(i_t, j_t)r_t^S(j_t) \\
&- \sum_{l=m+2}^{T-1} \sum_{k=0}^{l-1} (\phi\varphi)^k (r_k'^S(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)), \qquad (4.48)
\end{aligned}
$$

taking the conditional expectation on both sides of $Equation$(4.48) given $O_m = j_m$, $\Phi_m^{j_m}$ and W_m, we then have

$$
\begin{aligned}
& E\Big(E(W_T|\Phi_{m+1}^{j_{m+1}}, j_{m+1}, W_{m+1})|\Phi_m^{j_m}, j_m, W_m\Big) = E(W_T|\Phi_m^{j_m}, j_m, W_m) \\
=& A_m(j_m)\Big(\sum_{i_m=1}^{f}\phi_m^{j_m}(i_m)\sum_{j_{m+1}=1}^{g}\varphi_m(i_m, j_{m+1})A_{m+1}(j_{m+1})S_{m+1}(j_{m+1}) \\
& -\sum_{i_m=1}^{f}\phi_m^{j_m}(i_m)r'_m(i_m, j_m) \\
& \cdot\sum_{j_{m+1}=1}^{g}\varphi_m(i_m, j_{m+1})A_{m+1}(j_{m+1})S_{m+1}(j_{m+1})V_m^{-1}(j_m)r_m^{\lambda}(j_m)\Big)W_m \\
& + B_m(j_m)\Big(\sum_{i_m=1}^{f}\phi_m^{j_m}(i_m)\sum_{j_{m+1}=1}^{g}\varphi_m(i_m, j_{m+1})A_{m+1}(j_{m+1})S_{m+1}(j_{m+1}) \\
& -\sum_{i_m=1}^{f}\phi_m^{j_m}(i_m)r'_m(i_m, j_m) \\
& \cdot\sum_{j_{m+1}=1}^{g}\varphi_m(i_m, j_{m+1})A_{m+1}(j_{m+1})S_{m+1}(j_{m+1})V_m^{-1}(j_m)r_m^{\lambda}(j_m)\Big) \\
& + \frac{1}{2\omega}\phi_m^{j_m}(i_m)r'_m(i_m, j_m) \\
& \cdot\sum_{j_{m+1}=1}^{g}\varphi_m(i_m, j_{m+1})A_{m+1}(j_{m+1})S_{m+1}(j_{m+1})V_m^{-1}(j_m)r_m^{M}(j_m) \\
& + \sum_{t=m+1}^{T-1}(\phi\varphi)^{t-m}B_t(j_t)S_t(j_t) + \frac{\lambda}{2\omega}\sum_{t=m+1}^{T-1}(\phi\varphi)^{t-m}r_t'^{S}(j_t)V_t^{-1}(i_t, j_t)r_t^{S}(j_t) \\
& -\sum_{l=m+2}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^{k+1}(r_k'^{S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)) \\
=& A_m(j_m)S_m(j_m)W_m + B_m(j_m)S_m(j_m) + \frac{1}{2\omega}r_m'^{S}(j_m)V_m^{-1}(j_m)r_m^{M}(j_m) \\
& + \sum_{t=m+1}^{T-1}(\phi\varphi)^{t-m}B_t(j_t)S_t(j_t) + \frac{\lambda}{2\omega}\sum_{t=m+1}^{T-1}(\phi\varphi)^{t-m}r_t'^{S}(j_t)V_t^{-1}(i_t, j_t)r_t^{S}(j_t) \\
& -\sum_{l=m+2}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^{k+1}(r_k'^{S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)).
\end{aligned}
\tag{4.49}
$$

Substituting *Equation*(4.35) in *Lemma* 4.1 into *Equation*(4.49), we have

$$\begin{aligned}
&E(W_T|\Phi_m^{j_m}, j_m, W_m)\\
=&A_m(j_m)S_m(j_m)W_m + B_m(j_m)S_m(j_m) + \frac{\lambda}{2\omega} r_m^{'S}(j_m)V_m^{-1}(j_m)r_m^S(j_m)\\
&-\sum_{l=m+1}^{T-1} r_m^{'S}(j_m)V_m^{-1}(j_m)r_m^{\Delta,l}(j_m)\\
&+\sum_{t=m+1}^{T-1}(\phi\varphi)^{t-m}B_t(j_t)S_t(j_t) + \frac{\lambda}{2\omega}\sum_{t=m+1}^{T-1}(\phi\varphi)^{t-m}r_t^{'S}(j_t)V_t^{-1}(i_t,j_t)r_t^S(j_t)\\
&-\sum_{l=m+2}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^{k+1}(r_k^{'S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k))\\
=&A_m(j_m)S_m(j_m)W_m + \sum_{t=m}^{T-1}(\phi\varphi)^{t-m}B_t(j_t)S_t(j_t)\\
&+\frac{\lambda}{2\omega}\sum_{t=m}^{T-1}(\phi\varphi)^{t-m}r_t^{'S}(j_t)V_t^{-1}(i_t,j_t)r_t^S(j_t)\\
&-\sum_{l=m+1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^{k}(r_k^{'S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)).
\end{aligned} \tag{4.50}$$

Equation(4.50) shows that *Equation*(4.43) hold true for $h = 0, 1, ..., T-1$. Therefore, when $h = 0$, *Equation*(4.41) holds true.
Since the proof of *Equation*(4.42) is similar with the proof of *Equation*(4.41), we just omit the proof process. □

We then focus on the optimum policy and the corresponding efficient frontier for the problem $P(\omega)$. We early mentioned that the optimum policy for the original problem $P(\omega)$ is the one with special $\lambda^* = 1 + 2\omega E(W_T^{\pi^A}|\Phi_0^{j_0}, j_0, w_0)$ for the auxiliary problem if it exists, where $W_T^{\pi^A}$ is the terminal wealth under the optimum investment policy π^A in $A(\lambda,\omega)$. We have

$$\begin{aligned}\lambda^* =&1+2\omega\Big(A_0(j_0)S_0(j_0)w_0+\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))\\&+\frac{\lambda^*}{2\omega}\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))\\&-\sum_{l=1}^{T-1}\sum_{m=0}^{l-1}(\phi\varphi)^m(r_m^{\prime S}(j_m)V_m^{-1}(j_m)r_m^{\Delta,l}(j_m))\Big)\\=&1+2\omega A_0(j_0)S_0(j_0)w_0+2\omega\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))\\&+\lambda^*\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))\\&-2\omega\sum_{l=1}^{T-1}\sum_{m=0}^{l-1}(\phi\varphi)^m(r_m^{\prime S}(j_m)V_m^{-1}(j_m)r_m^{\Delta,l}(j_m)),\end{aligned}$$

then we have

$$\begin{aligned}\lambda^* =&\frac{1+2\omega A_0(j_0)S_0(j_0)w_0+2\omega\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))}{1-\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))}\\&\frac{-2\omega\sum_{l=1}^{T-1}\sum_{m=0}^{l-1}(\phi\varphi)^m(r_m^{\prime S}(j_m)V_m^{-1}(j_m)r_m^{\Delta,l}(j_m))}{1-\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))}.\end{aligned}\tag{4.51}$$

By *Equation*(4.51), *Theorem* 4.1 and *Lemma* 4.1, for $k=0,1,...,T-1$, we can derive the optimal strategy as follows:

$$\begin{aligned}\pi_k^A(j_k) =&(1-q_k)V_k^{-1}(j_k)\Big(\frac{\lambda^*}{2\omega}r_k^S(j_k)-\sum_{t=k+1}^{T-1}r_k^{\Delta,t}(j_k)-r_k^\lambda(j_k)B_k(j_k)\\&-r_k^\lambda(j_k)A_k(j_k)w_k\Big)\\=&(1-q_k)V_k^{-1}(j_k)\Big(\frac{r_k^S(j_k)}{2\omega(1-\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t)))}\\&+\frac{A_0(j_0)S_0(j_0)r_k^S(j_k)w_0+\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))r_k^S(j_k)}{1-\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))}\\&-\frac{\sum_{l=1}^{T-1}\sum_{m=0}^{l-1}(\phi\varphi)^m(r_m^{\prime S}(j_m)V_m^{-1}(j_m)r_m^{\Delta,l}(j_m))r_k^S(j_k)}{1-\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))}\\&-\sum_{t=k+1}^{T-1}r_k^{\Delta,t}(j_k)-r_k^\lambda(j_k)B_k(j_k)-r_k^\lambda(j_k)A_k(j_k)w_k\Big).\end{aligned}\tag{4.52}$$

Then by $Theorem$ 4.1, we have

$$
\begin{aligned}
&Var(W_T|\Phi_0^{j_0}, j_0, w_0) = E(W_T^2|\Phi_0^{j_0}, j_0, w_0) - (E(W_T|\Phi_0^{j_0}, j_0, w_0))^2 \\
=&A_0^2(j_0)\hat{A}_0^2(j_0)w_0^2 + 2A_0(j_0)\sum_{t=0}^{T-1}\Delta_0^t(j_0)w_0 + \sum_{t=0}^{T-1}(\phi\varphi)^t(B_t^2(j_t)\hat{A}_t^2(j_t)) \\
&+ \frac{1}{4\omega^2}\sum_{t=1}^{T-1}(\phi\varphi)^t(r_t^{\prime M}(j_t)V_t^{-1}(j_t)r_t^M(j_t)) + 2\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(B_k(j_k)\Delta_k^l(j_k)) \\
&+ \frac{1}{\omega}\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^M(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)) \\
&- (A_0(j_0)S_0(j_0)w_0)^2 - (\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t)))^2 \\
&- \frac{\lambda^2}{4\omega^2}(\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t)))^2 \\
&- (\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{\prime S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)))^2 \\
&- 2\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))A_0(j_0)S_0(j_0)w_0 \\
&- \frac{\lambda}{\omega}\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))A_0(j_0)S_0(j_0)w_0 \\
&+ 2\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{\prime S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k))A_0(j_0)S_0(j_0)w_0 \\
&- \frac{\lambda}{\omega}\sum_{k=0}^{T-1}(\phi\varphi)^k(B_k(j_k)S_k(j_k))\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t)) \\
&+ 2\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{\prime S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)) \\
&+ \frac{\lambda}{\omega}\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{\prime S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{\prime S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)).
\end{aligned}
$$

Then we have the following theorem:

Theorem 4.3. *For $k = 0, 1, ..., T-1$, $j_k \in G$, let $W_k = w_k$ and $W_0 = w_0$, the* *optimal investment strategy of problem $P(\omega)$ is*

$$\begin{aligned}\pi_k^A(j_k) =&(1-q_k)V_k^{-1}(j_k)\Big(\frac{r_k^S(j_k)}{2\omega(1-\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t'^S(j_t)V_t^{-1}(j_t)r_t^S(j_t)))}\\&+\frac{A_0(j_0)S_0(j_0)r_k^S(j_k)w_0+\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))r_k^S(j_k)}{1-\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t'^S(j_t)V_t^{-1}(j_t)r_t^S(j_t))}\\&-\frac{\sum_{l=1}^{T-1}\sum_{m=0}^{l-1}(\phi\varphi)^m(r_m'^S(j_m)V_m^{-1}(j_m)r_m^{\Delta,l}(j_m))r_k^S(j_k)}{1-\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t'^S(j_t)V_t^{-1}(j_t)r_t^S(j_t))},\end{aligned} \tag{4.53}$$

the corresponding efficient frontier is

$$
\begin{aligned}
&Var(W_T|\Phi_0^{j_0}, j_0, w_0)\\
=&A_0^2(j_0)(\hat{A}_0^2(j_0) - S_0^2(j_0))w_0^2\\
&+\Big(2A_0(j_0)\sum_{t=0}^{T-1}\Delta_0^t(j_0) - 2\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))A_0(j_0)S_0(j_0)\\
&-\frac{\lambda^*}{\omega}\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{'S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))A_0(j_0)S_0(j_0)\\
&+2\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{'S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k))A_0(j_0)S_0(j_0)\Big)w_0\\
&+\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t^2(j_t)\hat{A}_t^2(j_t))\\
&+\frac{1}{4\omega^2}\sum_{t=1}^{T-1}(\phi\varphi)^t(r_t^{'M}(j_t)V_t^{-1}(j_t)r_t^M(j_t)) + 2\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(B_k(j_k)\Delta_k^l(j_k))\\
&+\frac{1}{\omega}\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^M(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)) - (\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t)))^2\\
&-\frac{\lambda^{*2}}{4\omega^2}(\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{'S}(j_t)V_t^{-1}(j_t)r_t^S(j_t)))^2\\
&-(\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{'S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)))^2\\
&-\frac{\lambda^*}{\omega}\sum_{k=0}^{T-1}(\phi\varphi)^k(B_k(j_k)S_k(j_k))\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{'S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))\\
&+2\sum_{t=0}^{T-1}(\phi\varphi)^t(B_t(j_t)S_t(j_t))\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{'S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k))\\
&+\frac{\lambda^*}{\omega}\sum_{t=0}^{T-1}(\phi\varphi)^t(r_t^{'S}(j_t)V_t^{-1}(j_t)r_t^S(j_t))\sum_{l=1}^{T-1}\sum_{k=0}^{l-1}(\phi\varphi)^k(r_k^{'S}(j_k)V_k^{-1}(j_k)r_k^{\Delta,l}(j_k)),
\end{aligned}
\tag{4.54}
$$

where λ^* satisfies Equation(4.51).

