Higher-Order Optimality Conditions for Set-Valued Optimization[1]

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Abstract: This paper deals with higher-order optimality conditions of set-valued optimization problems. By virtue of the higher-order derivatives introduced in Ref. 1, higher-order necessary and sufficient optimality conditions are obtained for a set-valued optimization problem whose constraint condition is determined by a fixed set. Higher-order Fritz John type necessary and sufficient optimality conditions are also obtained for a set-valued optimization problem whose constraint condition is determined by a set-valued map.

Keywords: The m^{th}-order adjacent set, the m^{th}-order adjacent derivative, set-valued map, the m^{th}-order optimality condition.

1 Introduction

The study of vector optimization problems is very important since many optimization problems encountered in economics, engineering and other fields involve vector-valued maps (or set-valued maps) as constraints and objectives (see Refs. 2-3). First-order Fritz-John type necessary and sufficient optimality conditions of vector optimization problems with vector-valued maps have been extensively studied in the literature. See, for example, Refs. 4-7.

There has been a growing interest in second-order optimality conditions of vector optimization problems with vector-valued maps. In Ref. 8, Aghezzaf and Hachimi investigated second-order necessary and sufficient optimality conditions for vector optimization problems by virtue of second-order tangent sets. In Ref. 9, Jiménez and Novo obtained second-order Lagrange-Fritz John type optimality conditions by means of the generalized Motzkin alternative theorem. In Ref. 10, Jiménez and Novo studied second-order necessary and sufficient optimality conditions for a point to be an efficient element of a set with respect to a cone in a normed space by using common second-order tangent sets and asymptotic second-order cones. They also discussed second-order Lagrange-Fritz John type necessary conditions by virtue of the directional metric regularity condition and a second-order constraint qualification condition.

Recently, there are many optimality conditions to be obtained for vector optimization problems of set-valued maps (i.e., set-valued optimization problems). In Ref. 11, Luc studied necessary and sufficient conditions for both unconstrained and constrained vector optimization problems with objectives being set-valued maps in terms of contingent derivatives. In Ref. 12, Corley investigated first-order Fritz John necessary and sufficient conditions for general set-valued optimization problems by virtue of tangent derivative and contingent derivative. In Ref. 13, Li et al discussed necessary and sufficient optimality conditions for a general nonconvex set-valued optimization problem with the aid of the Gerstewitz's nonconvex separation functional. In Ref. 14, Jahn and Khan investigated the Fritz John type necessary optimality conditions of local proper minimizers, local weak minimizers and local strong minimizers for general set-valued optimization problems by using the generalized contingent epiderivative. They also obtained sufficient optimality conditions of local weak minimizers and local minimizers for quasi-convex set-valued optimization problems. In Refs. 15-16, Crespi et al. and Khan et al. obtained some optimization conditions to a set-valued optimization by using lower and upper Dini derivatives of set-valued maps, respectively. In Ref. 17, Jahn et al. investigated second-order necessary optimality conditions and sufficient optimality conditions in set-valued optimization by using two kinds of second-order epiderivatives for set-valued maps.

In Ref. 1, Aubin and Frankowska defined m^{th}-order tangent sets and then introduced m^{th}-order derivatives, where m is a positive integer. Since higher-order tangent sets, in general, are not cones and convex sets, there are some difficulties in studying higher-order optimality conditions for general set-valued optimization problems by virtue of the higher-order derivatives introduced by the higher-order tangent sets. Until now, there are no study yet for higher-order optimality conditions for set-valued optimization problems in terms of the higher-order derivatives. Motivated by the work reported in Refs. 1, 12 and 17, we investigate higher-order optimality conditions for general set-valued optimization problems. We discuss some properties of higher-order derivatives for S-concave set-valued maps. Then, we obtain the higher-order Fritz John type necessary and sufficient optimality conditions of set-valued optimization problems whose objective map and constraint map are S-concave.

The rest of the paper is organized as follows. In Section 2, we introduce two kinds of set-valued optimization models. In Section 3, we recall the m^{th}-order contingent set and the m^{th}-order adjacent set. Then, we discuss their properties and equivalent relations. In Section 4, we recall the m^{th}-order contingent derivative and the m^{th}-order adjacent derivative of a set-valued map introduced in Ref. 1. Then, we discuss their properties when the set-valued map is S-concave. In Section 5, we investigate a m^{th}-order necessary and sufficient optimality condition for a set-valued optimization problem whose constraint condition is determined by a fixed set. In Section 6, we obtain a m^{th}-order Fritz John type necessary and sufficient optimality conditions of a set-valued optimization problem whose constraint condition is determined by a set-valued map.

2 Set-Valued Optimization Problems and Preliminaries

Throughout this paper, let X, Y and Z be three real normed spaces, let $S \subseteq Y$ and $D \subseteq Z$ be pointed and convex cones with $\text{int}S \neq \emptyset$ and $\text{int}D \neq \emptyset$, and let A and E be subsets in X, $F : X \to 2^Y$ and $G : X \to 2^Z$. The domain of $F : X \to 2^Y$ is given by

$$\text{Dom}(F) = \{x \in X \mid F(x) \neq \emptyset\}.$$

Denote

$$F(A) = \bigcup_{x \in A} F(x) \text{ and } G^{-}(U) = \{x \mid G(x) \bigcap U \neq \emptyset\}.$$

Definition 2.1. Let $F : X \to 2^Y$ be a set-valued map. $F(\cdot)$ is said to be S-concave on X if, for any $x_1, x_2 \in X$ and $\lambda \in (0, 1)$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) - S.$$

Definition 2.2. Let $F : X \to 2^Y$ be a set-valued map. $F(\cdot)$ is said to be locally Lipschitz at $x_0 \in X$, if there exist $M > 0$ and a neighborhood W of x_0 such that

$$F(x_1) \subset F(x_2) + M||x_1 - x_2||B, \forall x_1, x_2 \in W,$$

where B denotes the unit ball of the origin in Y.

Now we introduce the (weak) maximal points of a set in real normed space Y and two set-valued optimization problems to be studied in this paper.

Definition 2.3. Let $B \subset Y$.

(i) $y_0 \in B$ is said to be a maximal point of B if

$$B \bigcap [y_0 + S] = \{y_0\}.$$

By $\max_S B$ we denote the set of all maximal points of B.

(ii) $y_0 \in B$ is said to be a weak maximal point of B if

$$B \bigcap [y_0 + \mathrm{int} S] = \{\emptyset\}.$$

By $\max_{\mathrm{int} S} B$ we denote the set of all weak maximal points of B.

