School of Electrical Engineering, Computing and Mathematical Sciences

Optimization of contracts and investment through various continuous-time dynamic principal-agent models

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Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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Abstract

Over the last few decades, the optimal contract problems in terms of principal-agent models have drawn great interest of scholars. Following Holmstrom and Milgrom [1], who apply an exponential-linear structure to the continuous-time principal-agent model, many scholars have investigated the optimal problems for various continuous-time dynamic models.

This thesis investigates three dynamic principal-agent models and an optimal investment problem of a defined contribution (DC) pension plan in continuous time. The main aspects of our work are as follows.

First of all, we build our model on top of that in [2]. Unlike the existing literature, we assume the volatility of the output diffusion process to satisfy a time-varying function, which is assumed to be a constant in [1,2]. We discuss the optimal contracts under three different information structures using the stochastic maximum principle, the first-order approach, etc. In the neoclassical setting, we give several examples of the time-varying volatility functions that could be implemented to describe different situations. In the case with hidden information, we show that moral hazard reduces effort but has little effect on the agent's consumption while hidden savings further reduces effort as well as consumption. We find that sufficiently large volatility or extremely volatile environment could make the contract meaningless.

The second aspect of our contribution is that we add ability uncertainty to the principal-agent model and give the general assumptions for the effort function, which could be implemented into the model. We derive the necessary and

sufficient conditions for the incentive contract and determine the optimal contract under a specific example of the effort function. The model suggests that as time goes by, the agent ability is gradually revealed and the ability uncertainty vanishes in the long-run. As a result, the optimal contracts under known and unknown ability become identical.

In the third model, under more realistic environment, we consider a dynamic model featured with firm size. Different from the work done by Chi and Choi [3], we assume that the diffusion processes of output and investment are simultaneously influenced by the firm size and an extra time-varying term, which is interpreted as external shocks, depicting stability of outside financial environment. Optimal contract is obtained under full information. The optimal compensation for the agent is decomposed into several terms and explained with the external shocks. Moreover, the optimal investment scheme is analyzed under certain assumptions. In addition, we find that large firms provide more protection for the principal's dividend and volatile environment is not preferable to the principal.

At last, we study the optimal asset allocation problem of a DC pension plan. Our main contribution in this part is that we extend the models studied in [4–7] by taking into account two administrative fees, which are the charge on balance and charge on flow, and the return of premium clause. In our setting, the evolution of the risky asset follows a constant elasticity of variance (CEV) model. Moreover, we consider the Weibull force function of mortality, which is not used by previous related literature. Using the stochastic control approach, we derive the explicit solutions to the optimal investment strategy under both CARA and CRRA utilities. Subsequently, we show how the certainty equivalent of the expected utility works for comparing the two fees. Numerical analysis is also provided regarding the fees scheme and the return of premium clause.

List of publications during PhD candidature

- Chong Lai, Rui Li, and Yonghong Wu. Optimal compensation and investment affected by firm size and time-varying external shocks. *Annals of Finance*, accepted, 2020.

- Chong Lai, Lishan Liu, and Rui Li. The optimal solution to a principal-agent problem with unknown agent ability. *Journal of Industrial and Management Optimization*, doi: 10.3934/jimo.2020084, 2020.

- Chong Lai, Rui Li, and Yonghong Wu. The investigation of optimal contracts in principal-agent model with time-varying volatility. *Journal of Economic Dynamics and Control*, under review, 2019.

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Contents

CHAPTER 1

Introduction

1.1 Background

The principal-agent problems emerged in the 1970s. Based on disciplines of economics and institutional theory, Stephen Ross and Barry Mitnick first proposed the agency theory [8, 9]. Subsequently, the principal-agent theory has been extended well beyond these two fields to all contexts of risk, uncertainty, and information asymmetry. For instance, it is studied in terms of law, game theory, employment contract, team production, etc.

Generally speaking, to solve a principal-agent problem, incentive provision is the main tool. In other words, how the principal motivates the agent so that the agent works as desired and how the contract functions well are the major concerns. When perfect information is available, this problem is quite simple. Nevertheless, when information friction exists, providing incentives becomes difficult. Milgrom and Roberts [10] state four principles of imperfect information contract design. The first one is the informativeness principle proposed by Holmstrom [11], suggesting that any measure of information about the agent's choice of effort should be included in the compensation scheme. This leads to highly incentive intensive contracts, however, it is not always optimal to set incentives as intense as possible. Thus, the incentive-intensity principle is introduced, which gives four factors that we should consider while designing incentive contracts: the agent's risk tolerance, the agent's sensitivity to incentives, criteria of desired actions, and incremental profits created by additional efforts. The third one is complementary to the second one and called monitoring-intensity principle. The last one is the compensation principle, stating that activities should be equally valuated by both contractual parties. Based on the four principles, the linear incentive compensation could be expressed as: wage = (base salary) + incentives × (unobservable effort + unobservable effects + (weight Z) × (observed endogenous effects)). Admittedly, the simple linear assumption has many shortcomings but it gives us a good start.

In the study of optimal contract, models can be built in discrete time and continuous time. We focus on continuous-time models for several reasons. Firstly, in discrete-time models, burdensome functions are used to map current output and continuation values into future consumption and values. In contrast, continuous-time models involve more natural descriptors of principal-agent dynamics, which are expressed by the volatility and drift term of the agent's continuation value or promised utility [12]. Secondly, differential equations can be used to clearly feature the factors that affect effort, consumption, and hidden states. We utilize the stochastic maximum principle, dynamic programming principle, and Hamilton-Jacobi-Bellman (HJB) equations to solve our problems regarding optimal contract, investment, and compensation. Lastly, continuous-time models highlight the essential characteristics of the optimal contracts.

The choice of utility functions directly influences the way to find optimal contracts. Holmstrom and Milgrom [1] cast the model under exponential utilities (also known as constant absolute risk aversion utility). This assumption simplifies the solution procedure and leads to explicit solutions of the principal-agent problem. Furthermore, exponential utility restrains wealth effect and makes it easier to solve the problem under information frictions (moral hazard, hidden savings, unknown ability, etc.). Thus, many researches regarding principal-agent problems take advantage of this form of utility function [13–16].

As the incentive contract under information frictions relies on the first-order conditions of the agent's optimization problem and the agent's promised utility or continuation value, the wage scheme will exhibit history dependence [17–20]. That is, the agent's wage depends on the whole past history of output, which is a state variable in the problem. To get rid of this kind of problems, we apply a change of measure process, which makes the density of the output a key state variable instead of the original output [21]. Consequently, the evolution of the state variables follows a stochastic differential equation characterized by random coefficients. In this way, we are able to deal with the agent's optimization problems.

Intuitively, information friction is the source of uncertainty, so solving the principal-agent problem under different information structures is of great significance. In reality, contracts with perfect information, in which the principal has all bargaining power, barely exist. Many researchers start with the problem under this assumption to create a first best benchmark. With this benchmark, the optimal contracts obtained under private information can be analyzed by comparing to the first best one. Many different information structures have been studied. For example, in an adverse selection model, the agent has private information about his type before the contract is signed. In a moral hazard model, the principal cannot observe the agent's choice of effort. In order to prevent the agent from shirking, an incentive compatible contract, which aligns the principal's expected utility with the agent's promised utility, should be provided. When the agent's wealth and consumption are private information, which hinders incentive provision, additional state variables should be introduced to restrict the agent's actions. Private information could also include the agent ability which is related to belief manipulation and affects the efficiency of the management. Moreover, investment decisions and timing of the salary distribution matter in some situations.

In real economy, the stability of outside environment directly influences the behavior of both contracting parties. However, very few work has been done relating to volatility or external shocks in principal-agent problems. In fact, external shocks vary with time and the volatility of shocks fluctuates during business cycles [22]. Investors and managers adjust their behaviors according to different situations. Therefore, it is worth adding the element of shocks into principal-agent models and analyzing its effects.

This thesis focuses on making reasonable extensions to a principal-agent model, solving the optimal contracts and analyzing investment decisions under different assumptions.

1.2 Objectives

This work uses stochastic calculus to find the optimal contracts and investment regarding the extended principal-agent models. The main objectives are as follows:

(1) Make reasonable assumptions about the economic environment and extensions on the existing continuous-time principal-agent model. Specifically, assume the volatility of shocks to be a function of time, find the explicit solutions to the incentive contracts and analyze the effect of external shocks by utilizing the agency theory, the stochastic maximum principle, dynamic programming principle, HJB equations, etc;

(2) Add ability uncertainty to a moral hazard model and solve the optimization problems of the principal and agent. Study the adverse effect caused by belief

manipulation and compare long-term (infinite time horizon) contracts with short-term ones;

(3) Investigate the model in a more specific financial environment by adding firm size as well as time-varying volatility to the output. Optimal investment and compensation are obtained and analyzed;

(4) Apply the stochastic calculus to a defined-contribution pension plan, in which two kinds of administrative fees and the return of premium clause are embedded. Then investigate the effects of the parameters on the optimal risky investment and obtain the corresponding numerical analysis.

1.3 Outline of the thesis

This thesis consists of seven chapters.

Chapter 1 introduces the research background and objectives.

Chapter 2 briefly reviews the previous works and results relating to principal-agent contract problems.

Chapter 3 solves a continuous-time principal-agent problem with time-varying volatility under three different information structures. Explicit solutions are obtained and the general form of the time-varying volatility is given.

Chapter 4 considers a principal-agent problem with ability uncertainty under the moral hazard environment. A learning process is embedded in the model and the solutions to the optimal contract are obtained. The effect of ability uncertainty on long-term and short-term contracts is compared and analyzed.

Chapter 5 investigates the optimal investment strategy and compensation scheme of a dynamic model under full information, which is associated with firm size and time-varying external shocks. Particularly, the drift of output is affected by the investment strategy as well as the agent's effort and the shock term is driven by firm size and external shocks.

Chapter 6 studies a defined contribution pension plan which is embedded with two administrative fees and the return of premium clause. The solution for optimal risky investment is obtained and the effects of fees and return of premium clause are analyzed. Numerical analysis are also provided.

Chapter 7 summarizes the main results of this thesis and provides several problems which could be further investigated.

CHAPTER 2

Literature Review

2.1 General

Over the last several decades, optimal contracting problems regarding principal-agent relationship have been studied worldwide. Due to the shortcomings of discrete-time models, which makes justifying the first order approach extremely complicate, many principal-agent problems are considered in continuous time, in which Brownian motions are applied. In this chapter, we will briefly review the emergence and development of the principal-agent theory mainly in continuous time, including problems about first-best contracts, compensation schemes, investment plans, etc.

The rest of this chapter is organized as follows. Section 2.2 reviews the emergence and development of the dynamic principal-agent models. Section 2.3 briefly describes some applications of the dynamic models. Finally, a concluding remark regarding the literature review is given in Section 2.4.

2.2 Dynamic principal-agent model

In early studies of contracting relationship, static models are the main focus [11, 23]. Usually, in a static setting, conflicts between the two parties and information are both assumed to be constant over time. A large number of investigations were carried out according to the work of Barry Mitnick [24]. In policy mechanism design, static models are well applied [25–27]. However, Waterman and Meier [28] argue that principal-agent models should be analyzed in a dynamic environment because they actually depict a relationship that changes over time.

In a basic dynamic principal-agent problem, both parties solve the optimization problems by maximizing their own expected utilities with the output process Y_t affected by the effort level:

$$dY_t = e_t dt + \sigma dZ_t,$$

where e_t is the agent's effort, σ is a constant representing the volatility of shocks and Z_t is a standard Brownian motion. From static to dynamic, Holmstrom and Milgrom [1] investigate the incentive compatible problem, prove that the linear rule plays an important role in the compensation scheme under the Brownian model, and show that the principal's expected payoff is independent of the timing of the agent's information. Jensen and Murphy [29] cast their empirical study on the pay-performance relation and indicate that dynamic settings are indeed crucial. Fudenberg et al. [30] make use of the knowledge of dynamic programming and propose that using the information implied by the agent's wealth to characterize incentive compatible contract is valid.

In a dynamic environment, it is challenging to solve the optimal contracts. As mentioned in Section 1.1, the agent's wage and consumption rely on the whole past history of output. This history dependence is confronted by many researchers. With imperfect information, Abreu et al. [17,18] deal with this problem under optimal cartel equilibria and discounted repeated games. Spear and Srivastrava [19] study the optimal contract of an infinitely repeated principal-agent model and also confront similar history dependence. Following the stochastic formulation and the change of measure in terms of density functions presented by Bismut [21,31], which are powerful tools for convex analysis in optimal stochastic control problems, we are able to simplify the problem and obtain the optimal solution. Williams [2] provides a solvable continuous-time model and shows that the first-order approach is valid, which is known to be invalid to contractual problems since Mirrlees [20] (also see [32, 33] in static environment). Moreover, this work demonstrates how explicit solutions to incentive contracts can be obtained in a full dynamic environment.

In general, it is difficult to solve a dynamic problem via discrete-time models. Although Phelan and Townsend [34] propose a method of iteration to deal with the discrete dynamic programming problem, it is rather complicated and requires solving a large number of linear programming problems. For other discrete-time dynamic principal-agent problems, readers are referred to [35–39] for more details. Compared with discrete-time models, continuous-time models simplify the process of solution. Taking advantage of the stochastic optimal control approach, one can find the optimal contracts through differential equations and gain more accuracy in terms of solutions. Sannikov [12] describes a new continuous-time model and explores the dynamics of the agent's salary and effort, and since then related literature emerges. For example, Ju and Wan [40] point out that the relationship between pay performance sensitivity and firm risk is non-monotonic under continuous setting. Mirrlees and Raimondo [41] present a universal method to construct strategies in continuous time. Under the moral hazard setting, Chang et al. [42] study the principal-agent problem with behavioral preferences. A continuous robust contracting problem is investigated by Miao and Rivera [43]. For the overview of the relating literature, readers are referred to [44].

As we mentioned in Section 1.1, private information is essential in dynamic principal-agent problems and is usually persistent in reality. In particular, an employee's income tends to vary from time to time and is hard for the employer to verify. Also, a manager's ability of management, a worker's productivity, etc. are all sources of the persistence of private information. The empirical results in [45, 46] show that idiosyncratic risk, which consists of private information, is highly persistent. In continuous time, Zhang [47] develops a method to solve the contracting problem where the persistent private information is the agent's productivity shocks. Ordinary differential equations (ODE) are used to solve the optimal contracts, which lead to a conclusion that the agent's utility converges to the lower bound. Williams [48] investigates the dynamic contracts under persistent private information and permanent shocks, and points out that increasing persistence enlarges the efficiency losses. The relatively general method for solving optimal contract problems in [48] gives a good start for later continuous-time researches.

Based on the linear results in [1], Williams [2] explicitly solves the optimal contracts under three different information structures: neoclassical, moral hazard, hidden savings. The mathematical approach developed in [2] provides a guideline for us to handle the models. Mitchell and Zhang [49] study the unemployment insurance problem under hidden savings and find the exact optimal contracts with exponential utilities, which suggest that applying a constant tax rate during employment and increasing benefits during unemployment would be efficient. Cvitanić et al. [50] consider a one-time payment to the agent in a moral hazard setting and present a general mathematical theory to these problems. They also provide a guideline for deriving the necessary and sufficient conditions of the optimal contracts via the stochastic maximum principle. A variational approach is introduced by Garrett and Paven [51] to deal with the incentive contract, where the agent generates stochastic cash flows, and explain the dynamics of average distortions.

In realistic environments, the agent ability, which is an indicator of the skills and productivity, is not always completely observed by the principal. This private information implicitly enhances the agent's bargaining power. That is, with unobservable ability, the agent can state that poor performance is a result of adverse shocks instead of his/her being shirking, distorting the principal's inference about output and ability downward. As a result, when the agent exerts efforts that are the principal's overestimates, the agent gets rewarded. This is called a belief manipulation, which is persistent and beneficial to the agent. To study the relevant cases, various models that feature learning process, which means updating information, have been developed. Relating problems are usually discussed in long-term contracts. Bergemann and Hege [52] study a dynamic principal-agent model about venture capital with long-last belief manipulation about the success rate of the project. Hörner and Samuelson [53] conclude that belief manipulation causes higher incentive costs, which may lead to a reduction on the project's continuation value. Unknown agent ability (or project quality) in dynamic contractual problems has drawn much attention from the research community. For example, Hopenhayn and Jarque [54] study the problem with persistent unknown ability and a single effort decision at the beginning. Adrian and Westerfield [55] assume that there is a disagreement between the contractual parties about the resolution of uncertainty. It is found in [55] that as the principal and the agent disagree about the belief of the agent ability, the belief manipulation vanishes. Giat et al. [56] build a structural model on top of that in [1] and analyze the effects of asymmetric beliefs. Prat and Jovanovic [57] consider a repeated principal-agent problem with unknown agent ability, where a risk-averse agent receives payments periodically. He et al. [58] focus on stationary learning and show that, on average, the recommended effort in the optimal contract decreases over time. Uğurlu [59] investigates the principal-agent problem with risk-averse principal and agent, and uses a non-stationary learning process to characterize the information of the ability.

2.3 Applications in Finance

The continuous-time dynamic models have been widely studied in finance, in which different financial elements are added to the model. For example, incentive contract about CEO's compensation scheme and firm size is an interesting topic. He [60] investigates a continuous-time principal-agent problem, in which the geometric Brownian motion is used to describe firm size, and finds that firm size has partial effect on incentives and large firms suffer from fewer agency problems. Following this paper, He [61] shows that smaller firms intend to take less leverage than their larger counterparts because of agency problems and debt-overhang. Chi and Choi [3] build on the model introduced in [60] and assume that the volatility of firm size is proportional to the square root of firm size. Under this assumption, they show that the firm's production efficiency improves with its size and come up with two interesting findings: (a)wages in large firms tend to be front-loaded while being back-loaded in their small counterparts; (b)large firms tend to make overinvestments while small ones have underinvestments. The dynamic contracts relating to the Q theory of investment are investigated in [62]. This article shows that output price fluctuation affects the agent's compensation scheme even though the agent cannot control the price movements. Earlier literature on contracting problems with corporate dynamics includes [63–65].

In a risk-neutral corporate setting, Hoffmann and Pfeil [66] apply the martingale techniques mentioned in [12] and study the reward for luck effect in a principal-agent model, in which the cash flow of the firm is controlled by the agent's effort, and their model suggests that as the mean of cash flow is affected by external shocks, incentive provision becomes cheaper and the agent's compensation decreases after an increase in the expected cash flow. Also, the marginal cost of compensation is smoothed by the optimal contract so that the agent is rewarded for good luck. Technically relating to [12], Bias et al. [67] analyze the agency model under large and infrequent risks, which is characterized by a Poisson process. They show that the agent can reduce the exogenous risks by increasing effort input. Clementi et al. [65] focus on finding several useful dynamic properties when the firm size of old firms begins to decrease. As is well known, the optimal contract for the principal-agent problem can be written recursively with the agent's promised utility being a state variable [19]. Combined with the financial constraints, different scenarios are modeled. Piskorski and Tchistyi [68] study the optimal mortgage problem. Zhu [69] points out that the first-best contract is not always optimal in the financial market and proposes a sticky incentive scheme as optimal. Biais et al. [70] give a brief introduction about the dynamic contracting problems, in which the agent's effort directly links to the downside risk, and conclude that the principal (firm) should pay the agent (manager) and make investment only after sufficiently high performance in order to prevent inappropriate incentives. Piskorsky and Westerfield [71] add monitoring, which reduces the intensity of performance-based incentives, to a moral hazard model.

In addition to optimal contract problems, optimal investment strategies of firms or portfolios are of great interest to the research community. Wang et al. [72] study the impact of optimism bias in delegated portfolio management, Cvitanić and Xing [73] discuss the equilibrium of asset prices, and Zhao et al. [74] investigate a hedge fund managers optimal effort under high-water mark, etc. The asset allocation strategies of the financial instruments or projects are usually determined by the manager according to the principal's risk tolerance, investment horizon, objectives, etc. There are many financial instruments in the market, so we just take one as an example.

In recent decades, human lifespans are increasing while the fertility rate decreases. As a result, social security system needs to provide supports for more people after their retirements. To relieve the governments of the burden, many different pension schemes are adopted. One of main pension schemes is the defined contribution (DC) pension plan, in which the pension member bares the longevity and financial risks. In the DC pension scheme, contributions are predetermined while the benefits are adjusted according to the return of pension fund portfolio during the accumulation phase.

By using stochastic calculus, dynamic programming principle, HJB equations, etc., the optimal investment strategies for DC pension plans have been investigated. For instance, Wang and Li [75] study a robust optimal investment problem of a DC plan for an ambiguity-averse member under stochastic interest rate and stochastic volatility, in which the stochastic dynamic programming approach is employed to find the exact robust optimal investment strategy and value function. Similarly, Guan and Liang [76] use the Cox-Ingersoll-Ross (CIR) model and the Vasicek model to characterize interest rates and let the contribution rate be stochastic. Different utility functions are important in analyzing principal-agent problems as well as the optimal investment strategy. When dealing with the solution of optimal contracts, the exponential utility, which is also known as constant absolute risk aversion (CARA) utility, is useful because it isolates wealth effect. However, the power utility, which is also called constant relative risk aversion (CRRA) utility, is applied to analyze the optimal investment. For instance, Boulier et al. [77], Vigna and Haberman [78] and Devolder et al. [79] use the stochastic optimal control theory to study the optimal investment strategy of a DC plan under both CARA and CRRA frameworks. Gao [80] applies the Legendre transformation and dual theory to analyze the optimal investment under the logarithm utility. By relaxing the assumption in terms of the price evolution of the risky asset, which is driven by the geometric Brownian motion, the constant elasticity of variance (CEV) model is introduced (see [5, 81, 82]). Besides, the Heston's stochastic volatility model (SV) is employed to analyze the effects of different risks of the optimal investment strategy. For example, Guan and Liang [83] consider the optimal investment in a stochastic interest rate framework where the price of the risky assets satisfies Heston's SV model. Sun et al. [84] add inflation risk and stochastic income to the model and derive the explicit solution of the equilibrium investment strategies via a set of extended HJB equations. Furthermore, the jump-diffusion risk process is widely applied in analyzing the optimal investment strategies of DC plans and other insurance problems [85–88].

As is well known, mean-variance framework is first employed in the optimal asset allocation problems by Markowitz [89]. In the light of the mean-variance process, He and Liang [6] add return of premium clause into the model and solve a relating continuous-time mean-variance optimal control problem, finding that this clause reduces the fund size and thus forces the manager to invest more in risky assets. Li et al. [7] consider a DC fund with default risk. Specifically, they explicitly solve the equilibrium investment strategies and value functions in pre- and post-default cases. Zhang et al. [90] study the optimal multi-period investment strategy for a DC plan under imperfect information, showing that unobservable state variable, which causes information frictions, indeed leads to a loss in investment return. Chen et al. [91] consider inflation risk, salary risk and longevity risk in their model, where the loss averse principal requires a minimum performance. The explicit solution of the optimal investment is obtained via the martingale approach and reveals that the optimal risky investment increases with lifespan but decreases with salary level.

In practice, the manager of the DC fund or the pension fund administrator

(PFA) charges administrative fees based either on the individual account (IA) assets (charge on balance) or the principal's income or contribution flow (charge on flow) [92]. These fees influence the asset allocation of the DC fund. The administrative fees in different countries are compared and illustrated in [93]. Shah [94] argues that administrative fees can cause high PFA set-up costs, hinder competition in the system, and lead to losses for older members. Whitehouse [95] finds that an annual levy of one per cent of assets adds up to nearly 20% of the final pension value. Chávez–Bedoya [4] considers a risk-averse member, who wants to maximize his/her expected utility of adjusted terminal wealth in a complete financial market, solves the corresponding stochastic control problems, and develops a methodology to compare the equivalent charges on balance and on flow.

2.4 Concluding remark

In this chapter, we briefly review the literature relating to dynamic principal-agent problems. Holmstrom and Milgrom [1] provide a tractable method under CARA utility for later researches to use. By assuming linearity on the optimal contracts, we are able to solve our extended models. The main issue in these models is incentive provision through different kinds of private information. We briefly reviewed the literature relating to moral hazard, hidden savings, persistent private information, ability uncertainty, belief manipulation, and optimal investment. In the upcoming chapters, specific extensions are made to the existing dynamic principal-agent models and the corresponding optimal contracts are obtained. Moreover, several analytical and numerical investigations are carried out.

CHAPTER 3

The investigation of optimal contracts in principal-agent model with time-varying volatility

3.1 General

In this chapter, we investigate a continuous time principal-agent model, in which the volatility of the output diffusion process depends on a positive function of time t, while the volatility is assumed to be a constant in [2]. Assuming that both the contracting parties have exponential utilities, and applying the stochastic maximum principle, we solve the corresponding HJB equation and find the exact optimal contracts under three different information structures. In the neoclassical environment, three concrete examples are given to illustrate the time-varying volatility and different functions of the time-varying volatility can be applied to fit different situations. Subsequently, we show that moral hazard reduces the agent's effort but has little effect on the agent's consumption. In the hidden savings case, both effort and consumption fall further than those do in moral hazard. For overly large volatility of shocks, the contract would be meaningless because the cost of information friction would be intolerable for the principal.

The organization of this chapter is as follows. Section 3.2 provides details of the basic model setting including the definition of incentive compatibility and the change of measure process. In Sections 3.3, 3.4, and 3.5, we manage to show the explicit optimal solutions in three different scenarios with varying degrees of information frictions. A concluding remark is given in Section 3.6.

3.2 The setting and benchmark

We consider a continuous time principal-agent model which is similar to the models in [2, 12, 96]. In the model, the principal has full commitment. To be more specific, the agent enters into a working contract provided by the principal and manages a production process as required. In the setting, the effort of the agent accumulates to the output (assets) and simultaneously, both parties draw consumption from it. In addition, the volatility of the diffusion process is assumed to be a time-varying variable instead of a constant. For simplification, we assume exponential utilities for both parties as in the literature [30].

3.2.1 The model setting

In continuous time, we define a standard Brownian motion Z on a probability space $(\Omega, \mathcal{F}, P)$ where the Brownian motion Z_t generates the information filtration $\mathcal{F}_t$, $0 \leq t \leq T$. As $0 \leq T < \infty$, the contract output Y_t accumulated up to T evolves according to

$$dY_t = Ae_t dt + B(t)dZ_t,$$

where A is the productivity of effort, $e_t \in \mathbb{A} \subset \mathbb{R}$ stands for the agent's choice of effort, and the positive bounded function $B(t)$ represents the time-varying volatility of shocks resulted from the Brownian increments. Moreover, we define

y_t as the principal's assets growing at a risk-free rate r, which satisfies

$$dy_t = (ry_t + Ae_t - p_t - d_t)dt + B(t)dZ_t, \tag{3.1}$$

in which $p_t \in \mathcal{P} \subset \mathbb{R}$ denotes the agent's salary (payment) and d_t represents the principal's own dividend (or consumption). We denote the agent's wealth by w_t, which also grows at rate r and supports the agent's consumption c_t according to:

$$dw_t = (rw_t + p_t - c_t)dt. \tag{3.2}$$

After the above settings, we assume risk aversion on both parties and investigate the principal-agent model in three different scenarios. To begin with, we deal with the optimal contract problem under neoclassical environment, in which no information friction exists. Because the principal obtains full information about the agent and the underlying output process, the only thing that the principal should focus on is to ensure that the agent participates in the contract. Thus, we set this scenario as our benchmark. Secondly, compared to our benchmark, the moral hazard setting shades the agent's effort so that the principal is unable to tell whether low output is caused by low effort or adverse shocks. Therefore, the principal has to offer the agent an ideal incentive payment p_t to achieve the desired $\hat{e}_t$. As described in [97], we also assume that the principal does all the savings. In both situations, the agent would not save or be better off because the total wealth $y_t + w_t$ determines the allocation. Under these circumstances, we set $w_t \equiv 0$ and thus $c_t = p_t$. Thirdly, we come to the scenario with hidden savings, in which the agent could save and the principal can no longer observe w_t or c_t for $t > 0$. With only the agent's initial wealth w_0 observable, the principal must set target conditions, in which $e_t = \hat{e}_t$, $\hat{w}_t \equiv 0$, and $\hat{c}_t = p_t$, given that risk is additive to this contract. Therefore, the payment p_t should offer reasonable incentives to prevent the agent from deviating effort or saving secretly. Consequently, the formula (3.1) becomes

$$dy_t = (ry_t + Ae_t - c_t - d_t)dt + B(t)dZ_t. \tag{3.3}$$

We assume exponential utilities for both parties. In particular, we choose the agent's utility $u(c, e) = -\exp(-\mu(c - \frac{e^2}{2}))$ and the terminal utility $\upsilon(p_T, w_T)$. Thus we have the expected utility over the whole process as

$$U(\bar{e}, \bar{c}) = E_0\left[\int_0^T e^{-\rho t}u(c_t, e_t)dt + e^{-\rho T}\upsilon(p_T, w_T)\right], \tag{3.4}$$

where ρ denotes the discount rate for both the agent and the principal. On the other hand, we choose the principal's flow utility $f(d) = -\exp(-\mu d)$ and the terminal utility $L(y_T, p_T)$. Thus the expected utility evolves by

$$F(\bar{d}) = E_0\left[\int_0^T e^{-\rho t}f(d_t)dt + e^{-\rho T}L(y_T, p_T)\right], \tag{3.5}$$

where a bar over a variable means that it happens on the entire time path up to T.

To find the explicit solution and handle our problem from a finite horizon to an infinite horizon, we make the same assumptions on the terminal utilities $\upsilon(p_T, w_T)$ and $L(y_T, p_T)$ as those in [2]. Namely, at terminal time T, the agent and the principal have the utilities as follows:

$$\upsilon(p_T, w_T) = V_T(p_T + w_T), \quad L(y_T, p_T) = V_T(y_T - p_T) \tag{3.6}$$

with

$$V_T(a) = -\frac{1}{r}\exp(\frac{r-\rho}{r} - \mu ra), \tag{3.7}$$

which is easily obtained from the consumption-saving problem at terminal date T.

3.2.2 Incentive compatibility

In order to define an incentive compatible contract, we let C be the space of continuous functions mapping $[0, T]$ into $\mathbb{R}$. In the scenarios of moral hazard and hidden savings, the principal's observation path is defined by $\bar{y} = \{y_t : t \in [0, T]\}$, which indicates an entire time path of output on $[0, T]$ and is an element of C. Filtration $\mathcal{Y}_t$ is the σ-algebra completion generated by y_t. The contract provided by the principal includes the recommended elements $(\hat{e}_t, \hat{c}_t)$ and a corresponding payment $p_t \in \mathcal{P}$, which depends on the relevant history of output. The admissible contracts belong to the set of $\mathcal{Y}_t$-predictable functions $(p, \hat{e}, \hat{c}) : [0, T] \times C \to \mathcal{P} \times \mathbb{A}$. Compared with the hidden savings case where the agent chooses both effort and consumption, the agent has no choice in the neoclassical environment and only controls effort in the moral hazard case. Then the agent's admissible sets are those $\mathcal{F}_t$-predictable functions $(\hat{e}_t, \hat{c}_t) : [0, T] \times C \to \mathbb{A}$. We make an assumption that $\mathbb{A}$ is the countable union of compact sets, which let us employ the maximum principle results from existing literature. At time 0, if the agent participates in the underlying contract and implements the desired actions: $(\hat{e}, \hat{c}) = (\bar{e}, \bar{c})$, we say that the contract $(p, \hat{e}, \hat{c})$ is incentive compatible.