4.5 Special Cases

4.5.1 The Case with Complete Information Market

If the market state is completely observable, then for $t = 0, 1, ..., T-1$, $U_t = O_t,\ f = g$. Then the hidden Markov chain becomes a Markov chain U. In order to clarify the relationship between the previous literature and this paper, in this case we assume that the return rate of the risk-less asset r_t^0 is deterministic, which is different from the original assumption in this paper. Therefore, when $U_t = i_t$, we have

$$\sigma_t(i_t, j_t) = P(O_t = j_t | U_t = i_t) = \begin{cases} 1.\ if\ i_t = j_t, \\ 0,\ if\ i_t \neq j_t, \end{cases} \tag{4.55}$$

and the emission matrix Σ_t reduces to the unit matrix I_f, and $Equation(4.3)$ becomes

$$\phi_{t+1}^{j_{t+1}}(i_{t+1}) = P(U_{t+1} = i_{t+1} | \overline{O}_t, O_{t+1} = j_{t+1}) = \begin{cases} 1.\ if\ i_{t+1} = j_{t+1}, \\ 0,\ if\ i_{t+1} \neq j_{t+1}, \end{cases} \tag{4.56}$$

and we also know that $\varphi_t(i_t, j_{t+1}) = \sum_{i_{t+1}=1}^{f} p_t(i_t, i_{t+1})\sigma_{t+1}(i_{t+1}, j_{t+1}) = p_t(i_t, j_{t+1})$. The wealth process reduces to

$$W_{t+1} = A_t W_t + B_t + \frac{R_t'(i_t)\pi_t(i_t)}{1-q_t}, \tag{4.57}$$

where $A_t = \frac{r_t^0}{1-q_t}$ and $B_t = \frac{C_t r_t^0 - a q_t \sum_{l=0}^{t} C_l}{1-q_t}$. Then we define

$$\begin{aligned} \eta_t(i_t) =& (1 - r_t'(i_t)\gamma_t^{-1}(i_t) r_t(i_t)) \sum_{i_{t+1}=1}^{f} p_t(i_t, i_{t+1})\eta_{t+1}(i_{t+1}) \\ =& (1 - r_t'(i_t)\gamma_t^{-1}(i_t) r_t(i_t)) Q_t(\eta_{t+1}(j_{t+1})), \end{aligned} \tag{4.58}$$

with terminal condition

$$\eta_T(i_T) = 1,$$

where $\eta_t = (\eta_t(1), \eta_t(2), ..., \eta_t(f))'$. Then for $t = 0, 1, ..., T-1,\ k \le t$, we have

$$\hat{A}_t^2(i_t) = \prod_{k=t+1}^{T-1} A_k^2 \eta(i_t), \tag{4.59}$$

$$r_t^{\lambda} = r_t(i_t) \prod_{l=t+1}^{T-1} A_l^2 Q_t(\eta_{t+1}), \tag{4.60}$$

$$r_t^{S} = r_t(i_t) \prod_{l=t+1}^{T-1} A_l Q_t(\eta_{t+1}), \tag{4.61}$$

$$r_k^{\Delta,t}(i_k) = r_k(i_k) \prod_{m=k+1}^{l} A_m B_l \prod_{h=l+1}^{T-1} A_h^2 Q_k(\eta_{k+1}), \tag{4.62}$$

$$\begin{aligned} r_k^M(j_k) &= \lambda r_k^S(j_k) - 2\omega \sum_{l=k+1}^{T-1} r_k^{\Delta,l}(j_k) \\ &= \lambda r_k(i_k) \prod_{l=k+1}^{T-1} A_l Q_k(\eta_{k+1}) - 2\omega \sum_{l=k+1}^{T-1} r_k(i_k) \prod_{m=k+1}^{l} A_m B_l \prod_{h=l+1}^{T-1} A_h^2 Q_k(\eta_{k+1}), \end{aligned} \tag{4.63}$$

$$V_t(i_t) = \gamma_t(i_t) \prod_{l=t+1}^{T-1} A_l^2 Q_t(\eta_{t+1}), \tag{4.64}$$

$$S_t(i_t) = \prod_{k=t+1}^{T-1} A_k \eta_t(i_t), \tag{4.65}$$

$$\Delta_k^t(i_k) = \prod_{m=k+1}^{t} A_m B_t \prod_{h=t+1}^{T-1} A_h^2 \eta_k(i_k). \tag{4.66}$$

Hence, for $t = 0, 1, ..., T-1$, the optimal strategy for the case without unobservable market state is as follows:

$$\pi_t^A(i_t) = \Big(\frac{\lambda^*}{2\omega \prod_{k=t+1}^{T-1} A_k} - \sum_{l=t}^{T-1} \frac{B_l}{\prod_{k=t+1}^{l} A_k} - A_t w_t\Big)(1-q_t)\gamma_t^{-1}(i_t) r_t(i_t), \tag{4.67}$$

where

$$\lambda^* = \frac{1 + 2\omega\Big(\sum_{k=0}^{T-1} A_k \eta_0(i_0) w_0 + \sum_{l=0}^{T-1} (\prod_{m=0}^{l-1} Q_m \eta_l)(i_0) B_l \prod_{n=l+1}^{T-1} A_n\Big)}{\eta_0(i_0)}$$
$$- \frac{2\omega \sum_{l=0}^{T-1} ((\prod_{m=0}^{l-1} Q_m) h_m Q_l \eta_{l+1})(i_0) \sum_{k=l+1}^{T-1} B_k \prod_{n=k+1}^{T-1} A_n}{\eta_0(i_0)}$$
$$= \frac{1 + 2\omega(\sum_{k=0}^{T-1} A_k \eta_0(i_0) w_0 + b_{T-1}(\beta, i_0))}{\eta_0(i_0)}, \tag{4.68}$$

$$b_k(\beta, i_0) = \sum_{l=0}^{k} (\prod_{m=0}^{l-1} Q_m \eta_l)(i_0) B_l \prod_{n=l+1}^{T-1} A_n$$
$$- \sum_{l=0}^{k} ((\prod_{m=0}^{l-1} Q_m) h_m Q_l \eta_{l+1})(i_0) \sum_{k=l+1}^{T-1} B_k \prod_{n=k+1}^{T-1} A_n, \tag{4.69}$$

Equations(4.67), (4.68) and (4.69) are consistent with *Equation*(23), (99), (29) in Bian et al.(2018) [72].

Then we have the corresponding efficient frontier given the initial wealth $W_0 = w_0$ and $U_0 = i_0$ as follows:

$$Var(W_T | i_0, w_0)$$
$$= \frac{\eta_0(i_0)}{\sum_{l=0}^{T-1} \Big((\prod_{m=0}^{l-1} Q_m) r'_m(i_m) \gamma_m^{-1}(i_m) r_m(i_m) Q_l \eta_{l+1}\Big)(i_0)}$$
$$\cdot \Big(E(W_T | i_0, w_0) - \prod_{k=0}^{T-1} A_k w_0 - \sum_{l=0}^{T-1} B_l \prod_{n=l+1}^{T-1} A_n\Big)^2$$
$$- \eta_0(i_0)\Big(\sum_{l=0}^{T-1} B_l \prod_{n=l+1}^{T-1} A_n\Big)^2 + \sum_{l=0}^{T-1} B_l^2 \prod_{n=l+1}^{T-1} A_n^2 \Big((\prod_{m=0}^{l-1} Q_m)\eta_l\Big)(i_0)$$
$$+ \sum_{l=0}^{T-1} B_l^2 \prod_{n=l+1}^{T-1} A_n^2 \Big((\prod_{h=0}^{l-1} Q_h)\eta_l\Big)(i_0)$$
$$+ \sum_{l=0}^{T-1} \Big(\sum_{m=l+1}^{T-1} B_m \prod_{n=m+1}^{T-1} A_n\Big)^2 \Big((\prod_{n=0}^{l-1} Q_n) h_n Q_l \eta_{l+1}\Big)(i_0)$$
$$+ 2\sum_{l=0}^{T-1} B_l \prod_{n=l+1}^{T-1} A_n b_{l-1}(\beta, i_0). \tag{4.70}$$

Equation(4.70) corresponds to *Equation*(36) in the work of Bian et al.(2018) [72].

4.5.2 The Case without Regime-Switching

Based on the assumption in the subsection above, we consider the case in which there is no regime-switching in the model, that is to say, the market is stable and no change in return rates of assets exist. The probability of the state transition from $U_t = i_t$ to $U_{t+1} = i_{t+1}$ at time $t+1$ is

$$p_t(i_t, i_{t+1}) = P\{U_{t+1} = i_{t+1} | U_t = i_t\} = \begin{cases} 1, \ if \ i_t = i_{t+1}, \\ 0, \ if \ i_t \neq i_{t+1}, \end{cases} \tag{4.71}$$

and the transition matrix $Q_t = Q(p_t(i_t, i_{t+1}))_{f\times f}$ reduces to the unit matrix I_f. Then for $k = 0, 1, ..., T-1$ and $U_k = i_k$, we have some simplified notations as follows:

$$\eta_k = \prod_{l=k}^{T-1}(1 - r_l' \gamma_l^{-1} r_l), \tag{4.72}$$

$$\hat{A}_k^2 = \prod_{l=k+1}^{T-1} A_l^2 \prod_{m=k}^{T-1}(1 - r_m' \gamma_m^{-1} r_m), \tag{4.73}$$

$$r_k^M = \lambda r_k^S - 2\omega \sum_{t=k+1}^{T-1} r_k^{\Delta,t}, \tag{4.74}$$

$$r_k^S = \prod_{l=k+1}^{T-1} A_l r_k \prod_{m=k+1}^{T-1} (1 - r_m' \gamma_m^{-1} r_m), \tag{4.75}$$

$$r_k^\lambda = \prod_{l=k+1}^{T-1} A_l^2 r_k \prod_{m=k+1}^{T-1} (1 - r_m' \gamma_m^{-1} r_m), \tag{4.76}$$

$$r_k^{\Delta,t} = \prod_{l=k+1}^{t} A_l B_t \prod_{h=t+1}^{T-1} A_h^2 \prod_{m=k+1}^{T-1} (1 - r_m' \gamma_m^{-1} r_m), \tag{4.77}$$

$$b_k(\beta) = \prod_{m=0}^{T-1}(1 - r_m' \gamma_m^{-1} r_m) \sum_{n=0}^{T-1} B_n \prod_{l=n+1}^{T-1} A_l - \prod_{m=k+1}^{T-1} (1 - r_m' \gamma_m^{-1} r_m) \sum_{n=k+1}^{T-1} B_n \prod_{l=n+1}^{T-1} A_l. \tag{4.78}$$

Then we have the optimal investment strategy and the corresponding efficient frontier as follows:

$$\begin{aligned}\pi_k^P =&\Big(\sum_{l=0}^{k-1} B_l \prod_{h=l+1}^{k} A_h + \frac{1}{2\omega \prod_{l=0}^{T-1}(1-r_l'\gamma_l^{-1}r_l)\prod_{h=k+1}^{T-1} A_h}\\&+\prod_{l=0}^{k} A_l w_0 - A_k w_k\Big)(1-q_t)\gamma_k^{-1}r_k,\end{aligned} \tag{4.79}$$

$$\begin{aligned}&Var(W_T|w_0)\\=&\frac{\eta_0}{1-\prod_{m=0}^{T-1}(1-r_m'\gamma_m^{-1}r_m)}\Big(E(W_T|w_0)-\prod_{l=0}^{T-1}A_l w_0-\sum_{l=0}^{T-1}B_l\prod_{h=l+1}^{T-1}A_h\Big)^2\\&-\eta_0(\sum_{l=0}^{T-1}B_l\prod_{h=l+1}^{T-1}A_h)^2+\sum_{l=0}^{T-1}B_l^2\prod_{h=l+1}^{T-1}A_h^2\prod_{m=l}^{T-1}(1-r_m'\gamma_m^{-1}r_m)\\&+\sum_{l=0}^{T-1}\Big(\sum_{m=l+1}^{T-1}B_m\prod_{h=m+1}^{T-1}A_m\Big)^2(1-r_l'\gamma_l^{-1}r_l)\prod_{n=l+1}^{T-1}(1-r_n'\gamma_n^{-1}r_n)\\&+2\sum_{l=0}^{T-1}B_l\prod_{h=l+1}^{T-1}A_h\Big(\prod_{m=0}^{T-1}(1-r_m'\gamma_m^{-1}r_m)\sum_{n=0}^{T-1}B_n\prod_{t=n+1}^{T-1}A_t\\&-\prod_{m=k+1}^{T-1}(1-r_m'\gamma_m^{-1}r_m)\sum_{n=k+1}^{T-1}B_n\prod_{t=n+1}^{T-1}A_t\Big).\end{aligned} \tag{4.80}$$

4.6 Numerical Examples

In this section, we provide some numerical examples to demonstrate how our results affect the efficient frontiers .

Suppose there are 4 time periods for the accumulation phase of a DC pension plan, i.e., $T=4$, assuming that the member in the plan has initial wealth $W_0=1$ and at each time point $t=0,1,2,3$, he contributes $C_t=1$ and the corresponding mortality probability is $q_0=0.00408,\ q_1=0.00534,\ q_2=0.00676,\ q_3=0.00837$. In addition, we assume the risk aversion parameter $\omega=2$.