In this paper, consider the following optimization problem:

$$\max_{x \in A} F(x), \tag{1}$$

i.e., to find all $x_0 \in A$ for which there exists a $y_0 \in F(x_0)$ such that $y_0 \in \max_S F(A)$ (or $y_0 \in \max_{\mathrm{int} S} F(A)$ if weak maximal solutions are desired). We also consider a special case of (1):

$$\begin{aligned} \max_{x \in E} \quad & F(x) \\ \text{s.t.} \quad & G(x) \bigcap D \neq \emptyset, \end{aligned} \tag{2}$$

i.e., to find all $x_0 \in E \bigcap G^-(D)$ for which there exists a $y_0 \in F(x_0)$ such that $y_0 \in \max_S F(E \bigcap G^-(D))$(or $y_0 \in \max_{\mathrm{int} S} F(E \bigcap G^-(D))$ if weak maximal solutions are desired).

Any x_0 solving (1) or (2) is called a (weak) maximal solution for the problem at y_0.

3 Higher-Order Tangent Sets

In this section, we shall recall the definitions of the m^{th}-order contingent set and the m^{th}-order adjacent set in Ref. 1. Then, we shall discuss their properties. Let X be a normed space supplied with a distance d and K be a subset of X. We denote by

$$d(x, K) = \inf_{y \in K} d(x, y)$$

the distance from x to K, where we set $d(x, \emptyset) = +\infty$.

Definition 3.1. Let x belong to a subset K of a normed space X and $v_1, \cdots, v_{m-1}$ be elements of X. We say that the subset

$$\begin{aligned} T_K^{(m)}(x, v_1, \cdots, v_{m-1}) &= \limsup_{h \to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m} \\ &= \{y \in X \mid \liminf_{h \to 0^+} d(y, \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}) = 0\} \end{aligned}$$

is the m^{th}-order contingent set of K at $(x, v_1, \cdots, v_{m-1})$.

Definition 3.2. Let x belong to a subset K of a normed space X and $v_1, \cdots, v_{m-1}$ be elements of X. We say that the subset

$$\begin{aligned} T_K^{\flat(m)}(x, v_1, \cdots, v_{m-1}) &= \liminf_{h \to 0^+} \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m} \\ &= \{y \in X \mid \lim_{h \to 0^+} d(y, \frac{K - x - hv_1 - \cdots - h^{m-1}v_{m-1}}{h^m}) = 0\} \end{aligned}$$

is the m^{th}-order adjacent set of K at $(x, v_1, \cdots, v_{m-1})$.

Proposition 3.1. If K is a convex subset and $v_1, \cdots, v_{m-1} \in K$, then

$$T_K^{\flat(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0) = T_K^{(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0)$$

$$= \mathrm{cl}\left(\bigcup_{h>0} \frac{K - x_0 - h(v_1 - x_0) - \cdots - h^{m-1}(v_{m-1} - x_0)}{h^m}\right).$$

Proof. We note that

$$\begin{aligned}
&T_K^{\flat(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0) \subseteq T_K^{(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0) \\
&\subseteq \mathrm{cl}\left(\bigcup_{h>0} \frac{K - x_0 - h(v_1 - x_0) - \cdots - h^{m-1}(v_{m-1} - x_0)}{h^m}\right).
\end{aligned}$$

So we only need to prove that for any $u_0 \in \mathrm{cl}\left(\bigcup_{h>0} \frac{K - x_0 - h(v_1 - x_0) - \cdots - h^{m-1}(v_{m-1} - x_0)}{h^m}\right)$,

$$u_0 \in T_K^{\flat(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0).$$

Let $\epsilon > 0$ be fixed. Then, there exist $y \in K$ and $\beta > 0$ such that

$$u_0 - \frac{y - x_0 - \beta(v_1 - x_0) - \cdots - \beta^{m-1}(v_{m-1} - x_0)}{\beta^m} \in \epsilon B,$$

where B is the unit ball of the origin. Let $h \in (0, \mu)$, where $0 < \mu \leq \beta$ and $\mu + \mu^2 + \cdots + \mu^{m-1} + \mu^m/\beta^m \leq 1$. Set

$$u = \frac{y - x_0 - \beta(v_1 - x_0) - \cdots - \beta^{m-1}(v_{m-1} - x_0)}{\beta^m}.$$

Then,

$$\begin{aligned}
&x_0 + h(v_1 - x_0) + \cdots + h^{m-1}(v_{m-1} - x_0) + h^m u = \\
&x_0 + \frac{h^m}{\beta^m}(y - x_0) + h(1 - \left(\frac{h}{\beta}\right)^{m-1})(v_1 - x_0) + \cdots + h^{m-1}(1 - \frac{h}{\beta})(v_{m-1} - x_0). \quad (3)
\end{aligned}$$

From $h \in (0, \mu)$ and the definition of μ, we have

$$\begin{aligned}
\frac{h^m}{\beta^m} + h(1 - \left(\frac{h}{\beta}\right)^{m-1}) + \cdots + h^{m-1}(1 - \frac{h}{\beta}) \quad &\leq \quad h^m/\beta^m + h + \cdots + h^{m-1} \leq \\
&\mu + \mu^2 + \cdots + \mu^{m-1} + \mu^m/\beta^m \leq 1.
\end{aligned}$$

It follows from $y, x_0, v_1, \cdots, v_{m-1} \in K$, the convexity of K and (3) that

$$x_0 + h(v_1 - x_0) + \cdots + h^{m-1}(v_{m-1} - x_0) + h^m u \in K.$$

Thus, $u_0 \in T_K^{\flat(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0)$ and the proof is complete. $\square$

Proposition 3.2. If K is convex, then $T_K^{\flat(m)}(x_0, v_1, \cdots, v_{m-1})$ is convex.

Proof. If $T_K^{\flat(m)}(x_0, v_1, \cdots, v_{m-1}) = \emptyset$, the result holds naturally. Then, we assume that there are $u_1, u_2 \in T_K^{\flat(m)}(x_0, v_1, \cdots, v_{m-1})$ and $\lambda \in (0, 1)$. It follows from the definition of the m^{th}-order adjacent subset that, for any $h_n \to 0^+$, there exist sequences $\{w_n^1\}$ and $\{w_n^2\}$ such that

$$\begin{aligned} w_n^1 &\to u_1 \\ w_n^2 &\to u_2, \end{aligned}$$

and

$$\begin{aligned} x_0 + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m w_n^1 &\in K, \\ x_0 + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m w_n^2 &\in K. \end{aligned}$$

From the convexity of K, we have

$$x_0 + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m(\lambda w_n^1 + (1-\lambda) w_n^2) \in K.$$

Thus, $\lambda u_1 + (1-\lambda) u_2 \in T_K^{\flat(m)}(x_0, v_1, \cdots, v_{m-1})$ and the proof is complete. $\square$

By Propositions 3.1 and 3.2, we have that the following corollary holds:

Corollary 3.1. If K is a convex subset and $v_1, \cdots, v_{m-1} \in K$, then sets $T_K^{(m)}(x_0, v_1 - x_0, \cdots, v_{m-1} - x_0)$ and $\mathrm{cl}\left(\bigcup_{h>0} \frac{K - x_0 - h(v_1 - x_0) - \cdots - h^{m-1}(v_{m-1} - x_0)}{h^m}\right)$ are convex.