3.2.3 Change of measure

Under the latter two scenarios with hidden states, we cannot simply solve the agent's problem to find incentive compatible conditions directly as the payment $p_t = p(t, \bar{y})$ depends on the entire past history of y_t. In order to solve this problem, we use the density of output, instead of the output, to be the key state variable [21]. Similar useful statements can be found in [12, 13, 98, 99].

For change of measure, the intuition is that different effort level has a different corresponding distribution of output. Thus, a specific probability measure over y_t corresponds to a specific effort choice. Therefore, the relative density Γ is chosen to be the key state variable instead of y_t itself.

To begin with, let Ω be the sample space and denote a Wiener process on C by Z_t^0, which generates the filtration $\mathcal{F}_t^0$. We define the basic probability space $(\Omega, \mathcal{F}_t, P)$, in which P is a Wiener measure and $\mathcal{F}_t$ is the completion of $\mathcal{F}_t^0$ with the null sets $\mathcal{F}_T^0$.

According to the payment $p(t, \bar{y})$ and the consumption of the principal $d(t, \bar{y})$, we denote the drift of output by

$$g(t, \bar{y}, e_t) = ry_t - Ae_t - p(t, \bar{y}) - d(t, \bar{y}).$$

According to the assumptions in [100], the continuity, predictability, and linear growth conditions are satisfied because of the linearity of the drift. Moreover, we denote the family of $\mathcal{F}_t$-predictable processes for $\bar{e} \in \mathbb{A}$ by

$$\Gamma_t(\bar{e}) = \exp\left(\int_0^t \frac{g(\tau, \bar{y}, e_\tau)}{B(\tau)} dZ_\tau^0 - \frac{1}{2} \int_0^t \left| \frac{g(\tau, \bar{y}, e_\tau)}{B(\tau)} \right|^2 d\tau \right),$$

where Γ_t is an $\mathcal{F}_t$-martingale, $E[\Gamma_T(\bar{e})] = 1$ and $B(t) > 0$. As a result, we are able to define a new measure $P_{\bar{e}}$ by the Girsanov Theorem as

$$\frac{dP_{\bar{e}}}{dP} = \Gamma_T(\bar{e}),$$

and we denote the new Brownian process $Z_t^{\bar{e}}$ under $P_{\bar{e}}$ by

$$Z_t^{\bar{e}} = Z_t^0 - \int_0^t \frac{g(\tau, \bar{y}, e_\tau)}{B(\tau)} d\tau.$$

Therefore, the output process evolves by

$$dy_t = g(t, \bar{y}, e_t)dt + B(t)dZ_t^{\bar{e}}.$$

Thus, suppressing $\bar{e}$, we denote the relative density by

$$d\Gamma_t = \frac{\Gamma_t}{B(t)}[ry_t + Ae_t - p(t, \bar{y}) - d_t]dZ_t^0, \tag{3.8}$$

where the initial value $\Gamma_0 = 1$.

When there is hidden information in the settings (the latter two scenarios), the co-variation between the observable and unobservable state variables plays an important role in our model. Therefore, we employ the density-weighted wealth $x_t = \Gamma_t w_t$ to be the relevant unobservable variable. Combining (3.2) and (3.8), we have the evolution

$$dx_t = \Gamma_t(\frac{rx_t}{\Gamma_t} + p(t, \bar{y}) - c_t)dt + \frac{x_t}{B(t)}(ry_t + Ae_t - p(t, \bar{y}) - d_t)dZ_t^0, \tag{3.9}$$

with initial condition $x_0 = w_0$.

Changing variables from (y_t, w_t) to (Γ_t, x_t), we obtain two stochastic differential equations (SDE) with random coefficients. After the evolution, the transformed coefficients now depend on $\bar{y}$ rather than the whole past history of y_t. That is, we analyze the agent's problem through a fixed outcome path $\bar{y}$ which is regarded as the likelihood of observing outcome affected by the agent's effort policy. As the density Γ_t substitutes for the entire past history of output, the problem becomes tractable.

3.3 The neoclassical problem

In this section, we investigate the neoclassical problem where the principal has all relevant information and observes the agent's effort, consumption, and wealth. Without information asymmetry, the principal can specify the values of these variables directly. In this case, the only target for the principal is to ensure the agent's participation at date zero. Moreover, we assume the agent's outside reservation utility to be U_0. Thus, the participation constraint is $U(\bar{e}, \bar{c}) \geq U_0$, which keeps the agent in the contract.

Before handling the optimal contract problems, as mentioned in Chapters 1 and 2, we introduce the agent's promised utility (also known as continuation value), which is an essential condition for the settings with information frictions (see [17–19]). As in [2], we let the expected discounted utility q_t be the agent's promised utility for contract continuation from time t, and it evolves by

$$q_t = E\left[\int_t^T e^{-\rho(\tau-t)} u(e_\tau, c_\tau) d\tau + e^{-\rho(T-t)} \upsilon(p_T, 0) \middle| \mathcal{F}_t\right]. \tag{3.10}$$

Using the martingale representation theorem, we obtain

$$dq_t = [\rho q_t - u(e_t, c_t)]dt + \gamma_t B(t) dZ_t, \quad q_T = \upsilon(p_T, 0), \tag{3.11}$$

where γ_t is the sensitivity of shocks and plays an important role in designing incentive compatible contract along with the time-varying volatility when information friction exists. Under our benchmark (neoclassical) setting, γ_t can be freely chosen as long as it is a solution of (3.11) and at the same time, the participation constraint $q_0 \geq U_0$ is satisfied. For the sake of maximizing the principal's expected utility (3.5) subject to (3.1), (3.11), and the participation constraint, we need to find the optimal terminal payment p_T and $(e_t, c_t, d_t, \gamma_t)$ for all t. Thus we define $G(t,y,q)$ as the value function of the principal to capture the maximal expected discount value at time t when $y_t = y$ and $q_t = q$. Employing (3.3) and (3.11), we get that the value function $G(t,y,q)$ satisfies the following HJB equation for $t \in (0,T)$

$$\begin{aligned}\rho G(t,y,q) - G_t(t,y,q) =& \max_{e,c,d,\gamma}\Big\{ -\exp(-\mu d) + G_y(t,y,q)(ry + Ae - c - d) \\ & + G_q(t,y,q)[\rho q + \exp(-\mu(c - \frac{e^2}{2}))] \\ & + \frac{1}{2}G_{yy}(t,y,q)B^2(t) + G_{yq}(t,y,q)\gamma B^2(t) \\ & + \frac{1}{2}G_{qq}(t,y,q)\gamma^2 B^2(t)\Big\},\end{aligned} \tag{3.12}$$

where the terminal conditions are $q_T = V_T(p_T)$ and $G(T, y_T, q_T) = V_T(y_T - p_T)$. Consequently, the first order conditions for maximizing the value function with respect to (e,c,d,γ) are

$$\begin{cases} \mu e \exp\left(-\mu(c - \frac{e^2}{2})\right) + A\frac{G_y}{G_q} = 0, \\ \mu \exp\left(-\mu(c - \frac{e^2}{2})\right) + \frac{G_y}{G_q} = 0, \\ \mu \exp\left(-\mu d\right) - G_y = 0, \\ \gamma + \frac{G_{yq}}{G_{qq}} = 0. \end{cases} \tag{3.13}$$

From (3.13), we find that $e = A$, indicating that the principal's optimal effort requirement is a constant A, which is equal to the agent's productivity, for the contract under neoclassical environment.

At terminal date T, from (3.6) and (3.11), we denote the final payment to the agent and terminal value function by

$$\begin{aligned} p_T &= -\frac{\log\left(-q_T/\xi\right)}{\mu r}, \\ G(T, y_T, q_T) = V_T(y_T - p_T) &= \frac{\xi^2}{q_T}\exp(-\mu r y_T), \end{aligned}$$

where $\xi = \frac{\exp((\rho-r)/r)}{r}$.

We conjecture that the value function $G(t,y,q)$ takes the form

$$G(t,y,q) = \frac{\exp\left(K(t)\right)}{q}\exp(-\mu r y), \tag{3.14}$$

where the exact expression of function $K(t)$ will be determined later. Using (3.14), we have

$$\begin{cases} G_t(t,y,q) = K'(t)G(t,y,q), \\ G_y(t,y,q) = -\mu r G(t,y,q), \\ G_q(t,y,q) = -\frac{1}{q}G(t,y,q), \\ G_{yy}(t,y,q) = \mu^2 r^2 G(t,y,q), \\ G_{qq}(t,y,q) = \frac{2}{q^2}G(t,y,q), \\ G_{yq}(t,y,q) = \frac{\mu r}{q}G(t,y,q). \end{cases}$$

Substituting the above identities into the first order conditions (3.13), we obtain the optimal policies for (e,c,d,γ) as follows

$$\begin{cases} e^{NS} = A, \\ c^{NS} = \frac{A^2}{2} - \frac{\log r}{\mu} - \frac{\log(-q)}{\mu}, \\ d^{NS} = -\frac{\log r}{\mu} - \frac{K(t)}{\mu} + \frac{\log(-q)}{\mu} + ry, \\ \gamma^{NS} = -\frac{\mu r q}{2}. \end{cases} \tag{3.15}$$

To find $K(t)$, we substitute the value function (3.14) and (3.15) into the HJB equation (3.12) and derive the following equation

$$\begin{aligned}\rho\frac{e^{K(t)}}{q}e^{-\mu ry}-\frac{e^{K(t)}}{q}e^{-\mu ry}K'(t) = &-e^{-\mu[-\frac{\log r}{\mu}-\frac{K(t)}{\mu}+\frac{\log(-q)}{\mu}+ry]}\\ &+\frac{-\mu re^{(K(t)-\mu ry)}}{q}[\frac{A^2}{2}+\frac{2\log r+K(t)}{\mu}]\\ &-\frac{e^{(K(t)-\mu ry)}}{q^2}(\rho q+\frac{G_y}{\mu G_q})+\frac{e^{(K(t)-\mu ry)}}{q}\frac{\mu^2r^2B^2(t)}{2}\\ &+\gamma\mu rB^2(t)\frac{e^{(K(t)-\mu ry)}}{q^2}+\gamma^2B^2(t)\frac{e^{(K(t)-\mu ry)}}{q^3}.\end{aligned}$$

Then we substitute $\gamma=-\frac{\mu rq}{2}$ and $\frac{G_y}{G_q}=\mu rq$ into the above equation and derive the following ODE

$$K'(t)=rK(t)+2(\rho-r)+\frac{\mu rA}{2}+2r\log r-\frac{1}{4}\mu^2r^2B^2(t). \tag{3.16}$$

With the terminal value $e^{K(T)}=\xi^2$, we obtain $K(t)$ in the form

$$K(t)=e^{-r(t-T)}(\log\xi^2+\frac{\alpha}{r})-\frac{\alpha}{r}+\frac{\mu^2r^2}{4}e^{rt}\int_t^T B^2(\tau)e^{-r\tau}d\tau, \tag{3.17}$$

where $\alpha=2(\rho-r)+\frac{\mu rA^2}{2}+2r\log r$. Thus, the optimal contract is solved explicitly with the exact expression of $K(t)$. As r, μ, α, ξ, and T are all known, for any bounded positive function $B(t)$, (3.17) makes sense.

After finding the exact form of the function $K(t)$, the optimal policies in (3.15) are explicitly obtained. The positive function $B(t)$ should be an bounded function that describes the volatility of shocks, helping us to find some insights of the principal-agent problems in arbitrary time. For $B^2(t)=\sigma^2$, our results reduce to those presented in [2], in which σ is a constant volatility of shocks.

According to (3.15), we have $d^{NS}=-\frac{\log r}{\mu}-\frac{K(t)}{\mu}+\frac{\log(-q)}{\mu}+ry$. If $B(t)>0$ is sufficiently large, $K(t)$ becomes very large. For example, assuming $B(t)=$

$\sqrt{M}e^{\frac{rt}{2}}$, in which $M > 0$ is sufficiently large, we derive

$$K(t) = \frac{\mu^2 r^2}{4} M e^{rt}(T-t) + e^{-r(T-t)}(\log \xi^2 + \frac{\alpha}{r}) - \frac{\alpha}{r},$$

from which we get

$$\begin{aligned} d^{NS} &= -\frac{\log r}{\mu} + \frac{\log(-q)}{\mu} + ry \\ &\quad - \frac{1}{\mu}\Big(\frac{\mu^2 r^2}{4} M e^{rt}(T-t) + e^{-r(T-t)}(\log \xi^2 + \frac{\alpha}{r}) - \frac{\alpha}{r}\Big), \end{aligned}$$

which implies that d^{NS} will be negative if t is fixed and $M > 0$ is chosen to be sufficiently large. In this situation, the principal is not able to accumulate dividend, make any consumption, or even has to contribute more to the project. However, if $B(t)$ remains very small, d^{NS} would be positive. We can infer that stable market and lower volatility fluctuations are more preferable for risk-averse principal (investor). From (3.15), we note that c^{NS} does not depend on $B(t)$, implying that the time-varying volatility term $B(t)$ does not affect the agent's consumption because the agent (CEO etc.) is getting some compensation specified beforehand (without considering bonus income).

Now, we discuss the optimal contracts from a finite to an infinite horizon. The infinite horizon problem is considered by choosing specific function $B(t)$. We study the limit of the solutions by letting $T \to \infty$ and present three examples with different functions of $B(t)$.

Example 1. Let $B(t) = \sqrt{\sigma_1^2 \sin^2 t + \sigma_2^2}$ (a periodic function) where $\sigma_1 \geq 0$ and $\sigma_2 > 0$ are two constants. Using (3.17), by computation, we obtain

$$\begin{aligned} K(t) = e^{-r(T-t)}\Big(\log \xi^2 + \frac{\alpha}{r} - \frac{\mu^2 r \sigma_2^2}{4}\Big) - \frac{\alpha}{r} + \frac{\mu^2 r \sigma_2^2}{4} + \frac{\mu^2 r^2 \sigma_1^2}{4}\Bigg(\frac{1}{2r} \\ - \frac{1}{2r}e^{-r(T-t)} + \frac{e^{-r(T-t)}(-r\cos 2T + 2\sin 2T) - (-r\cos 2t + 2\sin 2t)}{8 + 2r^2}\Bigg), \end{aligned}$$

which leads to

$$\lim_{T\to\infty} K(t) = -\frac{\alpha}{r} + \frac{\mu^2 r\sigma_2^2}{4} + \frac{\mu^2 r^2\sigma_1^2}{4}\Big(\frac{1}{2r} + \frac{r\cos 2t - 2\sin 2t}{8+2r^2}\Big).$$

In the infinite horizon, the state variable equations (3.3) and (3.11) become

$$\begin{aligned} dy_t &= \left[\frac{2(r-\rho)}{\mu r} + \frac{\mu r\sigma_2^2}{4} + \frac{\mu\sigma_1^2 r(8+2r^2+2r^2\cos 2t - 4r\sin 2t}{8(8+2r^2)}\right] dt \\ &\quad + \sqrt{\sigma_1^2\sin^2 t + \sigma_2^2} dZ_t, \\ dq_t &= (\rho - r)q_t dt - \frac{\mu r\sqrt{\sigma_1^2\sin^2 t + \sigma_2^2}}{2} q_t dZ_t, \end{aligned}$$

which show that the drift of the output depends on periodic functions of t (other periodic functions could also be applied instead of our example), indicating that the output process is affected by the business cycle with different volatility of shocks. If we let $\sigma_1 = 0$, then the above dy_t and dq_t reduce to the expressions in [2].

Example 2. If $B(t) = \sigma_3\sqrt{t}$ in which constant $\sigma_3 > 0$, it follows from (3.17) that

$$\begin{aligned} K(t) &= e^{-r(T-t)}\Big(\log\xi^2 + \frac{\alpha}{r}\Big) - \frac{\alpha}{r} \\ &\quad + \frac{\mu^2 r^2\sigma_3^2 e^{rt}}{4}\Big[\frac{1}{r}(te^{-rt} - Te^{-rT}) + \frac{1}{r^2}(e^{-rt} - e^{-rT})\Big], \end{aligned}$$

which results in

$$\lim_{T\to\infty} K(t) = -\frac{\alpha}{r} + \frac{\mu^2\sigma_3^2(rt+1)}{4}.$$

We then derive the state variable equations as follows

$$\begin{aligned} dy_t &= \left[\frac{2(r-\rho)}{\mu r} + \frac{\mu\sigma_3^2(rt+1)}{4}\right] dt + \sigma_3\sqrt{t} dZ_t, \\ dq_t &= (\rho - r)q_t dt - \frac{\mu r\sigma_3\sqrt{t}}{2} q_t dZ_t. \end{aligned}$$

Thus, the drift of the output depends on time t.

Example 3. If $B(t) = \sigma_4 t$ and $\sigma_4 > 0$, as T tends to infinity, we have

$$dy_t = \left[\frac{2(r-\rho)}{\mu r} + \frac{\mu\sigma_4^2(rt^2 + 2t + \frac{2}{r})}{4}\right] dt + \sigma_4 t dZ_t,$$
$$dq_t = (\rho - r)q_t dt - \frac{\mu r \sigma_4 t}{2} q_t dZ_t.$$

In examples 2 and 3, if $\sigma_3 = \sigma_4$, we see that larger volatility leads to larger drift of the output process since $(rt^2 + 2t + \frac{2}{r}) > rt + 1$. This is consistent with the theory that a high risk premium (partly a result of large volatility) could be a true profit as successful risky projects are inherently more profitable than low risk investments. However, there may be additional shortcomings when there are large adverse shocks.

3.4 The moral hazard problem

In this section, we study the moral hazard problem where the agent's effort cannot be observed while other elements are still observable. Therefore, we set $c_t = p_t$ and $w_t \equiv 0$ to embody the agent's observable consumption and wealth. Because of the unobservable effort, the principal cannot tell whether low output is caused by low effort or adverse shocks. Thus incentives must be provided to ensure that the agent does not deviate from the target effort.

To solve the optimal contract, we first derive the agent's optimal conditions which guarantee that the agent does not deviate from the desired effort and then manage to show the sufficiency of the first order conditions of the agent's optimal problem, which leads to implementable contract.

3.4.1 The agent's optimal problem

To design an incentive compatible contract, we have to find out the effort level chosen by the agent. As discussed in Section 3.2.3, we use the density process (3.8) to replace the variables which depend on the whole output history. Therefore the agent's expected utility function can be written as

$$\begin{aligned} U(\bar{e},\bar{p}) &= E^{P\bar{e}}\left[\int_0^T e^{-\rho t}u(p(t,\bar{y}),e_t)dt + \upsilon(p_T,0)\right] \\ &= E\left[\int_0^T \Gamma_t e^{-\rho t}u(p(t,\bar{y}),e_t)dt + \Gamma_T\upsilon(p_T,0)\right], \end{aligned}$$

where the agent's expected utility is derived by the density process under the measure $P_{\bar{e}}$. In this way, the problem of history dependence on y_t is solved and thus the agent's optimal problem becomes

$$\sup_{\bar{e}\in\mathbb{A}} U(\bar{e},\bar{p})$$

subject to (3.8) with a given $\bar{p}$.

According to the method in [31], we derive the agent's optimal conditions via the stochastic maximum principle. Specifically, differentiating the Hamiltonian, we can get the optimal conditions. Since the state variable Γ_t is stochastic and has no drift, the Hamiltonian does not contain the adjoint (or co-state) q. Using the state Γ_t and the adjoint state q_t, we define the Hamiltonian $\mathcal{H}$ for the agent's problem by

$$\mathcal{H} = \Gamma H(y,e,c,d,\gamma) = \Gamma[u(e,c) + \gamma(ry + Ae - c - d)]. \tag{3.18}$$

The optimal conditions can be found based on the (reduced) Hamiltonian H instead of $\mathcal{H}$. Using the maximum principle, we know that the differentials of the Hamiltonian determine the evolution of the adjoint variable. In particular, the drift of the adjoint variable involves $-\frac{\partial \mathcal{H}}{\partial \Gamma}$ and a discounting term. After a diffusion term is added, the adjoint q evolves by

$$\begin{aligned} dq_t &= [\rho q_t - u(e_t, c_t) - \gamma_t(ry + Ae_t - c_t - d_t)]dt + B(t)\gamma_t dZ_t^0, \\ &= [\rho q_t - u(e_t, c_t)]dt + B(t)\gamma_t dZ_t^{\bar{e}}, \\ q_T &= \upsilon(y_T, 0), \end{aligned} \tag{3.19}$$

in which the second line takes the same form as the promised utility in (3.11). In Section 3.3, the adjoint is introduced in the benchmark scenario just for our convenience. However, it occurs to be an element of agent's optimal condition in this moral hazard setting. If $E\int_0^T X_t^2 dt < \infty$, we say that the process X_t belongs to space L^2. In the following, we present the Proposition 3.4.1 which gives the necessary conditions for optimality.

Proposition 3.4.1. *Let (e^*, Γ^*) be an optimal control-state pair. Then there exists an $\mathcal{F}_t$-adapted process (q_t, γ_t) in L^2 that satisfies* (3.19) *with $e = e^*$. For almost every $t \in [0, T]$, the optimal effort e^* satisfies almost surely*

$$H(y_t, e_t^*, c_t, d_t, \gamma_t) = \max_{e_t \in \mathbb{A}} H(y_t, e_t, c_t, d_t, \gamma_t). \tag{3.20}$$

Furthermore, if $\mathbb{A}$ is a convex set, then for all $e_t \in \mathbb{A}$, an optimal control e^* must satisfy almost surely

$$H_e(y_t, e_t^*, c_t, d_t, \gamma_t)(e_t - e_t^*) \leq 0. \tag{3.21}$$

The proof of Proposition 3.4.1 is similar to the Proposition 4.1 in [2]. Here we omit its proof.

As described in [2], the set of implementable contracts is only a subset of that characterized by the first order conditions, so we have to set up the incentive constraint to make the contracts implementable.

3.4.2 Implementable contracts

Based on (3.21), we write the incentive constraint with a target effort $\hat{e}$ as

$$\gamma_t A = -u_e(\hat{e}_t, c_t) = \mu \hat{e}_t \exp\left(-\mu(c_t - \frac{\hat{e}_t^2}{2})\right), \tag{3.22}$$

from which the target $\hat{\gamma}_t$ can be defined. We can see from here that if $\hat{\gamma}_t$ is positive, there is a positive correlation between the promised utility and positive shocks.

If the first order condition (3.22) is satisfied, we say that a contract is locally incentive compatible. When a contract is incentive compatible, it prevents the agent from deviating from the target effort $\hat{e}$, which is also the optimal choice, and thus

$$H(y_t, \hat{e}_t, \hat{c}_t, d_t, \hat{\gamma}_t) = \max_{e_t \in \mathbb{A}} H(y_t, e_t, \hat{c}_t, d_t, \hat{\gamma}_t).$$

Under the moral hazard setting, because of the concave Hamiltonian in e, (3.22) is both the sufficient and necessary condition for local optimality. Further, if a contract implies a solution of (3.19), we say that the promise-keeping condition under the contract is satisfied. However, since the terminal condition (3.19) may not be satisfied at all times, the promise-keeping condition may not be valid for all contracts. If the solution of contract satisfies $q_0 \geq U_0$, we say that the participation constraint of this contract holds. We next present Proposition 3.4.2 which completely characterizes the set of implementable contracts under these conditions.

Proposition 3.4.2. *Under moral hazard, the implementable contract holds if and only if the following conditions are satisfied: (1) the contract is locally incen-*

tive compatible, (2) the participation constraint is valid, (3) the promise-keeping condition is satisfied.

We omit the proof of Proposition 3.4.2 since it is very similar to that for the Proposition 4.2 in [2].

3.4.3 The optimal solution

To find the optimal contract, the principal has to choose a contract that is implementable as discussed in Proposition 3.4.2. We also define the value function $G(t, y, q)$ as in the neoclassical scenario (NS). However, compared to the γ in NS, it becomes a function of effort and consumption as $\gamma(e, c)$ instead of a free choice. Thus, for $t \in (0, T)$, we acquire the HJB equation in the form

$$\begin{aligned}\rho G(t,y,q) - G_t(t,y,q) = \max_{e,c,d}\big\{ &-\exp(-\mu d) + G_y(t,y,q)[ry + Ae - c - d] \\ &+ G_q(t,y,q)[\rho q + \exp(-\mu(c - \frac{e^2}{2}))] + \frac{1}{2}G_{yy}(t,y,q)B^2(t) \\ &+ G_{yq}(t,y,q)\gamma(e,c)B^2(t) + \frac{1}{2}G_{qq}(t,y,q)\gamma^2(e,c)B^2(t)\big\}\end{aligned} \tag{3.23}$$

with the following terminal conditions

$$G(T, y_T, q_T) = V_T(y_T - p_T), \quad q_t = V_T(p_T).$$

Furthermore, from (3.23), we derive the first order conditions for optimization with respect to (e, c, d) as follows

$$\begin{cases} G_y A + G_q \mu e \exp(-\mu(c - \frac{e^2}{2})) \\ +G_{yq}B^2(t)\frac{1+\mu e^2}{e}\gamma(e,c) + G_{qq}B^2(t)\frac{1+\mu e^2}{e}\gamma^2(e,c) = 0, \\ -G_y - G_q\mu exp(-\mu(c - \frac{e^2}{2})) - G_{yq}B^2(t)\mu\gamma(e,c) - G_{qq}B^2(t)\mu\gamma^2(e,c) = 0, \\ -G_y + \mu\exp(-\mu d) = 0. \end{cases}$$

Because we make the specific assumption of exponential linearity, we know that the value function and optimal policies under the moral hazard scenario are similar to those in our benchmark. Similarly, we again assume the value function to be

$$G(t, y, q) = \frac{\exp(K_1(t))}{q} \exp(-\mu r y), \tag{3.24}$$

where the function $K_1(t)$ will be determined later.

Substituting (3.24) into the above first order conditions, we obtain the following optimal policies for (e, c, d, γ)

$$\begin{cases} e^{MH} = e^\star, \\ c^{MH} = \frac{(e^\star)^2}{2} - \frac{\log k}{\mu} - \frac{\log(-q)}{\mu}, \\ d^{MH} = -\frac{\log r}{\mu} - \frac{K_1(t)}{\mu} + \frac{\log(-q)}{\mu} + ry, \\ \gamma^{MH} = -\frac{\mu k q e^\star}{A}, \end{cases} \tag{3.25}$$

where $e^\star$ and k will be explained later. Using the first order conditions for (e, c) and the fact $u(e^\star, c^{MH}) = kq$, we find that $(e^\star, k)$ must satisfies

$$\begin{aligned} r - k + \frac{B^2(t) r \mu^2 e^\star k}{A} - \frac{2B^2(t)\mu^2 (e^\star)^2 k^2}{A^2} &= 0, \\ -rA + e^\star k - \frac{B^2(t) r\mu(1+\mu(e^\star)^2)k}{A} - \frac{2B^2(t)\mu e^\star(1+\mu(e^\star)^2)k^2}{A^2} &= 0. \end{aligned} \tag{3.26}$$

From (3.26), we obtain the expression of $e^\star(k)$ as

$$e^\star(k) = \frac{A^3 r + B^2(t)\mu A r k}{A^2 r + 2B^2(t)\mu k^2}. \tag{3.27}$$

Based on (3.26) and (3.27), we recognize that $e^\star$ and k depend on $B(t)$. In other words, both e^* and k vary with time t. In the following discussion, we denote $k = k(t)$ and $e^\star = e^\star(t)$. Using (3.26) and (3.27), we can solve k, which has a cumbersome form and thus we do not present the details of k here.

To determine $K_1(t)$, we substitute the optimal policies in (3.25) into (3.23) and obtain the following ODE

$$K_1'(t) = rK_1(t) + 2\rho - r + r\log r - M_1(t), \tag{3.28}$$

where

$$M_1(t) = -r\Big(A\mu e^\star(t) - \frac{\mu(e^\star(t))^2}{2} + \log k(t)\Big) + k(t) + r\mu^2 B^2(t)\Big(r - \frac{ke^\star}{A} + \frac{\mu k^2(e^\star)^2}{A^2 r}\Big).$$

Using the terminal value $K(T) = \log \xi^2$, which is discussed in the neoclassical scenario, we obtain the solution of $K_1(t)$

$$K_1(t) = e^{r(t-T)}\Big(\log\xi^2 + \frac{2\rho - r + r\log r}{r}\Big) - \frac{2\rho - r + r\log r}{r} + e^{rt}\int_t^T M_1(\tau)e^{-r\tau}d\tau. \tag{3.29}$$

The derivation process of $K_1(t)$ is similar to that of $K(t)$ in NS. As we can see, when $B^2(t) = 0$, the optimal policies will be the same as those in the case of NS, in which the optimal solutions are derived based on $e^\star = A$ and $k = r$. The intuition is that there should be no information friction without external shocks to output. Moreover, when $B^2(t)$ is small, we find that $e^\star \approx A$ and $k \approx r$ and consequently $\gamma^{MH} \approx -\mu r q = 2\gamma^{NS}$, which means that the agent's utility is more sensitive to new information under moral hazard than that under neoclassical environment.

In addition, we regard $B^2(t)$ as a variable and use the Taylor expression for $e^\star$ and k in $B^2(t) = B^2$ around zero. Thus, from (3.26) and (3.27), we have

$$\begin{cases} e^\star = A - B^2(t)\frac{\mu r}{A} + o(B^4(t)), \\ k = r - B^2(t)\mu^2 r^2 + o(B^4(t)), \end{cases}$$

from which we derive

$$c^{MH} = c^{NS} + o(B^4(t)),$$

implying that the consumption in the moral hazard setting is approximately the same as that in NS. However, information friction reduces the effort input. In particular, effort decreases with a larger volatility of shocks, return r, risk aversion parameter μ (which is also shown in details in [101]), or a lower productivity A. As for k, since it decreases with increasing volatility of shocks, we regard it as a after-tax return on savings. To sum up, the moral hazard problem reduces effort produced by the agent. However, this information friction hardly affects consumption because of the interaction between the decreasing effort and the effective rate of return k.

Applying the optimal policies in (3.25), we derive the following two state variable equations

$$\begin{aligned} dy_t &= \left[Ae^\star - \frac{(e^\star)^2}{2} + \frac{\log k}{\mu} + \frac{\log r}{\mu} + \frac{K_1(t)}{\mu}\right]dt + B(t)dZ_t, \\ dq_t &= (\rho - k)q_t dt - \frac{B(t)\mu e^\star k}{A} q_t dZ_t. \end{aligned}$$

The last term of $K_1(t)$ in (3.29) is very complex. Namely, we do not know $e^{rt}\int_t^T e^{-r\tau}M_1(\tau)d\tau$ as T tends to infinity. Thus, we cannot analyze how the state variable of y_t changes when $T \to \infty$. However, the drift of the promised utility is $\rho - k$. The fact that $k < r$ (at least for sufficiently small $B(t)$), and $\gamma^{MH} > \gamma^{NS}$, information friction increases both the expected growth rate and the volatility of the promised utility q_t.

