School of Electrical Engineering, Computing and Mathematical Sciences

Dynamical Properties of Solutions for Various Types of Nonlinear Partial Differential Equations

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Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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Abstract

In this thesis, we study the dynamic properties of solutions for various types of nonlinear partial differential equations, including a generalized Benjamin-Bona-Mahony-Burgers equation, three integrable non-evolutionary equations with quadratic nonlinearities, cubic nonlinearities and quartic nonlinearities, respectively, and a generalized Degasperis-Procesi equation for the motion of shallow water waves.

For the generalized Benjamin-Bona-Mahony-Burgers (GBBMB) equation, applying approximation approaches and several estimates derived from the equation, we obtain a space-time higher integrability estimate and a one-sided super bound estimate on the first order spatial derivative of the solution. By making some assumptions on initial data in the space $H^1(\mathbb{R})$, it is derived that the GBBMB equation has at least one global weak solution in the space $C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$. Using Kruzkov's technique of doubling the space variable, if the GBBMB equation has local or global strong solutions in $L^1(\mathbb{R}) \cap H^1(\mathbb{R})$, it is demonstrated that its strong solution has the property of $L^1(\mathbb{R})$ local stability.

For the integrable non-evolutionary equation with quadratic nonlinearities and quasi-local higher symmetries, we prove that its local strong solutions are well-posedness in the space $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ if the Sobolev index $s > \frac{3}{2}$. A condition for blow-up solutions is established under suitable assumptions. For the case of the Sobolev index $1 \le s \le \frac{3}{2}$, we show that the equation has local weak solutions in $H^s(\mathbb{R})$.

For the non-evolutionary equation with cubic nonlinearities and quasi-local higher symmetries, we find the $H^1(\mathbb{R})$ conservation law. By using Aubin's com-

pactness theorem and several estimates, we prove that the equation possesses local weak solutions in $L^2([0,T], H^s(\mathbb{R}))(1 \leq s \leq \frac{3}{2})$. If the Sobolev index $s > \frac{3}{2}$, by using approximation techniques and constructing a Cauchy sequence of the solutions in the space $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$, we derive that there exists a unique local strong solution to the equation.

For the nonlinear Camassa-Holm type equation possessing quartic nonlinearities, by supposing that its initial value $u_0(x) \in H^1(\mathbb{R})$ and $\| \frac{\partial u_0(x)}{\partial x} \|_{L^\infty(\mathbb{R})} < \infty$, we prove that the equation has at least one global weak solution in the space $C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$. Our assumptions about the initial value $u_0(x)$ are different from the sign condition which is often required to establish the well-posedness of global strong solutions and the existence of global weak solutions for the Camassa-Holm type equations. Our key contributions include establishing a space-time higher integrability estimate and a super bound estimate on the first order spatial derivative of the solution.

For the nonlinear shallow water wave equation, including the famous Degasperis-Procesi equation, several estimates are established to discuss the wave breaking of the solution. The sufficient and necessary conditions for the wave breaking are obtained.

List of publications during PhD candidature

- Rui Li, Chong Lai and Yonghong Wu. Global weak solutions to a generalized Benjamin-Bona-Mahony-Burgers equation. *Acta Mathematica Scientia*, vol.**38**(3), 915-925, 2018.
- Chong Lai, Rui Li and Kexin Luo. The L^1 stability to a generalized Benjamin-Bona-Mahony-Burgers model. *Dynamics of Continuous, Discrete and Impulsive Systems Series B*, vol.**26**(2), 123-132, 2019.

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Contents

CHAPTER 1

Introduction

1.1 Aims and outcomes of the thesis

The aim of this thesis is to investigate the dynamical properties of five nonlinear partial differential equations, including a generalized Benjamin-Bona-Mahony-Burgers model, a generalized Degasperis-Procesi equation and three integrable non-evolutionary equations, which are regarded as the generalizations of the Camassa-Holm type equations [87]. The three non-evolutionary equations possess quasi-local higher symmetries, and have quadratic, cubic and quartic nonlinearities, respectively.

The specific objectives of this study and the main outcomes achieved are as follows.

(1). We study a nonlinear generalized Benjamin-Bona-Mahony-Burgers (GBBM-B) equation, which takes the form

$$u_t - u_{txx} - au_{xx} + bu_x + u^p u_x + ku_{xxx} = 0, \tag{1.1}$$

where $p \geq 1$ is an integer, $a \geq 0$, b and k are constants. Eq.(1.1) becomes the nonlinear Benjamin-Bona-Mahony-Burgers equation (BBMB) if $k = 0$. By letting $a = 0, b = 1, p = 1, k = 0$, Eq.(1.1) is turned into the Benjamin-Bona-Mahony model (see [5, 90, 92]). For Eq.(1.1) with the coefficient $k = 0$, many scholars have investigated the existence of global weak solutions and strong solutions (see [1, 2, 8, 56]). For Eq.(1.1) with the coefficient $k \neq 0$, imposing certain restrictions on the initial value, we derive that there exists at least one global weak solution to the GBBMB equation (1.1). Applying the technique of doubling the space variables presented in Kruzkov's work [53], a few prior estimates are derived from Eq.(1.1). Assuming that the equation has strong solutions in $L^1(\mathbb{R}) \cap H^1(\mathbb{R})$, we prove that these strong solutions are local stable in $L^1(\mathbb{R})$. The acquired results for Eq.(1.1) are different from those in previous works.

(2). Novikov classified and generalized the Camassa-Holm-type models in [87]. The generalized equations possess integrability and quasi-local higher symmetries. One of the equations is in the form

$$u_t - \alpha^2 u_{txx} + 4uu_x - 2\alpha u_x^2 - 2\alpha u u_{xx} = 6\alpha^2 u_x u_{xx} + 2\alpha^2 u u_{xxx}, \qquad (1.2)$$

where $\alpha \neq 0$ is a constant. Eq.(1.2) is one of the equations of Theorem 3 in Novikov [87] under the scaling that x transforms $-x$ (see [87]). Eq.(1.2) is an integrable scalar evolution equation with quadratic nonlinearities.

Motivated by the works presented in [58, 59, 66] and the desire to probe and find the dynamical properties for Eq.(1.2), we investigate Eq.(1.2). Using the Kato theorem, it is proved that Eq.(1.2) possesses a unique local strong solution in the space $C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$ if the Sobolev index $s > \frac{3}{2}$. A sufficient and necessary condition for blow-up solutions is found. For the case of the Sobolev index $1 \leq s \leq \frac{3}{2}$, it is shown that Eq.(1.2) has local weak solutions in $H^s(\mathbb{R})$. As far as we know, these dynamical properties of Eq.(1.2) have not been found in the previous literature.