Now we recall a result of the page 172 in Ref. 1 as follows.

Proposition 3.3. For any $\lambda > 0$, we have

$$\begin{aligned} T_K^{(m)}(x, \lambda v_1, \cdots, \lambda^{m-1} v_{m-1}) &= \lambda^m T_K^{(m)}(x, v_1, \cdots, v_{m-1}), \\ T_K^{\flat(m)}(x, \lambda v_1, \cdots, \lambda^{m-1} v_{m-1}) &= \lambda^m T_K^{\flat(m)}(x, v_1, \cdots, v_{m-1}). \end{aligned}$$

4 Higher-Order Derivatives for Set-Valued Maps

In this section, we shall recall the definitions of the m^{th}-order contingent derivative and the m^{th}-order adjacent derivative for set-valued maps in Ref. 1. Then, we shall investigate their properties under the condition that the set-valued map is S-concave.

Definition 4.1. Let X, Y be normed spaces and $F : X \to 2^Y$ be a set-valued map. The m^{th}-order contingent derivative $D^{(m)}F(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x, y) \in \text{Graph}(F)$ for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} &\text{Graph}(D^{(m)}F(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1})) \\ &= T^{(m)}_{\text{Graph}(F)}(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1}), \end{aligned}$$

i.e.,

$$\begin{aligned} v_m \in\ & D^{(m)}F(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(u_m) \Leftrightarrow \\ & (u_m, v_m) \in T^{(m)}_{\text{Graph}(F)}(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1}), \end{aligned}$$

where $\mathrm{Graph}(H)$ denotes the graph of the set-valued map H, i.e., $\mathrm{Graph}(H) = \{(x, y) \mid y \in H(x), x \in \mathrm{Dom}(H)\}$.

Definition 4.2. Let X, Y be normed spaces and $F : X \to 2^Y$ be a set-valued map. The m^{th}-order adjacent derivative $D^{\flat(m)}F(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at $(x, y) \in \mathrm{Graph}(F)$ for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned}&\mathrm{Graph}(D^{\flat(m)}F(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1}))\\&= T^{\flat(m)}_{\mathrm{Graph}(F)}(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1}).\end{aligned}$$

Naturally, $T^{(m)}_{\mathrm{Graph}(F)}(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ or $T^{\flat(m)}_{\mathrm{Graph}(F)}(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ may be empty. From the necessary conditions that the m^{th}-order contingent and adjacent sets are not empty (see Section 4.7 in Ref. 1), we have that if the domain of the m^{th}-order contingent (adjacent) derivative of F at $(x, y) \in \mathrm{Graph}(F)$ for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is not empty, then necessaryly,

$$\begin{aligned}&(u_1, v_1) \in T^{(1)}_{\mathrm{Graph}(F)}(x, y), \cdots, (u_{m-1}, v_{m-1}) \in T^{(m-1)}_{\mathrm{Graph}(F)}(x, y, u_1, v_1, \cdots, u_{m-2}, v_{m-2})\\&\left((u_1, v_1) \in T^{\flat(1)}_{\mathrm{Graph}(F)}(x, y), \cdots, (u_{m-1}, v_{m-1}) \in T^{\flat(m-1)}_{\mathrm{Graph}(F)}(x, y, u_1, v_1, \cdots, u_{m-2}, v_{m-2})\right).\end{aligned}$$

For some basic calculus for the m^{th}-order derivative, see Section 5.6 in Ref. 1.

Remark 4.1. If F is a single-valued map which is 3^{th}-order continuously differentiable around a point $x_0 \in X$, then we have

$$D^{\flat(2)}F(x_0, F(x_0), u_1, v_1)(u_2) = \begin{cases} \emptyset, & \text{if } v_1 \neq \nabla F(x_0)(u_1), \\ \nabla F(x_0)(u_2) + \frac{1}{2}\nabla^2 F(x_0)(u_1, u_1), & \text{if } v_1 = \nabla F(x_0)(u_1), \end{cases}$$

and

$$D^{\flat(3)}F(x_0, F(x_0), u_1, v_1, u_2, v_2)(u_3) =$$

$$\begin{cases} \emptyset, & \text{if } v_1 \neq \nabla F(x_0)(u_1) \\ & \text{or } v_2 \neq \nabla F(x_0)(u_2) + \frac{1}{2}\nabla^2 F(x_0)(u_1, u_1), \\ \nabla F(x_0)(u_3) + \nabla^2 F(x_0)(u_1, u_2) & \\ +\frac{1}{3!}\nabla^3 F(x_0)(u_1, u_1, u_1), & \text{if } v_1 = \nabla F(x_0)(u_1) \text{ and} \\ & v_2 = \nabla F(x_0)(u_2) + \frac{1}{2}\nabla^2 F(x_0)(u_1, u_1), \end{cases}$$

where $\nabla^m F(x_0), (m = 1, 2, 3)$ denotes the m^{th}-order derivative of F at x_0.

Remark 4.2. Jahn et al. (Ref. 17) introduced the following second-order contingent set:

$$\begin{aligned} \tilde{T}^2_{\text{Graph}(F)}(x, y, u_1, v_1) \quad = \quad & \{(w, z) \in X \times Y \mid \exists\{(w_n, z_n)\} \subset X \times Y \text{ with} \\ & (w_n, z_n) \to (w, z) \text{ and } \lambda_n > 0, \forall n, \text{ with } \lambda_n \to 0^+ \text{ so that} \\ & (x, y) + \lambda_n(u_1, v_1) + \frac{\lambda_n^2}{2}(w_n, z_n) \in \text{Graph}(F)\}, \end{aligned}$$

and the second-order contingent derivative:

$$D^2_c F(x, y, u_1, v_1)(w) = \{z \in Y \mid (w, z) \in \tilde{T}^2_{\text{Graph}(F)}(x, y, u_1, v_1)\}.$$

It follows from Proposition 3.3 and the definition of the 2^{th}-order contingent set that

$$\tilde{T}^2_{\text{Graph}(F)}(x, y, u_1, v_1) = T^{(2)}_{\text{Graph}(F)}(x, y, \sqrt{2}u_1, \sqrt{2}v_1).$$

Then, we have

$$D^2_c F(x, y, u_1, v_1)(w) = D^{(2)}F(x, y, \sqrt{2}u_1, \sqrt{2}v_1)(w). \tag{4}$$

So, the 2^{th}-order contingent derivative introduced in this paper is different from the second-order contingent derivative introduced in Ref. 17. However, they have that the equivalent relation (4) holds. Obviously, the 2^{th}-order contingent derivative introduced in this paper is also different from second-order epiderivative and generalized second-order epiderivative introduced in Ref. 17.