3.5 The hidden savings problem

In this section, we further relax the assumption and assume that the agent could save and borrow money in an unobservable account. The problem becomes more complicated because the agent may not choose to consume all of his payment. Now the principal has to offer incentives not only to keep the agent making desired effort but also to prevent the agent from saving or borrowing. Thus the principal should set targets $w \equiv 0$ and $\hat{c} = p$. Compared to the case in moral hazard, the first order conditions in the agent's optimal problem can no longer directly specify the implementability of the contracts. To get rid of this problem, we first derive the necessary optimality conditions and obtain a candidate optimal contract. Secondly, we verify that this optimal contract is indeed incentive compatible and thus implementable. Similar approach is applied in [36, 37], in which an endogenous state variable is introduced. However, in our settings, the incentive compatibility is verified analytically without this additional state variable.

3.5.1 The agent's optimal problem

As discussed above, the agent is able to fully control his effort and consumption, so the optimal problem becomes

$$\sup_{(\bar{e},\bar{c})\in\mathbb{A}} U(\bar{e},\bar{c}),$$

subject to (3.8) and (3.9) with a given $\bar{p}$. Denoting the additional adjoint associated with the additional state variable x_t by (m_t, δ_t), we again define the Hamiltonian $\mathcal{H}$ by the stochastic maximum principle. Specifically, the new Hamiltonian $\mathcal{H} = \Gamma H$ where H evolves by

$$H(y,w,e,c,d,p,\gamma,m,\delta) = u(e,c)+(\gamma+\delta w)(ry+Ae-p-d)+m(rw+p-c). \quad (3.30)$$

Here we still take the advantage of the density-weighted wealth $x = \Gamma w$. We then differentiate the Hamiltonian and apply the change of measure in order to derive the adjoint (m_t, δ_t) related to x_t as follows

$$
\begin{aligned}
dm_t &= (\rho - r)m_t dt + \delta_t B(t) dZ_t^{\bar{e}}, \\
m_T &= \upsilon_w(p_T, w_T).
\end{aligned}
\tag{3.31}
$$

Note that the adjoint (q_t, γ_t) still follow (3.19) with a different terminal condition $q_T = \upsilon(p_T, w_T)$. Finally, we get the necessary conditions for optimality via the stochastic maximum principle.

Proposition 3.5.1. *Let $(e^\star, c^\star, x^\star, \Gamma^\star)$ be the optimal state control. Then in L^2, there exist adjoints (q_t, γ_t) and (m_t, δ_t) that are $\mathcal{F}_t$-adapted and satisfy* (3.19) *and* (3.31) *with $(e, c) = (e^\star, c^\star)$. In addition, for almost every $t \in [0, T]$, the optimal control $(e^\star, c^\star)$ satisfies almost surely*

$$
H(y_t, w_t^\star, e_t^\star, c_t^\star, d_t, p_t, \gamma_t, m_t, \delta_t) = \max_{(e,c)\in\mathbb{A}} H(y_t, w_t^\star, e, c, d_t, p_t, \gamma_t, m_t, \delta_t). \tag{3.32}
$$

Moreover, if $\mathbb{A}$ is a convex set, then for all $(e, c) \in \mathbb{A}$, an optimal control $(e^\star, c^\star)$ must satisfy almost surely

$$
\begin{aligned}
H_e(y_t, w_t^\star, e_t^\star, c_t^\star, d_t, p_t, \gamma_t, m_t, \delta_t)(e - e_t^\star) \leq 0, \\
H_c(y_t, w_t^\star, e_t^\star, c_t^\star, d_t, p_t, \gamma_t, m_t, \delta_t)(c - c_t^\star) \leq 0.
\end{aligned}
\tag{3.33}
$$

Here we state that the proof of Proposition 3.5.1 is similar to that for the Proposition 5.1 in [2]. Thus we omit the proof here.

Accordingly, we obtain the first order conditions for interior optima as

$$
\begin{aligned}
(\gamma + \delta w)A &= \mu e \exp(-\mu(c - \frac{e^2}{2})), \\
m &= \mu \exp(-\mu(c - \frac{e^2}{2})).
\end{aligned}
$$

From the above conditions, we find that $m_t = u_c(e_t, c_t)$, which tells us that the agent's consumption could increase marginally along with small increments of wealth. Moreover, we obtain the solution to (3.31)

$$m_t = E\left[e^{(r-\rho)(T-t)}\upsilon_w(p_T, w_T)\middle|\mathcal{F}_t\right]. \tag{3.34}$$

For $\tau > t$, we have the marginal utility of consumption

$$u_c(e_t, c_t) = E\left[e^{(r-\rho)(\tau-t)}u_c(e_\tau, c_\tau)\middle|\mathcal{F}_t\right].$$

To conclude, the expected marginal utility of consumption is ascertained by the agent's optimality conditions and the volatility term δ_t is influenced by the contract.

3.5.2 Necessary conditions for implementable contract

Generally, an optimal contract under hidden savings should depend on both the agent's marginal utility m_t and the promised utility q_t. However, thanks to the exponential utilities in our model, this problem is largely simplified. To be more specific, at terminal date, m_T is proportional to q_T as

$$m_T = \upsilon(p_T, 0) = V_T'(p_T) = -\mu r V_T(p_T) = -\mu r q_T.$$

In fact, for all $t \in [0, T]$, this relationship holds and we show the details below.

As mentioned before, the desired policies are $(\hat{e}, \hat{c})$ and $\hat{w} \equiv 0$. With $m_t = u_c(e_t, c_t) = -\mu u(e_t, c_t)$ and (3.34), we have

$$u(\hat{e}_t, \hat{c}_t) = E\left[e^{(r-\rho)(T-t)}\upsilon(p_T, 0)\middle|\mathcal{F}_t\right],$$

from which we have the evolution of (3.10) as

$$\begin{aligned}
q_t &= E\left[\int_t^T e^{-\rho(\tau-t)}u(\hat{e}_t,\hat{c}_t)d\tau + e^{-\rho(T-t)}\upsilon(p_T,0)\middle|\mathcal{F}_t\right] \\
&= E\left[r\int_t^T e^{-\rho(\tau-t)}E[e^{(r-\rho)(T-\tau)}\upsilon(p_T)|\mathcal{F}_\tau]d\tau + e^{-\rho(T-t)}\upsilon(p_T,0)\middle|\mathcal{F}_t\right] \\
&= E\left[\left(r\int_t^T e^{-\rho(\tau-t)+(r-\rho)(T-\tau)}d\tau + e^{-\rho(T-t)}\right)\upsilon(p_T,0)\middle|\mathcal{F}_t\right] \\
&= E[e^{(r-\rho)(T-t)]}\upsilon(p_T,0)|\mathcal{F}_t] \\
&= -\frac{1}{\mu r}E[e^{(r-\rho)(T-t)}V_T'(p_T)|\mathcal{F}_t] \\
&= -\frac{m_t}{\mu r}.
\end{aligned}$$

As we can see here, $m_t = -\mu r q_t$, which indicates that the marginal utility m_t contains no more information than q_t does. Furthermore, combining the agent's optimality conditions with $w = 0$ and $m = -\mu r q$, we obtain the following

$$\begin{cases}
\gamma(e,q) = \frac{-\mu r e q}{A}, \\
c(e,q) = \frac{e^2}{2} - \frac{\log(-rq)}{\mu}, \\
u(c(e,q),e) = rq.
\end{cases} \tag{3.35}$$

Thus, if the underlying contract satisfies the condition (3.35), it is locally incentive compatible. However, these are not enough for the contract to be globally incentive compatible, because we have to exclude situations where the agent would save ($w_t \neq 0$) or choose different effort inputs. As stated in [102], the concavity of Hamiltonian with respect to (w,e,c) is a sufficient condition for the contract to be implementable, which is similar to the statement of the maximum principal sufficiency in [103]. Nevertheless, since it is difficult to verify the assumption of concavity, we alternatively find a candidate optimal contract according to the necessary conditions and then verify that this contract is indeed globally incentive compatible [2]. Specifically, we present the necessary conditions in Proposition

3.5.2 as follows

Proposition 3.5.2. *In the scenario with hidden savings, an underlying contract* $(p, \hat{e}, \hat{c}) \in \mathcal{P} \times \mathbb{A}$ *with target wealth* $\hat{w} \equiv 0$ *is implementable if the following conditions are satisfied: (1) the contract is locally incentive compatible,(2) the participation constraint is met,(3) the promise-keeping condition is satisfied.*

The proof of Proposition 3.5.2 is similar to that of Proposition 3.4.2 and the necessity of the conditions is proved as it is a result of Proposition 3.5.1.

3.5.3 The optimal solution

We define the value function $G(t, y, q)$ again to find the implementable optimal contract that satisfies the conditions in Proposition 3.5.2. With the consumption $c(e, q)$ and volatility $\gamma(e, q)$, we derive the HJB equation for the principal in the form

$$\begin{aligned}\rho G(t,y,q) - G_t(t,y,q) = \max_{e,d}\big\{ &-\exp(-\mu d) + G_y(t,y,q)[ry + Ae - c(e,q) - d] \\ &+ G_q(t,y,q)(\rho - r)q + \frac{1}{2}G_{yy}(t,y,q)B^2(t) \\ &+ G_{yq}(t,y,q)\gamma(e,q)B^2(t) + \frac{1}{2}G_{qq}(t,y,q)\gamma^2(e,q)B^2(t)\big\}\end{aligned} \tag{3.36}$$

and we have the terminal condition $G(T, y_T, q_T) = V_T(y_T - p_T)$, in which $q_T = V_T(p_T)$. Then the first order conditions for (e, d) are

$$\begin{cases} G_y(A-e) - G_{yq}B^2(t)\frac{\mu r q}{A} + G_{qq}B^2(t)\frac{\mu^2 r^2 q^2 e}{A^2} = 0, \\ \mu \exp(-\mu d) - G_y = 0. \end{cases}$$

We assume that the value function possesses the form

$$G(t,y,q) = \frac{\exp(K_2(t))}{q}\exp(-\mu r y), \tag{3.37}$$

where the function $K_2(t)$ will be determined later. Again, we obtain the optimal policies (e, c, d, γ) as follows

$$\begin{cases} e^{HS} = \check{e}, \\ c^{HS}(q) = c(\check{e}, q) = \frac{(\check{e})^2}{2} - \frac{\log r}{\mu} - \frac{\log(-q)}{\mu}, \\ d^{HS}(t, y, q) = -\frac{\log r}{\mu} - \frac{K_2(t)}{\mu} + \frac{\log(-q)}{\mu} + ry, \\ \gamma^{HS}(q) = \gamma(\check{e}, q) = -\frac{\mu r q \check{e}}{A}, \end{cases} \tag{3.38}$$

where c^{HS} and γ^{HS} are results of the agent's necessary conditions given $e = \check{e}$. Using (3.36), we find that $K_2(t)$ satisfies the following ODE

$$K_2'(t) = rK_2(t) + \alpha_2 - M_2(t),$$

where

$$\alpha_2 = 2r \log r - 2(r - \rho)$$

and

$$M_2(t) = -r\Big(A\mu\check{e} - \frac{\mu\check{e}^2}{2}\Big) + \mu^2 r^2 B^2(t)\Big(1 - \frac{\check{e}}{A} + \frac{r\mu\check{e}^2}{A^2}\Big).$$

Letting $\exp(K_2(T)) = \xi^2$, we get the expression of $K_2(t)$ as

$$K_2(t) = e^{r(t-T)}(\log \xi^2 + \frac{\alpha_2}{r}) - \frac{\alpha_2}{r} + e^{-rt}\int_t^T M_2(\tau)e^{-r\tau}d\tau. \tag{3.39}$$

Similar to (3.27), $\check{e}$ has the form

$$\check{e} = e^\star(r) = \frac{A^3 + B^2(t)\mu A r}{A^2 + 2B^2(t)\mu r}, \tag{3.40}$$

indicating that if $r = k$, the scenarios of moral hazard and hidden savings will have the same results. Moreover, the effort and consumption fall more than those do in the moral hazard case because of the principal's inability of offering intertemporal incentives. However, this holds only for small $B(t)$. During times of large and volatile $B(t)$, according to the explicit solutions, the principal's dividends would be theoretically negative or zero in reality in all the three cases, and the agent's effort and consumption would fall dramatically in the cases with information friction. As a result, the contract would be meaningless.

For $B(t) = 0$, the result is consistent with that in the benchmark ($\check{e} = A$). Using the above obtained results, we derive that the state variables have the following forms

$$\begin{aligned} dy_t &= \left[A\check{e} - \frac{\check{e}^2}{2} + \frac{2\log r}{\mu} + \frac{K_2(t)}{\mu} \right] dt + B(t) dZ_t, \\ dq_t &= (\rho - r) q_t dt - \frac{B(t)\mu r \check{e}}{A} q_t dZ_t. \end{aligned}$$

As we can see, the expected growth of q_t is larger than that under the moral hazard setting due to the difference between k and r. This is because the principal is unable to provide inter-temporal incentives when the agent is able to save. Here we do not consider the situation $T \to \infty$ since we cannot calculate $\lim_{T\to\infty} K_2(t)$. However, if $B(t)$ is a positive constant, then the infinite horizon case has the same results as those presented in [2].

3.5.4 Verification of implementability

We now verify the global incentive compatibility of the candidate contract. To do so, we solve the agent's problem explicitly. The agent gets paid by $p_t = c^{HS}(q_t)$

and from the agent's point of view, q_t evolves by

$$\begin{aligned}
dq_t &= (\rho - r)q_t dt - \frac{B(t)\mu r\check{e}}{A} q_t dZ_t^{\check{e}} \\
&= (\rho - r)q_t dt - \frac{B(t)\mu r\check{e}}{A} q_t \left(\frac{dy_t - (ry_t + A\check{e} - c^{HS}(q_t) - d^{HS}(t, y_t, q_t))dt}{B(t)} \right) \\
&= [\rho - r - \mu r\check{e}(e_t - \check{e})]q_t dt - \frac{B(t)\mu r\check{e}}{A} q_t dZ_t, \\
q_T &= \upsilon(p_T, 0).
\end{aligned}$$

Here $Z_t^{\check{e}}$ is a Brownian motion in terms of the optimal contract and e_t is the real effort. Note that the $p_T(q_T)$ determined by the terminal condition is the inverse function of V_T.

At time 0, the agent's initial wealth can be observed, so we make the assumption that $w_0 = 0$ without loss of generality. Thus we have the evolution of the agent's wealth by

$$dw_t = (rw_t + c^{HS}(q_t) - c_t)dt. \tag{3.41}$$

We denote the value function for the agent by $U(t, w, q)$, and the corresponding HJB equation is

$$\begin{aligned}
\rho U(t, w, q) - U_t(t, w, q) = \max_{e,c} \{ &- \exp(-\mu(c - \frac{e^2}{2})) \\
&+ U_w(t, w, q)[rw + c^{HS}(q) - c] \\
&+ U_q(t, w, q)q[\rho - r - \mu r\check{e}(e - \check{e})] \\
&+ \frac{1}{2} U_{qq}(t, w, q) q^2 \frac{B^2(t)\mu^2 r^2 (\check{e})^2}{A^2} \}
\end{aligned} \tag{3.42}$$

along with $U(T, w_T, q_T) = V_T(p_T(q_T) + w_T) = q_T \exp(-\mu r w_T)$.

Apparently, for all t, we can easily verify that the value function for the agent has the form $U(t, w, q) = q \exp(-\mu r w)$. Substituting this value function back into

(3.42), we derive the first order conditions for optimality with respect to (e, c) by

$$\begin{cases} \mu \exp(-\mu(c - \frac{e^2}{2})) = U_w = -\mu r q \exp(-\mu r w), \\ \mu e \exp(-\mu(c - \frac{e^2}{2})) = -\mu r q \check{e} U_q = -\mu r q \check{e} \exp(-\mu r w), \end{cases}$$

from which we find that $e = \check{e}$. Therefore, the implementability of the target effort is verified. Also, from the above conditions, we denote the optimality of c by

$$c = \frac{(\check{e})^2}{2} - \frac{\log(-rq)}{\mu} + rw = c^{HS}(q) + rw.$$

By substituting the above c back into (3.41), we have $dw_t = 0$, which indicates that if the initial wealth starts at 0, then it will always be at 0. In other words, the agent will have the value $U(0, q, 0) = q$ and optimal consumption $c = c^{HS}(q)$. In this case, the implementability is verified and the contract is globally incentive compatible*.

3.6 Concluding remark

In this chapter, we have investigated a continuous time principal-agent model, in which the volatility of the output diffusion process relies on a positive function of time t. Applying exponential utilities and the stochastic maximum principle, we find the exact optimal contracts under three different information structures. In particular, under the neoclassical environment, three examples are given to illustrate the time-varying volatility and analyze the infinite limits of the contracts. We find that it is reasonable to assume that the volatility of shocks varies with time because volatility does change in different times of business cycles and influences the way that the principal and the agent behave. Specific functions

*For the design of how the optimal contracts could be implemented with the constant instruments of an equity, a flow of payment and a tax on savings, the reader is referred to [2].

of $B(t)$ could be applied to our model and implications similar to those in the Section 6 of [2] could be analyzed when concrete forms of $B(t)$ are chosen. After relaxing some assumptions, we find that moral hazard reduces effort and even further when the volatility of shocks gets larger, but the agent's consumption is hardly affected because of the interaction between the decreasing effort and the effective rate of return. In the hidden savings case, the effort distortions are further increased and the agent's consumption falls more significantly than it does in the moral hazard case. For very large volatility of shocks, the contract is meaningless because the cost of information friction is too high.

CHAPTER 4

The optimal solution to a principal-agent problem with unknown agent ability

4.1 General

In this chapter, we consider a principal-agent model featured with unknown a-gent ability. Under the assumption of exponential utilities, the necessary and sufficient conditions for optimality of the incentive contract are derived and the solutions of the optimal contracts are obtained. The uncertainty of the agent ability reduces the principal's ability of incentive provision. However, as time goes by, the information about ability accumulates, giving the agent less room for belief manipulation, and incentive provision becomes easier. As the contractual time tends to infinity (long-term), the agent ability is revealed completely, the ability uncertainty disappears, and the optimal contracts under known and unknown ability become identical.

This chapter is organized as follows. The basic settings are given in Section 4.2. Incentive compatible conditions are illustrated in Section 4.3. Section 4.4 solves the optimal contracts. A conclusion remark is given in Section 4.5.

4.2 The basic settings

The basic settings of the model are provided in this section. In particular, an agent is hired to manage a risky project whose output is affected by the agent's effort as well as the belief about the agent ability.

4.2.1 The model

$\{Z_t\}_{t\geq 0}$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}_t$ denotes the filtration generated by Z_t. In time interval $[0, t]$, we assume that the cumulative output Y_t satisfies the following stochastic integral equation

$$Y_t = \int_0^t \Big(f(e_s) + \eta\Big) ds + \int_0^t \sigma dZ_s, \tag{4.1}$$

where e_t is the agent's effort level. The function $f(e_t)$ satisfies $f(e) \geq 0$, its first order derivative $f'(e) > 0$ and the second order derivative $f''(e) \leq 0$. While in [57], the effort function $f(e) = e$. For $0 \leq t \leq T$, $e_t \in [0, M]$, in which $M > 0$ is the maximum effort that the agent can exert. The time-invariant agent ability is denoted by η. The constant σ is the volatility. In this model, the agent's effort choice affects the output without affecting its volatility. The differential form of the output process evolves by

$$dY_t = (f(e_t) + \eta)dt + \sigma dZ_t. \tag{4.2}$$

At time 0, the agent ability η is unknown. Both the principal and the agent have common priors about the ability which are normal with precision h_0 and mean m_0. The posteriors are denoted by $\hat{\eta}$ which rely on the output Y_t and the cumulative effort $\alpha_t \triangleq \int_0^t f(e_s)ds$. Thus, the mean of the normal posteriors $\hat{\eta}$

follows

$$\hat{\eta}(Y_t - \alpha_t, t) = E_t[\eta \mid Y_t, \alpha_t] \\ = \frac{h_0 m_0 + \sigma^{-2}(Y_t - \alpha_t)}{h_t} \tag{4.3}$$

and the precision evolves by

$$h_t = h_0 + \sigma^{-2} t. \tag{4.4}$$

At time 0, we have

$$\hat{\eta}(0, 0) = m_0.$$

If the priors over the mean of the normal distribution process are given, all the statistically information is reflected by the cumulative output Y_t, the cumulative effort α_t and time t. The fact that the beliefs depend on the history of the agent's effort through α_t alone is useful to characterize the incentive contract. Therefore, we only need to keep track of the information provided by the cumulative effort instead of by the whole effort path in solving the optimal contract.

In a moral hazard model, with the unobservable effort, the principal has to assume that the agent exerts the equilibrium effort e^*. The principal's beliefs satisfy (4.3) with $\alpha = \alpha^*$. However, the agent's belief about the actual level of effort e_t is private information. Therefore, the agent's belief satisfies (4.3) where the cumulative effort is α instead of α^*. We denote the filtration generated by (Y, e) by $\mathcal{F}_t^e \triangleq \sigma(Y_s, e_s; 0 \leq s \leq t)$ and let $\mathbb{F}^e = \{\mathcal{F}_t^e\}_{t \geq 0}$ be the P-augmentation filtration of $\mathcal{F}_t^e$. When the agent provides the effort sequence $\{e_s; 0 \leq s \leq t\}$, the principal believes that the output is Y_t with the cumulative external shocks Z_t. According to the filtering theorem provided by Fujisaki et al. [104], we have the following standard Brownian motion

$$dZ_t = \frac{1}{\sigma}\big[dY_t - \big(f(e_t) + \hat{\eta}(Y_t - \alpha_t, t)\big)dt\big] \tag{4.5}$$

on the probability space $(\Omega, \mathcal{F}_t^e, P)$. Moreover, $\hat{\eta}$, which is a P-martingale, evolves by

$$d\hat{\eta}(Y_t - \alpha_t, t) = \frac{\sigma^{-1}}{h_t} dZ_t \tag{4.6}$$

with decreasing variance. In our settings, we denote the principal's assets by y_t, which grows at the risk-free rate r. Furthermore, with the agent's payment $p_t \in \mathcal{P}$ and the principal's consumption (or dividend) d_t, the principal's assets y_t evolve by

$$dy_t = (ry_t + f(e_t) + \eta - p_t - d_t)dt + \sigma dZ_t. \tag{4.7}$$

As the assets y_t carry the same information as the output Y_t does, we regard the assets y_t as "output".

The agent's effort e belongs to the class of control processes $\mathcal{A} \triangleq \{e : [0,T] \times \Omega \to [0, M]\}$ which are $\mathbb{F}_t^e$-predictable. Although the principal does not observe the agent's actual effort, he observes the output of the project. Thus the principal's available information corresponds to the filtration $\mathcal{F}_t^y \triangleq \sigma\{y_s; 0 \le s \le t\}$ which is generated by output y. We define the $\mathcal{F}_t^y$-augmentation filtration as $\mathbb{F}^y = \{\mathcal{F}_t^y\}_{t\ge 0}$. Let C be the space of continuous functions mapping from $[0,T]$ into $\mathbb{R}$. The time path of the output $\bar{y} = \{y_t; t \in [0,T]\}$ is a random element in C, which defines the output path observed by the principal. We let $Z_t^0 = z_t$ be the family of coordinate functions with the filtration $\mathcal{F}_t^0 = \sigma\{Z_s^0, s \le t\}$. Let P^0 be the corresponding Wiener measure on $(\Omega, \mathcal{F}_t^0)$ and $\mathcal{F}_t$ be the completion of $\mathcal{F}_t^0$

with the null sets of $\mathcal{F}_T^0$. In the space $(\Omega, \mathcal{F}_t, P)$, we define Z_t^0 as the Brownian motion in (4.1). For any time t, both the agent's payment and the recommended effort $(\hat{e}, \hat{p}_t)$ depend on the whole history of output $\bar{y}_t$. In order to deal with moral hazard, we assume that the agent's consumption, the recommended consumption, and the agent's payment are identical, i.e., $\hat{c}_t = c_t = p_t$ (see Section 3.4). For a given contract, the agent chooses the level of effort. Consequently, the admissible set of agent's effort is a $\mathcal{F}_t$-measurable function $(\bar{e}, \bar{c}) : [0, T] \times C \to [0, M] \times \mathcal{P}$. After accepting the contract, the agent remains in the contract until the termination date T. As mentioned in Chapter 3, at time 0, if the agent participates in the contract and implements the desired action $(\hat{e}, \hat{c})$ during the contract period, i.e., $(e, c) = (\hat{e}, \hat{c})$, this contract is called an implementable contract.

Similar to the model in Chapter 3, both the principal and the agent are risk-averse and have exponential utilities, which are second-order continuous differential functions. Specifically, we assume that the agent's utility takes the following form

$$u(c, e) = -\exp\big(-\lambda(c - \mu e)\big), \tag{4.8}$$

where λ is the risk aversion coefficient, μ is a parameter related to endogenous technology and can be expressed as a function of volatility. To ensure that $e = M$ is the agent's optimal strategy, we restrict μ to the interval $[0, B_0)$. μe is the agent's cost associated with endogenous technology and effort. Again, taking advantage of the terminal condition of the agent's expected utility in [2], we have

$$\begin{aligned} v(p_T, w_T) &= V_T(p_T + w_T), \\ V_T(a) &= -\frac{1}{r}\exp\Big(\frac{r-\rho}{r} - \lambda r a\Big). \end{aligned}$$

In the moral hazard cases, we assume that the agent keeps the wealth w_t at zero and does not borrow or save from outside. That is, $w_t \equiv 0$ and $c_t = p_t$. Therefore, the agent's expected utility at terminal date can be written as $\upsilon(p_T) = V_T(p_T)$. For a given process $(\bar{e}, \bar{c})$, the agent's expected discounted utility at $t = 0$ evolves by

$$U(\bar{e}, \bar{c}) = E\Big[\int_0^T e^{-\rho t} u(c_t, e_t)dt + e^{-\rho T}\upsilon(p_T)\Big], \tag{4.9}$$

where ρ is the discount rate.

Similarly, the principal's utility evolves by

$$u(d) = -\exp(-\lambda d), \tag{4.10}$$

where d is the principal's consumption or dividend. The terminal condition of the principal's expected utility is $L(y_T, p_T) = V_T(y_T - p_T)$. For a given process $\bar{d}$, the principal's expected discounted utility at $t = 0$ evolves by

$$F(\bar{d}) = E\Big[\int_0^T e^{-\rho t} u(d_t)dt + e^{-\rho T} L(y_T, s_T)\Big]. \tag{4.11}$$

During the entire contract period, maximizing $F(\bar{d})$ is the principal's goal.

4.2.2 Change of measure

Similar to Section 3.2.3, since the whole history path of the output $\bar{y}$ is involved in the contract as a state variable, we cannot deal with the agent's problem directly. In order to overcome this problem, we replace the output process with the density of the output and use it as a key variable. Similarly, let Z_t^0 be a Wiener process on space C. The agent's different choice of effort changes the distribution of output. Hence, the agent's choice of effort is the choice of a probability measure over output. We regard the relative density Γ_t as a key state variable.

We denote the drift term of output by g, namely,

$$g(t, \bar{y}, e) = ry_t + f(e_t) + \hat{\eta}(Y_t - \alpha_t, t) - p_t - d_t, \tag{4.12}$$

in which $\bar{y}$ is the history path of output. The relative density $\Gamma_t(\bar{e})$, which depends on the effort $\bar{e}$, is a $\mathcal{F}_t$-measurable process and has the following form

$$\Gamma_t(\bar{e}) = \exp\Big(\int_0^t \sigma^{-1} g dZ_s^0 - \frac{1}{2}\int_0^t |\sigma^{-1} g|^2 ds\Big). \tag{4.13}$$

Since the settings of g guarantee the Novikov's condition, for all $\bar{e} \in \mathbb{A}$, Γ_t is a $\mathcal{F}_t$-measurable martingale with $E[\Gamma_T(\bar{e})] = 1$. Using Girsanov theorem, we have a new measure $P^{\bar{e}}$

$$\frac{dP^{\bar{e}}}{dP^0} = \Gamma_T(\bar{e}).$$

and consequently, the Brownian motion $Z_t^{\bar{e}}$ evolves by

$$Z_t^{\bar{e}} = Z_t^0 - \int_0^t \sigma^{-1} g(s, \bar{y}, e_s) ds. \tag{4.14}$$

In addition, we have the following stochastic differential equation (SDE)

$$\begin{aligned}
dy_t &= \sigma dZ_t^0 && (4.15)\\
&= \sigma \left[dZ_t^{\bar{e}} + \sigma^{-1} g(t, \bar{y}, e_t) dt\right] && (4.16)\\
&= g(t, \bar{y}, e_t) dt + \sigma dZ_t^{\bar{e}}. && (4.17)
\end{aligned}$$

For a given $\sigma > 0$, (4.15) indicates that the output is a P^0-martingale. When the agent exerts $\bar{e}^0$, the drift term of the output (4.15) equals zero for $t \in [0, T]$.

When the agent chooses $\bar{e}$, we have (4.17). Differentiating Γ_t, we obtain

$$d\Gamma_t = \Gamma_t \sigma^{-1}\big(ry_t + f(e_t) + \hat{\eta}(Y_t - \alpha_t, t) - p_t - d_t\big) dZ_t^0 \tag{4.18}$$

with the initial condition $\Gamma_0 = 1$. Different from the key state variable which depends directly on the whole history of output, the transformed state variable relies on a fixed path of the output $\bar{y}$.

As we can see, different choices of effort correspond to different Brownian motions. By Ito's lemma, the agent ability $\hat{\eta}$ under $P^{\bar{e}}$ has the following form

$$d\hat{\eta}(Y_t - \alpha_t, t) = \frac{\sigma^{-1}}{h_t} dZ_t^{\bar{e}}.$$

Under the relative density process Γ_t, the agent's problem can be written as

$$U(\bar{e}, \bar{c}) = E^0 \Big[\int_0^T \Gamma_t e^{-\rho t} u(c_t, e_t) dt + e^{-\rho T} \Gamma_T \upsilon(p_T) \Big], \tag{4.19}$$

where E^0 is the expectation under P^0.

4.3 Incentive compatible conditions

4.3.1 The agent's promised utility

For a given effort level, we first derive the necessary condition for the optimal contract. Subsequently, we impose restrictions on the necessary condition and obtain the sufficient condition of the optimal contract. At time 0, the only constraint of the contract is the participation constraint discussed in Section 3.3, which is $U(\bar{e}, \bar{c}) \geq U_0$.