(3). Novikov [87] derived many equations which are called the generalizations of the Camassa-Holm type models. Several models of the generalizations possess quasi-local higher symmetries. One of the equations with cubic nonlinearities takes the form

$$u_t - \alpha^2 u_{txx} = (1 + \alpha\frac{\partial}{\partial x})(\alpha u^2 u_{xx} + \alpha u u_x^2 - 2u^2 u_x), \tag{1.3}$$

where the constant $\alpha \neq 0$ (see Theorem 5 in [87]).

Motivated by the works made in Li and Olver [66] in which the existence of local weak solutions and the well-posedness of local strong solutions for an integrable non-linear dispersive wave equation including the standard Camassa-Holm equation, have been considered, we study Eq.(1.3). Imposing certain restrictions on its initial value and the coefficient α, several dynamical properties including the existence of local weak solutions, the existence and uniqueness of local strong solutions have been found. The results for Eq.(1.3) in this thesis have not been obtained in the previous literature.

(4). For the following nonlinear Camassa-Holm type equation with quartic nonlinearities (see [40, 41, 76])

$$u_t - u_{txx} + 5u^3 u_x = 4u^2 u_x u_{xx} + u^3 u_{xxx}, \tag{1.4}$$

we investigate its existence of global weak solutions. Assuming the initial value $u_0(x) \in H^1(\mathbb{R})$ and $\| \frac{\partial u_0(x)}{\partial x} \|_{L^\infty(\mathbb{R})} < \infty$, we prove that Eq.(1.4) has at least one global weak solution in the space $C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$. Here we mention that we do not need to assume that the $u_0(x)$ satisfies the sign condition. Our assumption is weaker than the sign condition. In fact, many researchers apply the sign condition to establish and derive the existence of global weak solutions of the Camassa-Holm type equations such as the Degasperis-Procesi equation and many other generalized Camassa-Holm equations(see [25, 26, 28, 35–39, 77]). The key contributions in our study of Eq.(1.4) include deriving a space-time higher integrability estimate and a super bound estimate about $\frac{\partial u(t,x)}{\partial x}$, which play a key role in demonstrating the existence of global weak solutions.

(5). A nonlinear shallow water wave equation, including the standard Degasperis-Procesi shallow water wave equation, is investigated. Several estimates, which are derived from the shallow water model itself, are established to discuss the wave breaking of the solutions. A necessary and sufficient condition is obtained when the wave breaking occurs.

1.2 Outline of the thesis

Eight chapters constitute the contents of this thesis.

In Chapter 1, the objectives and outcomes of this thesis are briefly illustrated.

In Chapter 2, we give a literature review of the previous works relating to the nonlinear Benjamin-Bona-Mahony-Burgers equations and the Camassa-Holm type equations.

In Chapter 3, by using the approaches in Xin and Zhang [101] (also see [9]), we prove that the generalized Benjamin-Bona-Mahony-Burgers equation Eq.(1.1) has at least one global weak solution in the space $C([0,\infty)\times\mathbb{R})\cap L^\infty([0,\infty);H^1(\mathbb{R}))$ under certain assumptions. Applying the technique of doubling the space variables provided in Kruzkov's work [53] and assuming that Eq.(1.1) has strong solutions, we prove that the strong solution is local stable in $L^1(\mathbb{R})$ by imposing restrictions on the initial value.

In Chapter 4, for the integrable non-evolutionary Eq.(1.2), which possesses quadratic nonlinearities, the existence of local weak solutions, the existence and uniqueness of local strong solution and blow-up criteria are investigated.

In Chapter 5, for the integrable non-evolutionary Eq.(1.3) with cubic nonlinearities, the existence of local weak solutions and the well-posedness of local strong solution are discussed.

In Chapter 6, for the nonlinear Camassa-Holm type Eq.(1.4) possessing quartic nonlinearities, supposing that the initial value $u_0(x)$ satisfies $u_0(x) \in H^1(\mathbb{R})$ and $\| \frac{\partial u_0(x)}{\partial x} \|_{L^\infty(\mathbb{R})} < \infty$, we establish the existence of global weak solutions in the space $C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$.

In Chapter 7, for a nonlinear shallow water wave model including the standard Degasperis-Procesi equation, we obtain the conditions to guarantee the wave breaking of the solutions.

In Chapter 8, we summarize the main results obtained in this thesis and discuss several problems to be considered in the future.

CHAPTER 2

Literature Review

In recent decades, many investigations have been carried out worldwide to study the Benjamin-Bona-Mahony model, the Benjamin-Bona-Mahony-Burgers equation, the Cammassa-Holm (CH) equation, the Degasperis-Procesi (DP) equation and the Novikov equation [3, 4, 17, 29–32]. Various approaches to probe the dynamical properties of these nonlinear partial differential equations and their generalizations have been established [11–16]. Based on the recent development in this field, our research focuses on the study of a generalized Benjamin-Bona-Mahony-Burgers model, a generalized Degasperis-Procesi model and three integrable non-evolutionary equations, which are related to the CH equation, the DP equation and the Novikov equation, because these equations have similar dynamical properties [20, 21, 43–45, 48–50, 52].

2.1 Background of the Benjamin-Bona-Mahony-Burgers type equations

The Benjamin-Bona-Mahony-Burgers equation takes the form

$$u_t - u_{txx} - au_{xx} + bu_x + u^p u_x = 0, \tag{2.1}$$

where $a \geq 0$ is a constant, b is an arbitrary constant, and integer $p \geq 1$. Setting $a = 0, b = 1$ and $p = 1$, Eq.(2.1) is turned into the Benjamin-Bona-Mahony model (see [5, 7, 89]).

Many researchers have investigated the Benjamin-Bona-Mahony-Burgers equation and its various generalizations. Benjamin et al. [5] discover the nonlinear stability of nonlinear periodic solutions of the regularized Benjamin-one equation and the BBM equation associated with perturbations of the wavelength. The long time existence result for the Cauchy problem related to the BBM-Boussinesq systems has been studied in [6]. Chen and Wang [8] give a stability criteria for the solitary wave solutions of the Benjamin-Bona-Mahony-Burgers equation which contains coupled nonlinear terms. The periodic initial value problem which contains the generalized Benjamin-Bona-Mahony equation with generalized damping on one dimensional torus is considered in Kang et al. [56]. It is pointed out in Mei [84] that the good predictive power in Eq.(2.1) is meaningful in the physical sense. Equation (2.1) and the Benjamin-Bony-Mahony equation possess the same dispersive effects [84]. The tanh technique with the aid of symbolic computational system is employed to find the exact solutions of BBMB-type equations [89]. The homogeneous balance method and symbolic computations are employed in [1, 89] to find the exact expressions of traveling wave solutions for the Benjamin-Bona-Mahoney equation. Other dynamical properties relating to Eq.(2.1) are discovered in [51, 92, 93].