As Ref. 12, we also define the S-directed m^{th}-order contingent derivative $D_S^{(m)}F(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$ of F at (x, y) for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ to be the m^{th}-order contingent derivative of the set-valued map

$$F(x) - S = \{y - s \mid y \in F(x), s \in S\}$$

at (x, y) for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$. The S-directed m^{th}-order adjacent derivative at (x, y) for vectors $(u_1, v_1), \cdots, (u_{m-1}, v_{m-1})$ is analogously defined to be $D_S^{\flat(m)}F$ $(x, y, u_1, v_1, \cdots, u_{m-1}, v_{m-1})$. By Proposition 3.1, we have the following result.

Proposition 4.1. Let F be S-concave on convex set $A \subset \mathrm{Dom}(F)$, $(x_0, y_0) \in \mathrm{Graph}(F)$ and let $u_1, \cdots, u_{m-1} \in A$ and $v_1 \in F(u_1) - S, \cdots, v_{m-1} \in F(u_{m-1}) - S$. Then

$$\begin{aligned} &D_S^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x) \\ &= D_S^{\flat(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x), \quad \text{for all } x \in A. \end{aligned}$$

Theorem 4.1. Let F be S-concave on convex set $A \subset \mathrm{Dom}(F)$. Then, for all $x', x'' \in A$ and any $y' \in F(x')$,

$$F(x'') - y' \subset D_S^{\flat(m)}F(x', y', u_1 - x', v_1 - y', \cdots, u_{m-1} - x', v_{m-1} - y')(x'' - x'),$$

where $u_1, \cdots, u_{m-1} \in A$ and $v_1 \in F(u_1) - S, \cdots, v_{m-1} \in F(u_{m-1}) - S$.

Proof. Let $x'', x' \in A$ and $y' \in F(x'), y'' \in F(x'')$. For $\lambda_n^m \in (0,1)$ and $\lambda_n^m \to 0, \forall m$, we have

$$x' + \frac{\lambda_n^m}{2}(x'' - x') = (1 - \frac{\lambda_n^m}{2})x' + \frac{\lambda_n^m}{2}x'' \in A,$$
$$x' + \lambda_n^{m-1}(u_{m-1} - x') = (1 - \lambda_n^{m-1})x' + \lambda_n^{m-1}u_{m-1} \in A,$$

and

$$y' + \frac{\lambda_n^m}{2}(y'' - y') = (1 - \frac{\lambda_n^m}{2})y' + \frac{\lambda_n^m}{2}y'' \in F(x' + \frac{\lambda_n^m}{2}(x'' - x')) - S,$$
$$y' + \lambda_n^{m-1}(v_{m-1} - y') = (1 - \lambda_n^{m-1})y' + \lambda_n^{m-1}v_{m-1} \in F(x' + \lambda_n^{m-1}(u_{m-1} - x')) - S.$$

Then,

$$x' + \frac{\lambda_n^{m-1}}{2}(u_{m-1} - x') + \frac{\lambda_n^m}{2^2}(x'' - x') \in A,$$

and

$$y' + \frac{\lambda_n^{m-1}}{2}(v_{m-1} - y') + \frac{\lambda_n^m}{2^2}(y'' - y') \in F(x' + \frac{\lambda_n^{m-1}}{2}(u_{m-1} - x') + \frac{\lambda_n^m}{2^2}(x'' - x')) - S.$$

So, we have the following result:

$$x_n \stackrel{def}{=} x' + \frac{\lambda_n}{2}(u_1 - x') + \cdots + \frac{\lambda_n^{m-1}}{2^{m-1}}(u_{m-1} - x') + \frac{\lambda_n^m}{2^m}(x'' - x') \in A,$$

and

$$y_n \stackrel{def}{=} y' + \frac{\lambda_n}{2}(v_1 - y') + \cdots + \frac{\lambda_n^{m-1}}{2^{m-1}}(v_{m-1} - y') + \frac{\lambda_n^m}{2^m}(y'' - y') \in F(x_n) - S.$$

Thus,

$$(x_n, y_n) \in \text{Graph}(F - S),$$

and

$$\left((x_n, y_n) - (x', y') - \lambda_n \left(\frac{u_1 - x'}{2}, \frac{v_1 - y'}{2}\right) - \cdots - \lambda_n^{m-1} \left(\frac{u_{m-1} - x'}{2^{m-1}}, \frac{v_{m-1} - y'}{2^{m-1}}\right)\right) \Big/ \lambda_n^m$$

$= \frac{1}{2^m}(x'' - x', y'' - y')$.

It follows readily that

$$\frac{1}{2^m}(x'' - x', y'' - y') \in T^{\flat(m)}_{\mathrm{Graph}(F-S)} \left((x', y'), \frac{1}{2}(u_1 - x', v_1 - y'), \cdots, \frac{1}{2^{m-1}}(u_{m-1} - x', v_{m-1} - y')\right).$$

Hence, from Proposition 3.3, we obtain

$$(x'' - x', y'' - y') \in T^{\flat(m)}_{\mathrm{Graph}(F-S)} \left((x', y'), (u_1 - x', v_1 - y'), \cdots, (u_{m-1} - x', v_{m-1} - y')\right),$$

and

$$y'' - y' \in D_S^{\flat(m)} F(x', y', u_1 - x', v_1 - y', \cdots, u_{m-1} - x', v_{m-1} - y')(x'' - x').$$

The proof of the result is complete. $\square$

From Proposition 4.1 and Theorem 4.1, we have the following corollary.

Corollary 4.1. Let F be S-concave on convex set $A \subset \mathrm{Dom}(F)$. Then, for all $x', x'' \in A$ and any $y' \in F(x')$,

$$F(x'') - y' \subset D_S^{(m)} F(x', y', u_1 - x', v_1 - y', \cdots, u_{m-1} - x', v_{m-1} - y')(x'' - x'),$$

where $u_1, \cdots, u_{m-1} \in A$ and $v_1 \in F(u_1) - S, \cdots, v_{m-1} \in F(u_{m-1}) - S$.

5 Optimality Conditions for Problem (1)

In this section, higher-order necessary and sufficient optimality conditions for problem (1) are investigated. The notation F_A is used to denote the restriction of F to A.