Suppose that there are two unobservable market state, i.e., bullish market state $U_t=1$ and bearish market state $U_t=2$, and the market transition matrixes at each time are shown as follows:

$$Q_0 = \begin{pmatrix} 0.60 & 0.40 \\ 0.50 & 0.50 \end{pmatrix}; \; Q_1 = \begin{pmatrix} 0.80 & 0.20 \\ 0.40 & 0.60 \end{pmatrix}; \tag{4.81}$$
$$Q_2 = \begin{pmatrix} 0.65 & 0.35 \\ 0.55 & 0.45 \end{pmatrix}; \; Q_3 = \begin{pmatrix} 0.60 & 0.40 \\ 0.70 & 0.30 \end{pmatrix}.$$

And assume that the initial probability of the unobservable market state $U_0 = 1$ is 0.6 and $U_0 = 2$ is 0.4. For $t = 0, 1, 2, 3$, suppose there are two observable market state, i.e., the positive perspective $O_t = 1$ and the negative perspective $O_t = 2$. The corresponding emission matrixes at each time point are shown as follows:

$$\Sigma_0 = \begin{pmatrix} 0.75 & 0.25 \\ 0.60 & 0.40 \end{pmatrix}; \; \Sigma_1 = \begin{pmatrix} 0.80 & 0.20 \\ 0.65 & 0.35 \end{pmatrix}; \tag{4.82}$$
$$\Sigma_2 = \begin{pmatrix} 0.60 & 0.40 \\ 0.70 & 0.30 \end{pmatrix}; \; \Sigma_3 = \begin{pmatrix} 0.55 & 0.45 \\ 0.70 & 0.30 \end{pmatrix}.$$

Suppose that there is one risk-less asset $r_t^0(O_t)$ and two risky assets $r_t^0(O_t) + R_t^1(U_t, O_t)$, $r_t^0(O_t) + R_t^2(U_t, O_t)$, the return rates of the risk-less asset at each time point is $r_0^0(1) = r_0^0(2) = 1.008$, $r_1^0(1) = r_1^0(2) = 1.007$, $r_2^0(1) = r_2^0(2) = 1.003$, $r_3^0(1) = r_3^{(}2) = 1.006$. The expected return rates of risky assets in different market states at each time point $t = 0, 1, 2, 3$ are shown as follows:

$E(asset_1(i,j))$	$j=1$	$j=2$	$E(asset_2(i,j))$	$j=1$	$j=2$
$i=1$	1.157	1.14		1.137	1.109
$i=2$	1.12	1.07		1.115	0.989

4.6.1 Initial Market States

In this section we consider the effect of initial observable market state on the corresponding efficient frontier. As is shown in Figure 4.1, the efficient frontier performs better when the initial observable market state $j_0 = 1$(at the top of Figure 4.1) and worse when $j_0 = 2$(at the bottom of the Figure 4.1). The reason is that an investor's perspective on the market is not always consistent with the change in the real market and the overall investors' perspectives have effect on the asset return rates in the market regardless of what level of the actual asset return rates are. When the initial observable market state $j_0 = 1$, investors believe that the market being in a good state will continue staying in that state in the future and they will obtain high investment returns from the market. Therefore, their

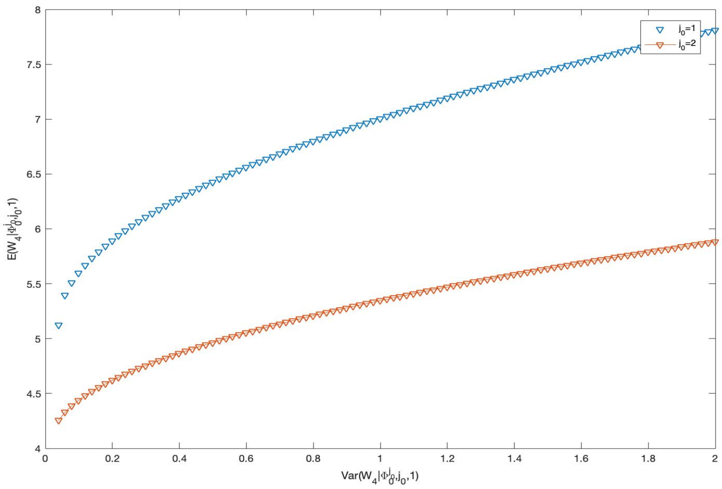

Figure 4.1: Efficient frontiers under different initial perspectives

optimistic attitude to the investment will keep the asset return rates being at a high level even when the real market is in the bearish state. On the other hand, when the initial observable market state $j_0 = 2$, the negative perspectives on the market reduce the asset return rates in the market even when the actual market state is bullish. In a word, the effect of the investors' perspectives on the market reflects in the actual asset return rates in the market, and an investor obtains a higher investment return under a positive initial observable market state than a negative initial observable market state. This result is consistent with the one in Bian et al.(2018)

4.6.2 Market Conditions

In this section, we focus on the efficient frontiers in three different markets, i.e, the incomplete information model(IIM), the complete information model(CIM) and the case with no regime-switching(CNR).

For CIM, We assume that the time-dependent market transition matrix Q_t in CIM is the same as the one in IIM. For $t = 0, 1, ..., T-1$, when the market state in CIM $i_t = 1$, we use the corresponding data in IIM under the unobservable market state $i_t = 1$ and $j_t = 1$. On the other hand, when the market state $i_t = 2$, the data given $i_t = 2$ and $j_t = 2$ in IIM are used in CIM. With regard to CNR, when the initial unobservable market state in IIM $j_0 = 1$, we use the data given $i_t = 1$ and $j_t = 1$ from IIM in CNR and the data with $i_t = 2$ and $j_t = 2$ from IIM are used when the initial market state $j_0 = 2$. Note once the data such as the asset return rates are used in CNR, they are unchanged as time goes by.

Figure 4.2 displays the efficient frontiers of IIM, CIM and CNR given an initial observable market state $j_0 = 1$. It turns out that the efficient frontier of CNR has the best performance and the efficient frontier of IIM is superior to the one of CIM. Note that when the initial market state $j_0 = 1$, the expected return rates of assets in CNR are unchanged during the investment horizon, which means an investor can obtain high return rates from the investment without consideration of the risks of the changing market state. In the CIM model, there are unexpected fluctuations in the market which an investor needs to be well aware of, the bad market condition can reduce the return rates of the assets which investors possess. Consequently, based on the same level of the investment risk, the expected terminal wealth in CNR is higher than that in CIM. Under the IIM model, however, investors' perspectives on the market are assumed to be an important factor that effects the asset return rates. The assumption that the initial observable market state $j_0 = 1$ means that investors have positive perspectives on the market, which influences the market and results in higher asset return rates. Compared with the efficient frontier in CIM, the IIM efficient frontier performs better under the initial positive perspectives on market.

Figure 4.3 illustrates the efficient frontier of IIM, CIM and CNR under the situation of a bad initial observable market state, i.e., $j_0 = 2$. Since the asset return rates in CNR is the lowest and unchanged during the investment horizon, the CNR efficient frontier performs the worst in Figure 4.3. At the top of Figure 4.3 lies the CIM efficient frontier because of the existence of the market fluctuation under which there is certain probability that the bad initial market state switches into the good state, which may result in higher return rates of the assets. In contrast, the efficient frontier under IIM lies below the CIM efficient frontier because the negative initial perspective impacts the return rates of assets in the market.

Based on Figure 4.2 and Figure 4.3, we can conclude that the regime-switching market can protect the investment return from extreme cases, that is, to maintain the return and prevent it from being too good or too bad. The observable market state serve as a kind of 'lubricant' in the regime-switching market. The market can be influenced by the perspectives from overall investors, which further changes the asset return rates in the market. Positive investment perspectives can 'pull up' the asset return rates in bad market conditions and therefore get investment gains, while negative investment perspectives can 'push down' the return rates under good market states and make investors loss money.

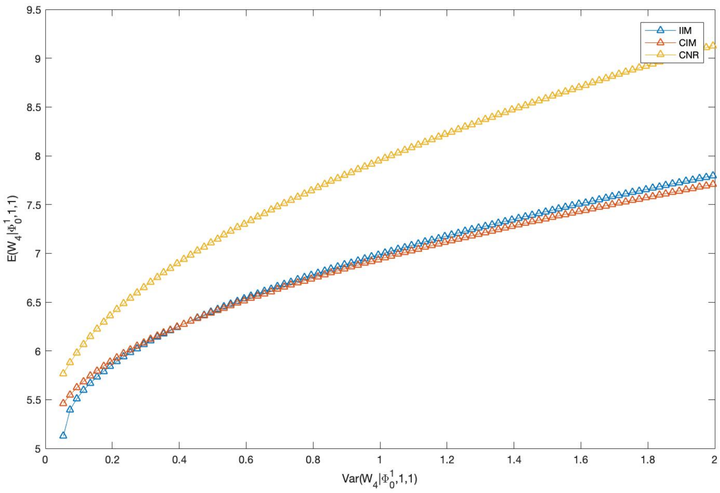

Figure 4.2: Efficient frontiers under the good initial perspective

4.6.3 Premiums Return Policy

This subsection investigates the effect of premiums return policy on the corresponding efficient frontier.

Figure 4.4 plots the case of premiums return policy under initial good observable market state while Figure 4.5 plots under initial bad observable state. We find that the efficient frontier with premiums return policy lies below the one without policy. For a DC pension plan with premiums return policy, if an investor dies during the accumulation phase, the amount of money the died investor contributed as premiums will be returned to his/her heirs, which leads to a loss in wealth level of the surviving members in the fund and increase the investment uncertainty as well. In this situation, the fund manager for the DC pension plan will bear more risks than the plan with no return policy.

4.7 Concluding Remarks

This chapter studies the portfolio selection for a DC pension plan with incomplete information where we use a discrete-time hidden Markov chain to represent the dynamics of the unobservable market state. In order to protect right of an investor, we introduce the premiums return policy in which the investor's heir will withdraw the contributions that the investor made before if the investor dies during the accumulation phase. The fund manager invests the contributions into a risk-free asset and n risky assets in the market where the return rate of the risk-free asset only depends on the observable market state and the risky assets

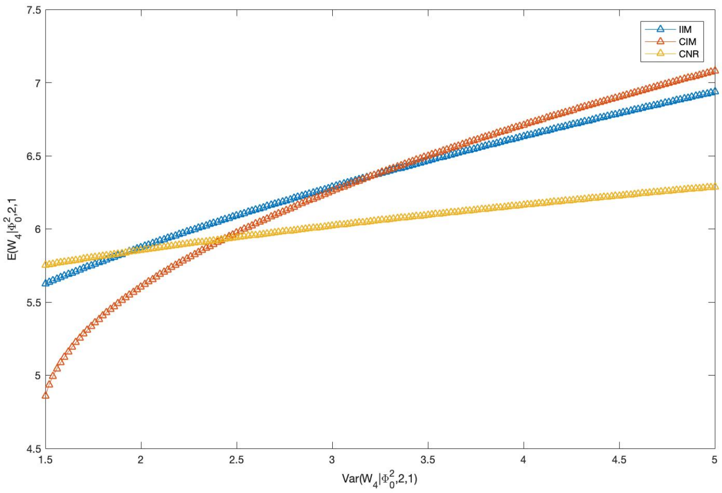

Figure 4.3: Efficient frontiers under the bad initial perspective

return rates depend on both the observable market state and the unobservable market state.

We use sufficient statistics to convert the optimization problem into a new problem which can be expressed by the observable state. Then we use the embedding technique to convert the original problem into an auxiliary problem in which we can use the Bellman equation and the dynamic programming approach to derive the optimal investment policy and the corresponding efficient frontier. Some special cases are given to demonstrate that our work generalizes some of the existing models in literature. In the numerical analysis, we plot figures to illustrate the effects of some important factors, such as the unobservable market state and the premiums return policy, on the corresponding efficient frontier. We find some interesting result: (i) The manager faces greater risks under the bad initial observable market state than the good initial state. (ii) The observable market state can be considered as 'lubricant' in the regime-switching market, it 'pulls up' the asset return rate when it is in a bad unobservable state and 'pushes down' the return rate in a good unobservable state. (iii) For the DC plan, the fund manager bears more risk in the plan with premiums return policy.

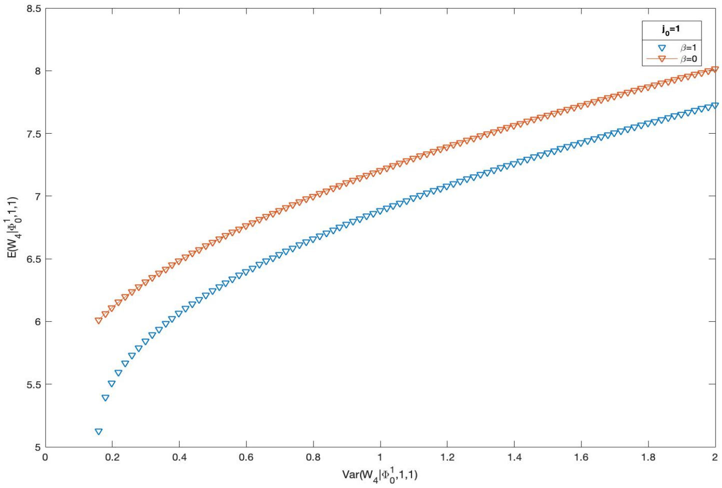

Figure 4.4: Efficient frontiers under the good initial perspective

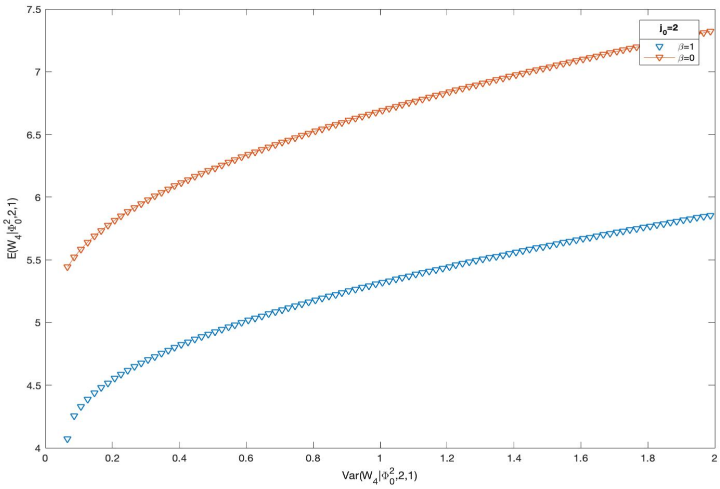

Figure 4.5: Efficient frontiers under the bad initial perspective

CHAPTER 5

Multi-Period Mean-Variance Portfolio Selection with State-Dependent Exit Probability and Bankruptcy State*

5.1 General Overview

Based on the portfolio selection model in a regime-switching market, this chapter focuses on the uncertain time horizon where the exit probability is dependent on the time and the market state. In addition, an extreme market state called the bankruptcy state is also considered. In such a state, a company in the market struggles with heavy liability and unhealthy financial state so that the bankruptcy is the best choice for the company itself. After the liquidation, people who invest in the bankrupt company can only get a partial amount of the money they invested before. There exist some previous works studying these two factors for the portfolio selection. Wu and Zeng(2013) [64] studied the portfolio selection with bankruptcy state, but did not consider the factor of exit probability. Based on the multi-period mean-variance portfolio selection, Wu et al.(2014) [55] assumed that the uncertain time horizon depends on the market state, however, the exit probability was not considered. This chapter generalizes the exiting literature and considers these two factors together under the mean-variance portfolio selection model. We assume that the conditional exiting probability depends on the current market state including the bankruptcy state which is assumed to be an absorbing state. To obtain the optimal strategy and the corresponding efficient frontier by the Lagrange multiplier method, some innovative expressions of the wealth formulation and the iteration process are derived, which is the main

$*$ *The content of this Chapter has been published.*

contribution in this chapter. Numerical examples illustrate that both of the factors affect the investment policy and the corresponding efficient frontier significantly.