2.2 Background of the related nonlinear equations

The standard Camassa-Holm(CH) equation takes the form

$$u_t - u_{txx} + \alpha u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad \alpha = \text{constant}, \tag{2.2}$$

which is first discovered by Fuchssteiner and Fokas [30]. Camassa and Holm later derived it as a water wave model [19]. Its alternative derivations are conducted in Constantin and Lannes [13], and Johnson [52]. Several conservation laws are found for Eq.(2.2) (see [24, 65, 78]). Eq.(2.2) has solitary wave solutions if $\alpha > 0$ or peaked solution if $\alpha = 0$ (see [14–16]). The geodesic flow properties of Eq.(2.2) are discussed in [18, 57, 74].

Degasperis et al. [22] investigate the integrability of the peaked Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{2.3}$$

which is derived in [23].

In fact, many scientists have been devoted to the investigations of various dynamical properties of the CH and DP equations. The global conservative solutions and dissipative solutions to Eq.(2.2) are discovered in Bressan and Constantin [3, 4]. The global strong and weak solutions, and blow-up phenomena for CH equation are studied in [12, 26, 27]. Sufficient conditions to guarantee the wave breaking for nonlinear nonlocal equations including the CH model are given in Constantin and Escher [14] in which several meaningful conclusions are drawn. Nonlinear dispersive wave equations relating to (2.2) and (2.3) are discussed in

Escher and Yin [26] while the global weak solutions and blow-up structures are considered in [25, 27, 69, 70] by imposing certain restrictions on the initial boundary data. The asymptotic stability and controllability for the CH equation are studied in Glass [42]. It is shown in Xin and Zhang [101] that Eq.(2.2) has global weak solutions in $H^1(\mathbb{R})$ without the assumption of sign condition (also see [9]). The existence of local weak solutions and the well-posedness of local strong solutions for Eq.(2.2) in the Sobolev space are established in Li and Olver [66] where a sufficient condition for blow-up solutions is found. The long time properties of low regularity solutions for Eq.(2.2) is given in Li [67]. The traveling wave solutions to the CH and DP models are classified in Lenells [63, 64]. Employing an appropriate Kodama transformation, Dullin et al. [24] derive Eq.(2.3) from the shallow water elevation model. Vakhnenko and Parkes [96] find many traveling wave solutions for Eq.(2.3). An inverse scattering technique to find n-peakon solutions to the DP model is employed in Lundmark and Szmigielski [71]. If the sign condition of the initial data holds, the stability of peakons for the DP equation is considered in Lin and Liu [68]. Matsuno [72] discusses multisoliton solutions and peakon limits for Eq.(2.3). The infinite speed of propagation for the smooth solutions to Eq.(2.3) is studied in Henry [46]. Coclite and Karlsen [11] obtain existence of entropy solutions for Eq.(2.3).

Novikov [87] derives the integrable equation with cubic nonlinearities

$$u_t - u_{txx} + 4u^2 u_x = 3u u_x u_{xx} + u^2 u_{xxx}. \tag{2.4}$$

Grayshan [41] investigates the peaked solutions of the Novikov model (2.4). The CH, DP and Novikov equations have similar peaked properties (see [75, 79, 95, 97–100]). The well-posedness for Eq.(2.4) in the Sobolev space are studied in Himonas and Holliman [47]. If the sign condition about the initial data holds, Lai and Wu [58] discuss global strong solutions to Eq.(2.4) in $C^1([0,\infty); H^s(\mathbb{R})) \cap$

$C([0,\infty); H^{s-1}(\mathbb{R}))$ if $s > \frac{3}{2}$. Applying the Kato theorem, Ni and Zhou [86] demonstrate the well-posedness of local strong solution of Eq.(2.4) in $H^s(\mathbb{R})$ if the index $s > \frac{3}{2}$. The persistence properties of the smooth solution to Eq.(2.4) are also found in [86]. Mi and Mu [85] consider the Cauchy for a modified Novikov equations and find its weak solution. A weakly dissipative Novikov equation is studied in Yan et al. [107].

In fact, establishing integrable equations is one of the important duties for mathematical experts in the field of partial differential equations [33, 34, 39, 73, 88,102–106,108]. Novikov [87] studies the problem of integrability to the following Camassa-Holm type equation

$$u_t - u_{txx} = F(u, u_x, u_{xx}, u_{xxx}, \cdots), \tag{2.5}$$

where F is a polynomial about $u, u_x, u_{xx}, u_{xxx} \cdots$. Using the existence of an infinite hierarchy of (quasi$-$) local higher symmetries as a definition of integrability, Novikov [87] derives and finds many integrable equations with quadratic, cubic and quartic nonlinearities which include Eqs.(2.2),(2.3), (2.4) and the equations which we will investigate in chapters 4, 5 and 6.

For other investigations on the Camassa-Holm equation, the Degasperis-Procesi model, the Novikov equation and several other related partial differential equations, we refer the readers to the literature [60–62, 80–83, 91] and the references therein.

CHAPTER 3

Global weak solutions and L^1 local stability to a generalized Benjamin-Bona-Mahony-Burgers equation

In this chapter, we discuss the Cauchy problem for a generalized Benjamin-Bona-Mahony-Burgers (GBBMB) equation. If the initial value is in $H^1(\mathbb{R})$, it is shown that the GBBMB model has at least one global weak solution in the space $C([0,\infty)\times\mathbb{R})\cap L^\infty([0,\infty);H^1(\mathbb{R}))$. The key elements in our proof of the existence of global weak solutions include establishing a space-time higher integrability estimate and a super bound estimate on the first order spatial derivatives of the solution. Subsequently, we investigate the local stability for the GBBMB equation. If the initial value belongs to the space $L^1(\mathbb{R})\cap H^1(\mathbb{R})$, assuming that the GBBMB equation has strong solutions, and employing the tool of doubling the space variables provided in Kruzkov's work [53], we prove that the strong solution possesses $L^1(\mathbb{R})$ local stability.

3.1 General

Firstly, we introduce some notations used in this chapter.

Let $L^\infty = L^\infty(\mathbb{R})$ represent all the functions $g(t,x)$ satisfying

$$\| g \|_{L^\infty} = \inf_{m(e)=0} \sup_{x\in\mathbb{R}\setminus e} |g(t,x)| < \infty,$$

where m denotes the measure.

Let $\mathbb{R} = (-\infty, \infty)$. We write notation C_0^∞ to represent all functions $g(t,x) \in C^\infty$ which have compact support in the domain $[0,+\infty) \times \mathbb{R}$.