Theorem 5.1. If x_0 is a weak maximal solution for (1) at y_0, then, for any $(u_i, v_i) \in X \times S,\ i = 1, \cdots, m-1$,

$$D^{(m)}F_A(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) \bigcap \mathrm{int}S = \emptyset, \quad \text{for all } x \in A,$$

and so

$$D^{\flat(m)}F_A(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x) \bigcap \mathrm{int}S = \emptyset, \quad \text{for all } x \in A.$$

Proof. Naturally, we only need to prove the first conclusion. Assume that the result does not hold. Then, there exist some $\hat{x} \in A$ and $\hat{y} \in D^{(m)}F(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(\hat{x})$ such that

$$\hat{y} \in \mathrm{int}S. \tag{5}$$

Hence, there exist $h_n \to 0^+$, $(x_n, y_n) \in \mathrm{Graph}(F)$ and $\{x_n\} \subset A$ such that

$$\frac{(x_n, y_n) - (x_0, y_0) - h_n(u_1, v_1) - \cdots - h_n^{m-1}(u_{m-1}, v_{m-1})}{h_n^m} \to (\hat{x}, \hat{y}).$$

So, it follows from (5) that when n is large enough, we have

$$\frac{y_n - y_0 - h_n v_1 - \cdots - h_n^{m-1} v_{m-1}}{h_n^m} \in \mathrm{int}S,$$

and then

$$y_n - y_0 - h_n v_1 - \cdots - h_n^{m-1} v_{m-1} \in \mathrm{int}S.$$

Since S is a convex cone and $v_1, \cdots, v_{m-1} \in S$,

$$h_n v_1 + \cdots + h_n^{m-1} v_{m-1} \in S.$$

Hence,

$$y_n - y_0 \in \mathrm{int} S,$$

which contradicts that x_0 is a weak maximal solution. □

Theorem 5.2. Let F be S-concave on the convex set $A \subset \mathrm{Dom}(F)$ and let $u_1, \cdots, u_{m-1} \in A$ and $v_1 \in F(u_1) - S, \cdots, v_{m-1} \in F(u_{m-1}) - S$. If

$$D_S^{\flat(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) \bigcap S = \{0\}, \forall x \in A,$$

then x_0 is a maximal solution for (1) at y_0. If

$$D_S^{\flat(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) \bigcap \mathrm{int} S = \emptyset,$$

then x_0 is a weak maximal solution for (1) at y_0.

Proof. It follows from Theorem 4.1 that for all $x \in A$,

$$\begin{aligned} &[F(x) - y_0] \bigcap S \\ &\subset S \bigcap D_S^{\flat(m)} F(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) = \{0\} \end{aligned} \quad (6)$$

Thus, x_0 is a maximal solution for (1) at y_0. Using the cone $\mathrm{int} S$ instead of S in (6), we similarly prove that the other conclusion holds. □

Now we give an example, which is similar to Example 3 in Ref. 18, to show a minimizer of the problem (1) which fails to satisfy the first-order assumption in Theorem 5.2, but satisfies the second-order one.

Example 5.1. Suppose that $X = \mathcal{R}^2, A = \mathcal{R}^2_+, Y = \mathcal{R}$ and $S = \mathcal{R}_+$. Let $F : X \to Y$ be a real-valued function with

$$F(x) = \begin{cases} -||x|| & \text{if } x_2 \geq 0 \\ ||x|| & \text{if } x_2 < 0. \end{cases}$$

Naturally, $F(x)$ is S-concave on the convex set A. Consider the following real optimization problem.

$$\max_{x \in A} F(x)$$

Assume that $x_0 = (0, 0)$ and $y_0 = F(x_0) = 0$. Then, x_0 is a global maximal solution of F on A. Choosing $\bar{x} = (1, 0) \in A$, we have

$$D^\flat_S F(x_0, y_0)(\bar{x} - x_0) = (-\infty, 1].$$

So,

$$D^\flat_S F(x_0, y_0)(\bar{x} - x_0) \bigcap \mathcal{R}_+ = [0, 1] \neq \{0\},$$

i.e., the first-order assumption in Theorem 5.2 is not satisfied. However, if we take $u = (0, 1)$ and $v = F(u) = -1$, then, for any $x = (x_1, x_2) \in A$, we have

$$D^{\flat(2)}_S F(x_0, y_0, u, v)(x - x_0) = (-\infty, -x_2].$$

Since $x_2 \geq 0$,

$$D^{\flat(2)}_S F(x_0, y_0, u, v)(x - x_0) \bigcap S = \emptyset \text{ or } \{0\}, \forall x \in A.$$

Hence, the second-order assumption in Theorem 5.2 is satisfied and, from Theorem 5.2, x_0 is a maximal solution of F on A.

6 Optimality Conditions for Problem (2)

Let Y^* denote the dual space of Y, and let

$$S^+ = \{\lambda \in Y^* \mid \lambda(y) \geq 0, \text{ for all } y \in S \subset Y\}$$

denote the nonnegative dual cone of S. $\lambda \in S^+$ is called to be definite positive if $\lambda(y) > 0$, for all $y \in \text{int}S$, and strictly positive if $\lambda(y) > 0$, for all $y \in S\backslash\{0\}$. The notation $(F,G)(x)$ is used to denote $F(x)\times G(x)$. In this section, necessary and sufficient optimality conditions are established for problem (2).

Theorem 6.1. Let F and G be S-concave and D-concave on the convex set E, respectively. Let $(u_i, v_i, w_i) \in X \times S \times D,\ i = 1, \cdots, m-1$. Suppose that x_0 is a (weak) maximal solution for (2) at y_0. Then, for any $z_0 \in G(x_0)\bigcap D$, there exist $\lambda \in S^+$ and $\mu \in D^+$, but not both zero functionals, such that

$$\mu(z_0) = 0, \tag{7}$$

$$\lambda(y) + \mu(z) \leq 0, \tag{8}$$

for all

$$(y, z) \in D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x),$$

and

$$x \in \text{Dom}[D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})].$$

Proof. Let $z_0 \in G(x_0)\bigcap D$ and

$$\Omega = \text{Dom}[D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})].$$

Define

$$B = \bigcup_{x \in \Omega} D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x) + (0, z_0).$$

It follows from the convexity of $\mathrm{Graph}(F_E - S, G_E - D)$ and Proposition 3.2 that

$$T^{\flat(m)}_{\mathrm{Graph}(F_E-S,G_E-D)}((x_0, y_0, z_0), (u_1, v_1, w_1), \cdots, (u_{m-1}, v_{m-1}, w_{m-1}))$$

is a convex set. Therefore, by similar proof method for the convexity of B in Theorem 5.1 in Ref. 12, we have that B is a convex set.