Since the participation constraint only needs to be satisfied at time 0, it can be obtained by the standard Lagrangian method. As the agent's effort is unobservable, the contracts should depend on the agent's promised utility. As that in Section 3.3, the agent's promised utility q_t is the remaining expected discounted utility in the contract from time t onward, which takes the following form

$$q_t = E\Big[\int_t^T e^{-\rho(\tau-t)}u(c_\tau, e_\tau)d\tau + e^{-\rho(T-t)}\upsilon(p_T, 0)|\mathcal{F}_t\Big]. \tag{4.20}$$

Through calculating the expectation under $\mathcal{F}_t$, we obtain the level of the agent's promised utility which varies with the cumulative effort. In addition, with unobservable effort, the principal should keep tract of all the levels of the promised utility. In continuous time, the first-order approach simplifies the problem of solving the contracts. We consider the agent's promised utility on the equilibrium path, and then show that the solution to the agent's problem is globally optimal so as to establish the incentive compatible conditions. By the martingale representation theorem, the backward stochastic differential form of the promised utility reads

$$\begin{aligned} dq_t &= [\rho q_t - u(c_t, e_t)]dt + \gamma_t \sigma dZ_t, \\ q_T &= \upsilon(p_T, 0), \end{aligned} \tag{4.21}$$

where γ_t is the sensitivity of the agent's promised utility to external shocks. In moral hazard environment, γ_t is closely linked to incentive provision and is generally referred as the incentive compatible parameter.

4.3.2 Necessary condition

To find the solutions of the optimal contract, we solve the agent's problem by maximizing the agent's expected utility. As the objective function (4.20) depends on the agent's payment, which is non-Markovian because of the history dependence, we cannot use the standard method to analyze the optimization problem.

For a given contract (c, e), the agent controls the distribution of the payment by changing the effort strategies. The agent also chooses the appropriate probability measure according to his actual payment p_t. Because the Radon-Nikodym derivative associated with any effort path is a Markovian process, we apply the martingale method to solve the agent's optimization problem. The idea of using the distribution function as a state variable to solve the principal-agent problem dates back to [23]. In our model, the agent ability is unknown and the principal gradually captures the information of it over time. The learning process complicates the optimal contract problem. Following the work of Cvitanić et al. [50] and Prat and Jovanovic [57], we give the necessary conditions for agent's problem. Before doing so, we define the following Hamiltonian

$$H(t, y, e, \alpha, \gamma) = u(c_t, e_t) + \big(ry_t + f(e_t) + \hat{\eta}(\alpha_t, Y_t) - p_t - d_t\big)\gamma_t. \tag{4.22}$$

The value function of the agent's problem is denoted by $U(t, e)$, which means that the maximum of agent's expected utility is $U(t, e)$.

Proposition 4.3.1. *Maximizing the Hamiltonian defined by (4.22) is sufficient for maximizing the agent's value function $U(t, e)$. For any $e_t \in [0, M]$, the necessary condition for the agent's optimal effort e_t^* is that the incentive compatible parameter γ_t satisfies*

$$\Big[\Big(\gamma_t + \frac{\sigma^{-2}}{h_t}\theta_t\Big) f'(e_t) + u_e(c_t, e_t)\Big](e_t - e_t^*) \leq 0, \tag{4.23}$$

where

$$\theta_t = h_t E\Big[-\int_t^T e^{-\rho(s-t)} \frac{\gamma_s}{h_s} ds | \mathcal{F}_t^e\Big]. \tag{4.24}$$

Proof. We denote the agent's optimal effort by $\hat{e}$. In the time interval $[t, T]$, integrating both sides of (4.21) yields

$$\begin{aligned} e^{-\rho t} q_T &= e^{-\rho t} \upsilon(p_T) \\ &= e^{-\rho t} U(t, \hat{e}) - \int_t^T e^{-\rho s} u(c_s, \hat{e}_s) ds + \int_t^T \hat{\zeta}_s \sigma dZ_s^{\hat{e}}, \end{aligned} \tag{4.25}$$

where $\hat{\zeta}_s = e^{-\rho s} \hat{\gamma}_s$. We denote any other effort level by $\bar{e}$. In the time interval $[t, T]$, integrating both sides of (4.21) gives rise to

$$\begin{aligned} e^{-\rho t} q_T &= e^{-\rho t} \upsilon(p_T) \\ &= e^{-\rho t} U(t, \bar{e}) - \int_t^T e^{-\rho s} u(c_s, \bar{e}_s) ds + \int_t^T \bar{\zeta}_s \sigma dZ_s^{\bar{e}}, \end{aligned} \tag{4.26}$$

where $\bar{\zeta}_s = e^{-\rho s} \bar{\gamma}_s$. According to the derivation process of the change of measure, from (4.14)-(4.17), we obtain

$$dy_t = \sigma dZ_t^0,$$

$$dZ_t^{\hat{e}} = dZ_t^0 - \frac{1}{\sigma} \left(r y_t + f(\hat{e}_t) + \hat{\eta}(Y_t - \hat{\alpha}_t, t) - p_t - d_t \right) dt, \tag{4.27}$$

$$dZ_t^{\bar{e}} = dZ_t^0 - \frac{1}{\sigma} \left(r y_t + f(\bar{e}_t) + \hat{\eta}(Y_t - \bar{\alpha}_t, t) - p_t - d_t \right) dt. \tag{4.28}$$

From (4.27) and (4.28), we have

$$dZ_t^{\hat{e}} = dZ_t^{\bar{e}} + \frac{1}{\sigma} \big[f(\bar{e}_t) - f(\hat{e}_t) + \big(\hat{\eta}(Y_t - \bar{\alpha}_t, t) - \hat{\eta}(Y_t - \hat{\alpha}_t, t) \big) \big] dt, \tag{4.29}$$

from which we derive

$$
\begin{aligned}
&U(t,\bar{e})-U(t,\hat{e})\\
=&e^{\rho t}E_t^{\bar{e}}\Big[\int_t^T e^{-\rho s}\big[u(c_s,\bar{e}_s)-u(c_s,\hat{e}_s)\big]ds+\int_t^T e^{-\rho s}(\hat{\gamma}_s-\bar{\gamma}_s)\sigma dZ_s^{\bar{e}}\\
&+\int_t^T e^{-\rho s}\hat{\gamma}_s\big[f(\bar{e}_s)-f(\hat{e}_s)+\big(\hat{\eta}(Y_s-\bar{\alpha}_s,s)-\hat{\eta}(Y_s-\hat{\alpha}_s,s)\big)\big]ds\Big] &(4.30)\\
=&e^{\rho t}E_t^{\bar{e}}\Big[\int_t^T e^{-\rho s}\big[H(\bar{e},\hat{\gamma})-H(\hat{e},\hat{\gamma})\big]ds+\int_t^T e^{-\rho s}\hat{\gamma}_s\sigma dZ_s^{\bar{e}}\Big] &(4.31)\\
\le&e^{\rho t}E_t^{\bar{e}}\Big[\int_t^T e^{-\rho s}\hat{\gamma}_s\sigma dZ_s^{\bar{e}}\Big]=0. &(4.32)
\end{aligned}
$$

According to (4.25), (4.26), (4.27) and $E_t^{\bar{e}}\Big[\int_t^T e^{-\rho s}\bar{\gamma}_s\sigma dZ_s^{\bar{e}}\Big]=0$, equation (4.30) holds. Using the definition of the Hamiltonian, equation (4.31) is established. Since $\hat{e}$ is the optimal effort, $H(t,y,\hat{e},\alpha,\gamma)$ is the maximum value of the Hamiltonian, we obtain inequality (4.32). As $\hat{\gamma}_t$ is square-integrable and $e\in[0,M]$, the last term is a martingale and its expectation is 0. Therefore, the sufficient condition for maximizing the agent's expected utility is obtained.

Next, we prove the necessary condition of agent's optimal effort. We define a control perturbation

$$\tilde{e}_t=e_t+\epsilon\Delta e_t,$$

where $\epsilon\in[0,\epsilon_0]$ and ϵ_0 is positive. Define

$$\nabla U_t(e)=\lim_{\epsilon\to 0}\frac{U(t,\tilde{e})-U(t,e)}{\epsilon}.$$

By small perturbation ϵ, we have

$$
\begin{aligned}
e^{-\rho t}\nabla U_t(e)=\lim_{\epsilon\to 0}\frac{1}{\epsilon}E_t^{\tilde{e}}\Big[&\int_t^T e^{-\rho s}\big[u(c_s,\tilde{e}_s)-u(c_s,e_s)\big]ds+\int_t^T \zeta_s\sigma dZ_s^{\tilde{e}}\\
&+\int_t^T \zeta_s\big[f(\tilde{e}_s)-f(e_s)+\big(\hat{\eta}(Y_s-\tilde{\alpha}_s,s)-\hat{\eta}(Y_s-\alpha_s,s)\big)\big]ds\Big].
\end{aligned}
\tag{4.33}
$$

On the basis of the proof of the sufficient condition, we have

$$E_t^{\tilde{e}}\Big[\int_t^T \zeta_s\big[f(\tilde{e}_s) - f(e_s) + \big(\hat{\eta}(Y_s - \tilde{\alpha}_s, s) - \hat{\eta}(Y_s - \alpha_s, s)\big)\big]ds + \int_t^T e^{-\rho s}\big[u(c_s, \tilde{e}_s) - u(c_s, e_s)\big]ds\Big] \leq 0.$$

As $\epsilon \to 0$, for any Δe_s, the following inequality holds

$$E_t^{\tilde{e}}\Big[\int_t^T \Big(e^{-\rho s}u_e \Delta e_s + \zeta_s\big(f'(e_s)\Delta e_s - \frac{\sigma^{-2}}{h_s}\int_t^s f'(e_\tau)\Delta e_\tau d\tau\big)\Big)ds\Big] \leq 0, \quad (4.34)$$

where u_e is the partial derivative of the agent's utility $u(c_t, e_t)$ with respect to e_t and $f'(e_s) = \frac{df(e_s)}{de_s}$. Changing the order of integral, we get

$$E_t^{\tilde{e}}\Big[\int_t^T \Big(e^{-\rho s}u_e + \zeta_s f'(e_s) - f'(e_s)\int_s^T \frac{\sigma^{-2}}{h_\tau}\zeta_\tau d\tau\Big)\Delta e_s ds\Big] \leq 0. \quad (4.35)$$

At time t, we denote the agent's optimal effort by e_t^* and any other effort level by e_t. As Δe_s is arbitrary, we obtain

$$\Big[\Big(E_t^{\tilde{e}}\Big[-\int_t^T \frac{\sigma^{-2}}{h_s}\zeta_s ds\Big] + \zeta_t\Big)f'(e_t) + e^{-\rho t}u_e\Big](e_t - e_t^*) \leq 0. \quad (4.36)$$

Multiplying both sides of (4.36) by $e^{\rho t}$, noticing $\zeta_t = e^{-\rho t}\gamma_t$ and letting

$$\theta_t = h_t E_t^{\tilde{e}}\Big[-\int_t^T e^{-\rho(s-t)}\frac{\gamma_s}{h_s}ds\Big],$$

we get

$$\Big[\Big(\gamma_t + \frac{\sigma^{-2}}{h_t}\theta_t\Big)f'(e_t) + u_e(c_t, e_t)\Big](e_t - e_t^*) \leq 0. \quad (4.37)$$

Thus, the necessary condition for agent's optimal effort is proved. □

An increase in the agent's current effort results in larger promised utility and more cumulative effort. The amount of the first effect is proportional to the sensitivity coefficient γ. The second effect is described by the expectation term in (4.24). When η is known, this expectation term disappears, i.e., $\theta_t = 0$. Consequently, we obtain

$$\big(\gamma_t f'(e_t) + u_e(c_t, e_t)\big)(e_t - e_t^*) \leq 0. \tag{4.38}$$

The Hamiltonian corresponding to the first-order condition (4.38) is

$$H(t, y, e, \gamma) = u(c_t, e_t) + (ry_t + f(e_t) - p_t - d_t)\,\gamma_t. \tag{4.39}$$

If we choose $f(e) = Ae$ (constant $A > 0$) and $\eta = 0$, our model is turned into the moral hazard model in [2]. The ability uncertainty leads to an additional expectation term in (4.23). Using (4.3) and differentiating $\hat{\eta}$ with respect to α_τ, we have

$$\frac{\partial \hat{\eta}(Y_\tau - \alpha_\tau, \tau)}{\partial \alpha_\tau} = -\frac{\sigma^{-2}}{h_\tau} < 0.$$

Thus an increase in the current cumulative effort reduces the value of the posteriors $\hat{\eta}$. Namely, for all future time $\tau > t$, an upward deviation from the recommended effort generates a negative output shock by σ^{-2}/h_τ. The expectation that σ^{-2}/h_τ multiplies the sensitivity γ_τ measures the effects induced by the ability uncertainty. Through summing and discounting all these marginal effects, the expected marginal returns of manipulating beliefs are acquired.

The θ_t defined by (4.24) is a stochastic process and measures the private information value. As θ_t is negative and satisfies (4.23), for any recommended effort e_t^* and any given payment p_t, the unknown agent ability leads to a higher volatility γ_t. In other words, with ability uncertainty, it is more difficult for the principal to provide incentives. This conclusion does not rely on any specific form of the utility.

The necessary condition (4.23) involves two stochastic variables: γ and θ. This is a general result of the dynamic contract with private information. We use the promised utility to characterize the past history. According to the incentive constraint (4.23) and the assumption $f'(e_t) > 0$, we obtain

$$\gamma_t \geq -\frac{u_e(c_t, e_t)}{f'(e_t)} - \frac{\sigma^{-2}}{h_t}\theta_t. \tag{4.40}$$

As the agent is risk-averse, the principal would like to minimize γ. Hence, for any time $\tau \in [t, T]$ and $e_t > 0$, the necessary condition (4.40) holds with equality almost everywhere on the equilibrium path. Namely,

$$\gamma_t = -\frac{u_e(c_t, e_t)}{f'(e_t)} - \frac{\sigma^{-2}}{h_t}\theta_t. \tag{4.41}$$

Taking advantage of the assumption of exponential utility, we know that (4.41) indeed holds. Using the definition of θ_t and equation (4.41), we obtain the expression of θ_t. We show the details of the derivation process below.

Letting $\tilde{\theta}_t = (\sigma^{-2}/h_t)\theta_t$ and using the definition of θ_t in (4.24), we get

$$\tilde{\theta}_t = E\Big[-\int_t^T e^{-\rho(s-t)}\frac{\sigma^{-2}}{h_s}\gamma_s ds\Big]. \tag{4.42}$$

Differentiating both sides of (4.42) with respect to t, we obtain the following equation

$$\begin{aligned}\frac{d\tilde{\theta}_t}{dt} &= \rho E\Big[-\int_t^T e^{-\rho(s-t)}\frac{\sigma^{-2}}{h_s}\gamma_s ds\Big] + \frac{\sigma^{-2}}{h_t}\gamma_t \\ &= \rho\tilde{\theta}_t + \frac{\sigma^{-2}}{h_t}\gamma_t. \end{aligned} \tag{4.43}$$

Substituting (4.41) into (4.43) gives rise to

$$\frac{d\tilde{\theta}_t}{dt} = \Big(\rho - \frac{\sigma^{-2}}{h_t}\Big)\tilde{\theta}_t - \frac{\sigma^{-2}}{h_t}\frac{u_e(c_t, e_t)}{f'(e_s)}. \tag{4.44}$$

Integrating both sides of (4.44) and noticing $\tilde{\theta}_T = 0$ yield

$$\tilde{\theta}_t = E\Big[\int_t^T e^{-\int_t^s (\rho - \frac{\sigma^{-2}}{h_\tau})d\tau}\frac{\sigma^{-2}}{h_s}\frac{u_e(c_s, e_s)}{f'(e_s)}ds\Big]. \tag{4.45}$$

As $h_\tau = h_0 + \sigma^{-2}\tau$, we have

$$\frac{\sigma^{-2}}{h_\tau} = \frac{\sigma^{-2}}{h_0 + \sigma^{-2}\tau} = \frac{d(\ln h_\tau)}{d\tau}. \tag{4.46}$$

Thus, we obtain

$$\exp\Big[\int_t^s \frac{\sigma^{-2}}{h_\tau}d\tau\Big] = e^{(\ln h_s - \ln h_t)} = \frac{h_s}{h_t}. \tag{4.47}$$

Substituting (4.47) into (4.45), the expression of $\tilde{\theta}_t$ becomes

$$\tilde{\theta}_t = \frac{\sigma^{-2}}{h_t}E\Big[\int_t^T e^{-\rho(s-t)}\frac{u_e(c_s, e_s)}{f'(e_s)}ds\Big]. \tag{4.48}$$

Combining the definition of $\tilde{\theta}_t$ and (4.48), the expression of θ_t is turned into

$$\theta_t = E\Big[\int_t^T e^{-\rho(s-t)}\frac{u_e(c_s, e_s)}{f'(e_s)}ds\Big]. \tag{4.49}$$

When the agent's optimal effort $e_t^* > 0$, for any time $t \in [0, T]$, the differential of θ_t is

$$\begin{aligned} d\theta_t &= \Big[\rho\theta_t - \frac{u_e(c_t, e_t)}{f'(e_t)}\Big]dt + \beta_t\sigma dZ_t, \\ \theta_T &= 0. \end{aligned} \tag{4.50}$$

Similarly, the volatility β is determined by the principal, which maximizes his own expected return. According to (4.49), θ_t is proportional to the expectation of the discounted value of the marginal cost of future efforts. Multiplying θ_t by σ^{-2}/h_t measures the effect of cumulative effort on promised utility. As time goes by, the principal gets to know the agent ability η more precisely and $-(\sigma^{-2}/h_t)\theta_t$ decreases, meaning that the principal's power to provide incentives for the agent becomes stronger over time.

4.3.3 Sufficient condition

The global concavity of the agent's objective function is crucial in characterizing the first-order conditions. However, the existence of persistent private information makes it difficult to ensure the concavity of agent's objective function. Because any agent's deviation from the recommended effort leads to a permanent gap of beliefs between the contracting parties, we should verify the sufficiency of the optimal strategy.

Compared to discrete-time models, continuous-time models make it possible for us to verify the sufficiency of the optimal strategy and incentive compatibility by the concavity of the agent's Hamiltonian function. Theorem 3.5.2 in [105] summarizes this general mathematical result. Proposition 4.3.2 gives the sufficient conditions for the agent's optimal effort.

The efforts on the equilibrium and on the arbitrary path are denoted by e_t^* and e_t, respectively. By comparing e_t^* and e_t, we obtain the sufficient conditions of agent's optimal effort. We define the current effort difference between the arbitrary and the recommended path as $\delta_t = f(e_t) - f(e_t^*)$. For the cumulative efforts, we let

$$\begin{cases} \alpha_t &= \int_0^t f(e_s)ds, \\ \alpha_t^* &= \int_0^t f(e_s^*)ds, \\ \Delta_t &= \int_0^t \delta_s ds = \alpha_t - \alpha_t^*. \end{cases}$$

Proposition 4.3.2. *For $t \in [0, T]$, if the matrix*

$$\begin{pmatrix} u_{ee}(c, e_t^*) - \frac{u_e(c,e_t^*)}{f'(e_t^*)} f''(e_t^*), & e^{\rho t}\xi_t f'(e_t^*) \\ e^{\rho t}\xi_t f'(e_t^*), & -2e^{\rho t}\xi_t \frac{\sigma^{-2}}{h_t} \end{pmatrix}$$

is negative semidefinite and (4.23) holds, then the control e_t^* is incentive compatible. ξ_t is the predictable process defined by

$$E\Big[-\int_0^T e^{-\rho s}\gamma_s \frac{\sigma^{-2}}{h_s} ds \Big| \mathcal{F}_t^{e^*}\Big] - E\Big[-\int_0^T e^{-\rho s}\gamma_s \frac{\sigma^{-2}}{h_s} ds \Big| \mathcal{F}_0^{e^*}\Big]$$
$$= \int_0^t \xi_s \sigma dW_s. \tag{4.51}$$

Proof. For simplicity, we abbreviate (e_t, e_t^*) to (e, e^*). For an arbitrary and recommended effort, the corresponding output processes become

$$\begin{cases} dy_t &= \big(ry_t + f(e_t) + \hat{\eta}(Y_t - \alpha_t, t) - p_t - d_t\big)dt + \sigma dZ_t^e, \\ dy_t &= \big(ry_t + f(e_t^*) + \hat{\eta}(Y_t - \alpha_t^*, t) - p_t - d_t\big)dt + \sigma dZ_t^{e^*}. \end{cases}$$

Thus, we get

$$\sigma dZ_t^{e^*} = \sigma dZ_t^e + \Big(\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t\Big)dt. \tag{4.52}$$

We denote the agent's reward by

$$\begin{aligned} I^e(t) &= E_t^e\Big[\int_0^T e^{-\rho t}u(c_t,e_t)dt + e^{-\rho T}\upsilon(p_T,0)\Big] \\ &= \int_0^t e^{-\rho s}u(c_s,e_s)ds + e^{-\rho t}U^e(t,e_t). \end{aligned} \tag{4.53}$$

According to the extended martingale representation theorem in [104], there is a process ζ in L^2 such that

$$I^e(T) = I^e(t) + \int_t^T \zeta_s \sigma dZ_s^e. \tag{4.54}$$

By (4.53), the agent's total reward on the equilibrium path is

$$\begin{aligned} I^{e^*}(T) &= \int_0^T e^{-\rho t}u(c_t,e_t^*)dt + e^{-\rho T}\upsilon(p_T,0) \\ &= U^{e^*}(0,e_t^*) + \int_0^T \zeta_t^* \sigma dZ_t^{e^*} \\ &= U^{e^*}(0,e_t) + \int_0^T \Big(\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t\Big)\zeta_s^* dt + \int_0^T \zeta_t^* \sigma dZ_t^e, \end{aligned} \tag{4.55}$$

where $U^{e^*}(t,e_t^*)$ represents the agent's value function at time t and the superscript indicates the agent's effort path. Similarly, the agent's total reward of arbitrary path is

$$\begin{aligned} I^e(T) &= \int_0^T e^{-\rho t}\big[u(c_t,e_t) - u(c_t,e_t^*)\big]dt + I^{e^*}(T) \\ &= \int_0^T e^{-\rho t}\big(u(c_t,e_t) - u(c_t,e_t^*)\big)dt + U^{e^*}(0,e_t) \\ &\quad + \int_0^T \Big(\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t\Big)\zeta_s^* dt + \int_0^T \zeta_t^* \sigma dZ_t^e. \end{aligned} \tag{4.56}$$

Calculating the third term of the second equality in (4.56), we get

$$\begin{aligned}
-\int_0^T \frac{\sigma^{-2}}{h_t}\Delta_t\delta_s^* dt &= -\int_0^T \frac{\sigma^{-2}}{h_t}\zeta_s^*\Big(\int_0^t \delta_s\Big)dt \\
&= \int_0^T \delta_t\Big(-\int_t^T \frac{\sigma^{-2}}{h_t}\zeta_s^* ds\Big)dt \\
&= \int_0^T \delta_t\Big(-\int_t^T e^{-\rho s}\frac{\sigma^{-2}}{h_t}\gamma_s^* ds\Big)dt. \qquad (4.57)
\end{aligned}$$

Substituting

$$E\Big[-\int_t^T e^{-\rho s}\frac{\gamma_s}{h_s}ds\Big] = \frac{1}{h_t}e^{-\rho t}\theta_t$$

and (4.51) into (4.57), we obtain

$$-\int_0^T \frac{\sigma^{-2}}{h_t}\Delta_t\zeta_s^* dt = \int_0^T \delta_t\Big(\frac{\sigma^{-2}}{h_t}e^{-\rho t}\theta_t^* + \int_t^T \xi_s^*\sigma dZ_s^{e^*}\Big)dt. \qquad (4.58)$$

Changing the Brownian motion and taking the expectation, we have

$$\begin{aligned}
&U^e(0,e) - U^{e^*}(0,e^*) \\
=&E_0^e\left[I^e(T)\right] - U^{e^*}(0,e_t^*) \\
=&E_0^e\Big[\int_0^T e^{-\rho t}\left(u(c_t,e_t) - u(c_t,e_t^*)\right)dt + \int_0^T \delta_t\zeta_t^* dt \\
&+ \int_0^T \delta_t\Big(\frac{\sigma^{-2}}{h_t}e^{-\rho t}\theta_t^* + \int_t^T \xi_s^*\sigma dZ_s^{e^*}\Big)dt\Big] \\
=&E_0^e\Big\{\int_0^T e^{-\rho t}\Big[\Big(u(c_t,e_t) - u(c_t,e_t^*)\Big) + \delta_t\Big(\gamma_t^* + \frac{\sigma^{-2}}{h_t}\theta_t^*\Big)\Big]dt\Big\} \\
&+ E_0^e\Big[\xi_t^*\Delta_t\int_0^T \Big(\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t\Big)\xi_t^*\Delta_t dt\Big]. \qquad (4.59)
\end{aligned}$$

Since e_t^* is the effort on the equilibrium path, the first term of the third equality in (4.59) is non-positive with a maximum of zero. However, we are not clear about the sign of the second term of the third equality. To check its sign, we introduce a predictable process χ^* which satisfies

$$\chi^* = \gamma_t^* - e^{\rho t}\xi_t^*\alpha_t^*. \tag{4.60}$$

In addition, we define a Hamiltonian function H by

$$\begin{aligned} H(t,e,\alpha,\chi^*,\xi^*,\theta^*) = u(c,e) + (\chi^* + e^{\rho t}\xi^*\alpha)f(e) - e^{\rho t}\xi^*\frac{\sigma^{-2}}{h_t}\alpha^2 \\ + \frac{\sigma^{-2}}{h_t}\theta^* f(e). \end{aligned} \tag{4.61}$$

In fact, (4.61) is the Hamiltonian function corresponding to the agent's optimal control problem. Taking a linear approximation of H around α^*, we have

$$\begin{aligned} & H_t(e_t,\alpha_t) - H_t(e_t^*,\alpha_t^*) - \frac{\partial H_t(e_t^*,\alpha_t^*)}{\partial\alpha}\Delta_t \\ & = u(c,e) - u(c,e^*) + \delta_t(\chi^* + e^{\rho t}\xi_t^*\alpha_t^* + \frac{\sigma^{-2}}{h_t}\theta_t^*) \\ & + e^{\rho t}\xi^*\Delta_t(\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t). \end{aligned} \tag{4.62}$$

Combining (4.59) and (4.62) yields

$$\begin{aligned} & U^e(0,e) - U^{e^*}(0,e^*) \\ = & E_0^e\Big[\int_0^T e^{-\rho t}\Big(H_t(e_t,\alpha_t) - H_t(e_t^*,\alpha_t^*) - \frac{\partial H_t(e_t^*,\alpha_t^*)}{\partial\alpha}\Delta_t\Big)\Big]. \end{aligned} \tag{4.63}$$

If (4.61) is concave, the value of (4.63) is negative, which turns out to be the sufficient condition. We obtain that the Hessian matrix of Hamiltonian function $H(\cdot)$ is

$$\mathcal{H}(t,e,\alpha) = \begin{pmatrix} u_{ee}(c,e_t) - \frac{u_e(c,e_t^*)}{f'(e_t)}f''(e_t), & e^{\rho t}\xi_t f'(e_t) \\ e^{\rho t}\xi_t f'(e_t), & -2e^{\rho t}\xi_t\frac{\sigma^{-2}}{h_t} \end{pmatrix}. \tag{4.64}$$

To ensure the concavity of (4.61), the Hessian matrix $\mathcal{H}(t,e,\alpha)$ should be negative semi-definite. □

According to [57], the process ξ_t measures the random fluctuation of the sum of the discounted marginal utility accumulated from time 0. Proposition 4.3.2 imposes rigorous restrictions on ξ_t. If a control process satisfies Proposition 4.3.2, it is incentive compatible. Further, for a given contract, the sufficient condition stated in Proposition 4.3.2 should be verified ex-post for (c_t, γ_t) being endogenous.

4.4 Incentive compatible contracts

4.4.1 Value function

In this section, incentive provision is considered. We discuss the optimal contracts under known and unknown agent ability, respectively. It is shown that when contract time tends to infinity, the case with unknown ability degenerates into the case with known ability. In order to maximize the expected utility, the principal chooses an appropriate strategies $(c_t, d_t, e_t, \gamma_t)$. From (4.10) and (4.11), we denote the principal's value function $J(t, y, q)$ at time t by

$$J(t, y, q) = \max_{\{c_t, d_t, e_t, \gamma_t\}} E\Big[\int_t^T e^{-\rho t} u(d_t) dt + e^{-\rho T} L(y_T, p_T)\Big]. \tag{4.65}$$

The principal's constraints are (4.7), (4.21), (4.41), (4.50). The reason why the principal's value function does not contain the variable $\hat{\eta}$ will be illustrated later. According to the terminal condition in Section 4.2.1, we obtain

$$L(y_T, p_T) = V_T(y_T - p_T) = -\frac{1}{r} \exp\Big[\frac{r-\rho}{r} - \lambda r(y_T - p_T)\Big]. \tag{4.66}$$

Using the terminal condition (4.21), we have

$$q_T = V_T(p_T) = \upsilon(p_T) = -\frac{1}{r}\exp\Big(\frac{r-\rho}{r} - \lambda r p_T\Big). \tag{4.67}$$

Combining (4.66) and (4.67), we get the principal's value function at time T, i.e.,

$$J(T, y_T, q_T) = L(y_T, p_T) = \frac{\frac{1}{r^2}\exp^2\left(\frac{r-\rho}{r}\right)}{q_T}\exp\left(-\lambda r y_T\right). \tag{4.68}$$

From (4.68), we conjecture that the expression of the principal's value function at time t is

$$J(t, y, q) = \frac{e^{K(t)}}{q}\exp\left(-\lambda r y\right), \tag{4.69}$$

where $K(t)$ is a function of time t.

Here we make an assumption about the contract with a specific effort function.

Assumption 4.4.1. *Let $f(e) = a_0 + a_1 e + a_2 e^2, e \in [0, M]$, where a_0, a_1, a_2 and $M > 0$ are constants, $f'(e) = a_1 + 2a_2 e > 0$, $f''(e) = 2a_2 \leq 0$ and $|a_2|$ is sufficiently small.*

4.4.2 Optimal contract under known ability

To intuitively understand the influence of agent ability on the optimal contract, we first analyze the optimal contract when the agent ability is known. Apparently, when the agent ability is known, belief manipulation does not exist. Thus the private information $\theta_t = 0$ and the necessary condition (4.41) is converted into

$$\gamma_t = -\frac{u_e(c_t, e_t)}{f'(e_t)}. \tag{4.70}$$

Under this assumption, the principal's problem is similar to those in [2,12]. As the model does not contain persistent private information, the necessary condition is also sufficient.

As the state variable q is a Markovian process, we solve the principal's problem via the HJB equation. According to the dynamic programming principle, we derive the HJB equation for the principal

$$\begin{aligned}&\rho J(t,y,q) - J_t(t,y,q)\\&= \max_{c,d,e}\Big\{ - \exp(-\lambda d) + J_y(t,y,q)(ry + f(e) + \eta - c - d)\\&+ J_q(t,y,q)\big[\rho q + \exp\big(- \lambda(c - \mu e)\big)\big] + \frac{1}{2}J_{yy}(t,y,q)\sigma^2 + \frac{1}{2}J_{qq}(t,y,q)\sigma^2\gamma^2\\&+ J_{yq}(t,y,q)\sigma^2\gamma\Big\}.\end{aligned} \tag{4.71}$$

Taking advantage of the exponential utility, the solutions of (4.71) can be obtained.