Assume that $L^p = L^p(\mathbb{R})$ $(1 \le p < \infty)$ contains all functions g with the norm

$$\| g \|_{L^p} = \Big(\int_R |g(t,x)|^p dx \Big)^{\frac{1}{p}} < \infty.$$

We let $H^s = H^s(\mathbb{R})$ represent the Sobolev space with norm satisfying

$$\| g \|_{H^s} = \left(\int_{\mathbb{R}} (1+|\eta|^2)^s |\hat{g}(t,\eta)|^2 d\eta \right)^{\frac{1}{2}} < \infty,$$

where s is an arbitrary real number, $\hat{g}(t,\eta) = \int_{-\infty}^{\infty} e^{-ix\eta} g(t,x)dx$. Namely, $\hat{g}(t,\eta)$ denotes the Fourier transformation of function $g(t,x)$ about variable x.

For $T > 0$ and $s > 0$, let $C([0,T); H^s(\mathbb{R}))$ represent the Frechet space of all continuous H^s-valued functions on the interval $[0,T)$ (see [54, 55]).

Set $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. For simplicity, let c represent any positive constant which does not rely on the parameter ε.

In this chapter, we consider the Cauchy problem of the following generalized Benjamin-Bona-Mahony-Burgers equation

$$u_t - u_{txx} - au_{xx} + bu_x + u^p u_x + ku_{xxx} = 0, \tag{3.1}$$

where $a \geq 0, b, k$ are constants, and the integer $p \geq 1$.

As illustrated in section 2.1 of this thesis, the existence of global weak solutions and strong solutions for Eq.(3.1) with $k = 0$ has been investigated by many scholars. One task of this chapter is to establish the existence of global weak solutions for (3.1) with the term ku_{xxx} $(k \neq 0)$ under the assumption that its initial value belongs to the space $H^1(\mathbb{R})$. The other is to consider the local stability of its strong solutions.

Here we address that the approaches in [101] will be used to prove our main result (also see [9]). After we obtain the higher integrability estimate (3.27) and the one-sided super bound estimate (3.33) in section 3.3, considering the derivative $q_\varepsilon = \frac{\partial u_\varepsilon(t,x)}{\partial x}$ (see (3.19)), which is only weakly compact, we will show that the derivative converges strongly. Namely, we will prove that this weak convergence is equivalent to strong convergence.

The structure of this chapter is as follows. We provide the main conclusions in section 3.2, and various lemmas about the viscous approximation problem are given in section 3.3. We prove strong compactness of the spatial derivative of solutions for the approximation problem and give the proof of existence of global weak solutions in section 3.4. The $L^1(\mathbb{R})$ local stability is investigated in section 3.5.

3.2 Main results

We write the Cauchy problem for equation (3.1) in the form

$$\begin{cases} u_t - u_{txx} - au_{xx} + bu_x + u^p u_x + ku_{xxx} = 0, \\ u(0, x) = u_0(x). \end{cases} \tag{3.2}$$

Applying the operator $\Lambda^{-2} = (1 - \frac{\partial^2}{\partial x^2})^{-1}$ on the first equation of problem (3.2), we have

$$\begin{cases} u_t - ku_x + au - a\Lambda^{-2}u + \Lambda^{-2}\partial_x\big[(b+k)u + \frac{1}{p+1}u^{p+1}\big] = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{3.3}$$

In fact, for function $g(t,x) \in L^p(\mathbb{R})$, $(1 \leq p < \infty)$ or $\| g(t,\cdot) \|_{L^\infty(\mathbb{R})} < \infty$, we have

$$\Lambda^{-2}g(t,x) = \frac{1}{2}\int_{\infty}^{\infty} e^{-|x-y|}g(t,y)dy. \tag{3.4}$$

Using integration by parts and Eq.(3.1), we derive that

$$\int_{-\infty}^{\infty}(u^2 + u_x^2)dx + 2a\int_0^t\int_{-\infty}^{\infty} u_x^2 dxdt = \int_{-\infty}^{\infty}(u_0^2 + u_{0x}^2)dx, \tag{3.5}$$

where $u_{0x} = \frac{\partial u(0,x)}{\partial x}$. From (3.5) and condition $a \geq 0$, we obtain

$$\| u \|_{L^\infty(\mathbb{R})} \leq c \| u_0 \|_{H^1(\mathbb{R})}, \tag{3.6}$$

where the constant $c > 0$.

We now introduce the definition of global weak solutions (see [101]) below.

Definition 3.1. *A function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is called to be a global weak solution to the Cauchy problem (3.2) or (3.3) if*

(i) $u \in C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$;

(ii) $\| u(t,.) \|_{H^1(\mathbb{R})} \leq \| u_0 \|_{H^1(\mathbb{R})}$;

(iii) $u(t,x)$ is the solution of (3.3) in the sense of distributions.

Now we state the main results in this chapter.

Theorem 3.2. *Let constant $a \geq 0$, $u_0(x) \in H^1(\mathbb{R})$ and $|\frac{\partial u_0(x)}{\partial x}| < \infty$. Then there exists at least one global weak solution $u(t,x)$ to problem (3.2) or (3.3)* *in the sense of the Definition 3.1. In addition, the global weak solution has the following properties.*

(a). There exists a positive constant c_0, which relies only on $\| u_0 \|_{H^1(\mathbb{R})}$ and the coefficients of Eq.(3.1) such that the one-sided estimate on the first order spatial derivative

$$\frac{\partial u(t,x)}{\partial x} \leq c_0(1+e^{-at}), \qquad for \quad (t,x) \in [0,\infty) \times \mathbb{R} \tag{3.7}$$

holds.

(b). For any $T \in (0,\infty)$, there exists a positive constant c_1, which relies only on $\| u_0 \|_{H^1(\mathbb{R})}$ and the coefficients of Eq.(3.1) such that the space higher integrability estimate

$$\int_0^t \int_{-\infty}^{\infty} \Big| \frac{\partial u(t,x)}{\partial x} \Big|^4 dxdt \leq c_1 T e^{c_1 T} \quad t \in [0,T] \tag{3.8}$$

is valid.

Theorem 3.3. *Assume that Eq.(3.1) has two strong solutions $u_1(t,x)$ and $u_2(t,x)$ associated with $u_i(0,x) = u_{i,0} \in L^1(\mathbb{R}) \cap H^1(\mathbb{R}), i = 1,2$. Suppose that both $u_1(t,x)$ and $u_2(t,x)$ have a maximum existence time T_0. Then, for every $t \in [0,T_0)$,*

$$\| u_1(t,\cdot) - u_2(t,\cdot) \|_{L^1(\mathbb{R})} \leq ce^{ct} \| u_{1,0} - u_{2,0} \|_{L^1(\mathbb{R})}, \tag{3.9}$$

where constant $c > 0$ depends on $\| u_{1,0} \|_{H^1(\mathbb{R})}$ and $\| u_{2,0} \|_{H^1(\mathbb{R})}$.

Theorem 3.4. *Suppose that $u(t,x)$ is a strong solution of problem (3.2) associated with initial value $u(0,x)$ belonging to the space $L^1(\mathbb{R}) \cap H^1(\mathbb{R})$. Then the strong solution is unique.*

Theorem 3.3 directly yields Theorem 3.4.