5.2 Model Formulation

This chapter assumes that an investor enters the market at time 0 with initial wealth w_0. Denote by S_t the market state at time t ($t = 0, 1, 2, ..., T-1$) and assume that S_t ($S_t = 1, 2, ..., L; t = 0, 1, 2, ..., T-1$) is a time-homogeneous Markov chain with a transition matrix Q. Let L denote the bankruptcy state in which the investor will get back δ of the wealth from the bankrupt company, where δ refers to the retrieval rate which is assumed to be a random variable ranging between $[0, 1]$. Let $R_t(S_t)$ and $r_t(S_t)$ signify the return of the risky asset and the risk-free asset respectively. With regard to the state-dependent exiting probability, suppose that the investor quits the market with the probability which depends on the current market state. Let τ represent the exit time and define

$$P_t(i) = P(\tau = t|S_t = i), t = 0, 1, .., T, \tag{5.1}$$

which satisfies $\sum_{t=0}^{T} P(\tau = t|S_0 = i) = 1, i = 1, 2, .., L$.
Throughout this chapter, some definitions need to be made, we summarize them as follows:

D5.1 It is unnecessary to define $R_t(L)$, because the company goes bankrupt and the risky asset has no investment value at state L. For $S_t = 1, 2, ..., L-1$, define $R_t^e(S_t) = R_t(S_t) - r_t(S_t)$, which means the difference between the return of the risky asset and risk-free asset at time t ($t = 0, 1, ..., T-1$), and for $S_t = 1, 2, ..., L-1$, denote by $r_t^e(S_t) = E(R_t^e(S_t))$ the expectation of $R_t^e(S_t)$ at time t ($t = 0, 1, ..., T-1$) and assume it to be nonnegative. Besides, we define $e_t(S_t) = (r_t(S_t), R_t(S_t))'$ ($t = 1, 2, ..., T-1$).

D5.2 π_t represents the investment amount into the risky asset at time t ($t = 0, 1, ..., T-1$) and π means a set of strategies $\pi_t(t = 0, 1, ..., T-1)$ within the investment time horizon.

D5.3 For any time-dependent matrix $A_t(L \times L)$ and any vector $a_t(L \times 1)$, define $\sum_{t=1}^{0} a_t = \mathbf{0}$ where $\mathbf{0}$ is the $L \times 1$ zero vector and $\prod_{t=1}^{0} A_t = \mathbb{I}$ where $\mathbb{I}$ is the $L \times L$ unit matrix.

D5.4 $Q(i,j)_{i,j=1,2,...,L}$ is the entry of transition matrix Q which means one-step transition probability from market state i to state j. $Q^k(i,j)$ signifies k-step transition probability which is the entry of Q^k, the kth power of transition matrix Q. Besides, define $\widehat{Q}(i,j) = Q(i,j)_{i,j=1,2,...,L-1}$ with Q^0 representing the identity matrix.

D5.5 For any matrix $A_{L\times L}$ and any vector $a_{L\times 1}$, $A(i)$ represents the ith row of A and $a(i)$ means the ith element of a, and we define $A_a(i,j)_{L\times L}$ to be a matrix in which $A_a(i,j) = A(i,j)a(j)$, and $\overline{A}$ is a column vector which is equivalent to $A \cdot \mathbb{I}_{L\times 1}$ where $\mathbb{I}_{L\times 1}$ is a column vector whose elements are all 1.

D5.6 $A_t^1(t = 1, 2, ..., T-1)$ is a column vector whose ith element is

$$A_t^1(i) = E(r_t^2(i))E(R_t^e(i)^2) - \left[E(r_t(i)R_t^e(i))\right]^2, \; i \neq L.$$

D5.7 $A_t^2(t = 1, 2, ..., T-1)$ is a column vector whose ith element is

$$A_t^2(i) = E(r_t(i))E(R_t^e(i)^2) - r_t^e(i)E(r_t(i)R_t^e(i)), \; i \neq L.$$

D5.8 $H_t(t = 1, 2, ..., T-1)$ is a column vector whose ith element is

$$\begin{aligned} H_t(i) = & \sum_{S_{t+1}=1}^{L-1} Q(i, S_{t+1})\Big[P_{t+1}(S_{t+1}) + \frac{H_{t+1}(S_{t+1})A_{t+1}^1(S_{t+1})}{E(R_{t+1}^e(S_{t+1})^2)}\Big] \\ & + Q(i,L)E(\delta^2)\Big[P_{t+1}(L) + \sum_{S_{t+2}=1}^{L} Q(L, S_{t+2})E(r(L)^2) \\ & \cdot \sum_{k=t+2}^{T} Q^{k-(t+2)}P_k(S_k) \prod_{j=1}^{k-(t+2)} E(r_{k-j}(S_{k-j})^2)\Big], \\ & i \neq L, \; t = 0, 1, ..., T-2, \end{aligned}$$

$$H_{T-1}(i) = Q(i,L)P_T(L)E(\delta^2) + \sum_{j=1}^{L-1} Q(i,j)P_T(j), \; i \neq L.$$

D5.9 $G_t(t = 1, 2, ..., T-1)$ is a column vector whose ith element is

$$G_t(i) = \sum_{S_{t+1}=1}^{L-1} Q(i, S_{t+1})\Big[P_{t+1}(S_{t+1}) + \frac{H_{t+1}(S_{t+1})A_{t+1}^2(S_{t+1})}{E(R_{t+1}^e(S_{t+1})^2)}\Big]$$
$$+ Q(i, L)E(\delta)\Big[P_{t+1}(L) + \sum_{S_{t+2}=1}^{L} Q(L, S_{t+2})E(r(L))$$
$$\cdot \sum_{k=t+2}^{T} Q^{k-(t+2)} P_k(S_k) \prod_{j=1}^{k-(t+2)} E(r_{k-j}(S_{k-j}))\Big],$$
$$i \neq L,\ t = 0, 1, ..., T-2,$$
$$G_{T-1}(i) = Q(i, L)P_T(L)E(\delta) + \sum_{j=1}^{L-1} Q(i, j)P_T(j),\ i \neq L.$$

D5.10 $K_t(t = 1, 2, ..., T-1)$ is a column vector whose ith element is

$$K_t(i) = \frac{(G_t(i)r_t^e(i))^2}{H_t(i)E(R_t^e(i)^2)},\ i \neq L,\ t = 0, 1, ..., T-1.$$

5.3 Wealth Process and Optimization Problem

Now define the dynamics of the wealth with π as follows:

$$w_{t+1}^{\pi} = \begin{cases} \Big(r_t(S_t)w_t^{\pi} + R_t^e(S_t)\pi_t\Big)\Big(1_{\{S_{t+1}\neq L\}} + \delta 1_{\{S_{t+1}=L\}}\Big), & S_t \neq L, \\ r_t(L)w_t^{\pi}, \quad S_t = L. \end{cases} \tag{5.2}$$

where $1_{\{S_{t+1}\neq L\}}$ represents the function whose value is 1 when $S_{t+1} \neq L$. Moreover, we generally suppose that the initial market state $S_0 \neq L$. Otherwise, the investor needs to save all the wealth into his bank account from the beginning, which is a trivial question.

For each time point, the investor makes rational strategy to optimize terminal wealth. This chapter studies this portfolio selection problem under mean-variance criterion in which the investment risk is measured by the terminal wealth variance. Considering the dependence of the exit probability on the market state and the fixed expectation of terminal wealth d, the objective function under a strategy π is presented as follows:

$$P(d) \begin{cases} min_\pi \ Var_{i_0,w_0}(w^\pi_{T\wedge\tau}) \\ s.t. \ E_{i_0,w_0}(w^\pi_{T\wedge\tau}) = d \ and \ (5.2), \end{cases}$$

where $E_{i_0,w_0}(w_{T\wedge\tau})$ denotes $E(w_{T\wedge\tau}|S_0 = i, w_0)$ and $Var_{i_0,w_0}(w_{T\wedge\tau})$ stands for $Var(w_{T\wedge\tau}|S_0 = i, w_0)$ respectively.

Let π^* be the optimal strategy. Then the portfolio is an efficient portfolio if there exists no strategy $\widehat{\pi}$ such that $E_{i_0,w_0}(w^{\pi^*}_{T\wedge\tau}) \le E_{i_0,w_0}(w^{\widehat{\pi}}_{T\wedge\tau})$ and $Var_{i_0,w_0}(w^{\pi^*}_{T\wedge\tau}) \ge Var_{i_0,w_0}(w^{\widehat{\pi}}_{T\wedge\tau})$ and at least one inequality holds true.
$\big(E_{i_0,w_0}(w^{\pi^*}_{T\wedge\tau}), Var_{i_0,w_0}(w^{\pi^*}_{T\wedge\tau})\big)$ is called an efficient point and the set of all efficient points is called the efficient frontier.

According to $Var(w_{T\wedge\tau}) = E(w_{T\wedge\tau} - E(w_{T\wedge\tau})^2 = E(w_{T\wedge\tau} - d)^2$, we have

$$\overline{P}(d) \begin{cases} min_\pi \ E_{i_0,w_0}(w^\pi_{T\wedge\tau} - d)^2 \\ s.t. \ E_{i_0,w_0}(w^\pi_{T\wedge\tau}) = d \ and \ (5.2). \end{cases}$$

By embedding a Lagrange multiplier λ into problem $\overline{P}(d)$ (Luenberger(1986) [76]), the original problem is transformed into an auxiliary problem $\overline{P1}(\lambda, d)$ which has no constraint condition,

$$\overline{P1}(\lambda, d) \begin{cases} min_\pi \ E_{i_0,w_0}\Big[(w^\pi_{T\wedge\tau} - d)^2 + 2\lambda(w^\pi_{T\wedge\tau} - d)\Big] \\ s.t. \ (5.2). \end{cases}$$

The Lemma below gives the relationship between the problem $\overline{P}(d)$ and $\overline{P1}(\lambda, d)$:

Lemma 5.1. Given that the value function of the problem $\overline{P1}(\lambda, d)$ at the beginning is $V_0(\lambda, d; i_0, w_0)$, and the corresponding optimal strategy is denoted by $\{\pi^*_t(\lambda, i_t, w_t)|t = 0, 1, ..., T-1\}$, then the value function of the problem $\overline{P}(d)$ at time 0 is $sup_\lambda V_0(\lambda, d; i_0, w_0)$, and its optimal strategy is $\{\pi^*_t(\lambda^*, i_t, w_t)|t = 0, 1, ..., T-1\}$ where λ^* is the one which satisfies $sup_\lambda V_0(\lambda, d; i_0, w_0)$.

Note that $[w_{T\wedge\tau} - d]^2 + 2\lambda[w_{T\wedge\tau} - d] = [w_{T\wedge\tau} - (d-\lambda)]^2 - \lambda^2$, and because λ has no effect on the choice of the optimal strategy, we neglect λ^2 to obtain the problem $\widehat{P1}(\lambda, d)$ as follows:

$$\widehat{P1}(\lambda, d) \begin{cases} min_{\pi}\ E_{i_0,w_0}\Big[(w_{T\wedge\tau} - (d-\lambda))^2\Big] \\ s.t.\ (5.2), \end{cases}$$

which is equivalent to problem $\overline{P1}(\lambda, d)$.
Throughout this chapter, we make the following important assumptions:

A5.1 Assume that for any $i, i = 1, 2, ..., L$, $P_T(i) > 0$ always holds true. Otherwise, we do not need to consider the problem in T time horizon.

A5.2 The market state $S_t, t = 0, 1, ..., T$, is independent of the returns of the risky asset and the risk-free asset, and the return of the risky asset and the risk-free asset are also independent of each other, which can be expressed as follows:

$$P_t(S_{t+1}, R_t(S_t) \in B) = P_t(S_{t+1})P_t(R_t(S_t) \in B),$$
$$P_t(R_t(S_t) \in B, r_t(S_t) \in B) = P_t(R_t(S_t) \in B)P_t(r_t(S_t)),$$

for all $B \in \mathbf{B}(\mathbb{R}), i = 1, 2, ..., L$ and $t = 0, 1, ..., T-1$, where P_t is the probability based on information up to time t and $\mathbf{B}(\mathbb{R})$ is the Borel $\sigma-$algebra on $\mathbb{R}$.

A5.3 $P(\tau = t|S_0, ..., S_t, ..., S_T) = P(\tau = t|S_t)$, which means that the exit probability is only dependent on the current market state S_t, and the time before and after the current market state has no effect on the current exit probability.

A5.4 For $S_t = 1, 2, ..., L-1$ and $t = 0, 1, ..., T-1$, assume that $E[e_t(S_t)e_t(S_t)']$ is positive definite.

A5.5 The investment strategy τ is self-finance, which means that there is no exogenous infusion or withdrawal of money during the investment time horizon.