Proposition 4.4.1. *Denote the optimal contract by* $(c^N, d^N, e^N, \gamma^N)$. *Assume that the contract expires at time* T *and the agent ability is known, i.e., for any time* $t \in [0,T]$, $h_t = \infty$ *holds. If assumption 4.4.1 holds and* $a_1 > \mu$, *then the optimal strategies at time* t *are*

$$\begin{cases} e^N &= M,\\ \gamma^N &= -\lambda\mu k q,\\ c^N &= \mu M - \frac{1}{\lambda}\left[\ln k + \ln(-q)\right],\\ d^N &= ry - \frac{1}{\lambda}\big[K(t) + \ln r - \ln(-q)\big],\end{cases}$$

in which k *is the root of the following equation*

$$2\lambda^2\mu^2\sigma^2k^2 + \Big([f'(M)]^2 - r\mu\lambda^2\sigma^2 f'(M)\Big)k - r[f'(M)]^2 = 0 \tag{4.72}$$

and $K(t)$ is expressed by

$$K(t) = e^{-r(T-t)}\Big(\frac{B}{r} + \frac{2(r-\rho)}{r} - 2\ln r\Big) - \frac{B}{r},$$

where

$$B = 2\rho - r - k - \frac{\lambda^2 r^2 \sigma^2}{2} + r\lambda\Big[f(M) + \eta - M\mu + \frac{1}{\lambda}\ln(rk)\Big] + \frac{r\mu k\sigma^2\lambda^2}{a_1 + 2a_2 M} - \frac{\lambda^2\mu^2 k^2\sigma^2}{(a_1 + 2a_2 M)^2}.$$

Proof. For simplicity, we drop out all the subscripts in the proof of this proposition. Differentiating the right hand side of (4.71) with respect to (c, e, d), we get the first-order conditions for (c, e, d)

$$-J_y - \lambda J_q e^{-\lambda(c-\mu e)} + J_{qq}\sigma^2\gamma\gamma_c + J_{yq}\sigma^2\gamma_c = 0, \tag{4.73}$$

$$f'(e)J_y + \lambda\mu J_q e^{-\lambda(c-\mu e)} + J_{qq}\sigma^2\gamma\gamma_e + J_{yq}\sigma^2\gamma_e \geq 0, \tag{4.74}$$

$$\lambda e^{-\lambda d} - J_y = 0. \tag{4.75}$$

To obtain the optimal contracts, we assume that the agent's utility satisfies

$$u(c, e) = kq.$$

From (4.70), γ satisfies

$$\gamma = -\frac{\lambda\mu u(c,e)}{f'(e)} = -\frac{\lambda\mu kq}{f'(e)}. \tag{4.76}$$

Differentiating γ with respect to (c, e), we have the first-order derivatives as follows

$$\frac{\partial \gamma}{\partial c} = -\lambda\gamma, \tag{4.77}$$

$$\frac{\partial \gamma}{\partial e} = (\lambda\mu - \frac{2a_2}{a_1 + 2a_2 e})\gamma. \tag{4.78}$$

Differentiating $J(t, y, q)$ with respect to (t, y, q), we have

$$\begin{aligned} J_t(t,y,q) &= K'(t)J(t,y,q), & J_y(t,y,q) &= -\lambda r J(t,y,q), \\ J_q(t,y,q) &= -\frac{1}{q}J(t,y,q), & J_{yy}(t,y,q) &= (\lambda r)^2 J(t,y,q), \\ J_{yq}(t,y,q) &= \frac{\lambda r}{q}J(t,y,q), & J_{qq}(t,y,q) &= \frac{2}{q^2}J(t,y,q). \end{aligned}$$

If (4.73) holds, from (4.74), we derive that

$$\Big(a_1 + 2a_2 e - \mu + \frac{2a_2}{\lambda(a_1 + 2a_2 e)}\Big)(-\lambda r) - \frac{2a_2}{q(a_1 + 2a_2 e)}e^{-\lambda(c-\mu e)} \leq 0,$$

If the Assumption 4.4.1 holds and $a_1 > \mu$, we can choose a very small $|a_2|$ such that

$$a_1 + 2a_2 e - \mu + \frac{2a_2}{\lambda(a_1 + 2a_2 e)} > 0,$$

then (4.74) holds. Therefore, we derive that the agent's optimal effort can only be obtained at the endpoint. If the agent puts forth zero effort, incentive provision is unnecessary. Since the right derivative of the necessary condition (4.70) with respect to e at $e = 0$ is positive, the value of γ_t increases with the effort level. As a result, the necessary condition (4.70) cannot be optimal at $e = 0$. Similarly, the left derivative of (4.70) with respect to e at $e^N = e^*$ is positive (see (4.78)) and the agent's effort satisfies $e^* \in [0, M]$, so the optimal effort $e^N = e^* = M$.

Thus, the agent's consumption is

$$c^N = \mu M - \frac{1}{\lambda}\big[\ln k + \ln(-q)\big]. \tag{4.79}$$

Substituting $J_y(t, y, q)$ into (4.75), we get the principal's optimal consumption (dividend)

$$d^N = ry - \frac{1}{\lambda}\big[K(t) + \ln r - \ln(-q)\big]. \tag{4.80}$$

In order to find k, we substitute the partial derivatives of $J(t, y, q)$ into the first-order condition (4.73) and obtain

$$2\lambda^2\mu^2\sigma^2k^2 + \Big([f'(M)]^2 - r\mu\lambda^2\sigma^2 f'(M)\Big)k - r[f'(M)]^2 = 0. \tag{4.81}$$

Letting

$$\Delta_1 = \Big([f'(M)]^2 - r\mu\lambda^2\sigma^2 f'(M)\Big)^2 + 8r\lambda^2\mu^2\sigma^2[f'(M)]^2,$$

we obtain

$$k = \frac{r\mu\lambda^2\sigma^2 f'(M) - [f'(M)]^2 + \sqrt{\Delta_1}}{4\lambda^2\mu^2\sigma^2} > 0. \tag{4.82}$$

Substituting $(c^N, d^N, e^N, \gamma^N)$ and (4.69) into (4.71), we derive the first-order ODE about $K(t)$, which satisfies

$$\begin{aligned} -K'(t) = &-rK(t) + r - 2\rho + k + \frac{r^2\lambda^2\sigma^2}{2} - r\mu[f(M) + \eta - M\mu + \frac{1}{r}\ln(rk)] \\ &+ \frac{k\lambda^2\mu^2\sigma^2}{(a_1 + 2a_2M)^2} - \frac{r\mu k\lambda^2\sigma^2}{a_1 + 2a_2M}. \end{aligned} \tag{4.83}$$

Setting

$$B = 2\rho - r - k - \frac{\lambda^2 r^2 \sigma^2}{2} + r\lambda\Big[f(M) + \eta - M\mu + \frac{1}{\lambda}\ln(rk)\Big] + \frac{r\mu k \sigma^2 \lambda^2}{a_1 + 2a_2 M} - \frac{\lambda^2 \mu^2 k^2 \sigma^2}{(a_1 + 2a_2 M)^2},$$

in which k satisfies (4.82), thus (4.83) is simplified to

$$K'(t) - rK(t) = B. \tag{4.84}$$

Using the terminal condition (4.68), we get the terminal condition of $K(t)$

$$K(T) = \frac{2(r-\rho)}{r} - 2\ln r. \tag{4.85}$$

By employing (4.85) and solving the first-order ODE, $K(t)$ becomes

$$K(t) = e^{-r(T-t)}\left(\frac{B}{r} + \frac{2(r-\rho)}{r} - 2\ln r\right) - \frac{B}{r}. \tag{4.86}$$

□

From Proposition 4.4.1, we see that the agent's effort is a constant. For any time, the agent exerts the maximum effort. Furthermore, from (4.79), the agent's consumption is also a constant. k is interpreted as the effective rate of return which varies with volatility. The agent's consumption is linear with the logarithm of both k and the promised utility. When the parameters of the model are given and k is fixed, the promised utility characterizes the dynamics of the agent's consumption. In the expression of d^N, we see that the principal's dividend contains the risk-free return and is proportional to the logarithm of r and the promised utility. Compared with the agent's consumption, the principal's consumption varies over time.

4.4.3 Optimal contract under unknown ability

In this subsection, we discuss the optimal contract in the case where the agent ability is unknown. The ability uncertainty makes the principal's problem complex. To find the solutions of optimal contract, we simplify the optimal control problem by eliminating two state variables. The first one is $\hat{\eta}$ and the second is the private information θ.

Since the constraints (4.21), (4.41) and (4.50) are not directly affected by the posterior mean $\hat{\eta}$, we can use the precision of beliefs to replace the posterior mean $\hat{\eta}$. Moreover, as h_t is a function of time t, t could be used to describe the precision of beliefs. In fact, for the principal, the expectation of η is of no significance, which means that the purpose of incentive provision is to reward the agent's effort instead of his ability.

When the agent's utility satisfies (4.8), we get $u_e(c,e) = \lambda\mu u(c,e)$, which simplifies the principal's problem substantially. For all time $\tau \in [t,T]$, the incentive constraint (4.41) always holds. By substituting $u_e(c,e) = \lambda\mu u(c,e)$ into (4.20) and (4.21), the private information can be expressed as

$$\theta_t = \frac{\lambda\mu}{f'(e_t)}\left(q_t - e^{-\rho(T-t)}E_t[q_T]\right). \tag{4.87}$$

When the expiration of the contract tends to infinity and the transversality condition $\lim_{T\to\infty} e^{-\rho T}E_t[q_T] = 0$ holds, we have $\theta_t = \frac{\lambda\mu}{f'(e_t)}q_t$. The fact that θ_t is proportional to q_t implies that these variables carry the same information. To find the quantitative relationship between θ_t and q_t, we suppose that

$$\theta_t = \frac{\lambda\mu\varphi^T(t)}{f'(e_t)}q_t, \tag{4.88}$$

where $\varphi^T(t)$ is a function of time t and its superscript T means the expiration date of contract. Simultaneously, we assume that

$$u(c, e) = k^T(t)q, \tag{4.89}$$

where $k^T(t)$ is a function of time t. Substituting (4.88) and (4.89) into the necessary condition (4.41), we have

$$\begin{aligned}\gamma_t &= -\frac{\lambda\mu q}{f'(e_t)}\Big(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\Big)\\ &= \Gamma_t^T q,\end{aligned} \tag{4.90}$$

where

$$\Gamma_t^T = -\frac{\lambda\mu}{f'(e_t)}\Big(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\Big).$$

Substituting (4.90) into (4.21), we denote the promised utility by

$$dq_t = q_t\big[\big(\rho - k^T(t)\big)dt + \Gamma_t^T \sigma dZ_t\big].$$

When Γ_t^T is bounded, we obtain

$$E_t[q_T] = q_t e^{\int_t^T \left(\rho - k^T(\tau)\right)d\tau}. \tag{4.91}$$

From (4.87) and (4.91), θ_t satisfies the following equality

$$\theta_t = \frac{\lambda\mu q_t}{f'(e_t)}\left(1 - e^{-\int_t^T k^T(\tau)d\tau}\right).$$

Therefore, the evolution of $\varphi^T(t)$ becomes

$$\varphi^T(t) = 1 - e^{-\int_t^T k^T(\tau)d\tau}. \tag{4.92}$$

From (4.92), we see that $\varphi^T(t)$ is a function of $k^T(t)$. Thus, the key to the principal's problem is to find the solution of $k^T(t)$. Because $k^T(t)$ is positive for all time $t < T$, we have $\varphi^T(t) \in (0, 1)$. Moreover, $k^T(t)$ is the effective rate of return and $k^T(t) \in (0, k)$. Therefore, Γ_t^T is indeed bounded. At time T, we have $\theta_T = \varphi^T(T) = 0$, which indicates that at time T, the agent ability is fully revealed and there is no longer information rent caused by belief manipulation. In other words, as time goes by, the principal's ability of providing incentives becomes better.

Applying the dynamic programming principle, we derive the following HJB equation corresponding to the principal's problem

$$
\begin{aligned}
&\rho J(t,y,q) - J_t(t,y,q) \\
&= \max_{c,d,e}\Big\{ -\exp(-\lambda d) + J_y(t,y,q)(ry + f(e) + \eta - c - d) \\
&+ J_q(t,y,q)\big[\rho q + \exp\big(-\lambda(c-\mu e)\big)\big] + \frac{1}{2}J_{yy}(t,y,q)\sigma^2 + \frac{1}{2}J_{qq}(t,y,q)\sigma^2\gamma^2 \\
&+ J_{yq}(t,y,q)\sigma^2\gamma\Big\}.
\end{aligned} \tag{4.93}
$$

Although (4.71) and (4.93) have the same form, their sensitivity coefficients are different. The following Proposition 4.4.2 gives the optimal policy of the incentive contract.

Proposition 4.4.2. *Denote the optimal contract by $(c^{un}, d^{un}, e^{un}, \gamma^{un})$. Let the Assumption 4.4.1 hold and $a_1 > \mu$. Assume that the expiration of the contract is T and the agent ability η is unknown. Then the optimal policies at time t are*

$$
\begin{cases}
e^{un} = M, \\
\gamma^{un} = -\frac{\lambda\mu q}{f'(M)}\Big(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\Big) \\
c^{un} = \mu M - \frac{1}{\lambda}\big[\ln k^T(t) + \ln(-q)\big], \\
d^{un} = ry - \frac{1}{\lambda}\big[K(t) + \ln r - \ln(-q)\big],
\end{cases}
$$

in which $k^T(t)$ is the root of the following equation

$$2\left(\lambda\mu\sigma k^T(t)\right)^2 + \left(\frac{2\lambda^2\mu^2\sigma^2\varphi^T(t)}{h_t} - r\mu\lambda^2\sigma^2 f'(M) + [f'(M)]^2\right) k^T(t) = [f'(M)]^2 r \tag{4.94}$$

and $K(t)$ satisfies

$$K'(t) - rK(t) = B_1(t)$$

with the corresponding terminal condition (4.85) and

$$\begin{aligned} B_1(t) = {} & 2\rho - r - \frac{1}{2}\lambda^2 r^2\sigma^2 + r\lambda[f(M) - M\mu + \eta] + r\ln(rk^T(t)) \\ & + \frac{\lambda^2\sigma^2\mu r}{f'(M)}\left(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\right) - \frac{\lambda^2\mu^2\sigma^2}{[f'(M)]^2}\left(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\right)^2. \end{aligned}$$

.

Proof. From (4.93), the first-order conditions with respect to (c, e, d) can be expressed as

$$-J_y - \lambda J_q e^{-\lambda(c-\mu e)} + J_{qq}\sigma^2\gamma\gamma_c + J_{yq}\sigma^2\gamma_c = 0, \tag{4.95}$$

$$f(e^*)J_y + \lambda\mu J_q e^{-\lambda(c-\mu e)} + J_{qq}\sigma^2\gamma\gamma_e + J_{yq}\sigma^2\gamma_e \geq 0, \tag{4.96}$$

$$\lambda e^{-\lambda d} - J_y = 0. \tag{4.97}$$

We have

$$\frac{\partial\gamma}{\partial e} = -\lambda\mu q\Big[\frac{\lambda\mu k^T(t)}{a_1 + 2a_2 e} - \frac{2a_2 k^T(t)}{(a_1 + 2a_2 e)^2}\Big] > 0$$

and

$$\gamma_c = \frac{\partial\gamma}{\partial c} = \frac{\lambda^2\mu k^T(t)q}{f'(e^*)} < 0.$$

Similar to the analysis of Proposition 4.4.1, if the first-order condition (4.95) holds, (4.96) is valid. Hence the agent's optimal effort is $e^* = M$, meaning that the agent provides the maximum effort. Since the agent's utility satisfies $u(c,e) = k^T(t)q$, the optimal consumption of agent is

$$c^{un} = \mu M - \frac{1}{\lambda}\big[\ln k^T(t) + \ln(-q)\big].$$

Substituting $J(t,y,q)$ into (4.97), we have the principal's consumption

$$d^{un} = ry - \frac{1}{\lambda}\big[K(t) + \ln r - \ln(-q)\big].$$

Substituting γ, γ_c and $J(t,y,q)$ into (4.95), we know that $k^T(t)$ satisfies

$$2\Big[\lambda\mu\sigma k^T(t)\Big]^2 + \Big(\frac{2\lambda^2\mu^2\sigma^2\varphi^T(t)}{h_t} - r\mu\lambda^2\sigma^2 f'(M) + [f'(M)]^2\Big)k^T(t) = [f'(M)]^2 r.$$

Substituting $(c^{un}, d^{un}, e^{un}, \gamma^{un})$ and $J(t,y,q)$ into (4.93), we obtain

$$\begin{aligned}
-K'(t) = {} & r + k^T(t) - 2\rho + \frac{1}{2}\lambda^2 r^2\sigma^2 + \frac{\lambda^2\mu^2\sigma^2}{[f'(M)]^2}\Big(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\Big)^2 \\
& - \Big[f(M) + \eta - M\mu + \frac{1}{\lambda}\ln(k^T(t)r) + \frac{1}{\lambda}K(t)\Big]\lambda r \\
& - \frac{\mu r\lambda^2\sigma^2}{f'(M)}\Big(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\Big).
\end{aligned} \tag{4.98}$$

Let

$$\begin{aligned}
B_1(t) = {} & 2\rho - r - \frac{1}{2}\lambda^2 r^2\sigma^2 + r\lambda[f(M) - M\mu + \eta] + r\ln(rk^T(t)) \\
& + \frac{\mu r\lambda^2\sigma^2}{f'(M)}\Big(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\Big) - \frac{\lambda^2\mu^2\sigma^2}{[f'(M)]^2}\Big(k^T(t) + \frac{\sigma^{-2}}{h_t}\varphi^T(t)\Big)^2.
\end{aligned}$$

Substituting $B_1(t)$ into (4.98), we get the following first-order ODE

$$K'(t) - rK(t) = B_1(t),$$

with terminal condition (4.85). Then

$$K(t) = e^{r(t-T)}\Big(\frac{2(r-\rho)}{r} - 2\ln r - \int_t^T B_1(\tau)e^{-(\tau-T)}d\tau\Big).$$

□

According to Proposition 4.4.2, we find that when the agent ability is unknown, the agent's optimal effort is still $e^{un} = M$. Compared with the known ability case, the agent's optimal effort does not change because the posteriors $\hat{\eta}$ has no effect on the first-order condition of e. The agent's optimal consumption is linear with the logarithm of $k^T(t)$ and $(-q)$. Since $k^T(t)$ is an increasing function of time t, c^{un} decreases over time. This result intuitively reflects that as time goes by, the principal knows the agent ability more clearly and as a result, the principal's ability for incentive provision improves so that the agent's consumption (or wage) decreases. Furthermore, from the expression of γ_t, we see that the volatility of the promised utility under unknown ability is larger than that under known ability because $\frac{\sigma^{-2}}{h_t}\varphi^T(t)$ is positive. This implies that the agent requires more incentives when the principal cannot observe his ability and can benefit from belief manipulation.

Now we show that ability uncertainty will eventually disappear in long-term contract. Specifically, when we substitute $\varphi^T(T) = 0$ into (4.94) at $t = T$, equation (4.94) degenerates into (4.72). That is, when the contract expires and the agent ability is revealed completely, the two cases become identical. We next show the features of the optimal contract when the expiration of the contract tends to infinity.

By the terminal condition $\varphi^T(T) = 0$ and (4.94), the solutions for $k^T(t)$ and $\varphi^T(t)$ are found by backward induction. We discuss the convergence of $k^T(t)$ and

$\varphi^T(t)$ as the terminal date $T \to \infty$. Let

$$\Delta_2 = \Big(\frac{2\lambda^2\mu^2\sigma^2\varphi^T(t)}{h_t} - \mu r\lambda^2\sigma^2 f'(M) + [f'(M)]^2\Big)^2 + 8\lambda^2\mu^2\sigma^2 B^2 r[f'(M)]^2,$$

then the solution of $k^T(t)$ can be expressed as

$$k^T(t) = \frac{-\Big(\frac{2\lambda^2\mu^2\sigma^2\varphi^T(t)}{h_t} - \mu r\lambda^2\sigma^2 f'(M) + [f'(M)]^2\Big) + \sqrt{\Delta_2}}{4\lambda^2\mu^2\sigma^2}.$$

Differentiating $k^T(t)$ with respect to $\varphi^T(t)$, we obtain

$$\frac{dk^T(t)}{d\varphi^T(t)} = \left(\frac{\frac{2\lambda^2\mu^2\sigma^2\varphi^T(t)}{h_t} - \mu r\lambda^2\sigma^2 f'(M) + [f'(M)]^2}{\sqrt{\Delta_2}} - 1\right)\frac{1}{2h_t} < 0.$$

For any time $t < T$, $k^T(t) > 0$ and $\varphi^T(t) \in (0, 1)$. Therefore, $k^T(t)$ is bounded and $k^T(t) > k(t)$. When $\varphi^T(t) = 1$, $k(t)$ becomes

$$k(t) = \frac{\sqrt{\Delta_3} - \Big(\frac{2\lambda^2\mu^2\sigma^2}{h_t} - \lambda^2\sigma^2\mu r f'(M) + [f'(M)]^2\Big)}{4\lambda^2\mu^2\sigma^2}$$

with

$$\Delta_3 = \Big(\frac{2\lambda^2\mu^2\sigma^2}{h_t} - \mu r\lambda^2\sigma^2 f'(M) + [f'(M)]^2\Big)^2 + 8\lambda^2\mu^2\sigma^2[f'(M)]^2 r.$$

Furthermore, differentiating $k(t)$ with respect to t, we have

$$\frac{dk(t)}{dt} = \frac{1}{2}\left[1 - \frac{\frac{2\lambda^2\mu^2}{h_t} - \lambda^2\sigma^2\mu r f(e^*) + [f(e^*)]^2}{\sqrt{\Delta_3}}\right]\Big(\frac{\sigma^{-2}}{h_t}\Big)^2 > 0,$$

in which we have used

$$\Big|\frac{2\lambda^2\mu^2\sigma^2}{h_t} - \mu r\lambda^2\sigma^2 f'(M) + [f'(M)]^2\Big| < \sqrt{\Delta_3}.$$

As a result, $k(t)$ is an increasing function of time t and

$$\int_t^T k^T(s)ds > \int_t^T k(s)ds > k(t)(T-t). \tag{4.99}$$

According to (4.99), we have $\lim\limits_{T\to\infty} \int_t^T k^T(s)ds = \infty$. Then we obtain

$$\varphi(t) = \lim_{T\to\infty} \varphi^T(t) = \lim_{T\to\infty} \left(1 - e^{-\int_t^T k^T(s)ds}\right) = 1. \tag{4.100}$$

Substituting (4.100) into (4.94), we have $\lim\limits_{T\to\infty} k^T(t) = k(t)$ and

$$2\big(\lambda\mu\sigma k(t)\big)^2 + \Big(\frac{2\lambda^2\mu^2\sigma^2}{h_t} - \mu r\lambda^2\sigma^2 f'(M) + [f'(M)]^2\Big)k(t) - r[f'(M)]^2 = 0. \tag{4.101}$$

When $t \to \infty$, $\lim\limits_{t\to\infty} h(t) = \infty$. Then (4.101) can be written as

$$2\left(\lambda\mu\sigma k(t)\right)^2 + \Big(-\mu r\lambda^2\sigma^2 f'(M) + [f'(M)]^2\Big)k(t) = r[f(M)]^2.$$

From the above derivation, we see that when the expiration of the contract tends to infinity, (4.101) becomes into (4.72). In other words, when the principal and the agent commit to a long-term contract, as time goes by, the principal gradually understands the agent ability η and the information rent caused by belief manipulation will eventually disappear. As $t \to \infty$, the two cases would become identical. We can also infer that long-term contracts make the principal better in providing incentives than short-term contracts do.

Now we verify that as $t \to \infty$ the optimal contract is indeed an incentive compatible contract, i.e., we should ensure that the contract satisfies the conditions given in Proposition 4.3.2. The following Corollary 4.4.1 gives the sufficient conditions for the incentive contract. As suggested in [57], this sufficient conditions can also be obtained directly in the settings of parameters in the model.

Corollary 4.4.1. *Let $a_1 > 0$, $a_2 = 0$, Assumption 4.4.1 hold and*

$$\frac{(2 - f'(M))\left[(\lambda^2\sigma^2\mu r - f'(M))f'(M) + \sqrt{\Delta_1}\right]}{4\lambda^2\mu^2} > \frac{1}{h_t}, \tag{4.102}$$

where

$$\Delta_1 = (f'(M) - \lambda^2\sigma^2\mu r)^2 + 8r\lambda^2\mu^2\sigma^2[f'(M)]^2,$$

then the agent's optimal effort $e = M$ is incentive compatible, i.e., $e^* = M$ satisfies (4.23) and Proposition 4.3.2. Moreover, since the precision h_t is an increasing function of time t, for any time $\tau \geq t$, (4.102) always holds.

Proof. When the expiration T tends to infinity, we can use a standard SDE instead of the BSDE used before. In order to derive the state equation of θ_t, we introduce an auxiliary variable, i.e., for any time $t \in [0, T]$,

$$\begin{aligned} b_t =& E\Big[- \int_0^T e^{-\rho s}\frac{\gamma_s}{h_s} ds | \mathcal{F}_t^e \Big] \\ =& b_0 + \int_0^t \xi_s \sigma dZ_s, \end{aligned}$$

where the second equality is obtained from (4.51). Consequently, the θ_t defined by (4.24) is turned into

$$\theta_t = e^{\rho t}\sigma^2 h_t\Big[b_t + \int_0^t e^{-\rho s}\gamma_s \frac{\sigma^{-2}}{h_s} ds\Big].$$

When t is sufficiently large, θ_t satisfies

$$\begin{aligned} d\theta_t =& \Big[\rho\theta_t + \frac{d(\sigma^2 h_t)}{dt}\frac{\sigma^{-2}}{h_t}\theta_t + \gamma_t\Big]dt + e^{\rho t}\sigma^2 h_t db_t \\ =& \Big[(\rho + \frac{\sigma^{-2}}{h_t})\theta_t + \gamma_t\Big]dt + \beta_t \sigma dZ_t, \end{aligned}$$

where $\beta_t = e^{\rho t}\sigma^2 h_t \xi_t$.

When the agent's optimal effort $e^* = M$, we have $u_{ee}(c_t, M) = \lambda^2\mu^2 qk(t)$ and

$$\beta_t^* = \lambda\mu\gamma_t^* = -\frac{\lambda^2\mu^2 q}{f'(M)}\Big[k(t) + \frac{\sigma^{-2}}{h_t}\Big].$$

Therefore, the sufficient condition of Proposition 4.3.2 becomes

$$-2k(t)q + f'(M)q\Big(k(t) + \frac{\sigma^{-2}}{h_t}\Big) > 0. \tag{4.103}$$

Since the solution of $k(t)$ is (4.82), we obtain the sufficient condition (4.102) by substituting $k(t)$ into (4.103). Because $\frac{dk(t)}{dt} > 0$ and $\frac{\sigma^{-2}}{h_t}$ decreases with time, for any time $\tau > t$, the condition (4.103) holds. $\square$

If both the principal and agent are very patient, the sufficient condition (4.102) holds almost surely. Especially, when the time lasts long enough, the sufficient condition holds. As the agent ability is revealed completely, the problem of asymmetric information disappears. Note that (4.102) is a sufficient condition, but may not be a necessary condition.

4.5 Concluding remark

In this chapter, we investigate a contracting problem with unknown agent ability. We derive the necessary and sufficient conditions for optimality of the incentive contracts when both the priors and the posteriors about the agent ability are normally distributed. We investigate a general form of the effort function which could be implemented in the model. Applying exponential utility and a specific effort function, we obtain the optimal contracts and analyze the influence of the uncertainty of agent ability on the incentive contract.

In our principal-agent problem featured with learning process, the uncertainty of the agent ability reduces the principal's ability to provide incentives. When the agent ability is unknown, the agent's optimal consumption is not only affected by the promised utility but also by the degree of the ability. Ability uncertainty leads to belief manipulation and benefits the agent. For long-term contract, the principal gradually understands the agent ability and thus, the principal's ability of incentive provision improves as time goes by. As a result, the agent's payment or wage is front-loaded. When time tends to infinity, the agent ability is completely revealed and the known and unknown ability cases become identical.

CHAPTER 5

Optimal compensation and investment affected by firm size and time-varying external shocks

5.1 General

In this chapter, we study a continuous dynamic model with a firm size term as well as an external shock term, which possesses the peculiarities: the drift term is dominated by the principal's investment strategy and the agent's effort, and the volatility term relies on the function $\sqrt{G^2(t) + z_t}$ in which $G^2(t)$ is a small continuously bounded function and is interpreted as external shocks or unpredictable factors, depicting the volatility of external environment, and z_t represents the firm size. The exact optimal contracts are obtained under full information. We find that the principal's dividends in large firms are at lower risk since the flow of dividends increases with firm size. Moreover, the optimal compensation scheme for the agent and investment plan for the principal are analyzed under certain assumptions. In extremely volatile environment with large $G^2(t)$, the compensation for the agent would become overly large and the optimal investment is not achievable. Under full information, the model suggests that small volatility environment is more preferable for the principal and the firm.

This chapter is organized as follows. Section 5.2 gives the model settings. Sections 5.3 and 5.4 solve the optimal contracts and compensation scheme, respectively. Section 5.5 presents a discussion on investment. Finally, a concluding remark is given in Section 5.6.

5.2 The model settings

A continuous time agency model is considered in this section. In the model, a principal (shareholders) hires an agent to be the executive to manage a firm. We employ the capital stock to represent firm size. In an arbitrary time interval, we assume that the firm production process is affected by the agent's effort e_t as well as the firm size z_t. In addition, we suppose that the firm's production exposes to risks whose volatility relies on $\sqrt{G^2(t) + z_t}$, where $G(t) \geq 0$ is a continuous bounded function about t, depicting the volatility of the external environment. Specifically, the cumulative output Y_t up to time t evolves in the form

$$dY_t = g(z_t, e_t)dt + \sigma\sqrt{G^2(t) + z_t}dW_t, \tag{5.1}$$

where the Brownian motion W_t belongs to the probability space $(\Omega, \mathbb{F}, P)$, $\sigma > 0$ is a constant, the drift term $g(z_t, e_t)$ denotes the firm's production technology and the term $\sigma\sqrt{G^2(t) + z_t}$, which varies with time t and the firm size z_t, represents the size-dependent production shock. A notation $\mathfrak{F}$, a compact set of progressively measurable processes of $\mathbb{F}_t$, is used to represent the set of feasible effort levels.