3.3 Viscous approximations

Set

$$\psi(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1 \end{cases} \tag{3.10}$$

and $\psi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\psi(\varepsilon^{-\frac{1}{4}}x)$ where $0 < \varepsilon < \frac{1}{4}$.

Define the convolution

$$u_{\varepsilon,0} = \psi_\varepsilon \star u_0 = \int_{-\infty}^{\infty} \psi(x-y)u_0(y)dy. \tag{3.11}$$

We know that $u_{\varepsilon,0} \in C^\infty$ for any $u_0 \in H^s(\mathbb{R})(s > 0)$ (see Lai and Wu [58, 59]) and

$$\| u_{\varepsilon,0} \|_{H^1(\mathbb{R})} \leq \| u_0 \|_{H^1(\mathbb{R})} \qquad \text{and} \quad u_{\varepsilon,0} \to u_0 \quad \text{in} \quad H^1(\mathbb{R}). \tag{3.12}$$

We aim to show that system (3.3) has global weak solution. Firstly, we handle the viscous approximation problem

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} - k\frac{\partial u_\varepsilon}{\partial x} + au_\varepsilon - a\Lambda^{-2}u_\varepsilon + \Lambda^{-2}\partial_x\Big((b+k)u_\varepsilon + \frac{1}{p+1}u_\varepsilon^{p+1}\Big) = \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x). \end{cases} \tag{3.13}$$

The convergence of smooth solution $\{u_\varepsilon\}$ and $\frac{\partial u_\varepsilon(t,x)}{\partial x}$ will be analyzed.

Now we give the well-posedness conclusion for system (3.13).

Lemma 3.5. *Let $u_0 \in H^1(\mathbb{R})$. Then, there exists a unique solution $u_\varepsilon \in$*

$C([0,\infty);H^\sigma(\mathbb{R}))$ $(\sigma \geq 2)$ to problem (3.13) with u_ε satisfying

$$\int_{\mathbb{R}} \Big(u_\varepsilon^2 + (\frac{\partial u_\varepsilon}{\partial x})^2\Big)dx + 2a\int_0^t\int_{-\infty}^{\infty}(\frac{\partial u_\varepsilon}{\partial x})^2 dxds \\ +2\varepsilon\int_0^t\int_{\mathbb{R}}\Big[(\frac{\partial u_\varepsilon}{\partial x})^2 + (\frac{\partial^2 u_\varepsilon}{\partial x^2})^2\Big]dxds = \| u_{\varepsilon,0} \|^2_{H^1(\mathbb{R})}, \tag{3.14}$$

or

$$\| u_\varepsilon(t,.) \|^2_{H^1(\mathbb{R})} + 2\varepsilon\int_0^t \| \frac{\partial u_\varepsilon}{\partial x}(s,.) \|^2_{H^1(\mathbb{R})} ds \\ +2a\int_0^t\int_{-\infty}^{\infty}(\frac{\partial u_\varepsilon}{\partial x})^2 dxds = \| u_{\varepsilon,0} \|^2_{H^1(\mathbb{R})} . \tag{3.15}$$

Proof. For $u_0 \in H^1(\mathbb{R})$ and every $\sigma \geq 2$, we know that function $u_{\varepsilon,0}$ belongs to the space $C([0,\infty);H^\sigma(\mathbb{R}))$. Using theorem 2.3 in [10], we derive that there exists a unique solution $u_\varepsilon(t,x) \in C([0,\infty);H^\sigma(\mathbb{R}))$ for system (3.13).

From system (3.13), we get

$$\frac{\partial u_\varepsilon}{\partial t} - \frac{\partial^3 u_\varepsilon}{\partial t x^2} - a\frac{\partial^2 u_\varepsilon}{\partial x^2} + b\frac{\partial u_\varepsilon}{\partial x} + u_\varepsilon^p\frac{\partial u_\varepsilon}{\partial x} + k\frac{\partial^3 u_\varepsilon}{\partial x^3} = \varepsilon\Big(\frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{\partial^4 u_\varepsilon}{\partial x^4}\Big). \tag{3.16}$$

Multiplying (3.16) by u_ε and then employing integration by parts, we have the following indentity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big(u_\varepsilon^2 + (\frac{\partial u_\varepsilon}{\partial x})^2\Big)dx + a\int_{-\infty}^{\infty}(\frac{\partial u_\varepsilon}{\partial x})^2 dx \\ +\varepsilon\int_{\mathbb{R}}\Big((\frac{\partial u_\varepsilon}{\partial x})^2 + (\frac{\partial^2 u_\varepsilon}{\partial x^2})^2\Big)(t,x)dx = 0. \tag{3.17}$$

Integrating (3.17) over $[0,t]$ gives rise to (3.14) or (3.15). The proof is completed. ■

Using the definition of $u_{\varepsilon,0}$ and Lemma 3.5 results in

$$\| u_\varepsilon \|_{L^\infty(\mathbb{R})} \leq \| u_\varepsilon \|_{H^1(\mathbb{R})} \leq \| u_{\varepsilon,0} \|_{H^1(\mathbb{R})} \leq \| u_0 \|_{H^1(\mathbb{R})} . \tag{3.18}$$

Writing $\frac{\partial u_\varepsilon}{\partial x} = q_\varepsilon$, we obtain

$$\begin{aligned} \frac{\partial q_\varepsilon}{\partial t} - k\frac{\partial q_\varepsilon}{\partial x} - \varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2} + aq_\varepsilon &= a\Lambda^{-2}q_\varepsilon + (b+k)u_\varepsilon + \frac{1}{p+1}u_\varepsilon^{p+1} \\ &\quad -\Lambda^{-2}\Big((b+k)u_\varepsilon + \frac{1}{p+1}u_\varepsilon^{p+1}\Big) \\ &= K_\varepsilon(t,x). \end{aligned} \tag{3.19}$$

Lemma 3.6. *For $K_\varepsilon(t,x)$, if $u_0 \in H^1(\mathbb{R})$, then*

$$\| K_\varepsilon(t,\cdot) \|_{L^\infty(\mathbb{R})} \leq c, \tag{3.20}$$

$$\| K_\varepsilon(t,\cdot) \|_{L^2(\mathbb{R})} \leq c, \tag{3.21}$$

$$\| \frac{\partial K_\varepsilon(t,\cdot)}{\partial x} \|_{L^2(\mathbb{R})} \leq c, \tag{3.22}$$

where c depends only on the coefficients of Eq.(3.1) and $\| u_0 \|_{H^1(\mathbb{R})}$.