Now we prove that

$$B \bigcap (\mathrm{int}S \times \mathrm{int}D) = \emptyset. \tag{9}$$

Assume that the result does not hold. Then, there exist $(\hat{x}, \hat{y}, \hat{z})$ and $\hat{x} \in \Omega$ such that

$$(\hat{y}, \hat{z}) \in D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(\hat{x}), \tag{10}$$

and

$$(\hat{y}, \hat{z} + z_0) \in \mathrm{int}S \times \mathrm{int}D. \tag{11}$$

It follows from (10) and the definition of the m^{th}-order adjacent derivative that for any sequence $\{h_n\}$ with $h_n \to 0^+$, there exists $\{(x_n, y_n, z_n)\}$ with

$$x_n \in E, \quad y_n \in F(x_n) - S, \quad z_n \in G(x_n) - D$$

such that

$$\frac{(x_n, y_n, z_n) - (x_0, y_0, z_0) - h_n(u_1, v_1, w_1) - \cdots - h_n^{m-1}(u_{m-1}, v_{m-1}, w_{m-1})}{h_n^m} \to (\hat{x}, \hat{y}, \hat{z}). \tag{12}$$

From (11) and (12), there exists $N > 0$ such that $h_n < 1$ and

$$\frac{(y_n, z_n) - (y_0, z_0) - h_n(v_1, w_1) - \cdots - h_n^{m-1}(v_{m-1}, w_{m-1})}{h_n^m} + (0, z_0) \in \mathrm{int}S \times \mathrm{int}D,$$

for $n \geq N$. Thus, we have

$$y_n - y_0 - h_n v_1 - \cdots - h_n^{m-1} v_{m-1} \in \mathrm{int}S, \text{ for } n \geq N,$$

and

$$z_n - z_0 - h_n w_1 - \cdots - h_n^{m-1} w_{m-1} + h_n^m z_0 \in \mathrm{int}D, \text{ for } n \geq N.$$

Since $z_0, w_1, \cdots, w_{m-1} \in D$ and $v_1, \cdots, v_{m-1} \in S$,

$$(1 - h_n^m) z_0 + h_n w_1 + \cdots + h_n^{m-1} w_{m-1} \in D,$$

and

$$h_n v_1 + \cdots + h_n^{m-1} v_{m-1} \in S.$$

Thus, $z_n \in \mathrm{int}D$ and $y_n - y_0 \in \mathrm{int}S$. Since $z_n \in G(x_n) - D$ and $y_n \in F(x_n) - S$, there exist $\bar{z}_n \in G(x_n), d_n \in D, \bar{y}_n \in F(x_n)$ and $s_n \in S$ such that

$$z_n = \bar{z}_n - d_n \text{ and } y_n = \bar{y}_n - s_n, \text{ for } n \geq N.$$

Naturally, $\bar{z}_n \in G(x_n) \bigcap D$ and $\bar{y}_n - y_0 \in \mathrm{int}S$, which contradicts that x_0 is a (weak) maximal point at y_0. Thus, (9) holds. It follows from a standard separation theorem of convex sets and similar proof method of Theorem 5.1 in Ref. 12 that there exist $\lambda \in S^+$and $\mu \in D^+$, not both zero functionals, such that

$$\mu(z_0) = 0,$$

$$\lambda(y) + \mu(z) \leq 0,$$

for all

$$(y,z) \in D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x),$$

and

$$x \in \mathrm{Dom}[D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})].$$

Thus, the proof is complete. □

Now we give an example to illustrate the necessary optimality conditions for m^{th}-order adjacent derivative, where we only take $m = 1.2$.

Example 6.1. Suppose that $X = Y = Z = \mathcal{R}$, $E = [-1, 1] \subset X$ and $S = D = \mathcal{R}_+$. Let $F : E \to 2^Y$ be a set-valued map with

$$F(x) = \{y \in \mathcal{R} \mid -1 \le y \le -x^4\},$$

and $G : E \to Z$ be a real-valued function with

$$G(x) = -2x + 1.$$

Naturally, F and G are two $\mathcal{R}_+$-concave functions on the convex set $[-1, 1]$, respectively. Consider the following constrained set-valued optimization problem (CSVOP):

$$\begin{aligned} \max \quad & F(x) \\ \text{s.t.} \quad & x \in E, G(x) \bigcap D \neq \emptyset. \end{aligned}$$

We have

$$E \bigcap G^{-}(D) = [-1, \frac{1}{2}], \text{ and } F(E \bigcap G^{-}(D)) = [-1, 0].$$

Let $(x_0, y_0) = (0, 0) \in \mathrm{Graph}(F)$. Since $(F(E \bigcap G^{-}(D)) - y_0) \bigcap \mathrm{int}\mathcal{R} = \emptyset$, (x_0, y_0) is a weak efficient maximal solution of (CSVOP). So, the conditions of Theorem 6.1 are satisfied at (x_0, y_0). It follow from the definitions of F and G that

$$\mathrm{Graph}(F - S, G - D) = \{(x, (y, z)) \in \mathcal{R} \times \mathcal{R}^2 \mid y \leq -x^4, z \leq -2x + 1, -1 \leq x \leq 1\}.$$

Take any $z_0 \in G(x_0) \bigcap \mathcal{R}_+$. Since $G(x_0) \equiv 1$, we have $z_0 = 1$. Then,

$$T^{\flat}_{\mathrm{Graph}(F-S,G-D)}(x_0, y_0, z_0) = \{(x, (y, z)) \in \mathcal{R} \times \mathcal{R}^2 \mid y \leq 0, z \leq 2x\},$$

and

$$D^{\flat}_{S \times D}(F, G)(x_0, y_0, z_0)(x) = \{(y, z) \in \mathcal{R}^2 \mid y \leq 0, z \leq 2x\}.$$

Take $\lambda > 0$ and $\mu = 0$. Thus, for any $(y, z) \in D^{\flat}_{S \times D}(x_0, y_0, z_0)(x)$ and $x \in \mathcal{R}$, we have

$$\lambda(y) + \mu(z) \leq 0 \text{ and } \mu(z_0) = 0,$$

which shows that the 1^{th}-order necessary optimality condition of Theorem 6.1 holds.

Take $u_1 = -1/4$, $v_1 = 0 \in S$ and $w_1 = 1/2 \in D$. Then, the conditions of Theorem 6.1 are satisfied at (x_0, y_0) for vector (u_1, v_1, w_1). Naturally, we have

$$T^{\flat(2)}_{\mathrm{Graph}(F-S,G-D)}(x_0, y_0, z_0, u_1, v_1, w_1) = \{(x, (y, z)) \in \mathcal{R} \times \mathcal{R}^2 \mid y \leq 0, z \leq 2x\},$$

and

$$\begin{aligned} D^{\flat(2)}_{S \times D}(F, G)(x_0, y_0, z_0, u_1, v_1, z_1)(x) &= D^{\flat}_{S \times D}(F, G)(x_0, y_0, z_0)(x) \\ &= \{(y, z) \in \mathcal{R}^2 \mid y \leq 0, z \leq 2x\}. \end{aligned}$$

Simultaneously, take $\lambda > 0$ and $\mu = 0$. We have that the 2^{th}-order necessary optimality condition of Theorem 6.1 holds.