A5.6 During the investment time horizon, short selling, borrowing and lending are not prohibited, and transaction costs are ignored.

Remark 5.1. The assumptions of **A**5.2 and**A**5.3 are strict. However, by using these assumptions, it is much more easy to obtain the closed form of the optimal strategy, which lays the root for the further study.

5.4 Solution for Problem $P1(\lambda, d)$

In order to derive the value functions and the optimal strategy of problem $\widehat{P}1(\lambda, d)$, we need to further transform the objective function.

By using the law of total probability, we have the following formulation:

$$
\begin{aligned}
&E_{i_0,w_0}\Big[(w_{T\wedge\tau} - (d-\lambda))^2\Big] \\
&= E_{i_0,w_0}\Big\{E\Big[(w_{T\wedge\tau} - (d-\lambda))^2 | S_0, S_1, ..., S_T\Big]\Big\} \\
&= E_{i_0,w_0}\Bigg\{\sum_{t=0}^{T} P(\tau = t | S_0, S_1, ..., S_T) E\Big[(w_t - (d-\lambda))^2 | S_0, S_1, ..., S_T\Big]\Bigg\} \\
&= E_{i_0,w_0}\Bigg\{\sum_{t=0}^{T} P(\tau = t | S_t) E\Big[(w_t - (d-\lambda))^2 | S_0, S_1, ..., S_T\Big]\Bigg\} \\
&= E_{i_0,w_0}\Bigg\{\sum_{t=0}^{T} P_t(S_t) E\Big[(w_t - (d-\lambda))^2 | S_0, S_1, ..., S_T\Big]\Bigg\} \\
&= E_{i_0,w_0}\Bigg\{\sum_{t=0}^{T} E\Big[P_t(S_t)(w_t - (d-\lambda))^2 | S_0, S_1, ..., S_T\Big]\Bigg\} \\
&= E_{i_0,w_0}\Bigg\{\sum_{t=0}^{T} P_t(S_t)(w_t - (d-\lambda))^2\Bigg\}.
\end{aligned} \tag{5.3}
$$

Note that the law of total probability is being applied on the first line and the assumption **A**5.2 on the third line of $Equation$(5.3). Therefore, by using the above formulation, the problem $\widehat{P1}(\lambda, d)$ can be rewritten as follows:

$$
P1(\lambda, d)\begin{cases} min_\pi\ E_{i_0,w_0}\Big\{\ \sum_{t=0}^{T} P_t(S_t)(w_t - (d-\lambda))^2\Big\} \\ s.t.\ (5.2). \end{cases}
$$

Now we consider the optimal strategy for the problem $P1(\lambda, d)$ by using the dynamic programming approach. Define the value function as follows:

$$
\begin{aligned}
&V_t(i, w_t) = min_{\pi_t,\pi_{t+1},...,\pi_{T-1}}\ E_{i,w_0}\Big[\sum_{n=t}^{T} P_n(S_n)(w_n - (d-\lambda))^2\Big],\ t = 0, 1, ..., T-1, \\
&V_T(i, w_T) = w_T - (d-\lambda)^2,\ i = 1, 2, ..., L.
\end{aligned} \tag{5.4}
$$

When $S_n \neq L$, we have the following Bellman's equations:

$$\begin{aligned} V_t(i, w_t) &= P_t(i)(w_t - (d-\lambda))^2 + min_{\pi_t}\ E_{i,w_t}\Big[V_{t+1}(S_{t+1}, w_{t+1})\Big] \\ &= P_t(i)(w_t - (d-\lambda))^2 \\ &+ \sum_{j=1}^{L} Q(i,j) E\Big[V_{t+1}(j, (r_t(i)w_t + \pi_t R_t^e(i))\big(1_{\{S_t \neq L\}} + \delta 1_{\{S_t = L\}}\big)\Big], \\ &t = 0, 1, ..., T-1. \end{aligned} \tag{5.5}$$

When $S_t = L$, an investor will invest all the money into the bank account and has wealth of $r_t(L)w_t$ at time $t+1$. Therefore, the investor will possess the terminal wealth of $\prod_{k=t+1}^{T-1} r_k(S_k) r_t(L) w_t$ that only depends on the return of the risk-free asset after time t. Based on the state-dependent exit probability, the value function then can be shown to be as follows:

$$\begin{aligned} V_t(L, w_t) &= P_t(L)(w_t - (d-\lambda))^2 \\ &+ E_{L,w_t}\Big[\sum_{k=t+1}^{T} P_k(S_k)\big[\prod_{j=1}^{k-(t+1)} r_{k-j}(S_{k-j}) r_t(L) w_t - (d-\lambda)\big]^2\Big], \\ &t = 0, 1, ..., T-1. \end{aligned} \tag{5.6}$$

Lemma 5.2. For $i = 1, 2, .., L-1$ and $t = 0, 1, ..., T-1$, $A_t^1(i) > 0$, $H_t(i) > 0$ and $K_t(i) > 0$.

Proof. From **A**5.4, we have

$$E(e_t(S_t)e_t(S_t)') = \begin{pmatrix} E(r_t^2(S_t)), E(r_t(S_t)R_t(S_t)) \\ E(r_t(S_t)R_t(S_t)), E(R_t^2(S_t)) \end{pmatrix} > 0, \tag{5.7}$$

from which, we get

$$\begin{pmatrix} E(r_t^2(S_t)), E(r_t(S_t)R_t^e(S_t)) \\ E(r_t(S_t)R_t^e(S_t)), E(R_t^e(S_t)^2) \end{pmatrix} = \begin{pmatrix} 1, 0 \\ -1, 1 \end{pmatrix} E(e_t(S_t)e_t(S_t)') \begin{pmatrix} 1, -1 \\ 0, 1 \end{pmatrix} > 0,$$

Then we obtain the following from the above equation:

$$E(R_t^e(S_t)^2) > 0,$$

and

$$E(r_t^2(S_t)) - \frac{[E(r_t(S_t)R_t^e(S_t))]^2}{E(R_t^e(S_t)^2)} > 0, \tag{5.8}$$

which can be rewritten as follows:

$$E(r_t^2(S_t))E(R_t^e(S_t)^2) - [E(r_t(S_t)R_t^e(S_t))]^2 = A_t^1(S_t) > 0. \tag{5.9}$$

According to *Equation*(5.9) and **D**5.8, we get that $H_t(S_t) > 0$. Then we can deduce that $K_t(S_t) > 0$ from **D**5.10. $\square$

Note that the above lemma is used to guarantee the existence of the optimal strategy for problem $P1(\lambda, d)$ which is presented in the following theorem:

Theorem 5.1. The optimal investment strategy for $P1(\lambda, d)$ is of the following form:

$$\begin{aligned}
&\pi_t^*(i, w_t) = \frac{G_t(i) r_t^e(i)}{H_t(i)E(R_t^e(i)^2)}(d-\lambda) - \frac{E(r_t(i)R_t^e(i))}{E(R_t^e(i)^2)} w_t,\ i \neq L,\ t = 0, 1, ..., T-1,\\
&\pi_t^*(L, w_t) = 0,\ t = 0, 1, ..., T-1.
\end{aligned} \tag{5.10}$$

The corresponding optimal value functions are as follows:

$$\begin{aligned}
V_t^*(i, w_t) =& \Big[\sum_{k=t}^{T} \overline{Q}_{P_k}^{k-t}(i) - (\sum_{m=t}^{T-1} \widehat{Q}^{m-t} K_m)(i)\Big](d-\lambda)^2 + \Big[P_t(i) + \frac{H_t(i)A_t^1(i)}{E(R_t^e(i)^2)}\Big] w_t^2\\
&- 2\Big[P_t(i) + \frac{G_t(i)A_t^2(i)}{E(R_t^e(i)^2)}\Big](d-\lambda)w_t,\ i \neq L,\ t = 0, 1, ..., T-1,
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
V_t^*(L, w_t) =& Q(L, S_{t+1})\Big[E(r_t(L)^2) \sum_{k=t+1}^{T} \overline{Q}_{P_k}^{k-(t+1)}(S_k) \prod_{j=1}^{k-(t+1)} E(r_{k-j}(S_{k-j})^2) w_t^2\\
&+ \sum_{k=t+1}^{T} \overline{Q}_{P_k}^{k-(t+1)}(S_k)(d-\lambda)^2 - 2E(r_t(L)) \sum_{k=t+1}^{T} \overline{Q}_{P_k}^{k-(t+1)}(S_k)\\
&\cdot \prod_{j=1}^{k-(t+1)} E(r_{k-j}(S_{k-j}))(d-\lambda)w_t\Big] + P_t(L)(w_t - (d-\lambda))^2,\\
&i = L,\ t = 0, 1, ..., T-1.
\end{aligned} \tag{5.12}$$

Proof. (i) First, we prove the expression of *Equation*(5.12) to be right based on

$Equation(5.6)$, for $i = L, t = 0, 1, ..., T-1$, we have

$$\begin{aligned} V_t(L, w_t) =& P_t(L)(w_t - (d-\lambda))^2 \\ &+ E_{L,w_t}\Big[\sum_{k=t+1}^{T} P_k(S_k)\big[\prod_{j=1}^{k-(t+1)} r_{k-j}(S_{k-j})r_t(L)w_t - (d-\lambda)\big]^2\Big] \\ =& P_t(L)(w_t - (d-\lambda))^2 \\ &+ E_{L,w_t}\Big[\sum_{k=t+1}^{T} P_k(S_k)\big[\prod_{j=1}^{k-(t+1)} r^2_{k-j}(S_{k-j})r^2_t(L)w^2_t \\ &+ (d-\lambda)^2 - 2\prod_{j=1}^{k-(t+1)} r_{k-j}(S_{k-j})r_t(L)(d-\lambda)w_t\big]\Big]. \end{aligned} \tag{5.13}$$

Based on **A**5.2, we rewrite $Equation(5.13)$ as follows:

$$\begin{aligned} V_t(L, w_t) =& P_t(L)(w_t - (d-\lambda))^2 + Q(L, S_{t+1}) \sum_{k=t+1}^{T} \overline{Q}_{P_k}^{k-(t+1)}(S_k) \\ &\cdot \Big[\prod_{j=1}^{k-(t+1)} E(r^2_{k-j}(S_{k-j}))E(r^2_t(L))w^2_t + (d-\lambda)^2 \\ &- 2\prod_{j=1}^{k-(t+1)} E(r_{k-j}(S_{k-j}))E(r_t(L))(d-\lambda)w_t\Big]. \end{aligned} \tag{5.14}$$

Note that $Equation(5.14)$ is equivalent to $Equation(5.12)$, which means that $Equation(5.12)$ holds true for $t = 0, 1, ..., T-1$.

Then we prove the expressions(10)-(11) for $i \neq L$ by using mathematical induction. First, we show that the expressions are true for $t = T-1$ in (ii), and then in (iii) prove that if the expressions hold true for $t = n+1$, then they are also true for $t = n$.

(ii) For $t = T-1$, based on $Equation(5.4)$ and (5.5) and (5.12), we have

$$
\begin{aligned}
&V_{T-1}(i, w_{T-1})\\
=&P_{T-1}(i)(w_{T-1}-(d-\lambda))^2 + min_{\pi_{T-1}}E_{i,w_{T-1}}\Big[P_T(S_T)(w_T-(d-\lambda))^2\Big]\\
=&P_{T-1}(i)(w_{T-1}-(d-\lambda))^2\\
&+min_{\pi_{T-1}}\Bigg\{Q(i,L)E\Big[P_T(L)[\delta(w_{T-1}r_{T-1}(i)+\pi_{T-1}R^e_{T-1}(i))-(d-\lambda)]^2\Big]\\
&+(1-Q(i,L))E\Big[[(w_{T-1}r_{T-1}(i)+\pi_{T-1}R^e_{T-1}(i))-(d-\lambda)]^2P_T(S_T)\Big]\Bigg\}\\
=&P_{T-1}(i)(w_{T-1}-(d-\lambda))^2 + min_{\pi_{T-1}}\Bigg\{\overline{Q}_{P_T}(i)(d-\lambda)^2 - 2G_{T-1}(i)\\
&\cdot[E(r_{T-1}(i))w_{T-1}+r^e_{T-1}(i)\pi_{T-1}](d-\lambda)+H_{T-1}(i)[E(r^2_{T-1}(i))w^2_{T-1}\\
&+E(R^e_{T-1}(i)^2)\pi^2_{T-1}+2E(r_{T-1}(i)R^e_{T-1}(i))\pi_{T-1}w_{T-1}]\Bigg\}.
\end{aligned}
$$

As $H_{T-1}(i) > 0$ in $Lemma 5.2$, the optimal solution π^*_{T-1} for V_{T-1} exists and can be obtained by setting $\frac{dV_{T-1}(i,w_{T-1})}{d\pi_{T-1}} = 0$ to yield

$$\pi^*_{T-1}(\lambda, i, w_{T-1}) = \frac{G_{T-1}(i)r^e_{T-1}(i)}{H_{T-1}(i)E(R^e_{T-1}(i)^2)}(d-\lambda) - \frac{E(r_{T-1}(i)R^e_{T-1}(i))}{E(R^e_{T-1}(i)^2)}w_{T-1}. \quad (5.15)$$

Substituting $Equation(5.15)$ into the expression for $V_{T-1}(i, w_{T-1})$, we have

$$
\begin{aligned}
V^*_{T-1}(i, w_{T-1}) =&\Big[\overline{Q}_{P_T}(i) + P_{T-1}(i) - K_{T-1}(i)\Big](d-\lambda)^2\\
&+\Big[P_{T-1}(i) + \frac{H_{T-1}(i)A^1_{T-1}(i)}{E(R^e_{T-1}(i)^2)}\Big]w^2_{T-1}\\
&-2\Big[P_{T-1}(i) + \frac{G_{T-1}(i)A^2_{T-1}(i)}{E(R^e_{T-1}(i)^2)}\Big](d-\lambda)w_{T-1}.
\end{aligned} \quad (5.16)
$$

Hence, $Equations(5.10)$ and (5.11) are true for $t = T-1$.