Proof. Applying the property of operator Λ^{-2}, we obtain

$$\begin{aligned} &\Lambda^{-2}\Big((b+k)u_\varepsilon + \frac{1}{k+1}u_\varepsilon^{p+1}\Big) \\ &\quad = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\Big((b+k)u_\varepsilon + \frac{1}{k+1}u_\varepsilon^{p+1}\Big)dy \end{aligned} \tag{3.23}$$

and

$$
\begin{aligned}
|\Lambda^{-2}q_\varepsilon| &= |\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\frac{\partial u_\varepsilon}{\partial y}dy| \\
&= |\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\frac{\partial u_\varepsilon}{\partial y}dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}\frac{\partial u_\varepsilon}{\partial y}dy| \\
&= |-\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}u_\varepsilon dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}u_\varepsilon dy| \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}|u_\varepsilon(t,y)|dy \\
&\le c. \qquad (3.24)
\end{aligned}
$$

From (3.18), (3.23) and (3.24), we get that (3.20) holds.

Since

$$\| \Lambda^{-2}q_\varepsilon \|_{L^2(\mathbb{R})} \le c \| q_\varepsilon \|_{L^2(\mathbb{R})} \le c \qquad (3.25)$$

and

$$
\begin{aligned}
&\| (b+k)u_\varepsilon + \frac{1}{p+1}u_\varepsilon^{p+1} - \Lambda^{-2}\Big((b+k)u_\varepsilon + \frac{1}{p+1}u_\varepsilon^{p+1}\Big) \|_{L^2(\mathbb{R})} \\
&\le c, \qquad (3.26)
\end{aligned}
$$

using (3.25)-(3.26), we conclude that (3.21) is valid.

Applying the similar proof for (3.23)-(3.26), we derive (3.22). The proof is completed. ■

Lemma 3.7. *Let $t \in [0,T]$ and $u_0 \in H^1(\mathbb{R})$. Then, it holds that*

$$\int_0^t \int_{-\infty}^{\infty} \Big(\frac{\partial u_\varepsilon(t,x)}{\partial x}\Big)^4 dxdt \le c_1 T e^{c_1 T}, \qquad (3.27)$$

where $c_1 > 0$ depends only on the coefficients of Eq.(3.1) and $\| u_0 \|_{H^1(\mathbb{R})}$.

Proof. Multiplying (3.19) by q_ε^3 gives rise to

$$\frac{1}{4}\frac{d}{dt}\Big(\int_{-\infty}^{\infty} q_\varepsilon^4 dx\Big) - k\int_{-\infty}^{\infty} q_\varepsilon^3 \frac{\partial q_\varepsilon}{\partial x}dx - \varepsilon\int_{-\infty}^{\infty} q_\varepsilon^3\frac{\partial^2 q_\varepsilon}{\partial x^2}dx + a\int_{-\infty}^{\infty} q_\varepsilon^4 dx$$
$$= \int_{-\infty}^{\infty} q_\varepsilon^3 K_\varepsilon(t,x)dx. \tag{3.28}$$

Using integration by parts, we derive that

$$\int_{-\infty}^{\infty} q_\varepsilon^3 \frac{\partial q_\varepsilon}{\partial x}dx = 0, \tag{3.29}$$

$$-\int_{-\infty}^{\infty} q_\varepsilon^3 \frac{\partial^2 q_\varepsilon}{\partial x^2}dx = 3\int_{-\infty}^{\infty} q_\varepsilon^2 (\frac{\partial q_\varepsilon}{\partial x})^2 \geq 0 \tag{3.30}$$

and

$$\Big|\int_{-\infty}^{\infty} q_\varepsilon^3 K_\varepsilon(t,x)dx\Big| \leq \Big(\int_{-\infty}^{\infty}|q_\varepsilon^4|dx\Big)^{\frac{3}{4}}\Big(\int_{-\infty}^{\infty} K_\varepsilon(t,x)^4 dx\Big)^{\frac{1}{4}}$$
$$\leq c\Big(\int_{-\infty}^{\infty}|q_\varepsilon^4|dx\Big)^{\frac{3}{4}}$$
$$\leq c\Big(1+\int_{-\infty}^{\infty}|q_\varepsilon^4|dx\Big). \tag{3.31}$$

From (3.28)-(3.31), we get

$$\frac{d}{dt}\Big(\int_{-\infty}^{\infty} q_\varepsilon^4 dx\Big) \leq c\Big(1+\int_{-\infty}^{\infty}|q_\varepsilon^4|dx\Big). \tag{3.32}$$

Integrating (3.32) on the interval $[0,t]$ and using the Gronwall inequality, we get that (3.27) holds. ■

Lemma 3.8. *Assume $a \geq 0$, $u_0(x) \in H^1(\mathbb{R})$ and $|\frac{\partial u_0(x)}{\partial x}| < \infty$. Let $u_\varepsilon = u_\varepsilon(t,x)$ be the solution of (3.13). Then, the following one-sided L^∞ norm estimate on the* *first order spatial derivative holds*

$$\frac{\partial u_\varepsilon(t,x)}{\partial x} \le c(1+e^{-at}) \le 2c, \quad \textit{if} \quad (t,x) \in [0,\infty)\times\mathbb{R}, \tag{3.33}$$

where the constant $c > 0$ depends only on the coefficients of Eq.(3.1) and $\| u_0 \|_{H^1(\mathbb{R})}$.

Proof. From Lemma 3.6 and (3.19), we have

$$\frac{\partial q_\varepsilon}{\partial t} - k\frac{\partial q_\varepsilon}{\partial x} - \varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2} + aq_\varepsilon = K_\varepsilon(t,x) \le c, \tag{3.34}$$

where $c > 0$ is defined in Lemma 3.6 and $a \ge 0$ is assumed in (3.1).

Let $f = f(t)$ satisfy the problem

$$\frac{df}{dt} + ag = c, \qquad f(0) = \| \frac{\partial u_{\varepsilon,0}}{\partial x} \|_{L^\infty(\mathbb{R})}, \tag{3.35}$$

where $\sup_{x\in R} q_\varepsilon(t,x) = f(t)$. Using the comparison principle to parabolic equations(see [101]), we conclude that

$$q_\varepsilon(t,x) \le f(t). \tag{3.36}$$

Solving the ordinary differential equation (3.35), we have

$$f = e^{-at}(g(0) + \int_0^t ce^{e^{-a\tau}} d\tau) \le c(1+e^{-at}) \le 2c.$$

The proof is completed. ■

Lemma 3.9. *Let $u_0(x) \in H^1(\mathbb{R})$ and $|\frac{\partial u_0(x)}{\partial x}| < \infty$. There exists a sequence*

$\{\varepsilon_j\}_{j\in N}$ *tending to zero and a function* $K \in L^\infty([0,\infty)\times\mathbb{R})$ *such that*

$$K_{\varepsilon_j} \to K \qquad \textit{strongly in} \quad C([0,\infty)\times\mathbb{R}), \tag{3.37}$$

where

$$\begin{aligned} K(t,x) = a\Lambda^{-2}q + (b+k)u + \frac{1}{p+1}u^{p+1} \\ -\Lambda^{-2}\Big((b+k)u + \frac{1}{p+1}u^{p+1}\Big). \end{aligned}$$