Remark 6.1. From the properties of higher-order contingent and adjacent sets (see the page 172 of Ref. 1), we deduce

$$\mathrm{Dom}[D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})] \neq \emptyset$$

if and only if

$$(u_j, v_j, w_j) \in T^{\flat(j)}_{\mathrm{Graph}(F_E - S, G_E - D)}((x_0, y_0, z_0), (u_1, v_1, w_1), \cdots, (u_{j-1}, v_{j-1}, w_{j-1})),$$

for $j = 1, \cdots, m-1$. Furthermore, by Proposition 3.1, if $\mathrm{Graph}(F_E - S, G_E - D)$ is a convex set, then we have

$$T^{\flat}_{\mathrm{Graph}(F_E - S, G_E - D)}(x_0, y_0, z_0) = \mathrm{cl}\left(\bigcup_{h>0} \frac{\mathrm{Graph}(F_E - S, G_E - D) - (x_0, y_0, z_0)}{h}\right).$$

Thus, if F and G are S-concave and D-concave on X, respectively, then

$$\mathrm{Dom}[D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})] \neq \emptyset$$

if and only if

$$(u_1, v_1, w_1) \in \mathrm{cl}\left(\bigcup_{h>0} \frac{\mathrm{Graph}(F_E - S, G_E - D) - (x_0, y_0, z_0)}{h}\right),$$

and

$$(u_j, v_j, w_j) \in T^{\flat(j)}_{\mathrm{Graph}(F_E - S, G_E - D)}((x_0, y_0, z_0), (u_1, v_1, w_1), \cdots, (u_{j-1}, v_{j-1}, w_{j-1})),$$

for $j = 2, \cdots, m-1$.

Note that the following equation may not hold:

$$D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x)$$

$$= D^{\flat(m)}_{S} F_E(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)$$

$$\times D^{\flat(m)}_{D} G_E(x_0, z_0, u_1, w_1, \cdots, u_{m-1}, w_{m-1})(x). \tag{13}$$

Indeed, when F and G are S-concave and D-concave, respectively, (13) may also not hold. The following example explains the case.

Example 6.2. Suppose $E = [0, +\infty), S = D = [0, +\infty), m = 1,\ G(x) = \sqrt[3]{x},\ x \in (-\infty, +\infty)$ and

$$F(x) = \begin{cases} \sqrt{x}, & x \in [0, +\infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

Then, F and G are $\mathcal{R}^+$-concave on E. We have

$$D^{\flat}_{S} F_E(0,0)(0) = \mathcal{R} \text{ and } D^{\flat}_{D} G_E(0,0)(0) = \mathcal{R},$$

namely,

$$D^{\flat}_{S} F_E(0,0)(0) \times D^{\flat}_{D} G_E(0,0)(0) = \mathcal{R}^2. \tag{14}$$

However,

$$D^{\flat}_{S\times D}(F_E, G_E)(0,0,0)(0) = \left\{ (y,z) \in \mathcal{R}^2 \;\middle|\; \begin{array}{ll} y \in (-\infty, 0], z \in (-\infty, +\infty) & \text{or} \\ y \in (-\infty, +\infty), z \in (-\infty, 0]. & \end{array} \right\} \tag{15}$$

It follows from (14) and (15)) that

$$D^{\flat}_{S} F_E(0,0)(0) \times D^{\flat}_{D} G_E(0,0)(0) \neq D^{\flat}_{S\times D}(F_E, G_E)(0,0,0)(0).$$

Now we give the following proposition for explaining that (13) holds when F or G is locally Lipschitz.

Proposition 6.1. If either F or G is locally Lipschitz at x_0, then,

$$\begin{aligned}
&D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x)\\
&= D^{\flat(m)}_{S} F_E(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)\\
&\quad \times D^{\flat(m)}_{D} G_E(x_0, z_0, u_1, w_1, \cdots, u_{m-1}, w_{m-1})(x). \qquad (16)
\end{aligned}$$

Proof. Naturally, we only need to prove

$$\begin{aligned}
&D^{\flat(m)}_{S} F_E(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)\\
&\quad \times D^{\flat(m)}_{D} G_E(x_0, z_0, u_1, w_1, \cdots, u_{m-1}, w_{m-1})(x)\\
&\subseteq D^{\flat(m)}_{S\times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x). \qquad (17)
\end{aligned}$$

Without loss of generality, suppose that G is locally Lipschitz at x_0 and

$$\begin{aligned}
(y, z) \in \quad & D^{\flat(m)}_{S} F_E(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x)\\
& \times D^{\flat(m)}_{D} G_E(x_0, z_0, u_1, w_1, \cdots, u_{m-1}, w_{m-1})(x).
\end{aligned}$$

Then, for any $h_n \to 0^+$, there exist $(x_n, y_n) \to (x, y)$ and $x_n \in E$ such that

$$y_0 + h_n v_1 + \cdots + h_n^{m-1} v_{m-1} + h_n^m y_n \in F(x_0 + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n) - S. \quad (18)$$

Similarly, for any $h_n \to 0^+$, there exist $(\bar{x}_n, \bar{z}_n) \to (x, z)$ and $\bar{x}_n \in E$ such that

$$z_0 + h_n w_1 + \cdots + h_n^{m-1} w_{m-1} + h_n^m \bar{z}_n \in G(x_0 + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m \bar{x}_n) - D. \quad (19)$$

It follows from locally Lipschitz continuity of G that there exist a constant $M > 0$ and a neighborhood W of x_0 such that

$$G(x_1) \subset G(x_2) + M||x_1 - x_2||B, \ \forall x_1, x_2 \in W. \tag{20}$$

Naturally, there exists $N > 0$ satisfying

$$x_0 + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n, x_0 + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m \bar{x}_n \in W, \ \forall n \geq N.$$

It follows from (20) that

$$\begin{aligned} & G(x_0 + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m \bar{x}_n) \\ & \subset G(x_0 + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n) + h_n^m M ||\bar{x}_n - x_n|| B, \ \forall n \geq N. \end{aligned} \tag{21}$$

From (19) and (21), there exists $z_n \to z$ such that for any $n \geq N$,

$$z_0 + h_n w_1 + \cdots + h_n^{m-1} w_{m-1} + h_n^m z_n \in G(x_0 + h_n u_1 + \cdots + h_n^{m-1} u_{m-1} + h_n^m x_n) - D. \tag{22}$$

It follows from (18) and (22) that

$$(y, z) \in D^{\flat(m)}_{S \times D}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})(x),$$

and (16) holds. $\square$

From Theorem 6.1 and Proposition 6.1, we have the following result.