(iii)Assume that $Equations(5.10), (5.11)$ and (5.12) hold true for $t = n+1$, then when $t = n$, we have

$$
\begin{aligned}
&V_n(i, w_n) \\
=&P_n(i)(w_n - (d-\lambda))^2 + min_{\pi_n} E_{i,w_n}\Big[V_{n+1}(S_{n+1}, w_{n+1})\Big] \\
=&P_n(i)(w_n - (d-\lambda))^2 + min_{\pi_n} \sum_{j=1}^{L-1} Q(i,j) \\
&\cdot E\Bigg\{\Big[\sum_{k=n+1}^{T} \overline{Q}_{P_k}^{k-(n+1)}(j) - [\sum_{m=n+1}^{T-1} \widehat{Q}^{m-(n+1)} K_m](j)\Big](d-\lambda)^2 \\
&+ \Big[P_{n+1}(j) + \frac{H_{n+1}(j)A_{n+1}^1(j)}{E(R_{n+1}^e(j)^2)}\Big] w_{n+1}^2 \\
&- 2\Big[P_{n+1}(j) + \frac{G_{n+1}(j)A_{n+1}^2(j)}{E(R_{n+1}^e(j)^2)}\Big](d-\lambda) w_{n+1}\Bigg\} \\
&+ min_{\pi_n} Q(i, L) E\Bigg\{Q(L, S_{n+2}) \\
&\cdot \Big[E(r_{n+1}^2(L)) \sum_{k=n+2}^{T} Q^{k-(n+2)} P_k(S_k) \prod_{j=1}^{k-(n+2)} E(r_{k-j}^2(S_{k-j})) w_{n+1}^2 \\
&+ \sum_{k=n+2}^{T} \overline{Q}_{P_k}^{k-(n+2)}(S_k)(d-\lambda)^2 - 2E(r_{n+1}(L)) \sum_{k=n+2}^{T} Q^{k-(n+2)} P_k(S_k) \\
&\cdot \prod_{j=1}^{k-(n+2)} E(r_{k-j}(S_{k-j}))(d-\lambda) w_{n+1}\Big] + P_{n+1}(L)(w_{n+1} - (d-\lambda))^2\Bigg\}.
\end{aligned}
\tag{5.17}
$$

Substituting $Equation(5.2)$ into $Equation(5.17)$, we have

$$
\begin{aligned}
& V_n(i, w_n) \\
=& P_n(i)(w_n - (d-\lambda))^2 \\
& + min_{\pi_n} \sum_{j=1}^{L-1} Q(i,j) E\Bigg\{ \Big[\sum_{k=n+1}^{T} \overline{Q}_{P_k}^{k-(n+1)}(j) - [\sum_{m=n+1}^{T-1} \widehat{Q}^{m-(n+1)} K_m](j) \Big] (d-\lambda)^2 \\
& + \Big[P_{n+1}(j) + \frac{H_{n+1}(j) A^1_{n+1}(j)}{E(R^e_{n+1}(j)^2)} \Big] \Big(r_n(i) w_n + R^e_n(i)\pi_n \Big)^2 \\
& - 2\Big[P_{n+1}(j) + \frac{G_{n+1}(j) A^2_{n+1}(j)}{E(R^e_{n+1}(j)^2)} \Big] (d-\lambda) \Big(r_n(i) w_n + R^e_n(i)\pi_n \Big) \Bigg\} \\
& + min_{\pi_n} Q(i,L) E\Bigg\{ Q(L, S_{n+2}) \Big[E(r^2_{n+1}(L)) \sum_{k=n+2}^{T} Q^{k-(n+2)} P_k(S_k) \\
& \cdot \prod_{j=1}^{k-(n+2)} E(r^2_{k-j}(S_{k-j})) \Big(r_n(i) w_n + R^e_n(i)\pi_n \Big)^2 \delta^2 \\
& + \sum_{k=n+2}^{T} \overline{Q}_{P_k}^{k-(n+2)}(S_k)(d-\lambda)^2 - 2E(r_{n+1}(L)) \sum_{k=n+2}^{T} Q^{k-(n+2)} P_k(S_k) \\
& \cdot \prod_{j=1}^{k-(n+2)} E(r_{k-j}(S_{k-j}))(d-\lambda) \Big(r_n(i) w_n + R^e_n(i)\pi_n \Big) \delta \Big] \\
& + P_{n+1}(L) \Big[\Big(r_n(i) w_n + R^e_n(i)\pi_n \Big) \delta - (d-\lambda) \Big]^2 \Bigg\}.
\end{aligned}
\tag{5.18}
$$

Note that in $Equation(5.18)$, $w_{n+1} = r_n(i)w_n + R^e_n(i)\pi_n$ when $S_{n+1} \neq L$; when $S_{n+1} = L$, $w_{n+1} = \big(r_n(i)w_n + R^e_n(i)\pi_n\big)\delta$ where δ is the retrieval rate of the money that has been invested into the bankrupt company.

Then we rewrite $Equation(5.18)$ as follows:

$$
\begin{aligned}
&V_n(i, w_n)\\
=&min_{\pi_n}\Bigg\{P_n(i)(w_n-(d-\lambda))^2+\Big\{\sum_{j=1}^{L-1}Q(i,j)\Big[\sum_{k=n+1}^{T}\overline{Q}_{P_k}^{k-(n+1)}(j)\\
&-[\sum_{m=n+1}^{T-1}\widehat{Q}^{m-(n+1)}K_m](j)\Big]\\
&+Q(i,L)\Big[Q(L,S_{n+2})\sum_{k=n+2}^{T}\overline{Q}_{P_k}^{k-(n+2)}(S_k)+P_{n+1}(L)\Big]\Big\}(d-\lambda)^2\\
&+\Big\{\sum_{j=1}^{L-1}Q(i,j)\Big[P_{n+1}(j)+\frac{H_{n+1}(j)A_{n+1}^1(j)}{E(R_{n+1}^e(j)^2)}\Big]\\
&+E(\delta^2)Q(i,L)\Big[Q(L,S_{n+2})E(r_{n+1}^2(L))\sum_{k=n+2}^{T}Q^{k-(n+2)}P_k(S_k)\\
&\cdot\prod_{j=1}^{k-(n+2)}E(r_{k-j}^2(S_{k-j}))+P_{n+1}(L)\Big]\Big\}E\Big((r_n(i)w_n+R_n^e(i)\pi_n)^2\Big)\\
&-2\Big\{\sum_{j=1}^{L-1}Q(i,j)\Big[P_{n+1}(j)+\frac{G_{n+1}(j)A_{n+1}^2(j)}{E(R_{n+1}^e(j)^2)}\Big]\\
&+E(\delta)Q(i,L)\Big[Q(L,S_{n+2})E(r_{n+1}(L))\\
&\cdot\sum_{k=n+2}^{T}Q^{k-(n+2)}P_k(S_k)\prod_{j=1}^{k-(n+2)}E(r_{k-j}(S_{k-j}))+P_{n+1}(L)\Big]\Big\}\\
&\cdot(d-\lambda)E\Big((r_n(i)w_n+R_n^e(i)\pi_n)\Big)\Bigg\}.
\end{aligned}
\tag{5.19}
$$

According to **D**5.8 and **D**5.9, we rewrite *Equation*(5.19) as follows:

$$
\begin{aligned}
&V_n(i, w_n) \\
=&min_{\pi_n}\Bigg\{P_n(i)(w_n-(d-\lambda))^2 \\
&+\Big\{\sum_{j=1}^{L-1}Q(i,j)\Big[\sum_{k=n+1}^{T}\overline{Q}_{P_k}^{k-(n+1)}(j)-[\sum_{m=n+1}^{T-1}\widehat{Q}^{m-(n+1)}K_m](j)\Big] \\
&+Q(i,L)\Big[Q(L,S_{n+2})\sum_{k=n+2}^{T}\overline{Q}_{P_k}^{k-(n+2)}(S_k)+P_{n+1}(L)\Big]\Big\}(d-\lambda)^2 \\
&+H_n(i)\Big[E(r_n^2(i))w_n^2+E(R_n^e(i)^2)\pi_n^2+2E(r_n(i)R_n^e(i))\pi_n w_n\Big] \\
&-2G_n(i)(d-\lambda)\Big[E(r_n(i))w_n+r_n^e(i)\pi_n\Big]\Bigg\}.
\end{aligned}
\tag{5.20}
$$

As $H_n(i) > 0$ in *Lemma* 5.2, the optimal solution π_n^* for V_n exists and can be obtained by $\frac{dV_n(i,w_n)}{d\pi_n} = 0$, we then have

$$
\pi_n^*(i, w_n) = \frac{G_n(i)r_n^e(i)}{H_n(i)E(R_n^e(i)^2)}(d-\lambda) - \frac{E(r_n(i)R_n^e(i))}{E(R_n^e(i)^2)}w_n. \tag{5.21}
$$

Substituting *Equation*(5.21) into *Equation*(5.20), we obtain

$$
\begin{aligned}
&V_n(i, w_n) \\
=&\Big\{P_n(i)+\sum_{j=1}^{L-1}Q(i,j)\Big[\sum_{k=n+1}^{T}\overline{Q}_{P_k}^{k-(n+1)}(j)-[\sum_{m=n+1}^{T-1}\widehat{Q}^{m-(n+1)}K_m](j)\Big] \\
&+Q(i,L)\Big[Q(L,S_{n+2})\sum_{k=n+2}^{T}\overline{Q}_{P_k}^{k-(n+2)}(S_k)+P_{n+1}(L)\Big]-\frac{G_n^2(i)r_n^e(i)^2}{H_n(i)E(R_n^e(i)^2)}\Big\}(d-\lambda)^2 \\
&+\Big\{P_n(i)+H_n(i)\Big[E(r_n^2(i))-\frac{[E(r_n(i)R_n^e(i))]^2}{E(R_n^e(i)^2)}\Big]\Big\}w_n^2 \\
&-2\Big\{P_n(i)+G_n(i)\Big[E(r_n(i))-\frac{E(r_n(i)R_n^e(i))r_n^e(i)}{E(R_n^e(i)^2)}\Big]\Big\}(d-\lambda)w_n \\
=&\Big\{P_n(i)+\overline{Q}_{P_{n+1}}(i)+\sum_{k=n+2}^{T}\overline{Q}_{P_k}^{k-n}(i)-\widehat{Q}[\sum_{m=n+1}^{T-1}\widehat{Q}^{m-(n+1)}K_m](i)-K_n(i)\Big\}(d-\lambda)^2 \\
&+\Big\{P_n(i)+H_n(i)\Big[E(r_n^2(i))-\frac{[E(r_n(i)R_n^e(i))]^2}{E(R_n^e(i)^2)}\Big]\Big\}w_n^2 \\
&-2\Big\{P_n(i)+G_n(i)\Big[E(r_n(i))-\frac{E(r_n(i)R_n^e(i))r_n^e(i)}{E(R_n^e(i)^2)}\Big]\Big\}(d-\lambda)w_n.
\end{aligned}
\tag{5.22}
$$

Based on **D**5.6, **D**5.7 and **D**5.10, we know that *Equation*(5.21) and (5.22) are equivalent to *Equation*(5.10) and (5.11) respectively, which means that *Equation* (5.10) and (5.11) hold true when $t = n$. $\square$

5.5 Solution for Problem $\overline{P}(d)$

Based on *Theorem* 5.1 and the relationship between the problem $P1(\lambda, d)$ and $\overline{P1}(\lambda, d)$, we can immediately obtain the value function $V_0(\lambda, d; i_0, w_0)$ for $\overline{P1}(\lambda, d)$ as follows:

$$\begin{aligned}
& V_0(\lambda, d; i_0, w_0) \\
=& V_0^*(i_0, w_0) - \lambda^2 \\
=& \Big[\sum_{k=0}^{T} \overline{Q}_{P_k}^k(i_0) - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)\Big](d-\lambda)^2 \\
& + \Big[P_0(i_0) + \frac{H_0(i_0)A_0^1(i_0)}{E(R_0^e(i_0)^2)}\Big]w_0^2 - 2\Big[P_0(i_0) + \frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\Big](d-\lambda)w_0 - \lambda^2.
\end{aligned} \tag{5.23}$$

From *Lemma* 5.1, in order to obtain the optimal strategy and the corresponding value function for the problem $\overline{P}(d)$, we first maximize *Equation*(5.23) with respect to λ. Note that $V_0(\lambda, d; i_0, w_0)_{\lambda\lambda} = 2[\sum_{k=0}^{T} \overline{Q}_{P_k}^k(i_0) - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)] - 2 \leq 0$, which suggests that the optimal λ in *Equation*(5.23) exists.
Let $V_0(\lambda, d; i_0, w_0)_\lambda = 0$, then we have

$$\lambda^* = \frac{\Big[\sum_{k=0}^{T} \overline{Q}_{P_k}^k(i_0) - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)\Big]d - \Big[P_0(i_0) + \frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\Big]w_0}{\Big[\sum_{k=0}^{T} \overline{Q}_{P_k}^k(i_0) - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)\Big] - 1}. \tag{5.24}$$

Substituting $Equation(5.24)$ into (5.23), we have the maximum of $V_0(\lambda, d; i_0, w_0)$ as follows:

$$
\begin{aligned}
&V_0(\lambda^*, d; i_0, w_0)\\
=&\frac{\Big[\sum_{k=0}^{T}\overline{Q}_{P_k}^k(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)\Big]d^2}{1-\Big[\sum_{k=0}^{T}\overline{Q}_{P_k}^k(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)\Big]}\\
&+\frac{2\Big[P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\Big]d}{1-\Big[\sum_{k=0}^{T}\overline{Q}_{P_k}^k(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)\Big]}w_0 \qquad (5.25)\\
&+\left[P_0(i_0)+\frac{H_0(i_0)A_0^1(i_0)}{E(R_0^e(i_0)^2)}+\frac{\Big[P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\Big]^2}{\sum_{k=0}^{T}\overline{Q}_{P_k}^k(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)}\right]w_0^2.
\end{aligned}
$$