Proof. For simplicity, we use notations $u = u_\varepsilon$ and $q = q_\varepsilon$. By calculation, we obtain

$$\begin{aligned} \frac{dK_\varepsilon}{dt} &= a\Lambda^{-2}q_t + (b+k)u_t + u^p u_t - \Lambda^{-2}\Big[(b+k)u_t + u^p u_t\Big] \\ &= a\Lambda^{-2}[kq_x + \varepsilon q_{xx} - aq + K_\varepsilon(t,x)] + (b+k+u^p) \\ &\times\Big(ku_x - au + a\Lambda^{-2}u - \Lambda^{-2}\partial_x[(b+k)u + \frac{1}{p+1}u^{p+1}] + \varepsilon u_{xx}\Big) \\ &-\Lambda^{-2}\Big\{(b+k+u^p)(ku_x - au + a\Lambda^{-2}u \\ &-\Lambda^{-2}\partial_x\Big((b+k)u + \frac{1}{p+1}u^{p+1}\Big) + \varepsilon u_{xx}\Big\} \\ &= a\Lambda^{-2}[kq_x + \varepsilon q_{xx} - aq + K_\varepsilon(t,x)] + I_1. \end{aligned} \tag{3.38}$$

We derive that

$$\begin{aligned} \| \Lambda^{-2}q \|_{L^2(\mathbb{R})} \le \| q \|_{L^2(\mathbb{R})} \le c, \\ \| \Lambda^{-2}q_x \|_{L^2(\mathbb{R})} \le \| q \|_{L^2(\mathbb{R})} \le c \end{aligned} \tag{3.39}$$

and

$$\begin{aligned}\| \Lambda^{-2} q_{xx} \|_{L^2(\mathbb{R})} &\leq \| q - \Lambda^{-2} q \|_{L^2(\mathbb{R})} \\ &\leq c \| q \|_{L^2(\mathbb{R})} + \| \Lambda^{-2} q \|_{L^2(\mathbb{R})} \\ &\leq c. \end{aligned} \tag{3.40}$$

From Lemmas 3.6 and 3.7, we obtain

$$\| \Lambda^{-2} K_\varepsilon \|_{L^2(\mathbb{R})} \leq \| K_\varepsilon \|_{L^2(\mathbb{R})} \leq c. \tag{3.41}$$

Similar to the proof of (3.39)-(3.41), we have

$$\| I_1 \|_{L^2(\mathbb{R})} \leq c. \tag{3.42}$$

It follows from (3.38)-(3.42) that

$$\| \frac{dK_\varepsilon}{dt} \|_{L^2(\mathbb{R})} < \infty. \tag{3.43}$$

Since every term of $K_\varepsilon(t,x)$ possesses the same property as the function $K_\varepsilon(t,x)$ does, making use of Corollary 4 on page 85 in Simon [94], we can find a subsequence of $\varepsilon_j \to 0$, still represented by ε_j, such that

$$\begin{aligned} K_{\varepsilon_j} &= a\Lambda^{-2} q_{\varepsilon_j} + (b+k) u_{\varepsilon_j} + \frac{1}{p+1} u_{\varepsilon_j}^{p+1} - \Lambda^{-2}\Big((b+k) u_{\varepsilon_j} + \frac{1}{p+1} u_{\varepsilon_j}^{p+1} \Big) \\ &\to \quad K \qquad \text{strongly in} \quad C([0,\infty) \times \mathbb{R}). \end{aligned}$$

It completes the proof. ■

Lemma 3.10. *If $u_0(x) \in H^1(\mathbb{R})$, there exist a function $u \in L^\infty([0,\infty); H^1(\mathbb{R})) \cap$*

$H^1([0,T]\times\mathbb{R})$ *and a sequence* $\{\varepsilon_j\}_{j\in N}$, $\varepsilon_j\to 0$, *such that*

$$u_{\varepsilon_j} \rightharpoonup u \quad in \quad H^1([0,T]\times\mathbb{R}), \quad for\ each \quad T>0, \tag{3.44}$$

$$u_{\varepsilon_j} \to u \qquad in \quad L^\infty_{loc}([0,\infty)\times\mathbb{R}). \tag{3.45}$$

We notice that the proof of Lemma 3.10 is similar to that of Lemma 5.2 in [9]. We thus omit the proof.

We employ overbars to represent weak limits which are taken in the space $L^r_{loc}([0,\infty)\times\mathbb{R})$ with $1<r<2$.

Lemma 3.11. *Let* $u_0(x)\in H^1(\mathbb{R})$. *There exist two functions* $q\in L^p_{loc}([0,\infty)\times\mathbb{R}), 1<p<4$, $\overline{q^2}\in L^r_{loc}([0,\infty)\times\mathbb{R}), 1<r<2$ *and a sequence* $\{\varepsilon_j\}_{j\in N}$, $\varepsilon_j\to 0$, *such that*

$$\begin{cases} q_{\varepsilon_j} \rightharpoonup q \quad in \quad L^p_{loc}([0,\infty)\times\mathbb{R}), \\ q_{\varepsilon_j} \overset{\star}{\rightharpoonup} q \quad in \quad L^\infty_{loc}([0,\infty); L^2(\mathbb{R})), \end{cases} \tag{3.46}$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \qquad in \quad L^r_{loc}([0,\infty)\times\mathbb{R}), \tag{3.47}$$

$$q^2(t,x) \le \overline{q^2}(t,x) \quad for\ almost\ every \quad (t,x)\in[0,\infty)\times\mathbb{R} \tag{3.48}$$

and

$$\frac{\partial u}{\partial x} = q \quad in\ the\ sense\ of\ distributions\ on\ the\ domain \quad [0,\infty)\times\mathbb{R}. \tag{3.49}$$

Proof. Using Lemmas 3.5 and 3.7, we obtain (3.46) and (3.47). Using weak convergence in (3.47) results in inequality (3.48). From the definition of q_ε, Lemma 3.10 and (3.46), we get that (3.49) holds. ■

Choosing an arbitrary convex function $\phi \in C^1(\mathbb{R})$ satisfying that ϕ' is bounded and Lipschitz continuous on $\mathbb{R}$, and making use of (3.46), we acquire that

$$\phi(q_\varepsilon) \rightharpoonup \overline{\phi(q)} \quad \text{in} \quad L^p_{loc}([0,\infty)\times\mathbb{R}), \quad 1<p<4, \tag{3.50}$$

$$\phi(q_\varepsilon) \overset{\star}{\rightharpoonup} \overline{\phi(q)} \quad \text{in} \quad L^\infty_{loc}([0,\infty);L^2(\mathbb{R})). \tag{3.51}$$