In fact, the principal can observe and verify the realized output and monitor the agent's choice of effort $e_t \in \mathfrak{F}$ in the full information case. In order to make the agent work as desired, the principal must provide a committed contract for the agent at the beginning time $t = 0$. The underlying contract should clearly state the agent's compensation c_t and the principal's dividends d_t at any time in the interval $[t, t+\triangle t]$. The agent's utility $u(c_t, e_t)$ depends on the compensation c_t and agent's effort choice e_t. The amount of dividends gives rise to the principal's utility $v(d_t)$. We assume that the agent cannot borrow or save money from outside and his only income source is c_t.

We use a capital accumulation model to describe the evolution of the firm's capital stock z_t. Specifically, z_t accumulates by investment dI_t, in which I_t represents the cumulative investment at time t associated with the initial value $I_0 = 0$, but depreciates at a rate $0 \leq r < 1$. Namely, z_t evolves by

$$dz_t = dI_t - rz_t dt.$$

As described by Chi and Choi [3] and Greenwood et al. [106], dI_t is a result of the remaining output after giving the compensations to the principal and agent. Therefore, a unified framework is applied in our model. Namely, we assume that the corporate growth path and the agent's incentive plan are simultaneously determined by a suitable simple contract. Noticing (5.1), we get that z_t satisfies

$$dz_t = \Big(g(z_t, e_t) - c_t - d_t - rz_t\Big)dt + \sigma\sqrt{G^2(t) + z_t}dW_t, \tag{5.2}$$

in which we define $dI_t = dY_t - (c_t + d_t)dt$. As shown above, the production shock is turned into a function of t and z_t. In other words, the production shock term (volatility of z_t) depends on $\sqrt{G^2(t) + z_t}$. In our setting, we suppose that the volatility of z_t increases with $G^2(t)$ and the firm size z_t. To check the existence and uniqueness to the SDE (5.2), the reader may refer to [107].

There are three reasons why we choose the square root $\sqrt{G^2(t) + z_t}$ to be a component of the volatility. Firstly, according to realistic economic environments, it is reasonable to assume that the volatility or the production shock directly depends on time t and thus, we add a continuous bounded time function $G(t)$ to the volatility term. Secondly, the square-root process does not admit z_t to go below zero. Thirdly, the square-root term embodies empirical patterns about the dynamic properties of a firm if $G(t) = 0$. Existing literature has applied the idea of square-root process. For instance, Hymer and Pashigian [108] show that compared to small firms, large firms possess lower standard deviation of growth rate, but suffer from larger cash flow swings induced by an aggregate underlying shock. The similar statements can be found in [3, 109, 110].

We define a long-term contract $(c_t, d_t, e_t, \Upsilon_t), t \in [0, \infty)$, where Υ_t is defined later. We mention that a contract is feasible in the condition that all the terms $(c_t, d_t, e_t, \Upsilon_t)$ belong to the completion of the σ-algebra which is generated by all records of $\{z_s\}_{s \leq t}$. Let the notation $\mathbb{S}$ represent the set of these feasible contracts.

Suppose that the constant $\beta \in (0, 1)$ is the discount rate of future payoffs for both contracting parties. We write the expected lifetime utilities as

$$V_0(c, d, e, \Upsilon) = \mathbb{E}\Big[\int_0^\infty e^{-\beta t} v(d_t) dt\Big]$$

for the principal and

$$U_0(c, d, e, \Upsilon) = \mathbb{E}\Big[\int_0^\infty e^{-\beta t} u(c_t, e_t) dt\Big]$$

for the agent.

According to the firm size process (5.2), the objective of the principal is to maximize the expected lifetime utility via a feasible contract that satisfies the following participation restriction:

Individual Rationality (IR): An expected utility higher than the agent's reservation utility q_0 is guaranteed when the agent participates in the contract.

Mathematically, the optimal problem that we want to solve becomes

$$\max_{(c,d,e,\Upsilon)\in\mathbb{S}} V_0(c,d,e,\Upsilon),$$

which obeys the output process (5.2) and

$$U_0(c,d,e,\Upsilon) \geq q_0, \tag{IR}$$

which is similar to the participation constraint in Chapters 3 and 4.

5.2.1 Description of contracts

For a long-term contract $(c_t, d_t, e_t, \Upsilon)$ and a filtration $\mathbb{F}_t$, we define the continuation value (promised utility) q_t, which is the promised expected future payoff to the agent, in the form of

$$q_t = \mathbb{E}\Big(\int_t^\infty e^{-\beta(s-t)} u(c_s, e_s) ds \Big| \mathbb{F}_t\Big),$$

where $\mathbb{E}$ represents the expectation under the probability measure P. Using the expression of q_t, we are able to obtain the agent's expected lifetime utility U_t, which evolves by

$$U_t = \mathbb{E}\Big(\int_0^\infty e^{-\beta s} u(c_s, e_s) ds \Big| \mathbb{F}_t\Big) = \int_0^t e^{-\beta s} u(c_s, e_s) ds + e^{-\beta t} q_t. \tag{5.3}$$

Applying the martingale approach, we derive that the process U_t is a P-martingale. Specifically, $\mathbb{E}[U_T|F_t] = U_t$ for each $0 \leq t \leq T$. The martingale representation theorem is used to obtain

$$U_t = U_0 + \int_0^t e^{-\beta s} \Upsilon_s \sigma \sqrt{G^2(t) + z_s} dW_s, \tag{5.4}$$

where Υ_t is a progressively measurable process and W is a Brownian motion under measure P. Consequently, we have the following proposition in terms of q_t.

Proposition 5.2.1. *For (c, d, e), there must exist a process $\{\Upsilon_t\}_{t\in[0,\infty)}$, which is progressively measurable, such that the agent's continuation value q_t satisfies*

$$dq_t = \Big(\beta q_t - u(c_t, e_t)\Big)dt + \Upsilon_t \sigma \sqrt{G^2(t) + z_t} dW_t, \tag{5.5}$$

where $\mathbb{E}\Big[\Upsilon_s^2 ds\Big] < \infty$ for $0 < t < \infty$.

Proof. The identity (5.5) is derived from (5.3) and (5.4). Differentiating (5.3) and (5.4) with respect to the variable t, we obtain

$$dU_t = \Big[e^{-\beta t} u(c_t, e_t) - \beta e^{-\beta t} q_t\Big] dt + e^{-\beta t} dq_t = e^{-\beta t} \Upsilon_t \sigma \sqrt{G^2(t) + z_t} dW_t,$$

which results in (5.5). □

The drift of q_t is a result of the promise-keeping condition. The total flow of utility $u(c_t, e_t)dt + dq_t$ during the interval $[t, t + dt)$ has an increasing rate of βq_t over time. Moreover, the volatility term $\Upsilon_t \sigma \sqrt{G^2(t) + z_t}$ motivates the agent to work as desired. Also the volatility term measures the sensitivity of the changes in the output process dY_t (see (5.1)). The size-time-dependent production technology implies that the time function $G(t)$ and z_t affect the agent's effort and the consumption scheme, which we show in the later sections.

5.2.2 The Hamilton-Jacobi-Bellman equation

As our model contains time-variant-firm-size and the external shock term $G(t)$, theoretically speaking, the principal's value function relies on z_t, q_t and time t or function $G(t)$. Consequently, we define the continuation value function for the principal as $J(t, z, q)$, which represents the expected maximum payoff at the state (t, z_t, q_t). Let q_0 represent the agent's reservation utility and z_0 denote the initial firm size. Then the optimization problem becomes

$$\max_{(c,d,e,\Upsilon)} V_0(c, d, e, \Upsilon),$$

which is subject to (5.2) and (5.5).

We derive the HJB equation

$$\begin{aligned}\beta J - J_t = \max_{(c,d,e,\Upsilon)\in S} \Bigg\{ & v(d) + J_z[g(z,e) - c - d - rz] + J_q[\beta q - u(c,e)] \\ & + \frac{1}{2}\Big[J_{zz} + 2J_{zq}\Upsilon + J_{qq}\Upsilon^2\Big]\Big(G^2(t) + z\Big)\sigma^2\Bigg\},\end{aligned} \tag{5.6}$$

where Υ is a volatility element of q and satisfies the assumption in Proposition 5.2.1. J_z and J_q represent the first order partial derivatives of J with respect to z and q, respectively. J_{zz}, J_{qq} and J_{zq} denote the second order partial differential derivatives of J.

5.3 Optimal contracts

The aim of this section is to solve the optimal contract problem. As in [3], we denote the production function by

$$g(z_t, e_t) = (z_t + h)e_t,$$

where the positive constant h denotes the agent's human capital or working skills. h is assumed to be a constant which means that the learning effect does not depend on experience.

Similar to utilizing the CARA utilities, we assume the principal's utility function to be

$$v(d) = -\exp(-\lambda d)$$

and the agent's utility function to be

$$u(c, e) = -\exp\Big(-\lambda(c - \frac{(z+h)e^2}{2a})\Big),$$

where the positive constant λ denotes the risk aversion coefficients for both the principal and the agent, the factor e^2 represents the agent's monetary cost of effort, and the positive constant a is the optimal effort under full information environment (see Proposition 5.3.1).

To ensure that the discussion of our problem is reasonable, we need to add assumptions to restrict the parameters.

Assumption 5.3.1. *(similar Feller condition). Let $a/2 > r + \beta(4 + \sigma^2)/4$ and $\beta^2 J_0 < 1$, where the positive constant J_0 is expressed in Proposition 5.3.1.*

This similar Feller condition is modified to fit our firm-size process associated with the volatility term $\sqrt{G^2(t) + z_t}$, which is a key element in investigating the optimal contracts under full information. Specifically, this condition guarantees that the drift term of size z_t is positive if $z_t > 0$. This avoids the firm-size process arriving its boundary zero or triggering liquidation of the firm. For the rest of this section, we assume that Assumption 5.3.1 always holds.

5.3.1 A solvable differential equation

In this subsection, by using the first order optimality condition and conjecturing the form of $J(t, z_t, q_t)$, we turn the HJB equation into a solvable first order ODE whose solution is tractably obtained with the CARA utility.

We assume that the continuation value function $J(t, z_t, q_t)$ has the following form

$$J = J(t, z, q) = \frac{J_0}{q} \exp(-Az) \exp[K(t)], \tag{5.7}$$

where the constants $J_0 > 0, A > 0$ and the function $K(t)$ is determined later with the boundary condition. From the exact expression of $K(t)$, we see that the principal's value function negatively correlates with the volatility term $G(t)$. Using (5.7), we are able to solve the HJB equation and obtain the exact expression of the optimal contract.

We discuss the possibility for liquidating a firm or terminating a contract as in [12]. Particularly, we need to specify a boundary condition for the ODE that we will obtain later. For this purpose, at the initial time when signing a contact, we suppose that the two parties agree that the firm will be liquidated if $z_t = 0$. When facing liquidation at time $T = \inf\{t > 0 \big| z_t = 0\}$, the principal must provide the agent with a flow of severance pay c, which is determined by the agent's continuation value at the liquidation date. In this case, the agent can choose zero effort at no cost. Thus, the utility function for the agent becomes $u(c, 0) = -\exp(-\lambda c)$. Namely, the agent receives $c_t = c$ for any $t \geq T$, which is equal to his expected payoff

$$\mathbb{E}\Big[\int_T^\infty e^{-\beta(s-T)}u(c,0)ds\Big] = q_T \quad \text{or} \quad c = -\frac{\ln(-q_T\beta)}{\lambda}.$$

As for the principal, the utility function is transformed into $v(-c) = -e^{\lambda c}$ from the liquidation date onward. In this case, the principal's continuation value $J(T,0,q_T)$ must satisfy

$$J(T,0,q_T) = \mathbb{E}\Big[\int_T^\infty e^{-\beta(s-T)}v(-c)ds\Big] = \frac{1}{\beta^2 q_T}.$$

Together with the form of $J(t,z_t,q_t)$ gives rise to

$$\frac{J_0}{q_T}e^{K(T)} = \frac{1}{\beta^2 q_T},$$

which yields that the function $K(t)$ must satisfies $K(T) = -\ln(\beta^2 J_0)$.

Proposition 5.3.1. *(optimal contracts under full information). Let the Assumption 5.3.1 be satisfied and the boundary condition hold. Then the value function*

$$J = \frac{J_0}{q}\exp(-Az)\exp[K(t)], \quad 0 \le t \le T$$

solves the HJB equation (5.6), and the optimal contracts are

$$\begin{cases} e^* = a, \\ c^* = \frac{a(h+z)}{2} + \frac{1}{\lambda}\ln\left(\frac{4+\sigma^2}{2a-4r}\right) - \frac{\ln(-q)}{\lambda}, \\ d^* = \frac{1}{\lambda}\ln\left(\frac{4+\sigma^2}{2a-4r}\right) - \frac{\ln J_0}{\lambda} + \frac{\ln(-q)}{\lambda} + \frac{Az}{\lambda} - \frac{K(t)}{\lambda}, \\ \Upsilon = -\frac{Aq}{2}, \end{cases} \tag{5.8}$$

where

$$\begin{cases} A = \dfrac{4\lambda(a/2 - r)}{4+\sigma^2} > 0, \\ J_0 = (\lambda/A)^2 \exp\Big(2 - 2\lambda\beta/A - \lambda a h/2\Big), \\ K(t) = -\ln(\beta^2 J_0)\exp[4(a/2-r)(t-T)/(4+\sigma^2)] \\ \quad + (A^2\sigma^2/4)\exp[\dfrac{4(a/2-r)t}{4+\sigma^2}]\displaystyle\int_t^T G^2(s)\exp[-\dfrac{4(a/2-r)s}{4+\sigma^2}]ds. \end{cases} \tag{5.9}$$

From the exact expressions of the optimal polices, the two state variable equations are turned into

$$\begin{cases} dz_t = \Big([\frac{a}{2} - r - \frac{A}{\lambda}]z_t + \frac{2}{\lambda} - \frac{2\beta}{A} + \frac{K(t)}{\lambda}\Big)dt + \sigma\sqrt{G(t)^2 + z_t}dW_t, \\ dq_t = [\beta - \frac{A}{\lambda}]q_t dt - \frac{\sigma A}{2} q_t \sqrt{G(t)^2 + z_t}dW_t. \end{cases} \tag{5.10}$$

Remark. With the exact expressions of A, J_0 and $K(t)$ in (5.9), the principal's continuation value function J is known. The Assumption 5.3.1 ensures that the drift term of the first equation in (5.10) is positive and $-\ln(\beta^2 J_0) > 0$.

Proof. Using (5.6) and the first order approach, we obtain

$$\begin{cases} \lambda\exp(-\lambda d) = J_z, \\ -J_z - \lambda\exp(-\lambda\Big(c - \frac{(z+h)e^2}{2a})\Big)J_q = 0, \\ (z+h)J_z + \frac{\lambda(z+h)e}{a}\exp\Big(-\lambda(c - \frac{(z+h)e^2}{2a})\Big)J_q = 0, \\ \Upsilon = -J_{zq}/J_{qq}. \end{cases}$$

We conjecture that $J(t,z,q) = \frac{J_0}{q}\exp(-Az)\exp[K(t)]$. Subsequently, we need to find the expressions of A, J_0, and the function $K(t)$ so that the optimal contract can be solved explicitly.

First of all, the following identities hold:

$$\begin{cases} J_t = K'(t)J, \\ J_z = -\frac{J_0 A}{q} e^{-Az} e^{K(t)} = -AJ, \\ J_q = -\frac{J_0}{q^2} e^{-Az} e^{K(t)} = -\frac{1}{q} J, \\ J_{zq} = \frac{AJ_0}{q^2} e^{-Az} e^{K(t)} = \frac{A}{q} J, \\ J_{zz} = \frac{A^2 J_0}{q} e^{-Az} e^{K(t)} = A^2 J, \\ J_{qq} = \frac{2J_0}{q^3} e^{-Az} e^{K(t)} = \frac{2}{q^2} J. \end{cases}$$

With the first order conditions, the above identities, and the HJB equation, we derive that

$$\begin{aligned} 2\beta - \frac{dK(t)}{dt} = \frac{2A}{\lambda} - A\Big[(\frac{a}{2} - \frac{A}{\lambda} - r)z + \frac{ha}{2} + \frac{2}{\lambda}\ln(\frac{A}{\lambda}) + \frac{\ln J_0}{\lambda} + \frac{K(t)}{\lambda}\Big] \\ + \frac{1}{4}A^2\sigma^2 G^2(t) + \frac{1}{4}A^2\sigma^2 z. \end{aligned} \tag{5.11}$$

Letting $-A(a/2 - A/\lambda - r) + A^2\sigma^2/4 = 0$ yields

$$A(\frac{1}{\lambda} + \frac{\sigma^2}{4}) = \frac{a}{2} - r > 0,$$

which results in

$$A = \frac{4\lambda(a/2 - r)}{4 + \sigma^2}.$$

Thus, (5.11) becomes

$$2\beta - \frac{dK(t)}{dt} = \frac{2A}{\lambda} - A\Big[\frac{ha}{2} + \frac{2}{\lambda}\ln(\frac{A}{\lambda}) + \frac{\ln J_0}{\lambda} + \frac{K(t)}{\lambda}\Big] + \frac{1}{4}A^2\sigma^2 G^2(t). \tag{5.12}$$

We split (5.12) into the following two equations

$$2\beta = \frac{2A}{\lambda} - A\Big[\frac{ha}{2} + \frac{2}{\lambda}\ln(\frac{A}{\lambda}) + \frac{\ln J_0}{\lambda}\Big], \tag{5.13}$$

$$\frac{dK(t)}{dt} - A\frac{K(t)}{\lambda} = -\frac{1}{4}A^2\sigma^2G^2(t). \tag{5.14}$$

By (5.13), we obtain $J_0 = (\lambda/A)^2\exp(2 - 2\lambda\beta/A - \lambda ah/2)$. The equation (5.14) is a standard ODE of order one. Using the boundary condition for the function $K(T)$, namely $K(T) = -\ln(\beta^2 J_0)$, we obtain the expression of $K(t)$ in (5.9). Using the first order conditions and the exact expressions of J_0 and A, we obtain the optimal policies in (5.8). Applying (5.8) and (5.9), we obtain that (5.10) holds. □

Now we give the following two examples to illustrate how the function $G^2(t)$ affects $K(t)$ and the dividend d^*.

Example 4: The volatility of external shocks varies with business cycles, so it is reasonable to assume that $G(t)$ is periodically variant. If $G^2(t) = \cos^2 t$, we have

$$\begin{aligned}\int_t^T e^{-\frac{A}{\lambda}s}\cos^2 t ds = e^{-\frac{A}{\lambda}T}\Big[-\frac{\lambda}{2A} + \frac{2\lambda^2\sin(2T) - A\lambda\cos(2T)}{2(A^2+4\lambda^2)}\Big] \\ -e^{-\frac{A}{\lambda}t}\Big[-\frac{\lambda}{2A} + \frac{2\lambda^2\sin(2t) - A\lambda\cos(2t)}{2(A^2+4\lambda^2)}\Big].\end{aligned}$$

We handle the infinite limit for the optimal contracts and the forms of the state variable equations. Using (5.9) and letting $T \to \infty$ yield

$$K(t) = \frac{A^2\sigma^2}{4}\Big[\frac{\lambda}{2A} - \frac{2\lambda^2\sin(2t) - A\lambda\cos(2t)}{2(A^2+4\lambda^2)}\Big],$$

from which we acquire the state variable equations as

$$\begin{cases} dz_t = \left([\frac{a}{2} - r - \frac{A}{\lambda}]z_t + \frac{2}{\lambda} - \frac{2\beta}{A} + \frac{A\sigma^2}{8} + \frac{A^2\sigma^2[A\cos 2t - 2\lambda \sin 2t]}{8(A^2 + 4\lambda^2)}\right)dt \\ \quad + \sigma\sqrt{\cos^2 t + z_t}dW_t, \\ dq_t = [\beta - \frac{A}{\lambda}]q_t dt - \frac{\sigma A}{2} q_t \sqrt{\cos^2 t + z_t}dW_t \end{cases}$$

as $T \to \infty$. In this example, when T tends to infinity, we get that $K(t) > 0$, the drift term of the firm size is positive if $z_t > 0$ and the dividend d^* is a periodic function about time t.

Example 5: As $G(t)$ may also be a monotone function, we assume that $G^2(t) = \exp(Am_0 t/\lambda)$ where m_0 is a constant. If $m_0 = 1$, we have

$$K(t) = -\ln(\beta^2 J_0)\exp[\frac{4(\frac{a}{2} - r)}{4 + \sigma^2}(t - T)] + \frac{A^2\sigma^2}{4}\exp[\frac{4(\frac{a}{2} - r)t}{4 + \sigma^2}](T - t). \quad (5.15)$$

If $m_0 \neq 1$, we obtain

$$\begin{aligned} K(t) = &-\ln(\beta^2 J_0)\exp[\frac{4(\frac{a}{2} - r)}{4 + \sigma^2}(t - T)] + \frac{A^2\sigma^2}{4}\exp[\frac{At}{\lambda}] \\ &\times \frac{(4 + \sigma^2)}{4(\frac{a}{2} - r)(m_0 - 1)}\left(\exp[\frac{A(m_0 - 1)T}{\lambda}] - \exp[\frac{A(m_0 - 1)t}{\lambda}]\right). \end{aligned} \quad (5.16)$$

It derives from (5.15) and (5.16) that if $m_0 \geq 1$ and for some fixed t, $K(t) \to \infty$ as $T \to \infty$. In this situation, the principal's own consumption d^* in (5.8) becomes negative infinity, implying that there is no dividend at all for the principal. It concludes that in the long run, increasing volatility of shocks is not preferable for the principal because it not only increases the risk compensation for the agent (see Section 4), but also wipes out the principal's dividends.

If $0 < m_0 < 1$, which indicates that the shock $G^2(t)$ is an increasing function, we have

$$K(t) \to \frac{A^2\sigma^2(4+\sigma^2)}{16(a/2-r)(1-m_0)} \exp(\frac{Am_0 t}{\lambda}), \quad \text{as} \quad T \to \infty. \tag{5.17}$$

For some large t, the dividend d^* may possibly be negative but it may be acceptable in the short run because it is optimal to increase investment.

If $m_0 \leq 0$, $G^2(t)$ is a decreasing function of time t. Noticing the Assumption 5.3.1 and the exact expression of $K(t)$, we derive that the positive $K(t)$ is a decreasing function which is uniformly bounded by a positive constant, implying that dividends are stable and increase as time goes by. With decreasing volatility of shocks, the risk compensation for the agent is relatively lower compared to that under increasing volatile environment (see Section 5.4).

From (5.8), we know that the flow of dividend increases with firm size, indicating that the principal's dividend is at relatively lower risk in a large firm than that in a small firm when the volatility of external shocks $G(t)$ increases. In other words, large firms have a better ability to protect dividends than small firms do in a volatile environment.

5.4 Optimal compensation scheme

Now, we apply the optimal rate of instantaneous payment to the optimal compensation plan. From Proposition 5.3.1, we have

$$c_t^* = \frac{a(h+z_t)}{2} - \frac{\ln(-q_t)}{\lambda} + \frac{1}{\lambda} \ln\Big(\frac{4+\sigma^2}{2a-4r}\Big), \tag{5.18}$$

which, apparently, is influenced by the firm size z_t as well as the volatility term $G(t)$ and we will study the effect of them in this section. In fact, the payment plan relies on the continuation value q_t, which depends on z_t and $G(t)$. In order to fulfill the promise-keeping condition, the principal must pay an instantaneous value q_t to the agent. In order to investigate how c_t^* varies with z_t and $G(t)$, we have to find out the direct effect from z_t and the indirect effect from $G(t)$, which affects q_t. The first term in (5.18) shows that a change in z_t directly leads to a change in c_t^* with a linear growth rate $a/2$. To analyze the indirect effect, we apply Itô's lemma to the term dq_t/q_t in the second equation of (5.10) and obtain the following expression of $\ln(-q_t)$.

$$\begin{aligned}\ln(-q_t) = \ln(-q_0) + (\beta - \frac{A}{\lambda})t - \frac{\sigma A}{2}\int_0^t \sqrt{G^2(t) + z_t}dW_t \\ -\frac{\sigma^2 A^2}{8}\int_0^t \Big(G^2(s) + z_s\Big)ds,\end{aligned} \tag{5.19}$$

in which q_t keeps track of firm size up to t. As a result, formula (5.19) includes the agent's reservation utility $\ln(-q_0)$ and three other terms. Substituting (5.19) into (5.18), the optimal payment plan can be discussed in the sense of firm size z_t and $G(t)$.

Proposition 5.4.1. *The optimal instantaneous payment under full information has the expression*

$$\begin{aligned}c^*(t, z_t) = & \underbrace{-\frac{\ln(-q_0)}{\lambda}}_{L_1} + \underbrace{\frac{a(h+z_t)}{2}}_{L_2} + \underbrace{\frac{1}{\lambda}\ln\Big(\frac{4+\sigma^2}{2a-4r}\Big)}_{L_3} + \underbrace{\frac{1}{\lambda}(\frac{A}{\lambda} - \beta)t}_{L_4} \\ & + \underbrace{\frac{\sigma A}{2\lambda}\int_0^t \sqrt{G^2(s) + z_s}dW_s}_{L_5} \\ & + \underbrace{\frac{\sigma^2 A^2}{8\lambda}\int_0^t z_s ds}_{L_6} + \underbrace{\frac{\sigma^2 A^2}{8\lambda}\int_0^t \Big(G^2(s)\Big)ds}_{L_7}.\end{aligned} \tag{5.20}$$

Inspired by the ideas in [3], we interpret the meanings of each term in (5.20) as follows:

L_1 : reservation utility,

L_2 : compensation for the agent's cost of effort,

L_3 : compensation adjustment for future production,

L_4 : allocation of compensation over time,

L_5 : compensation risk in terms of external shocks and firm size,

L_6 : firm size-wage premium,

L_7 : risk premium due to external shocks.

Proposition 5.4.1 illustrates the indirect effects of firm size z_t and $G(t)$. Here, the idea of decomposing c_t^* is similar to the compensation rules in [1, 3, 13], in which both the principal and the agent have exponential utility. In particular, the agent's compensation is paid at expiration in [13]. In contrast, in [3] and our work, compensation is made continuously over time. However, we derive that compensation is affected by both the firm size and the continuously bounded function $G(t)$, which represents the volatility of external shocks, whereas Chi and Choi [3] only focus on analysing the firm size effect.

Now we give some explanations of each term from L_1 to L_7. Firstly, L_1 and L_2 represent the agent's reservation utility and compensation for the cost of effort e^*, respectively. Secondly, L_3 is regarded as the compensation adjustment for future production and it is a constant under full information. Thirdly, by the Feller condition in Assumption 5.3.1, we know that the term $L_4 = (t/\lambda)(4(a/2 - r)/(4+\sigma^2) - \beta)$ is a positive function, implying that the allocation of compensation increases as t increases because the agent requires risk compensation for increasing variability. This tells us that without information friction, the wage is likely to be back-loaded regardless of firm size or external shocks. Fourthly, L_5 is interpreted as compensation risk in terms of external shocks and firm size. Fifthly, L_6 stands for the firm size-wage premium, which is positively correlated with the firm size, indicating that large firms pay higher wages than small firms do. This is consistent with the empirical results in [111, 112]. Lastly, from the risk premium term L_7, we derive that if the external shock $G(t)$ is larger, the risk premium would be larger. The model suggests that in unstable environments, it costs more to reach the same level of results compared to stable environments with small $G(t)$.

5.5 Investment affected by external shock and firm size

We now consider the effects of external shock and firm size on investment. The residual cash flow after paying compensation and dividends determines the instantaneous investment dI_t. Therefore, for the optimal contract, we have $dI^* = dY_t - (c_t^* + d_t^*)dt$. For the state (z_t, q_t), employing the conditional expectation, we denote expected investment, which is equivalent to the optimal investment scheme, by

$$I_t^* = \frac{d}{dt}\mathbb{E}[T_t^*|z_t, q_t] = (z_t + h)a - c^*(z_t, q_t) - d^*(z_t, q_t). \tag{5.21}$$

From the exact expressions of $c^*(z_t, q_t)$ and $d_t^*(z_t, q_t)$ in Proposition 5.3.1, we see that the sum $c^*(z_t, q_t) + d^*(z_t, q_t)$ does not rely on q_t. However, this sum depends on the function $K(t)$ and the firm size z_t ($K(t)$ depends on $G(t)$). We recognize that q_t does not influence the optimal investment plan because there is no wealth effect under the CARA utility [1]. Therefore, we are able to analyze the optimal investment scheme in terms of the current firm size and the external shock term $G(t)$.

Applying Proposition 5.3.1, we have

$$\begin{aligned} I_t^* &= (z_t + h)a - c^*(z_t, q_t) - d^*(z_t, q_t) \\ &= (\frac{a}{2} - \frac{2a - 4r}{4 + \sigma^2})z_t + \frac{2}{\lambda} - \frac{2\beta}{A} + \frac{K(t)}{\lambda}. \end{aligned} \tag{5.22}$$

The Assumption 5.3.1 ensures that $I_t^* > 0$ in stable environments. According to (5.22), when the external environment becomes volatile, $K(t)$ would increase as $G(t)$ increases. As a result, the optimal investment would increase. We can see from the model that firms may reduce dividends to increase investment in order to stabilize the firm. Moreover, under extreme cases, the time-dependent function $G(t)$ becomes overly large such that $K(t)$ tends to positive infinity (also shown in the last equation in formula (5.9)) and consequently, the corresponding I^* would tend to positive infinity, which is not achievable. In this situation, firms are unable to achieve optimal investment plan regardless of the firm sizes. Moreover, the agent requires large compensation for carrying out the target effort and the principal's dividend vanishes. Thus, small volatility environments may be more preferable.

5.6 Concluding remark

In this chapter, we investigate a continuous time dynamic model with the process of firm size and external shock, whose volatility term relies on the square root $\sqrt{G^2(t) + z_t}$ in which $G^2(t)$ is a small continuously bounded function, which may be interpreted as external shocks or unpredictable factors, and z_t represents the firm size. $G(t)$ could take many different forms under different assumptions, and we give two examples of $G(t)$ to analyze the effect of external shocks. The explicit solution of the optimal contract is obtained under full information. From the optimal policies, we observe a positive linear relationship between the flow of dividends and firm size, indicating that large firms provide more protection for the principal's dividends than small firms do when the external environment becomes volatile. The optimal compensation scheme and investment plan are discussed under certain assumptions. We derive mathematically that if the external shock term $G^2(t)$ is sufficiently large, the optimal investment would tend to positive infinity, which is not realistic for the firm to invest, and the firm distributes no dividends to the principal while the agent is still in the contract. Under the risk aversion assumption, the model suggests that small volatility environments are more preferable for the principal.

CHAPTER 6

Optimal investment strategies for DC pension plan with two administrative fees and the return of premium clause

6.1 General

This chapter studies a defined contribution (DC) pension system with the return of premium clause and two administrative fees: the charge on balance and the charge on flow. In the DC system, a constant elasticity of variance (CEV) model is employed to depict the evolution of the risky asset. Moreover, we use the Weibull model to characterize the force function of mortality. The stochastic control approach is applied to derive the explicit solutions in the cases where a pension member has constant absolute risk aversion (CARA) utility and constant relative risk aversion (CRRA) utility, respectively. We also illustrate how the certainty equivalent (CE) of the expected utility works for comparing the two fees. Finally, we give several examples of numerical analysis to highlight our results.