Theorem 6.2. Let F and G be S-concave and D-concave on the convex set E, respectively and either F or G be locally Lipschtiz at x_0. Let $(u_i, v_i, w_i) \in X \times S \times D, \ i = 1, \cdots, m-1$. Suppose that x_0 is a (weak) maximal solution for (2) at y_0. Then, for any

$z_0 \in G(x_0) \bigcap D$, there exist $\lambda \in S^+$ and $\mu \in D^+$, but not both zero functionals, such that

$$\mu(z_0) = 0,$$

$$\lambda(y) + \mu(z) \leq 0,$$

for all

$$y \in D_S^{\flat(m)} F_E(x_0, y_0, u_1, v_1, \cdots, u_{m-1}, v_{m-1})(x),$$

$$z \in D_D^{\flat(m)} G_E(x_0, z_0, u_1, w_1, \cdots, u_{m-1}, w_{m-1})(x)$$

and $x \in \mathrm{Dom}[D_{S\times D}^{\flat(m)}(F_E, G_E)(x_0, y_0, z_0, u_1, v_1, w_1, \cdots, u_{m-1}, v_{m-1}, w_{m-1})]$.

Theorem 6.3. Let F and G be S-concave and D-concave respectively on the convex set $E \subset \mathrm{Dom}(F) \bigcap \mathrm{Dom}(G)$, and let $A = E \bigcap G^-(D)$. Suppose that there exist $x_0, u_1, \cdots, u_{m-1} \in A, y_0 \in F(x_0), v_1 \in F(u_1) - S, \cdots, v_{m-1} \in F(u_{m-1}) - S, z_0 \in G(x_0) \bigcap D, w_1 \in G(u_1) \bigcap D, \cdots, w_{m-1} \in G(u_{m-1}) \bigcap D$, strictly (definite) positive $\lambda \in S^+$, and $\mu \in (T_D^{\flat(m)}(z_0, w_1 - z_0, \cdots, w_{m-1} - z_0))^+$ such that

$$\lambda(y) + \mu(z) \leq 0, \tag{23}$$

for all

$$\begin{aligned}(y, z) \in \quad & D_S^{\flat(m)} F_A(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) \\ & \times D_D^{\flat(m)} G_A(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0),\end{aligned}$$

and $x \in A$. Then, x_0 is a (weak) maximal solution for (2) at y_0.

Proof. Suppose that

$$H_A(x) = G_A(x) \bigcap D, \forall x.$$

Now we prove that H_A is a D-concave function on A. In fact, suppose that $x_1, x_2 \in \mathrm{Dom}(H_A)$, $z_1 \in H_A(x_1), z_2 \in H_A(x_2)$ and $\beta \in (0,1)$. It follows readily that $z_1 \in G_A(x_1), z_2 \in G_A(x_2)$ and $z_1, z_2 \in D$. From the concavity of G_A and the convexity of D, we have

$$\beta z_1 + (1-\beta)z_2 \in G_A(\beta x_1 + (1-\beta)x_2) - D, \tag{24}$$

and

$$\beta z_1 + (1-\beta)z_2 \in D. \tag{25}$$

It follows from (24) that there exist $\bar{z} \in G_A(\beta x_1 + (1-\beta)x_2)$ and $\bar{d} \in D$ such that

$$\beta z_1 + (1-\beta)z_2 = \bar{z} - \bar{d}. \tag{26}$$

By (25) and (26), we obtain

$$\bar{z} \in D$$

Thus, we have

$$\beta z_1 + (1-\beta)z_2 \in \Big(G_A(\beta x_1 + (1-\beta)x_2) \bigcap D\Big) - D,$$

and H_A is D-concave. Naturally, $\mathrm{Graph}(H_A - D) \subset \mathrm{Graph}(G_A - D)$. It follows from Table 4.7 in Ref. 1 that

$$\begin{aligned} &D_D^{\flat(m)} H_A(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0) \\ &\subset D_D^{\flat(m)} G_A(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0), \end{aligned} \tag{27}$$

for all $x \in A$. From the definition of the set-valued map H_A, we have

$$D_D^{\flat(m)} H_A(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0)$$

$$\subset T_D^{\flat(m)}(z_0, w_1 - z_0, \cdots, w_{m-1} - z_0).$$

By Theorem 4.1, we get

$$D_D^{\flat(m)} H_A(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0) \neq \emptyset, \forall x \in A$$

So,

$$\mu(D_D^{\flat(m)} H_A(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0)) \geq 0. \tag{28}$$

It follows from (23), (27) and (28) that

$$\lambda(D_S^{\flat(m)} F_A(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0)) \leq 0,$$

for any $x \in A$. Thus, by the definition of λ and Theorem 5.2, we have that x_0 is a (weak) maximal solution for (2) at y_0. □

From the proof of Theorem 6.3, we have that the following corollary holds.

Corollary 6.1. Let F and G be S-concave and D-concave respectively on the convex set $E \subset \mathrm{Dom}(F) \bigcap \mathrm{Dom}(G)$, and let $A = E \bigcap G^{-}(D)$. Suppose that there exist $x_0, u_1, \cdots, u_{m-1} \in A, y_0 \in F(x_0), v_1 \in F(u_1), \cdots, v_{m-1} \in F(u_{m-1}), z_0 \in G(x_0) \bigcap D, w_1 \in G(u_1) \bigcap D, \cdots, w_{m-1} \in G(u_{m-1}) \bigcap D$, strictly (definite) positive $\lambda \in S^+$, and $\mu \in (T_D^{\flat(m)}(z_0, w_1 - z_0, \cdots, w_{m-1} - z_0))^+$ such that

$$\lambda(y) + \mu(z) \leq 0, \tag{29}$$

for all

$$\begin{aligned}(y, z) \in \quad & D_S^{\flat(m)} F_A(x_0, y_0, u_1 - x_0, v_1 - y_0, \cdots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) \\ & \times D_D^{\flat(m)} H_A(x_0, z_0, u_1 - x_0, w_1 - z_0, \cdots, u_{m-1} - x_0, w_{m-1} - z_0)(x - x_0),\end{aligned}$$

and $x \in A$. Then, x_0 is a (weak) maximal solution for (2) at y_0, where $H_A : A \to 2^D$ with $H_A(x) = G_A(x) \bigcap D, \forall x \in A$.

Remark 6.2. If we use the m^{th}-order contingent derivatives for F_A and G_A (F_A and H_A) instead of their m^{th}-order adjacent derivatives in Theorem 6.3 (Corollary 6.1), then, the result of Theorem 6.3 (Corollary 6.1) also holds.

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