Therefore, based on $Equation(5.25)$ and $Lemma$ 5.1, we have the following theorem:

Theorem 5.2. The optimal investment strategy $\pi_t^*(\lambda^*, i, w_t)$ for $\overline{P}(d)$ is of the following form:

$$
\begin{aligned}
\pi_t^*(\lambda^*, i, w_t)=&\frac{G_t(i)r_t^e(i)\Big[d-(P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)})w_0\Big]}{H_t(i)E(R_t^e(i)^2)\Big[1-\sum_{k=0}^{T}\overline{Q}_{P_k}^k(i_0)+(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)\Big]}\\
&-\frac{E(r_t(i)R_t^e(i))}{E(R_t^e(i)^2)}w_t, \qquad (5.26)\\
&i\neq L, t=0,1,...,T-1,\\
\pi_t^*(\lambda^*, L, w_t)=&0, t=0,1,...,T-1.
\end{aligned}
$$

The corresponding efficient frontier is of the following form:

$$
\begin{aligned}
&Var_{i_0,w_0}(w^{\pi^*}_{T\wedge\tau})\\
=&\frac{\sum_{k=0}^{T}\overline{Q}^k_{P_k}(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)}{1-\left[\sum_{k=0}^{T}\overline{Q}^k_{P_k}(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)\right]}\\
&\cdot\left[E_{i_0,w_0}(w^{\pi^*}_{T\wedge\tau})-\frac{\left[P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\right]w_0}{\sum_{k=0}^{T}\overline{Q}^k_{P_k}(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)}\right]^2\\
&+\left[P_0(i_0)+\frac{H_0(i_0)A_0^1(i_0)}{E(R_0^e(i_0)^2)}-\frac{\left[P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\right]^2}{\sum_{k=0}^{T}\overline{Q}^k_{P_k}(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)}\right]w_0^2,\\
&E_{i_0,w_0}(w^{\pi^*}_{T\wedge\tau})\geq\frac{\left[P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\right]w_0}{\sum_{k=0}^{T}\overline{Q}^k_{P_k}(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)}.
\end{aligned}
\tag{5.27}
$$

From $Equation$(5.27), the variance has the global minimum

$$
Var_{min}=\left[P_0(i_0)+\frac{H_0(i_0)A_0^1(i_0)}{E(R_0^e(i_0)^2)}-\frac{\left[P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\right]^2}{\sum_{k=0}^{T}\overline{Q}^k_{P_k}(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)}\right]w_0^2,
$$

which occurs at $E_{min}=\frac{\left[P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\right]w_0}{\sum_{k=0}^{T}\overline{Q}^k_{P_k}(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)}$.

The corresponding optimal strategy is as follows:

$$
\begin{aligned}
\pi_t^{min}(\lambda^*,i,w_t)=&\frac{G_t(i)r_t^e(i)\Big(P_0(i_0)+\frac{G_0(i_0)A_0^2(i_0)}{E(R_0^e(i_0)^2)}\Big)w_0}{H_t(i)E(R_t^e(i)^2)\Big(\sum_{k=0}^{T}\overline{Q}^k_{P_k}(i_0)-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)\Big)}\\
&-\frac{E(r_t(i)R_t^e(i))}{E(R_t^e(i)^2)}w_t,\ i\neq L,\ t=0,1,...,T-1,\\
\pi_t^{min}(\lambda^*,L,w_t)=&0,\ t=0,1,...,T-1.
\end{aligned}
$$

5.6 Special Cases

In this section, we prove that the model in this paper is a general form of those in the existing literature and includes the existing models as special cases.

5.6.1 The Case with Fixed Time Horizon

In this case, we assume that the investment time horizon is fixed, which means that the probability of exit $P_t = 0$ at time $t, t = 0, 1, ..., T-1$, but $P_T = 1$. Then the important parameters are shown as follows:

$$H_t(i) = \sum_{S_{t+1}=1}^{L-1} Q(i, S_{t+1}) \frac{H_{t+1}(S_{t+1}) A^1_{t+1}(S_{t+1})}{E(R^e_{t+1}(S_{t+1})^2)}$$
$$+ Q(i,L)E(\delta^2)\Big[\sum_{S_{t+2}=1}^{L} Q(L, S_{t+2})E(r(L)^2)Q^{T-(t+2)} \prod_{j=1}^{T-(t+2)} E(r_{k-j}(S_{k-j})^2)\Big],$$
$$t = 0, 1, ..., T-2,$$
$$H_{T-1}(i) = Q(i,L)E(\delta^2) + 1 - Q(i,L),\ i \neq L.$$
$$G_t(i) = \sum_{S_{t+1}=1}^{L-1} Q(i, S_{t+1}) \frac{H_{t+1}(S_{t+1}) A^2_{t+1}(S_{t+1})}{E(R^e_{t+1}(S_{t+1})^2)}$$
$$+ Q(i,L)E(\delta)\Big[\sum_{S_{t+2}=1}^{L} Q(L, S_{t+2})E(r(L))Q^{T-(t+2)} P_k(S_k) \prod_{j=1}^{T-(t+2)} E(r_{k-j}(S_{k-j}))\Big],$$
$$t = 0, 1, ..., T-2,$$
$$G_{T-1}(i) = Q(i,L)E(\delta) + 1 - Q(i,L),\ i \neq L.$$
$$K_t(i) = \frac{(G_t(i) r^e_t(i))^2}{H_t(i) E(R^e_t(i)^2)},\ i \neq L,\ t = 0, 1, ..., T-1.$$

The corresponding efficient frontier is

$$\begin{aligned} &Var_{i_0,w_0}(w^{\pi^*}_T) \\ =&\frac{1 - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)}{(\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)} \left[E_{i_0,w_0}(w^{\pi^*}_{T\wedge\tau}) - \frac{\frac{G_0(i_0)A^2_0(i_0)}{E(R^e_0(i_0)^2)} w_0}{1 - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)} \right]^2 \\ &+ \left[\frac{H_0(i_0)A^1_0(i_0)}{E(R^e_0(i_0)^2)} - \frac{\left[\frac{G_0(i_0)A^2_0(i_0)}{E(R^e_0(i_0)^2)}\right]^2}{1 - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)} \right] w_0^2. \end{aligned} \tag{5.28}$$

5.6.2 The Case with Deterministic Risk-Free Asset

Based on 5.6.1, by further assuming that the risk-free asset is independent of the market state, we have the following equations:

$$
\begin{aligned}
&A_t^1(i) = r^2 E(R_t^e(i)^2) - r^2(r_t^e(i))^2 = r^2 Var(R_t^e(i)),\ i \neq L.\\
&A_t^2(i) = rE(R_t^e(i)^2) - r(r_t^e(i))^2 = rVar(R_t^e(i)),\ i \neq L.\\
&H_t(i) = r^2 \sum_{S_{t+1}=1}^{L-1} Q(i, S_{t+1}) \frac{H_{t+1}(S_{t+1}) Var(R_t^e(i))}{E(R_{t+1}^e(S_{t+1})^2)} + r^2[T-(t+1)]Q(i,L)E(\delta^2),\\
&t = 0, 1, ..., T-2,\\
&H_{T-1}(i) = Q(i,L)E(\delta^2) + 1 - Q(i,L),\ i \neq L.\\
&G_t(i) = r \sum_{S_{t+1}=1}^{L-1} Q(i, S_{t+1}) \frac{H_{t+1}(S_{t+1}) Var(R_t^e(i))}{E(R_{t+1}^e(S_{t+1})^2)} + r^{T-(t+1)} Q(i,L)E(\delta),\\
&t = 0, 1, ..., T-2,\\
&G_{T-1}(i) = Q(i,L)E(\delta) + 1 - Q(i,L),\ i \neq L.\\
&K_t(i) = \frac{(G_t(i) r_t^e(i))^2}{H_t(i) E(R_t^e(i)^2)},\ i \neq L,\ t = 0, 1, ..., T-1.
\end{aligned}
$$

We obtain the corresponding efficient frontier from $Equation(5.28)$ as follows:

$$
\begin{aligned}
&Var_{i_0,w_0}(w_T^{\pi^*})\\
=&\frac{1 - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)}{(\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)} \left[E_{i_0,w_0}(w_{T\wedge\tau}^{\pi^*}) - \frac{\frac{G_0(i_0) Var(R_0^e(i_0))}{E(R_0^e(i_0)^2)} r^T w_0}{1 - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)} \right]^2\\
&+ r^{2T} \left[H_0(i_0) - \frac{G_0(i_0)^2 Var(R_0^e(i_0))}{E(R_0^e(i_0)^2)[1 - (\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0)]} \right] \frac{Var(R_0^e(i_0))}{E(R_0^e(i_0)^2)} w_0^2,
\end{aligned}
$$

which coincides with that in Wu and Zeng(2013) [64] whose research does not involve uncertain time horizons.

5.6.3 The Case with No Bankruptcy State

Based on the two cases above, we assume that there is no bankruptcy state, which means that it is impossible to reach market state L from state i ($i \neq L$). Therefore, according to Wu and Zeng(2013) [64] $H_t(i) = G_t(i) = \overline{Q_{1-h}^{T-t-1}}(i)$ and $K_t(i) = G_t(i)h(i)$ where $h_t(i) = \frac{(r_t^e(i))^2}{E((R_t^e(i))^2)}$. Based on the assumptions in this paper, some important notations in Çakmak and Özekici(2006) [8] are shown as follows:

$$\begin{aligned}
&g(i) = r(1-h(i)),\ f(i) = r^2(1-h(i)),\ Q_g^t = r^t Q_{1-h}^t,\ Q_f^t = r^{2t} Q_{1-h}^t,\\
&a_1(i) = \overline{Q_{1-h}^{T-1}}(i)(1-h(i))r^T, a_2(i) = \overline{Q_{1-h}^{T-1}}(i)(1-h(i))r^{2T},\\
&2b(i) = \sum_{t=0}^{T-1} Q^t(\overline{Q_{1-h}^{T-t-1}h})(i).
\end{aligned}$$

Based on the equations above, we have the following equations:

$$\begin{aligned}
&\sum_{m=0}^{T-1} \widehat{Q}^m K_m)(i_0) = 2b(i_0),\ \frac{G_0(i)Var(R_0^e(i_0))}{E(R_0^e(i_0)^2)} r^T w_0 = a_1(i_0)w_0,\\
&r^{2T}\left[H_0(i_0) - \frac{G_0(i_0)^2 Var(R_0^e(i_0))}{E(R_0^e(i_0)^2)[1-(\sum_{m=0}^{T-1}\widehat{Q}^m K_m)(i_0)]}\right]\frac{Var(R_0^e(i_0))}{E(R_0^e(i_0)^2)}\\
&= a_2(i_0) - \frac{a_1(i_0)}{1-2b(i_0)}.
\end{aligned}$$

Then based on the equations above and $Equation(5.29)$, we have the following corresponding efficient frontier when there is no bankruptcy state:

$$Var_{i_0,w_0} = (a_2(i_0) - \frac{a_1(i_0)}{1-2b(i_0)})w_0^2 + \frac{1-2b(i_0)}{2b(i_0)}\left[E_{i_0,w_0} - \frac{a_1(i_0)}{1-2b(i_0)}w_0\right],$$

which is in accordance with that in Çakmak and Özekici(2006) [8].

5.7 Numerical Examples

This section has three parts and focuses on the changes in the efficient frontier. The first part studies the different transition probabilities to bankruptcy state $Q(i, L)$ $(i \neq L)$. In the second part we consider the effect of various state-dependent exit probability $P_t(S_t)$ $(S_t = 1, 2, ..., L)$. The third part focuses on the effect of the factor of the retrieval rate δ $(\delta \in [0, 1])$.