6.2 The model settings

Assume that $(\Omega, \mathbb{F}, \mathbb{P})$ is a complete probability space in which the filtration $\mathbb{F} = \{\mathbb{F}_t\}$ is generated by the standard Brownian motion $W(t)$. Let all the stochastic processes used in this chapter be well defined on this probability space.

6.2.1 Financial market

Let the financial market only consist of two assets, a risky asset (i.e., stock) and a risk-free asset (i.e., monetary account). We assume that short selling is allowed and there are no transaction costs or taxes in the market. At time t, we let $A(t)$ be the price of the risk-free asset and $r > 0$ be the risk-free interest rate. Thus, we have

$$dA(t) = rA(t)dt. \tag{6.1}$$

The geometric Brownian motion (GBM) is often used to describe the price evolution of the risky asset in the study of DC pension plan (see [4,77]). However, in our setting, the CEV model is applied to this process. Let $B(t)$ be the price of the risky asset at time t. Then we have

$$\frac{dB(t)}{B(t)} = \lambda dt + \sigma B(t)^{\beta} dW(t), \tag{6.2}$$

where λ denotes the expected return of the risky asset and satisfies the general restriction $\lambda > r$. $\sigma B(t)^{\beta}$ represents the instantaneous volatility, where β is the elasticity parameter. In [5], $\beta \leq 0$ indicates that as the stock price decreases, the instantaneous volatility increases. In [7], $\beta > 0$ is applied to illustrate an inverse relationship between the volatility and stock prices. In addition, when $\beta = 0$, (6.2) reduces to the standard GBM model.

6.2.2 Wealth process

In our setting, the accumulation phase starts from age w_0 and ends at $w_0 + T$. In other words, the age interval is $[w_0, w_0 + T]$. The contribution premium is θ per unit time. Let $X^\delta(t)$ denote the pension member's wealth in the individual account (IA) at time t. The return of premium clause is added to the model which means that if a member dies before retirement, the contributed premiums could be withdrawn.

According to [6], we denote the differential form of the wealth process in time interval $[t, t + \frac{1}{m}]$ by

$$\begin{aligned} X^\delta\left(t + \frac{1}{m}\right) = \frac{1}{1 - {}_{\frac{1}{n}}\bar{q}_{w_0+t}} \Bigg\{ & X^\delta(t) \left[\delta(t)\frac{B(t+\frac{1}{m})}{B(t)} + \big(1 - \delta(t)\big)\frac{A(t+\frac{1}{m})}{A(t)}\right] \\ & + \theta\frac{1}{m} - h\theta t\,{}_{\frac{1}{m}}\bar{q}_{w_0+t}\Bigg\}, \end{aligned} \tag{6.3}$$

where $\delta(t)$ denotes the investment proportion of the risky asset and $1 - \delta(t)$ represents the proportion of the risk-free asset. The notation ${}_{\frac{1}{m}}\bar{q}_{w_0+t}$ is an actuarial symbol implying the probability that a pension member, who is alive at the age of $w_0 + t$, will die in the following $\frac{1}{m}$ time period. θt denotes the accumulated premiums at time t. In order to study the effect of the return of premium clause, we introduce the indicator variable h. Particularly, $h = 1$ indicates that the premiums are returned upon the death of the pension member while $h = 0$ means that the member gets nothing upon death. Thus, the term $h\theta t\,{}_{\frac{1}{m}}\bar{q}_{w_0+t}$ is the amount that should be refunded to the representative of the dead member from time t to $t+\frac{1}{m}$. After returning the premiums, the surviving members would share remaining accumulation equally, indicated by $\frac{1}{1-{}_{\frac{1}{m}}\bar{q}_{w_0+t}}$.

Furthermore, according to (6.3), we let

$$\Delta\eta_t^{\frac{1}{m}} = \delta(t)\frac{B(t+\frac{1}{m})-B(t)}{B(t)} + \big(1-\delta(t)\big)\frac{A(t+\frac{1}{m})-A(t)}{A(t)}. \tag{6.4}$$

The conditional death probability satisfies ${}_t\bar{q}_x = 1 -{}_t p_x = 1 - e^{-\int_0^t \mu(x+\tau)d\tau}$, in which the function $\mu(t)$ is the force function of mortality at time t. Thus we have

$$\begin{aligned}
{}_{\frac{1}{m}}\bar{q}_{w_0+t} &= 1 - e^{-\int_0^{\frac{1}{m}} \mu(w_0+t+\tau)d\tau} \\
&\approx \mu(w_0+t)\frac{1}{m} \\
&= o\left(\frac{1}{m}\right), \quad as \quad m \to \infty
\end{aligned}$$

and

$$\begin{aligned}
\frac{{}_{\frac{1}{m}}\bar{q}_{w_0+t}}{1-{}_{\frac{1}{m}}\ \bar{q}_{w_0+t}} &= \frac{1-e^{-\int_0^{\frac{1}{m}} \mu(w_0+t+\tau)d\tau}}{e^{-\int_0^{\frac{1}{m}} \mu(w_0+t+\tau)d\tau}} \\
&= e^{\int_0^{\frac{1}{m}} \mu(w_0+t+\tau)d\tau} - 1 \\
&\approx \mu(w_0+t)\frac{1}{m} \\
&= o\left(\frac{1}{m}\right), \quad as \quad m \to \infty.
\end{aligned}$$

Substituting the above equations into (6.3), we obtain

$$\begin{aligned}
X^\delta\left(t+\frac{1}{m}\right) = &\left(1+\frac{{}_{\frac{1}{m}}\bar{q}_{w_0+t}}{1-{}_{\frac{1}{m}}\ \bar{q}_{w_0+t}}\right)\left\{X^\delta(t)\left(1+\Delta\eta_t^{\frac{1}{m}}\right)\right. \\
&\left. + \theta\frac{1}{m} - h\theta t\,{}_{\frac{1}{m}}\bar{q}_{w_0+t}\right\} \\
=&X^\delta(t)\left(1+\Delta\eta_t^{\frac{1}{m}}\right) + X^\delta(t)\mu(w_0+t)\frac{1}{m} \\
&+ X^\delta(t)\Delta\eta_t^{\frac{1}{m}}\mu(w_0+t)\frac{1}{m} + \theta\frac{1}{m} \\
&- h\theta t\mu(w_0+t)\frac{1}{m} + o\left(\frac{1}{m}\right)
\end{aligned}$$

and

$$1 + \Delta\eta_t^{\frac{1}{m}} \to [(\lambda - r)\delta(t) + r]dt + \delta(t)\sigma dW_t \quad \text{as} \quad m \to \infty.$$

As $m \to \infty$, the dynamics of $X^\delta(t)$ becomes

$$\begin{aligned} dX^\delta(t) = &\left[X^\delta(t)[\delta(t)(\lambda - r) + r + \mu(w_0 + t)] + \theta - h\theta t\mu(w_0 + t)\right] dt \\ &+ X^\delta(t)\delta(t)\sigma B(t)^\beta dW(t). \end{aligned} \tag{6.5}$$

Subsequently, we apply the Weibull formula to the force function of mortality $\mu(t)$ [113], namely

$$\mu(t) = kt^n, \quad 0 \le t \le T, \quad k > 0, \quad n > 0, \tag{6.6}$$

where n is the failure rate or shape parameter and k is the scaled parameter (characteristic life). Therefore, we have

$$\begin{cases} dX^\delta(t) = \Big[X^\delta(t)[\delta(t)(\lambda - r) + r + k(w_0 + t)^n] \\ \qquad\qquad + \theta[1 - hkt(w_0 + t)^n]\Big] dt + X^\delta(t)\delta(t)\sigma B(t)^\beta dW(t), \\ X^\delta(0) = x_0. \end{cases} \tag{6.7}$$

6.2.3 Optimization

Our goal is to maximize the expected utility of the terminal wealth. Namely, we aim to solve the optimal investment strategy $\delta^*(t)$, which satisfies

$$\max_\delta \; E\left[U\big(X^\delta(T)\big)\right],$$

where the function $U(\cdot)$ is strictly concave and the Inada conditions hold, i.e., $U'(0) = +\infty$ and $U'(+\infty) = 0$.

If a person participates in the DC pension plan at time w_0 and retires at time $w_0 + T$, we write $t \in [0, T]$, which is the accumulation phase. Suppose that the initial wealth in the IA is x_0. The pension manager invests the premiums contributed by the member in either a risk-free asset or a risky asset. Thus, our optimal problem is

$$
\begin{aligned}
&\max_{\delta} \quad E\left[U(X^{\delta}(T))\right] \\
&s.t. \quad \begin{cases} dX^{\delta}(t) = \Big[X^{\delta}(t)\left[\delta(t)(\lambda - r) + r + k(w_0+t)^n\right] \\ \qquad\qquad + \theta[1 - hkt(w_0+t)^n]\Big] dt + X^{\delta}(t)\delta(t)\sigma B(t)^{\beta} dW(t), \\ X^{\delta}(0) = x_0. \end{cases}
\end{aligned}
\tag{6.8}
$$

We deal with the following two utility functions

$$
CARA \quad utility: \quad U(x) = -\frac{1}{q}e^{-qx}, \quad \text{with} \quad q > 0, \tag{6.9}
$$

where q is the CARA coefficient.

$$
CRRA \quad utility: \quad U(x) = \frac{x^p}{p}, \quad \text{with} \quad p < 1,\ p \neq 0, \tag{6.10}
$$

where p is the CRRA coefficient.

Using the approaches of guessing the form of solutions for differential equations, we will find a close-form solution of our optimal problem in the following sections.

6.3 Optimal investment with charge on balance

6.3.1 Charge on balance

According to [4], we introduce a fee based on the value of assets under management, which is defined by a positive constant ζ. Specifically, the charge on balance is a percentage of the asset value. Under this assumption, our optimal problem becomes

$$
\begin{aligned}
&\max_{\delta} \quad E\left[U(X_b^{\delta}(T))\right]\\
&s.t. \quad \begin{cases} dX_b^{\delta}(t) = \Big[X_b^{\delta}(t)\left[\delta(t)(\lambda - r) + r - \zeta + k(w_0+t)^n\right] \\ \quad +\theta[1 - hkt(w_0+t)^n]\Big]dt + X_b^{\delta}(t)\delta(t)\sigma B(t)^{\beta}dW(t), \\ X_b^{\delta}(0) = x_0, \end{cases}
\end{aligned}
\tag{6.11}
$$

under which we define the value function

$$
J(t,b,x) = \max_{\delta} \; E\left[U(X_b^{\delta}(T)) | B(t) = b, X_b^{\delta}(t) = x\right].
$$

Using Ito's lemma, the dynamic programming principle, (6.2) and (6.11), we derive the HJB equation in the form

$$
\begin{aligned}
&J_t + \lambda b J_b + \Big(\Big(r - \zeta + k(w_0+t)^n\Big)x + \theta[1 - hkt(w_0+t)^n]\Big) J_x \\
&+ \frac{1}{2}\sigma^2 b^{2\beta+2} J_{bb} + \max_{\delta} \; \Big\{\frac{1}{2}\delta^2\sigma^2 b^{2\beta} x^2 J_{xx} + \delta x(\lambda - r)J_x \\
&+ \delta\sigma^2 b^{2\beta+1} x J_{xb}\Big\} = 0,
\end{aligned}
\tag{6.12}
$$

where J_t, J_b, J_x, J_{bb}, J_{xx}, J_{xb} are the first and second order partial derivatives of J with respect to t, b, x, respectively.

Using the first order conditions of the maximum principle, we obtain the optimal investment strategy of $\delta_b^*(t)$

$$
\delta_b^*(t) = -\frac{(\lambda - r)J_x + \sigma^2 b^{2\beta+1} J_{xb}}{\sigma^2 b^{2\beta} x J_{xx}}. \tag{6.13}
$$

Substituting (6.13) into (6.12), we obtain the following PDE

$$\begin{aligned} & J_t + \lambda b J_b + \Big(\Big(r - \zeta + k(w_0+t)^n\Big)x + \theta[1 - hkt(w_0+t)^n]\Big) J_x \\ & + \frac{1}{2}\sigma^2 b^{2\beta+2} J_{bb} - \frac{[(\mu - r)J_x + \sigma^2 b^{2\beta+1} J_{xb}]^2}{2\sigma^2 b^{2\beta} J_{xx}} = 0 \end{aligned} \tag{6.14}$$

with the boundary condition $J(T,b,x) = U(x)$. By the technique of guessing solutions, we shall solve the nonlinear PDE (6.14) and find the exact solutions under certain assumptions.

6.3.2 The solution under CARA utility

For a member who has CARA utility, according to [5] and [114], we conjecture that (6.14) has a solution in the form of

$$J(t,b,x) = -\frac{1}{q} e^{-q[a(t)(x-f(t))+g(t,b)]}, \tag{6.15}$$

with the boundary conditions $a(T) = 1$, $f(T) = 0$, $g(T,b) = 0$.

Proposition 6.3.1. *Under the CARA utility, the optimal strategy is*

$$\begin{aligned} \delta_b^*(t) = & \frac{(\lambda - r)\Big(1 + \frac{\lambda - r}{2r}[1 - \exp(2r\beta(t-T))]\Big)}{xq\sigma^2 b^{2\beta}} \\ & \times \exp\Big[-(r-\zeta)(T-t) - \frac{k}{n+1}[(w_0+T)^{n+1} - (w_0+t)^{n+1}]\Big]. \end{aligned}$$

In (6.15)*, the function* $a(t), f(t)$ *and* $g(t,b)$ *have the following expressions*

$$\begin{aligned}
a(t) =& \exp\Big[(r-\zeta)(T-t) + \frac{k}{n+1}[(w_0+T)^{n+1} - (w_0+t)^{n+1}]\Big],\\
f(t) =& \exp\Big[-(r-\zeta)(T-t) - \frac{k}{n+1}[(w_0+T)^{n+1} - (w_0+t)^{n+1}]\Big]\\
&\times \int_t^T \theta\Big[1 - hk\tau(w_0+\tau)^n\Big]a(\tau)d\tau,\\
g(t,b) =& \frac{(2\beta+1)(\lambda-r)^2}{4rq}\Big[T - t - \frac{(1-e^{2r\beta(t-T)})}{2r\beta}\Big]\\
&+ \frac{(\lambda-r)^2}{4r\beta q b^{2\beta}\sigma^2}\big(1 - e^{2r\beta(t-T)}\big).
\end{aligned}$$

Proof. Substituting the partial derivatives of $J(t,b,x)$ into (6.14), we obtain

$$\begin{aligned}
a'(t)\big[x - f(t)\big] - a(t)f'(t) + g_t + rbg_b + \Big[(r - \zeta + k(w_0+t)^n)\, x\\
+\theta[1 - hkt(w_0+t)^n]\Big]a(t) + \frac{1}{2}\sigma^2 b^{2\beta+2} g_{bb} + \frac{(\lambda-r)^2}{2\sigma^2 b^{2\beta} q} = 0.
\end{aligned} \tag{6.16}$$

We decompose (6.16) into three equations in order to eliminate the dependence on x. Namely, we let

$$a'(t) + [r - \zeta + k(w_0+t)^n]a(t) = 0, \quad a(T) = 1, \tag{6.17}$$

$$a'(t)f(t) + a(t)f'(t) - \theta[1 - hkt(w_0+t)^n]a(t) = 0, \quad f(T) = 0, \tag{6.18}$$

$$g_t + rsg_b + \frac{1}{2}\sigma^2 b^{2\beta+2} g_{bb} + \frac{(\lambda-r)^2}{2\sigma^2 b^{2\beta} q} = 0, \quad g(T,b) = 0. \tag{6.19}$$

Solving the ODE (6.17), we get

$$a(t) = \exp\Big[(r-\zeta)(T-t) + \frac{k}{n+1}[(w_0+T)^{n+1} - (w_0+t)^{n+1}]\Big]. \tag{6.20}$$

Substituting $a(t)$ into (6.18) yields

$$f'(t) + (\zeta - r - k(w_0 + t)^n)f(t) = \theta[1 - hkt(w_0 + t)^n], \quad f(T) = 0. \qquad (6.21)$$

Solving (6.21) gives rise to

$$f(t) = \exp\Big[- (r - \zeta)(T - t) - \frac{k}{n+1}[(w_0 + T)^{n+1} - (w_0 + t)^{n+1}]\Big] \\ \times \int_t^T \theta\Big[1 - hk\tau(w_0 + \tau)^n\Big]a(\tau)d\tau.$$

To solve the nonlinear PDE (6.19), we transform it into a linear PDE via the method of power transformation and variable change according to [5, 114]. Specifically, setting $g(t, b) = G(t, y)$, $y = b^{-2\beta}$, along with (6.19), we get

$$G_t + \beta\big[(2\beta + 1)\sigma^2 - 2ry\big]G_y + 2\beta^2\sigma^2 yG_{yy} + \frac{(\lambda - r)^2}{2q\sigma^2}y = 0 \qquad (6.22)$$

with the boundary condition $G(T, y) = 0$. Subsequently, we conjecture that the solution of (6.22) takes the following form

$$G(t, y) = G_1(t) + G_2(t)y \qquad (6.23)$$

with the boundary condition $G_1(T) = 0$ and $G_2(T) = 0$. Substituting (6.23) into (6.22) yields

$$G_1'(t) + \beta(2\beta + 1)\sigma^2 G_2(t) + \left[G_2'(t) - 2r\beta G_2(t) + \frac{(\lambda - r)^2}{2q\sigma^2}\right] y = 0. \qquad (6.24)$$

To eliminate the variable y, we split (6.24) into the following two equations

$$G_1'(t) + \beta(2\beta + 1)\sigma^2 G_2(t) = 0,$$
$$G_2'(t) - 2r\beta G_2(t) + \frac{(\lambda - r)^2}{2q\sigma^2} = 0.$$

Solving these ODEs yields

$$
\begin{aligned}
G_1(t) &= \frac{(2\beta+1)(\lambda-r)^2}{4rq}\left[T-t-\frac{(1-e^{2r\beta(t-T)})}{2r\beta}\right],\\
G_2(t) &= \frac{(\lambda-r)^2}{4r\beta q\sigma^2}(1-e^{2r\beta(t-T)}),
\end{aligned}
$$

from which we derive the expression of function $g(t,b)$ by

$$
\begin{aligned}
g(t,b) = {} & \frac{(2\beta+1)(\lambda-r)^2}{4rq}\left[T-t-\frac{(1-e^{2r\beta(t-T)})}{2r\beta}\right]\\
& + \frac{(\lambda-r)^2}{4r\beta q b^{2\beta}\sigma^2}\big(1-e^{2r\beta(t-T)}\big).
\end{aligned}
$$

Finally, substituting the derivatives of $J(t,b,x)$ with respect to t, b, x into (6.13) gives

$$
\begin{aligned}
\delta_b^*(t) &= -\frac{(\lambda-r)J_x+\sigma^2 b^{2\beta+1}J_{xb}}{\sigma^2 b^{2\beta}xJ_{xx}}\\
&= \frac{(\lambda-r)qaJ-\sigma^2 b^{2\beta+1}q^2 a g_b J}{\sigma^2 b^{2\beta}xq^2a^2J}\\
&= \frac{(\lambda-r)-\sigma^2 b^{2\beta+1}q g_b}{xq\sigma^2 b^{2\beta}a}\\
&= \frac{(\lambda-r)\Big(1+\frac{\lambda-r}{2r}[1-\exp(2r\beta(t-T))]\Big)}{xq\sigma^2 b^{2\beta}}\\
&\qquad \times \exp\Big[-(r-\zeta)(T-t)-\frac{k}{n+1}[(w_0+T)^{n+1}-(w_0+t)^{n+1}]\Big].
\end{aligned}
$$

$\square$

From the exact expression of $\delta_b^*(t)$ in Proposition 6.3.1, we recognize that the optimal strategy does not depend on the contribution rate θ or the control parameter h, which indicates that the return of premium clause has no effect on optimal investment strategy for a member who has CARA utility. However, the charge on balance ζ does have influence on the optimal strategy, we will illustrate this effect in our numerical analysis.

Corollary 6.3.1. *If $\beta = 0$, the CEV model reduces to the GBM model. Then the optimal investment strategy is*

$$\delta_b^*(t) = \frac{(\lambda - r)}{xq\sigma^2} \exp\Big[-(r-\zeta)(T-t) - \frac{k}{n+1}[(w_0+T)^{n+1} - (w_0+t)^{n+1}]\Big].$$

6.3.3 The solution under CRRA utility

Under the CRRA utility, we again guess that (6.14) has the following form of solution

$$J(t,b,x) = \frac{(x - v(t))^p}{p} F(t,b), \quad p < 1, \quad p \neq 0 \tag{6.25}$$

with the boundary conditions $v(T) = 0$, $F(T,b) = 1$.

Proposition 6.3.2. *Under the CRRA utility, the optimal strategy is*

$$\delta_b^*(t) = \frac{(\lambda - r)}{(1-p)\sigma^2 b^{2\beta}} - \frac{(\lambda - r)v(t)}{x(1-p)\sigma^2 b^{2\beta}} - \frac{2\beta\psi(t)}{\sigma^2 b^{2\beta}} + \frac{2\beta v(t)\psi(t)}{x\sigma^2 b^{2\beta}},$$

where

$$v(t) = -\int_t^T \theta\Big[1 - hk\tau(w_0+\tau)^n\Big] \exp\Big(-\int_t^\tau (r - \zeta + k(w_0+\xi)^n) d\xi\Big) d\tau$$

and

$$\psi(t) = \frac{I_1 - I_1 e^{2\beta^2(I_1 - I_2)(T-t)}}{1 - \frac{I_1}{I_2} e^{2\beta^2(I_1-I_2)(T-t)}},$$

in which

$$\begin{cases} I_1 = \frac{(\lambda - rp) - \sqrt{(1-p)(\lambda^2 - r^2 p)}}{2\beta(1-p)}, \\ I_2 = \frac{(\lambda - rp) + \sqrt{(1-p)(\lambda^2 - r^2 p)}}{2\beta(1-p)}. \end{cases}$$

$F(t,b)$ in (6.25) evolves by

$$F(t,b) = \exp\Big[\frac{pk}{n+1}\Big((w_0+T)^{n+1} - (w_0+t)^{n+1}\Big)\Big] \times \Big\{E(t)\exp\Big\{\Big[I_1\beta(2\beta+1) + \frac{(r-\delta)p}{1-p}\Big](T-t) + \frac{\psi(t)}{\sigma^2 b^{2\beta}}\Big\}\Big\}^{1-p},$$

where

$$E(t) = \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1 e^{2\beta^2(\lambda_2-\lambda_1)(T-t)}}\right)^{\frac{2\beta+1}{2\beta}}.$$

Proof. Substituting the derivatives of $J(t,b,x)$ into (6.14), we get

$$\Big\{F_t + \frac{\lambda - rp}{1-p}F_b + \frac{1}{2}\sigma^2 b^{2\beta+2}F_{bb} + \frac{p\sigma^2 b^{2\beta+2}}{2(1-p)}\frac{F_b^2}{F} + \frac{p(\lambda-r)^2}{2\sigma^2 b^{2\beta}(1-p)}F + (r-\zeta+k(w_0+t)^n)\,ph\Big\}(x-v)^p + ph\Big\{\theta[1-hkt(w_0+t)^n] + (r-\zeta)v - v_t\Big\}(x-v)^{p-1} = 0. \tag{6.26}$$

We then split (6.26) into the following two equations

$$v'(t) - [r-\zeta+k(w_0+t)^n]\,v(t) - \theta[1-hkt(w_0+t)^n] = 0, \tag{6.27}$$

$$F_t + \frac{\lambda - rp}{1-p}F_b + \frac{1}{2}\sigma^2 b^{2\beta+2}F_{bb} + \frac{p\sigma^2 b^{2\beta+2}}{2(1-p)}\frac{F_b^2}{F} + \frac{p(\lambda-r)^2}{2\sigma^2 b^{2\beta}(1-p)}F + (r-\zeta+k(w_0+t)^n)\,pF = 0. \tag{6.28}$$

Solving the first order ODE (6.27) with the boundary condition $v(T) = 0$ gives rise to

$$v(t) = -\int_t^T \theta[1-h\tau\mu(w_0+\tau)]\exp\Big(-\int_t^\tau (r-\zeta+\mu(w_0+\xi))d\xi\Big)d\tau.$$

To solve (6.28), we let $F(t,b) = K(t,y)^{1-p}$, $y = b^{-2\beta}$ and obtain

$$
\begin{aligned}
&K_t + \beta\left[(2\beta+1)\sigma^2 - \frac{2(\lambda - rp)y}{1-p}\right]K_y + 2\delta^2\beta^2 y K_{yy} + \frac{p(\lambda-r)^2}{2\delta^2(1-p)^2}yK \\
&+ \frac{p}{1-p}\left(r - \zeta + k(w_0+t)^n\right)K = 0.
\end{aligned} \tag{6.29}
$$

Assuming $K(t,y) = K_1(t)e^{K_2(t)y}$ results in

$$
\begin{aligned}
&\left\{\frac{K_1'(t)}{K_1(t)} + \beta(2\beta+1)\sigma^2 K_2(t) + \frac{p}{1-p}\left(r-\zeta+k(w_0+t)^n\right)\right\} + \left\{K_2'\right. \\
&\left. - \frac{2\beta(\lambda-rp)}{1-p}K_2 + 2\sigma^2\beta^2K_2^2 + \frac{p(\lambda-r)^2}{2\sigma^2(1-p)^2}\right\}y = 0.
\end{aligned} \tag{6.30}
$$

To eliminate the variable y in (6.30), we let

$$
\frac{K_1'(t)}{K_1(t)} + \beta(2\beta+1)\sigma^2K_2(t) + \frac{p}{1-p}\left(r-\zeta+k(w_0+t)^n\right) = 0, \tag{6.31}
$$

$$
K_2'(t) - \frac{2\beta(\lambda-rp)}{1-p}K_2(t) + 2\sigma^2\beta^2K_2(t)^2 + \frac{p(\lambda-r)^2}{2\sigma^2(1-p)^2} = 0. \tag{6.32}
$$

From the Riccati equation (6.32), we derive

$$
K_2(t) = \frac{1}{\sigma^2}\psi(t), \tag{6.33}
$$

where

$$
\psi(t) = \frac{I_1 - I_1e^{2\beta^2(I_1-I_2)(T-t)}}{1-\frac{I_1}{I_2}e^{2\beta^2(I_1-I_2)(T-t)}}
$$

with

$$
\begin{cases}
I_1 = \frac{(\lambda-rp)-\sqrt{(1-p)(\lambda^2-r^2p)}}{2\beta(1-p)}, \\
I_2 = \frac{(\lambda-rp)+\sqrt{(1-p)(\lambda^2-r^2p)}}{2\beta(1-p)}.
\end{cases}
$$

The expression of $K_1(t)$ is

$$K_1(t) = \exp\Big[\frac{pk}{(n+1)(1-p)}\Big((w_0+T)^{n+1} - (w_0+t)^{n+1}\Big)\Big]$$
$$\times E(t)\exp\left\{\left[I_1\beta(2\beta+1) + \frac{(r-\zeta)p}{1-p}\right](T-t)\right\},$$

where

$$E(t) = \left(\frac{I_2 - I_1}{I_2 - I_1 e^{2\beta^2(I_2-I_1)(T-t)}}\right)^{\frac{2\beta+1}{2\beta}}.$$

Finally, the explicit solution of $F(t,b)$ evolves by

$$F(t,b) = \exp\Big[\frac{pk}{n+1}\Big((w_0+T)^{n+1} - (w_0+t)^{n+1}\Big)\Big]$$
$$\times \left\{E(t)\exp\left\{\left[I_1\beta(2\beta+1) + \frac{(r-\delta)p}{1-p}\right](T-t) + \frac{\psi(t)}{\sigma^2 b^{2\beta}}\right\}\right\}^{1-p}.$$

Substituting the derivatives of $J(t,b,x)$ with respect to t, b, x into (6.13), we obtain the optimal investment strategy

$$\begin{aligned}
\delta_b^*(t) &= -\frac{(\lambda - r)J_x + \sigma^2 b^{2\beta+1}J_{xb}}{\sigma^2 b^{2\beta} x J_{xx}} \\
&= \frac{(\lambda-r)b(x-v)^{p-1} + \sigma^2 b^{2\beta+1}F_b(x-v)^{p-1}}{\sigma^2 b^{2\beta} x(1-p)(x-v)^{p-2}} \\
&= \frac{(\lambda-r)}{(1-p)\sigma^2 b^{2\beta}} - \frac{(\lambda-r)v(t)}{x(1-p)\sigma^2 b^{2\beta}} - \frac{2\beta\psi(t)}{\sigma^2 b^{2\beta}} + \frac{2\beta v(t)\psi(t)}{x\sigma^2 b^{2\beta}}.
\end{aligned}$$

□

From the results of Proposition 6.3.2, we see that $v(t)$ contains the contribution rate θ, the control parameter h and the charge on balance ζ. Thus, different from the CARA case, θ, h and ζ have effects on the optimal strategy under the CRRA utility. We will show the details of the influences in our numerical analysis.

Corollary 6.3.2. *If $\beta = 0$, the CEV model reduces to the GBM model and the* *optimal investment strategy becomes*

$$\delta_b^*(t) = \frac{(\lambda - r)}{(1-p)\sigma^2} - \frac{(\lambda - r)v(t)}{x(1-p)\sigma^2}.$$

6.4 Optimal investment with charge on flow

6.4.1 Charge on flow

The charge on flow is denoted by $\alpha > 0$ and is a proportion of the member's contribution. Specifically, when the member contributes c in the IA in a particular month, he/she pays a proportion of the contribution as the commission to the pension fund administrator, i.e., $(1-e^{-\alpha})c$, which could have been invested in the fund. Therefore, we write $e^{-\alpha}c$ as the fee-adjusted contribution. With a constant rate of contribution θ, we denote the wealth process as $X_f^\delta(t)$ which satisfies the SDE

$$\begin{cases} dX_f^\delta(t) = \Big[X_f^\delta(t)\left(\delta_f(t)(\lambda - r) + r + k(w_0+t)^n\right) \\ \qquad\qquad + e^{-\alpha}\theta[1 - hkt(w_0+t)^n]\Big]dt + X_f^\delta(t)\delta_f(t)\sigma B(t)^\beta dW(t), \\ X_f^\delta(0) = e^{-\alpha}x_0. \end{cases} \tag{6.34}$$

From (6.34), the optimization problem is

$$\max_\delta \quad E\left[U(X_f^\delta(T))\right].$$

Again, we define the value function

$$J(t,b,x) = \max_\delta \; E\left[U(X_f^\delta(T))|B(t) = b, X_f^\delta(t) = x\right]$$

and similarly, derive the following HJB equation

$$
\begin{aligned}
J_t + \mu b J_b + \Big(\Big(r + k(w_0+t)^n\Big)x + e^{-\alpha}\theta[1 - hkt(w_0+t)^n]\Big)\, J_x \\
+\frac{1}{2}\sigma^2 b^{2\beta+2} J_{bb} + \max_{\delta}\ \Bigg\{\frac{1}{2}\delta^2\sigma^2 b^{2\beta}x^2 J_{xx} + \delta x(\lambda - r)J_x \\
+\delta\sigma^2 b^{2\beta+1} x J_{xb}\Bigg\} = 0.
\end{aligned}
\tag{6.35}
$$

Subsequently, according to the first order optimality conditions derived from (6.35), we obtain the optimal fraction of the risky asset

$$
\delta_f^*(t) = -\frac{(\lambda - r)J_x + \sigma^2 b^{2\beta+1} J_{xb}}{\sigma^2 b^{2\beta} x J_{xx}}. \tag{6.36}
$$

It follows from (6.35) and (6.36) that

$$
\begin{aligned}
J_t + \mu b J_b + \Big(\Big(r + k(w_0+t)^n\Big)x + e^{-\alpha}\theta[1 - hkt(w_0+t)^n]\Big)\, J_x \\
+\frac{1}{2}\sigma^2 b^{2\beta+2} J_{bb} - \frac{[(\lambda - r)J_x + \sigma^2 b^{2\beta+1} J_{xb}]^2}{2\sigma^2 b^{2\beta} J_{xx}} = 0
\end{aligned}
\tag{6.37}
$$

with the boundary condition $J(T, b, x) = U(x)$.