In numerical analysis, we assume that there are three market states during the four-period time horizon $T = 4$ including the bullish state, the bearish state and the bankrupt state represented by $i = 1, 2$ and 3 respectively. Note that the investment condition under the bullish market state $(i = 1)$ is better than that under the bearish market state $(i = 2)$ which in turn is superior to the one under the bankrupt state ($i = 3$). In the bankrupt state, the company invested goes bankrupt and the investor can get back δ ($E(\delta) = 0.3$, $Var(\delta) = 0.21$) of the wealth. Besides, assume that the initial wealth $w_0 = 1$ and the initial market state $i_0 = 1$, and other parameters are as follows:

$$E(r_t(1)) = 1.162,\ E(r_t(2)) = 1.03,\ E(r_t(3)) = 1.01,$$
$$Var(r_t(i)) = 0, i = 1, 2, 3; t = 0, 1, 2, 3,$$
$$E(R_t(1)) = 1.246,\ E(R_t(2)) = 1.14,$$
$$Var(R_t(1)) = 0.0154,\ Var(R_t(2)) = 0.0312.$$

We define the transition matrix Q as follows:

$$Q = \begin{pmatrix} 0.5 & 0.5 - \frac{1}{n+2} & \frac{1}{n+2} \\ 0.5 - \frac{1}{n+2} & 0.5 & \frac{1}{n+2} \\ 0.2 & 0.3 & 0.5 \end{pmatrix}.$$

Besides, we define the state-dependent exit probability at each time point as follows:

Table 5.1: three different sets of probabilities

	$P_0(i)$	$P_1(i)$	$P_2(i)$	$P_3(i)$	$P_4(i)$
$i = 1$	0.1	0.1	0.1	0.1	0.6
$i = 2$	0.15	0.15	0.15	0.15	0.4
$i = 3$	0.2	0.2	0.2	0.2	0.2

5.7.1 Regime-Switching

In this case, we focus on the effects of different transition probabilities from state i ($i \neq L$) to L ($L = 3$) on the corresponding efficient frontiers. Assume $n = 8, 18, 48, 98$ in Q and we then calculate the efficient frontier respectively. Note that when n increases, the probability of entering the bankrupt state decreases, which suggests that the company tends to be more safer. Besides, in order to make comparison we also consider the case with no bankrupt state ($L = 2$). Figure 5.1 shows the corresponding efficient frontiers based on different n.
From Figure 5.1 we can see that when n increases, namely, the transition probability to bankrupt state L decreases, the corresponding efficient frontier moves down to get closer to the efficient frontier with no bankrupt state which is at

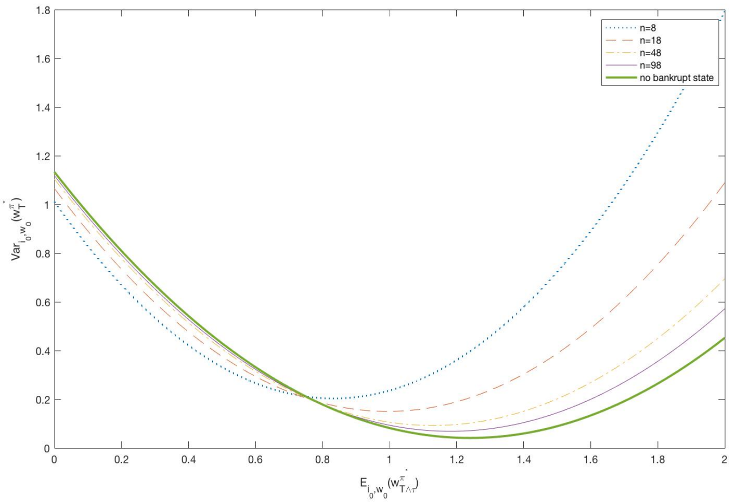

Figure 5.1: Mean-variance efficient frontiers for Case 1

the lowest position in Figure 5.9, and the slope of the efficient frontier also gets smaller. What the figure shows is reasonable because in order to achieve the same level of the final wealth expectation, the investor will take more risks if there is a bankrupt state in the market, namely, they have to face the possibility that the invested company goes bankrupt and they lost their investment. Otherwise, the investor does not need to worry about the bankruptcy risks and consequently has relatively low variance of the investment.

5.7.2 State-Dependent Exit Probability

This case underlines the importance of the exit probability which depends on the market state. Besides the state-dependent exit probability mentioned above, we define another two sets of probabilities as follows in order to make a comparison and show the differences: Table 5.2 shows indifference between the market state

Table 5.2: the same set of probabilities

	$P_0(i)$	$P_1(i)$	$P_2(i)$	$P_3(i)$	$P_4(i)$
$i = 1$	0.10	0.10	0.10	0.10	0.60
$i = 2$	0.10	0.10	0.10	0.10	0.60
$i = 3$	0.10	0.10	0.10	0.10	0.60

and the exit probability, namely, the exit probability becomes deterministic and is just like that in literature. In Table 5.3, we underlines the difference between

Table 5.3: two different sets of probabilities

	$P_0(i)$	$P_1(i)$	$P_2(i)$	$P_3(i)$	$P_4(i)$
$i = 1$	0.10	0.10	0.10	0.10	0.60
$i = 2$	0.15	0.15	0.15	0.15	0.40
$i = 3$	0.15	0.15	0.15	0.15	0.40

the bullish state and the bearish state. The bankrupt state, however, has the same exit probability with that in the bearish state, which suggests that the exit probability is partially dependent on the market state. On the other hand, Table 1 shows the total dependence of the exit probability on the market state. Besides, we assume that $n = 8$ and the other parameters are the same as those at the beginning of this section. We then obtain the results as shown in Figure 5.2.

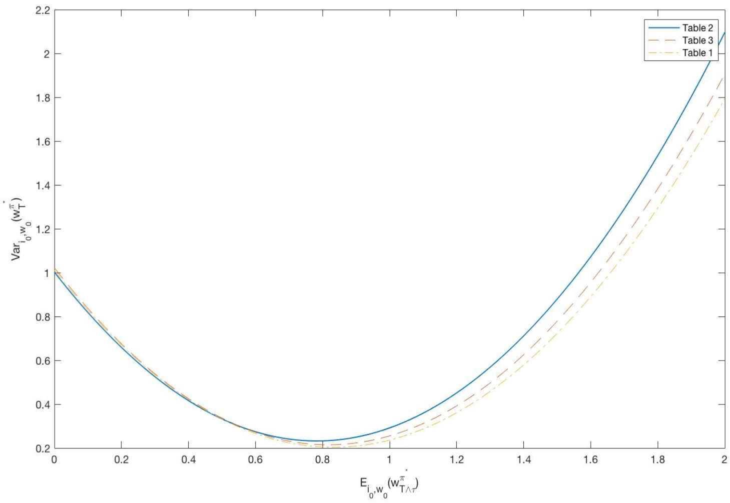

Figure 5.2: Mean-variance efficient frontiers for Case 2

With an increase in the exit probability under the bankruptcy state during the time horizon, the efficient frontier moves downward. However, the slope of the efficient frontier with Table 5.2 is larger than those with Table 5.1 and Table 5.3. This result is realistic and logical. When the exit probability is deterministic and independent of the market state, people will take higher investment risks for the sake of the same terminal wealth expectation. This is because when the exit probability depends on the market state, for example, Table 5.1, the investor quits the market with higher exit probability when the market is being the bankruptcy state. Regardless of whatever reasons the investor quits the market, and the bankruptcy market state makes it easier for the company to go bankrupt and this high exit probability indeed provides more possibility for the investor to quit the market and prevent further investment loss. Therefore, the dependency of exit probability on the market state is like a 'fuse' that indirectly protects people from further investment loss. On the other hand, investors who are under a situation like Table 5.2 or Table 5.3, when the market state is being the worst, their exit probability is always unchanged. Hence, the exit probability makes no extra contribution to the protection for the investment.

5.7.3 Retrieval Rate

This case considers the impact of δ on the corresponding efficient frontier. Note that δ stands for the retrieval part of the wealth after bankruptcy happens. We assume that there are five sets of data about the expectation of δ and are shown as follows:

Table 5.4: the expectation of δ

$E(\delta)$	0.1	0.3	0.5	0.7	0.9

Note that we assume that the variance of δ is unchanged ($Var(\delta) = 0.21$). Besides, let $n = 8$ and use the state-dependent exit probability as in Table 5.1, and the other parameters remain unchanged. We then have the results as shown in Figure 5.3.

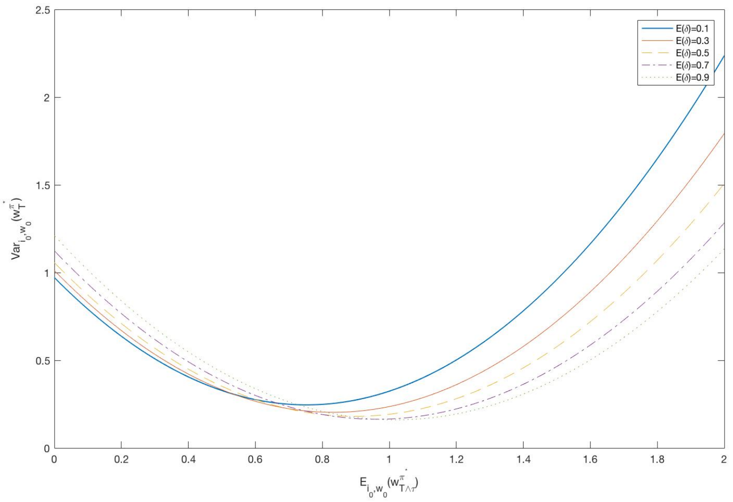

Figure 5.3: Mean-variance efficient frontiers for Case 3

From Figure 5.3 we see that when the expectation of δ increases, the efficient frontier moves downward and its slope also decreases. When the expectation of δ is large, one will have high expectation of terminal wealth which can obtain the global minimum variance, which is not surprising, as one knows that bankruptcy has little impact on their investment since they can get more money back from the bankrupt company. Therefore, one has smaller global minimum variance and larger corresponding expectation. This result is also consistent with the one in Wu and Zeng(2013). Besides, the efficient frontier with larger δ has smaller slope, this is because when people get the money back from the bankrupt company, they invest it into the risk-less asset. Hence, this kind of investment situation is always of relatively small investment risks from the beginning to the end.

5.8 Concluding Remarks

Based on multi-period mean-variance portfolio selection under the regime-switching framework, we consider two fairly new and important factors: the bankruptcy state and the state-dependent exit probability, both of which are fairly meaningful in the real world. When the market state is in the bankruptcy state, the company goes bankrupt and the investor can only get part of the investment back. On the other hand, there are exit probabilities that depend on the market state: the exit probabilities under bad states tend to be larger than that in good states. We then create an innovative expression for the wealth process and value functions by dynamic programming, and then we obtain the closed-form solution of the optimal strategy and the efficient frontier. Our study shows that (i) the model in this paper generalizes those in literature, when ignoring certain factors, we can derive the previous models and results; (ii) in order to obtain the same level of terminal wealth expectation, investors will take more investment risks when it is easier for the market state to enter the bankruptcy state; (iii) when the exit probability depends on the bad market state such as the bearish state and the bankruptcy state, the efficient frontier moves down and its slope gets larger, which suggests that the high exit probability provides more possibility for the investor to quit the market and prevent further investment loss; (iv) when the retrieval parameter δ becomes larger, the efficient frontier moves down and its slope decreases, which suggests that investors will take less risks to obtain the same level of expected terminal wealth.

CHAPTER 6

Conclusion

6.1 Summary of the Main Contributions

In this thesis, we study the mean-variance portfolio selection optimization under the regime-switching framework with mainly two kinds of constraints: uncertain time horizon and special market conditions. We use the embedding technique and the Lagrange multiplier method to solve the inseparability of the objective function in the problem. Then by the dynamic programming approach, we derive the closed form solution of the optimal strategy and the corresponding efficient frontier. In each chapter, we introduce new expressions for the value function and some iteration processes to deal with the computation complexity during the dynamic programming, which is considered as the main difficulties in this thesis. Specifically, the results obtained are concluded in three parts as follows.

(1) Mean-variance portfolio selection for a defined contribution plan with regime-switching and mortality risk

(i) Due to the computational complexity, the Lagrange method is usually used to transform the problem. By introducing innovative expressions for the value function and the corresponding iteration process, we firstly use the embedding technique to obtain the analytical solution and the efficient frontier for the problem, which fills in the gap in this area.

(ii) The model discussed in this thesis generalizes the results of Chen and Yang(2011) [48], Li and Ng(2000) [2] and Wu and Li(2011) [52].

(iii) The numerical example demonstrates that the initial market state has significant impact on the efficient frontier, that is, under the same variance, the expected terminal wealth with a bullish initial market state is higher than the one under a bearish market state.

(iv) With the increase of the transition probability from the bad market state into the good state, investors bear more investment risks under the same level of the expected terminal wealth.

(v) Numerical analysis also illustrates that the investor who has lower mortality risk gets a higher expected terminal wealth.

(2) Portfolio selection for a DC pension plan with premiums return policy and incomplete information

(i) We investigate the effect of both the premiums return policy and incomplete information market on the portfolio selection for a DC pension plan, which was not considered before in the exiting research.

(ii) The studies of some special cases suggests that when we do not consider the factor of incomplete information market, the result obtained is consistent with the work of Bian et al.(2018) [72].

(iii) In order to obtain the same expected terminal wealth, the fund manager bears more investment risks under a bad initial market mode.

(iv) The observable market state can be seen as a kind of 'lubricant' in the regime-switching market, it pulls up the asset return rate when it is in a bad unobservable state and pushes down the return rate in a good unobservable state.

(v) The fund manager assumes more risks in the case with a premiums return policy.

(3) Multi-period mean-variance portfolio selection with state-dependent exit probability and bankruptcy state

(i) We consider both the factor of state-dependent exit probability and bankruptcy state in the portfolio selection problem, which was not studied before.

(ii) Investors will bear more investment risk when the bankruptcy state is more easily accessible given the same level of the expected terminal wealth.

(iii) Bad market state results in higher exit probability, which makes investors more easily to have investment loss.

(iv) Investors take less risks given the same expected terminal wealth when the retrieval rate is higher.

6.2 Future Research Directions

Based on the current work, further possible directions are as follows,

(1) This thesis mainly focuses on the portfolio selection under regime-switching with uncertain time horizon and some special market conditions. Based on the existing research, we can consider further constraints including asset-liability management, risk control over bankruptcy, jump diffusion in asset price and inflation.

(2) The work can be extended to the continuous-time version and use the method exclusive to the continuous-time version to solve the problem. In addition, we can also apply the model to other practical problems such as the defined benefit pension plan and insurance/reinsurance products.

(3) In this thesis, we consider the portfolio selection optimization for a DC pension plan in the accumulation phase. We can also apply the model into the DC pension plan in the decumulation phase.

(4) This thesis adopts the embedding technique and the Lagrange method to transform the problem into the auxiliary problem, and uses the dynamic programming approach to obtain the solution. Further studies include using other methods such as the Nash equilibrium strategy to get the optimal investment strategy, which is also a very interesting topic in portfolio selection.

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Appendix1. Statement of Candidate's Contributions to Joined Authored Published

To whom it may concern,

I, Yang Wang, made the major contributions in the design of the research work, development of theories, analysis of results drafting of the paper entitled 'Multi-period Mean-Variance Portfolio Selection with State-Dependent Exit Probability and Bankruptcy State, Journal of Mathematical Finance, 9(2), 2019.'

<u>Yang Wang</u>

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

<u>Yonghong Wu</u>

Xinguang Zhang

PA

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