6.4.2 The solution under CARA utility

Similar to Section 6.3.2, we conjecture the value function J to take the form

$$
J(t, b, x) = -\frac{1}{q} e^{-q[a_1(t)[x - f_1(t)] + g_1(t,b)]}, q > 0 \tag{6.38}
$$

with the boundary conditions $a_1(T) = 1,\ f_1(T) = 0,\ g_1(T, b) = 0.$

Proposition 6.4.1. *The optimal strategy is*

$$\begin{aligned}\delta_f^*(t) =& \frac{(\lambda-r)\Big(1+\frac{\lambda-r}{2r}[1-\exp(2r\beta(t-T))]\Big)}{xq\sigma^2 b^{2\beta}}\\ &\times \exp\Big[-r(T-t)-\frac{k}{n+1}[(w_0+T)^{n+1}-(w_0+t)^{n+1}]\Big].\end{aligned}$$

In (6.38), $a_1(t), f_1(t)$ *and* $g_1(t,b)$ *have the following expressions*

$$\begin{aligned}a_1(t) =& \exp\Big[r(T-t)+\frac{k}{n+1}[(w_0+T)^{n+1}-(w_0+t)^{n+1}]\Big],\\ f_1(t) =& \exp\Big[-r(T-t)-\frac{k}{n+1}[(w_0+T)^{n+1}-(w_0+t)^{n+1}]\Big]\\ &\times\int_t^T e^{-\alpha}\theta\Big[1-hk\tau(w_0+\tau)^n\Big]\exp[a_1(\tau)]d\tau,\\ g_1(t,b) =& \frac{(2\beta+1)(\lambda-r)^2}{4rq}\Big[T-t-\frac{1-e^{2r\beta(t-T)}}{2r\beta}\Big]\\ &+\frac{(\lambda-r)^2}{4r\beta q b^{2\beta}\sigma^2}\big(1-e^{2r\beta(t-T)}\big).\end{aligned}$$

Proof. The proof is quite similar to that of Proposition 6.3.1. Hence, we omit the proof here. □

From the exact expression of $\delta_f^*(t)$ in Proposition 6.4.1, we recognize that under the CARA utility, the optimal strategy again does not depend on the contribution rate θ, the indicator h or charge on flow α, implying that the return of premiums clause and charge on flow α make no difference on the optimal investment strategy.

Corollary 6.4.1. *If $\beta=0$, the CEV model reduces to the GBM model. Then the optimal investment strategy is*

$$\delta_f^*(t) = \frac{(\lambda-r)}{xq\sigma^2}\exp\Big[-r(T-t)-\frac{k}{n+1}[(w_0+T)^{n+1}-(w_0+t)^{n+1}]\Big].$$

6.4.3 The solution under CRRA utility

Similar to Section 6.3.3, we guess that (6.37) takes the following form of solution

$$J(t,b,x) = \frac{(x - v_1(t))^p}{p} F_1(t,b), \quad p < 1, \quad p \neq 0 \tag{6.39}$$

with the boundary conditions $v_1(T) = 0$, $F_1(T,b) = 1$.

Proposition 6.4.2. *Under the CRRA utility, the optimal strategy is*

$$\delta_f^*(t) = \frac{(\lambda - r)}{(1-p)\sigma^2 b^{2\beta}} - \frac{(\lambda - r)v_1(t)}{x(1-p)\sigma^2 b^{2\beta}} - \frac{2\beta\psi(t)}{\sigma^2 b^{2\beta}} + \frac{2\beta v_1(t)\psi(t)}{x\sigma^2 b^{2\beta}}$$

with

$$v_1(t) = -\int_t^T e^{-\alpha}\theta\Big[1 - hk\tau(w_0+\tau)^n\Big]\exp\Big(-\int_t^\tau (r + k(w_0+\xi)^n)d\xi\Big)d\tau$$

and

$$\psi(t) = \frac{I_1 - I_1 e^{2\beta^2(I_1-I_2)(T-t)}}{1 - \frac{I_1}{I_2}e^{2\beta^2(I_1-I_2)(T-t)}},$$

in which

$$\begin{cases} I_1 = \frac{(\lambda - rp) - \sqrt{(1-p)(\lambda^2 - r^2 p)}}{2\beta(1-p)}, \\ I_2 = \frac{(\lambda - rp) + \sqrt{(1-p)(\lambda^2 - r^2 p)}}{2\beta(1-p)}. \end{cases}$$

For the $J(t,b,x)$ in (6.39), the expression of $F_1(t,b)$ is

$$\begin{aligned} F_1(t,b) &= \exp\left[\frac{pk}{n+1}\Big((w_0+T)^{n+1} - (w_0+t)^{n+1}\Big)\right] \\ &\quad \times \left\{E(t)\exp\left\{\left[I_1\beta(2\beta+1) + \frac{(r-\delta)p}{1-p}\right](T-t) + \frac{\psi(t)}{\sigma^2 b^{2\beta}}\right\}\right\}^{1-p}, \end{aligned}$$

where

$$E(t) = \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1 e^{2\beta^2(\lambda_2-\lambda_1)(T-t)}}\right)^{\frac{2\beta+1}{2\beta}}.$$

Proof. The proof is similar to that of the Proposition 6.3.2. Hence, we omit the proof here. □

From the results of Proposition 6.4.2, we see that the function $v_1(t)$ depends on the contribution rate θ, the indicator h and the charge on flow α. Thus, θ, h and α do have effects on the optimal strategy. Specific relationships will be demonstrated in our numerical analysis.

Corollary 6.4.2. *If $\beta = 0$, the CEV model reduces to the GBM model and the optimal investment strategy becomes*

$$\delta_f^*(t) = \frac{(\lambda - r)}{(1-p)\sigma^2} - \frac{(\lambda - r)v_1(t)}{x(1-p)\sigma^2}.$$

6.5 Comparison between "charge on balance" and "charge on flow"

The aim of this section is to provide a method for comparing the "charge on balance" and "charge on flow" and finding the relationship between the two charges under the same utility. Under certain assumptions, we aim to know which charge for the DC pension system with the return of premium clause is better for the pension members. As the pension members aim to maximize their expected terminal utilities, we should compare the expected utility with the two types of charges under the same utility. As mentioned in [4], the comparison could be done via the maximum certainty equivalent (CE). Hence, we introduce the following ratio in terms of different CEs.

$$R_{bf} \equiv \frac{CE\left(\overline{X}_b^{\delta}(t)\right)}{CE\left(\overline{X}_f^{\delta}(T)\right)}. \tag{6.40}$$

Comprehensibly, $R_{bf} > 1$ implies that the charge on balance is more preferable for the members. On the contrary, $R_{sf} < 1$ means that the charge on flow is better. When $R_{sf} = 1$, the member is indifferent between the two types of fees.

6.5.1 Comparison under CARA utility

For the pension member who has the CARA utility, when the investment proportion approaches to the optimal strategy $\delta_b^*(t)$, we use $\overline{X}_b^\delta(T)$ to denotes the member's terminal wealth under charge on balance, and

$$\begin{aligned} E[U(\overline{X}_b^\delta(T))] =& V(0, b, x_0) \\ =& -\frac{1}{q}\exp\Big(- q[a(0)(x_0 - f(0)) + g(0,b)]\Big), \end{aligned}$$

where the expressions of functions $a(0), f(0)$ and $g(0, b)$ can be found in Proposition 6.3.1.

Similarly, let $\overline{X}_f^\delta(T)$ denote the member's terminal wealth under charge on flow when the investment proportion approaches to the optimal strategy $\delta_f^*(t)$. Then, we have

$$E[U(\overline{X}_f^\delta(T))] = -\frac{1}{q}\exp\Big(- q[a_1(0)(x_0 - f_1(0)) + g_1(0,b)]\Big),$$

where $a_1(0), f_1(0)$ and $g_1(0, b)$ are defined in Proposition 6.4.1.

Since $U[CE(\overline{X}^\delta(T))] = E[U(\overline{X}^\delta(T))]$ and $g(0, b) = g_1(0, b)$, we have

$$CE[\overline{X}_b^\delta(T)] = a(0)\Big(x_0 - f(0)\Big) + g(0,b)$$

and

$$CE[\overline{X}_f^\delta(T)] = a_1(0)\Big(x_0 - f_1(0)\Big) + g(0,b).$$

Thus, we obtain

$$R_{bf} = \frac{a(0)\Big(x_0 - f(0)\Big) + g(0,b)}{a_1(0)\Big(x_0 - f_1(0)\Big) + g(0,b)}. \tag{6.41}$$

The expression of $a(0)$ and $f(0)$ relies on ζ. $f_1(0)$ contains $e^{-\alpha}\theta$. We see that R_{bf} is determined by α, ζ, θ, h, T, β, x_0, r, q, b, k and n. If all the parameters are fixed, we can calculate R_{bf}. This may be a cumbersome calculation which could be done by mathematical software.

6.5.2 Comparison under CRRA utility

Under the CRRA utility, we denote the terminal wealth with charge on balance under the optimal strategy $\delta_b^*(t)$ by $\overline{X}_b^\delta(T)$. Then, we have

$$E[U(\overline{X}_b^\delta(T))] = V(0,b,x_0) = \frac{(x_0 - v(0))^p}{p} F(0,b),$$

in which $v(0)$ and $F(0,b)$ are defined in Proposition 6.3.2.

Similarly, in the case of charge on flow, $\overline{X}_f^\delta(T)$ is the terminal wealth under the optimal strategy $\delta_f^*(t)$ and

$$E[U(\overline{X}_f^\delta(T))] = \frac{(x_0 - v_1(0))^p}{p} F_1(0,b),$$

where $v_1(0)$ and $F_1(0,b)$ can be found in Proposition 6.4.2. As we can see from Propositions 6.3.2 and 6.4.2, $F(t,b) = F_1(t,b)$.

Thus, the CEs are

$$CE[\overline{X}_b^\delta(T)] = (x_0 - v(0))F^{\frac{1}{p}}(0,b)$$

and

$$CE[\overline{X}_f^{\delta}(T)] = (x_0 - v_1(0))F_1^{\frac{1}{p}}(0, b).$$

Finally, the ratio evolves by

$$R_{bf} = \frac{x_0 - v(0)}{x_0 - v_1(0)}. \tag{6.42}$$

From Propositions 6.3.2 and 6.4.2, we know that $v(0)$ depends on the parameters $\zeta, \theta, h, w_0, T, k, n$ and $v_1(0)$ relies on $\alpha, \theta, h, w_0, T, k, n$. When all these parameters are chosen, we can calculate the value of R_{bf}. Differently, R_{bf} under CRRA utility does not depends on β.

6.6 Numerical Analysis

In this section, we present our numerical analysis in terms of different utility functions. Throughout the analysis, unless otherwise stated, the basic values of the parameters are given by: $\lambda = 0.12$, $r = 0.03$, $\sigma = 0.1$, $w_0 = 20$, $\beta = 0.7$, $\theta = 1$, $x = 10$, $b = 10$, $k = 0.01$, $n = 0.001$, $p = -1$, $q = 0.5$, $T = 40$.

6.6.1 Analysis under CARA utility

A Charge on balance

As shown in Fig.6.1, we present the evolution of the optimal fraction of the risky asset over time and over charge on balance. Note that the return of premium clause has no effect on the optimal investment under the CARA utility. In particular, in Fig.6.1(a), the optimal investment in the risky asset $\delta_b^*(t)$ increases for approximately 25 to 28 years from early to late middle period of the accumulation phase, but as retirement is approaching, $\delta_b^*(t)$ falls. The intuition is that in the

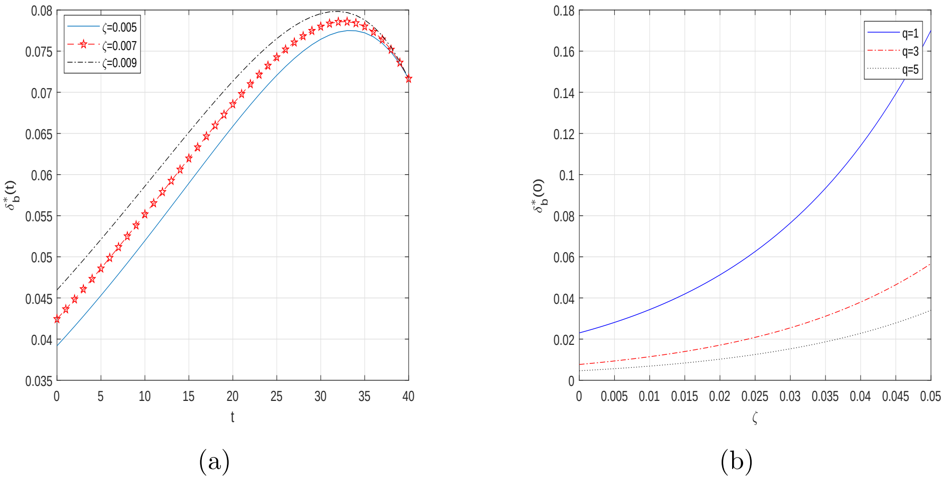


Figure 6.1: (a) The evolution of $\delta_b^*(t)$ over time under different charges on balance ζ. (b) The evolution of $\delta_b^*(t)$ over ζ under different risk-aversion parameter q at $t = 0$.

early accumulation phase, the fund size increases with time and its ability to bare risk increases, so the proportion invested in risky assets grows to obtain higher return. Subsequently, as investment horizon shortens, the ability for the fund to bare risk will decrease because when the pre-retirement phase approaches, the fund should have low level of risk and be ready for distribution. Moreover, as we can see from Fig.6.1(a), higher charge on balance results in larger proportion invested in risky assets. On the one hand, the member requires higher returns as compensation for higher administrative fees. On the other hand, as the manager of the IA, the pension fund administrator (PFA) is willing to allocate more in risky assets to gain more fees, which are based on the size (balance) of the account. That's why we see that $\delta_b^*(t)$ grows rapidly when the charge on balance increases and this can also be observed in Fig.6.1(b). Meanwhile, Fig.6.1(b) shows that larger risk-aversion parameter q leads to smaller $\delta_b^*(t)$ (for $t > 0$, the results are similar), indicating that members who are more risk-averse would like less portion of the fund invested in risky assets.

Fig.6.2 plots the effects of return of the risky asset λ and risk elasticity parameter β on optimal investment with $\zeta = 0.005$ at $t = 0$ (when $t > 0$, similar

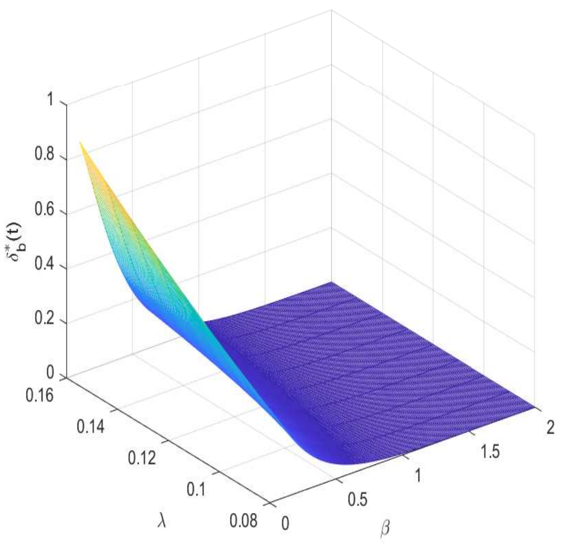


Figure 6.2: Effects of λ and β on the optimal investment in risky assets with $t = 0$ and $\zeta = 0.005$.

results are applicable). We can see that $\delta_b^*(t)$ increases with λ and decreases with β. Comprehensibly, higher return makes the risky asset more attractive to both the pension manager and the member if the level of volatility keeps invariant. Moreover, larger β makes the risky asset price more vulnerable to adverse shocks. In order to protect the fund and reduce risk, the member and the manager would reduce the portion allocated in the risky asset as β increases.

B Charge on flow

Since $\delta_f^*(t)$ under the CARA utility does not depend on the charge on flow and behaves quite similar to $\delta_b^*(t)$, we omit the numerical analysis here, and readers are referred to the exact expression of $\delta_f^*(t)$ in Proposition 6.4.1 and the analysis in Section 6.6.1A.

6.6.2 Analysis under CRRA utility

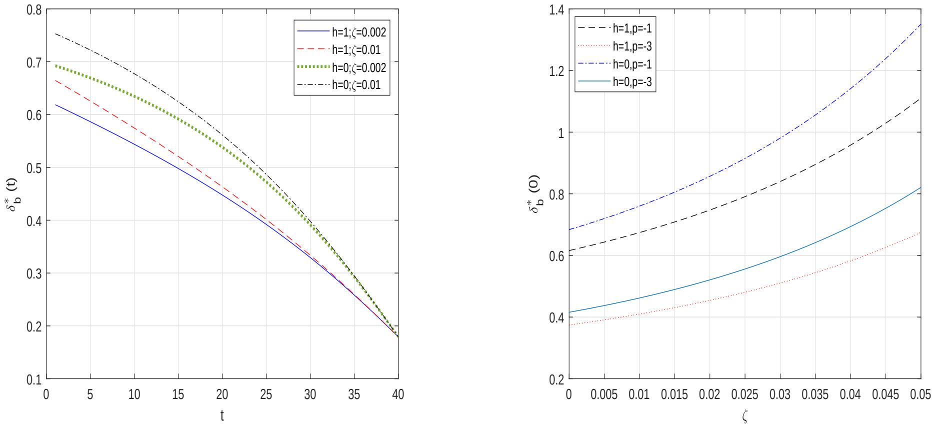
A Charge on balance

Fig.6.3(a) gives the evolution of $\delta_b^*(t)$ over time under the CRRA utility in terms of different charge on balance. As we can see, the optimal investment in the risky asset decreases over time because the ability for the fund to take risk decreases as the investment horizon shortens (longer horizon means more time for the fund to recover from adverse events). In addition, $\delta_b^*(t)$ is lower when $h = 1$ than that when $h = 0$. This can be explained by that the return of premiums would reduce the fund size when the premiums are withdrawn upon the death of a member, so the manager is forced to allocate less into the risky asset as the return of premiums is regarded as a risk factor to the fund. From Fig.6.3(b), we see that the optimal risky investment increases with the charge on balance. As mentioned earlier, higher return is required by the member as compensation for higher fees. From the manager's perspective, higher return also means larger fund size and more fees. In this case, both the member and the manager have the motivation to increase risky allocation (in a reasonable range) as charge on balance increases. Moreover, we observe that larger absolute value of p results in less investment in the risky asset. In other words, the members who are more risk-averse and conservative would hold less risky positions. As for $t > 0$, the results are similar, we omit the explanations.

Fig.6.4 shows the effects of λ and β on $\delta_b^*(t)$ at time $t = 0$. The results and explanation are similar to those of Fig.6.2, and readers are referred to Section 6.6.1A.

B Charge on flow

Fig.6.5(a) plots the relationship between the optimal investment in the risky asset and time t for different charges on flow with $h = 1$ and $h = 0$. Similar to



(a) (b)

Figure 6.3: (a) The evolution of $\delta_b^*(t)$ over time under different charges on balance ζ. (b) The evolution of $\delta_b^*(t)$ over ζ under different risk-aversion parameter p at $t = 0$.

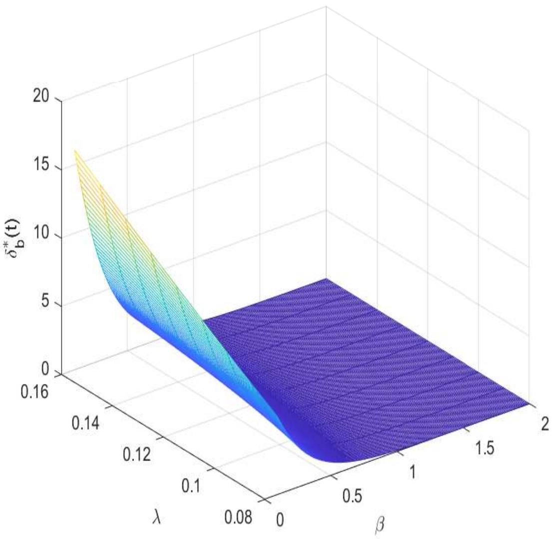
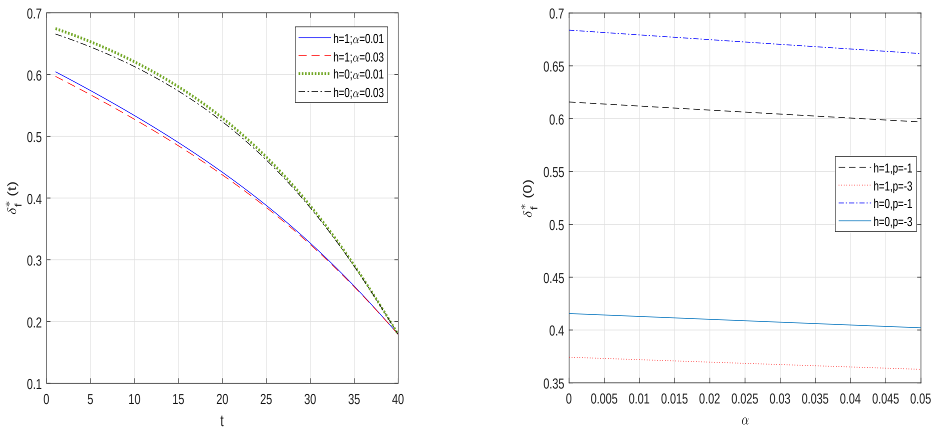


Figure 6.4: Effects of λ and β on the optimal investment in risky assets with $t = 0$ and $\zeta = 0.005$.



(a) (b)

Figure 6.5: (a) The evolution of $\delta_f^*(t)$ over time under different charges on flow α. (b) The evolution of $\delta_b^*(t)$ over α under different risk-aversion parameter p at $t = 0$.

Fig.6.3(a), not surprisingly, $\delta_f^*(t)$ decreases with time. Also, $\delta_f^*(t)$ is lower when $h = 1$ than that when $h = 0$.

Indicated by Fig.6.5(b), we find that the major difference between the cases of charge on balance and on flow is that the optimal investment in the risky asset exhibits a slightly negative relation with the charge on flow but a positive one with the charge on balance. This could be explained by the following reason. The charge on balance is based on the fund size, which aligns the member's interest with the manager's. In this case, the manager is more likely to recommend for risky investments to achieve higher returns. In contrast, the charge on flow is usually a fixed fee as a proportion of the contribution, which provides no motivation for the manager to invest more in the risky asset. In this case, achieving stability is the primary goal.

Fig.6.5(b) also shows that $\delta_f^*(t)$ decreases as the risk-aversion parameter increases in absolute value, which is consistent with the previous examples.

In Fig.6.6, $\delta_f^*(t)$ behaves similarly to $\delta_b^*(t)$ in Fig.6.2 and Fig.6.4, hence we omit the analysis here.

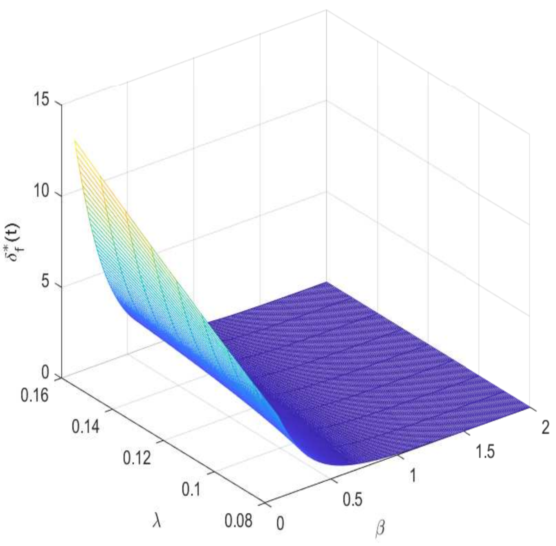


Figure 6.6: Effects of λ and β on the optimal investment in risky assets with $t = 0$ and $\alpha = 0.03$.

6.7 Concluding remark

This chapter studies the optimal investment strategy for a DC pension plan embedded with the return of premium clause as well as two types of administrative fees: the charge on balance and the charge on flow. In our settings, the pension fund could be invested in either a risk-free asset or a risky asset. The CEV model is applied to depict the evolution of the risky asset. In addition, we use the Weibull model to characterize the force function of mortality. The explicit solutions for the optimal strategy under the CARA and CRRA utilities are derived via the maximum principle and the change of variable method. Subsequently, we illustrate how the certainty equivalent of the expected utility works for comparing the two types of fees. Finally, we have the following findings from our numerical analysis. (i) The proportion of fund invested in risky asset will fall when the retirement is approaching; (ii) larger absolute values of the risk-aversion parameters (q and p) will lead to less risky investment; (iii) the optimal investment in the risky asset increases as the return of the risky asset increases and decreases as the elasticity parameter β increases under all the cases discussed; (iv) the return of premium clause has no effect on the optimal investment under the CARA utility while it leads to less risky investment under the CRRA utility; (v) under the CARA utility, the optimal investment in the risky asset increases as the charge on balance increases but is not affected by the charge on flow; (vi) under the CRRA utility, the optimal risky investment behaves the same over time as it does under the CARA utility. However, it slowly decreases as the charge on flow increases.

CHAPTER 7

Summary and future research

7.1 Summary

In this thesis, we investigate several extended continuous-time dynamic models. First of all, we consider a model with time-varying volatility of shocks in chapter 3. Secondly, we investigate a moral hazard model with unknown agent ability in chapter 4. Thirdly, we study an agency model featured with time-varying firm size and volatility of external shocks under full information in chapter 5. Finally, in chapter 6, we deal with the optimal investment problem of a defined contribution pension plan, in which two types of administrative fees and the return of premium clause are taken into account. The main results are summarized as follows.

(i) In the first principal-agent model, we add the time-varying volatility to the output process and derive the optimal contracts under neoclassical, moral hazard, and hidden saving environments. The necessary and sufficient conditions of the implementable contracts are obtained. In our analysis, the volatility function $B(t)$ could be any deterministic function satisfying our restrictions. The model suggests that information frictions reduce effort. When the volatility of shocks increases, effort is further reduced. For extremely volatile environment, the contract would be meaningless or terminated by the principal because the cost of information friction is too high.

(ii) In the second model, a learning process and the agent ability, whose uncertainty would lead to belief manipulation, are considered. The necessary and sufficient conditions for the incentive contracts with normally distributed priors and posteriors about the agent ability are derived. We also give the general assumptions about the effort function that could be applied to the model. The ability uncertainty hinders the principal's incentive provision and makes the agent's wage front-loaded. However, as time goes to infinity, the agent ability will be revealed completely and as a result, belief manipulation disappears. Namely, the unknown ability case degenerates into the known one.

(iii) The optimal compensation and investment are considered in a dynamic model where the cumulative output diffusion process is affected by the time-varying firm size and a non-negative continuously bounded function of time, representing external shocks. We find that large firms provide more protection for the principal's dividend. We establish mathematically that if the external shock term $G^2(t)$ is sufficiently large, the optimal investment plan would tend to positive infinity, which is not realistic for the firm to invest, and the firm distributes no dividends to the principal. Under the risk aversion assumption, small volatility environments are more preferable for the principal.

(iv) The optimal asset allocation problem for a DC pension plan is solved in the last part of this thesis. We add two types of administrative fees and the return of premium clause into the model. The evolution of the risky asset is characterized by a CEV model. And the force of mortality function is described by the Weibull model. After solving the optimal investment in risky asset, we provide numerical analysis in terms of charge on balance, charge on flow, rate of return, risk aversion and elasticity parameters, etc.

7.2 Future research

In this thesis, we study four extended dynamic continuous time models regarding principal-agent problems, optimal compensation and investment. We add various new features such as time-varying volatility, ability uncertainty, external shocks, and additional fees to the existing models to analyze problems under more realistic environments. However, there are many limitations and further studies could be made as follows.

(i) In order to obtain the explicit solution to the optimal contracts, we assume both the principal and the agent believe in the CARA (exponential) utilities. However, in practice, the two parties probably have different utility functions, which lead to different types of value functions. So various types of utility functions could be considered in future researches.

(ii) In our settings, we assume that the time-varying volatility is a deterministic function of time as in our examples. In our opinion, volatility of business cycles should be modeled as being stochastic, which is difficult to deal with. Hopefully, we could overcome the technical barriers and solve the problem featured with random and persistent shocks.

(iii) In our third model, due to the time-varying external shock term, we only solve the optimal compensation and investment under full information case, where there is no information frictions. Information friction could be added to fit more realistic environments.

(iv) In some DC pension plans, the administrative fees could alter the behaviors of the managers to a large extent, as their objective is to maximize the utility of total rewards. The pension manager's utility could be considered in the objective function in our future research.

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Appendix 1. Statement of Candidate's Contributions to Joint-Authored Papers

To Whom It May Concern,

I, Chong Lai, made major contributions in the design of the research work, development of theories, analysis of results drafting of the paper entitled *'Optimal compensation and investment affected by firm size and time-varying external shocks.' Annals of Finance,* **accepte***d, 2020.*

Chong Lai

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Rui Li

YongHong Wu

To Whom It May Concern,

I, Chong Lai, made major contributions in the design of the research work, development of theories, analysis of results drafting of the paper entitled *'The optimal solution to a principal-agent problem with unknown agent ability.' Journal of Industrial and Management Optimization,* doi: *10.3934/jimo.2020084, 2020.*

Chong Lai

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Lishan Liu

Rui Li

To Whom It May Concern,

I, Chong Lai, made major contributions in the design of the research work, development of theories, analysis of results drafting of the paper entitled *'The investigation of optimal contracts in principal-agent model with time-varying volatility.' Journal of Economic Dynamics and Control, under review, 2019*.

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I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

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