Multiplying Eq.(3.34) by $\phi'(q_\varepsilon)$ yields

$$\begin{aligned} &\frac{\partial}{\partial t}\phi(q_\varepsilon) - k\frac{\partial\phi(q_\varepsilon)}{\partial x} - \varepsilon\frac{\partial^2\phi(q_\varepsilon)}{\partial x^2} + \varepsilon\phi''(q_\varepsilon)\Big(\frac{\partial q_\varepsilon}{\partial x}\Big)^2 \\ &\qquad = -aq_\varepsilon\phi'(q_\varepsilon) + K_\varepsilon(t,x)\phi'(q_\varepsilon). \end{aligned} \tag{3.52}$$

Lemma 3.12. *Let $a \ge 0$, $u_0(x) \in H^1(\mathbb{R})$ and $|\frac{\partial u_0(x)}{\partial x}| < \infty$. For an arbitrary convex function $\phi \in C^1(\mathbb{R})$ with ϕ' being bounded and Lipschitz continuous on $\mathbb{R}$, it holds that*

$$\overline{\frac{\partial\phi(q)}{\partial t}} - k\frac{\partial}{\partial x}\overline{\phi(q)} \le -a\overline{q\phi'(q)} + K(t,x)\overline{\phi'(q)}, \tag{3.53}$$

where (3.53) holds in the sense of distributions on $[0,\infty)\times\mathbb{R}$ and $\overline{q\phi'(q)}$ represents the weak limits of $q_\varepsilon\phi'(q_\varepsilon)$ in $L^r_{loc}([0,\infty)\times\mathbb{R})$, $1<r<2$.

Proof. In (3.52), applying Lemmas 3.9, 3.10, 3.11 and the convexity of function ϕ, letting $\varepsilon \to 0$, we completes the proof. ■

From (3.46) and (3.47), we have

$$\begin{cases} q = q_- + q_+ = \overline{q_-} + \overline{q_+}, \\ q^2 = (q_-)^2 + (q_+)^2, \\ \overline{q^2} = \overline{(q_-)^2} + \overline{(q_+)^2}, \end{cases} \tag{3.54}$$

where (3.54) holds almost everywhere in $[0,\infty) \times \mathbb{R}$ and the notations $\eta_+ := \eta\chi_{[0,+\infty)}(\eta)$, $\eta_- := \eta\chi_{(-\infty,0]}(\eta)$ for $\eta \in (-\infty,\infty)$ (Here χ_E denotes the characteristic function of the set E, namely, $\chi_E(x) = 1$ if $x \in E$; $\chi_E(x) = 0$ if $x \overline{\in} E$).

Using (3.46) and Lemma 3.8, we obtain

$$q_\varepsilon(t,x), \qquad q(t,x) \leq c(1+e^{-at}), \quad t>0, \quad x \in \mathbb{R}. \tag{3.55}$$

Lemma 3.13. *In the sense of distributions on the domain* $[0,\infty) \times \mathbb{R}$*, if* $u_0(x) \in H^1(\mathbb{R})$*, then*

$$\frac{\partial q}{\partial t} - k\frac{\partial q}{\partial x} = -aq + K(t,x). \tag{3.56}$$

Proof. Using Lemmas 3.9-3.11 and (3.19), by letting $\varepsilon \to 0$ in (3.19), we get that (3.56) holds. ■

For a generalized formulation of (3.56), we have the following conclusion.

Lemma 3.14. *For each* $\phi \in C^1(\mathbb{R})$ *with* $\phi \in L^\infty(\mathbb{R})$*, in the sense of distributions on the domain* $[0,\infty) \times \mathbb{R}$*, if* $u_0(x) \in H^1(\mathbb{R})$*, it holds that*

$$\frac{\partial \phi(q)}{\partial t} - k\frac{\partial \phi(q)}{\partial x} = -aq\phi'(q) + K(t,x)\phi'(q). \tag{3.57}$$

Proof. We choose $\{\psi_\delta\}$ such that it is a family of mollifiers on $\mathbb{R}$. Denote the convolution about variable x as $q_\delta(t,x) := (q(t,\cdot) \star \psi_\delta)(x)$. Multiplying (3.56) by $\phi'(q_\delta)$, we obtain

$$\frac{\partial \phi(q_\delta)}{\partial t} = \phi'(q_\delta)\frac{\partial q_\delta}{\partial t} = \phi'(q_\delta)\Big[- aq_\delta + K(t,x) \star \psi_\delta + k\frac{\partial q}{\partial x} \star \psi_\delta\Big]. \quad (3.58)$$

Making use of the boundedness for ϕ, ϕ' and taking $\delta \to 0$ in (3.58), we conclude that (3.57) holds. ■

3.4 Strong convergence and proof of existence of global weak solutions

In this section, we will show that the weak convergence q_ε in (3.47) converges strongly and then prove our main results. Several lemmas will be established to handle $\left(\overline{q^2} - q^2\right) = 0$ almost everywhere in $[0,\infty) \times (-\infty,\infty)$.

Lemma 3.15. **[101]** *Assume* $u_0 \in H^1(\mathbb{R})$*. Then*

$$\lim_{t\to 0}\int_{-\infty}^{\infty} q^2(t,x)dx = \lim_{t\to 0}\int_{-\infty}^{\infty} \overline{q^2}(t,x)dx = \int_{-\infty}^{\infty} \Big(\frac{\partial u_0}{\partial x}\Big)^2 dx. \quad (3.59)$$

Lemma 3.16. **[101]** *If* $u_0 \in H^1(\mathbb{R})$*, for any constant* $B > 0$*, then*

$$\lim_{t\to 0}\int_{-\infty}^{\infty} \Big(\overline{\phi_B^{\pm}(q(t,x))} - \phi_B^{\pm}(q(t,x))\Big)dx = 0, \quad (3.60)$$

in which

$$\phi_B(\eta) := \begin{cases} \frac{1}{2}\eta^2, & \text{if} \quad |\eta| \le B, \\ B\eta - \frac{1}{2}B^2, & \text{if} \quad |\eta| > B, \end{cases} \tag{3.61}$$

and $\phi_B^-(\eta) := \phi_B(\eta)\chi_{(-\infty,0]}(\eta)$, $\phi_B^+(\eta) := \phi_B(\eta)\chi_{[0,+\infty)}(\eta)$, $\eta \in \mathbb{R}$.

Lemma 3.17. [101] *For any constant* $B > 0$ *and each* $\eta \in \mathbb{R}$, *we have*

$$\begin{cases} \phi_B(\eta) = \frac{1}{2}\eta^2 - \frac{1}{2}(B - |\eta|)^2\chi_{(-\infty,-B)\cap(B,\infty)}(\eta), \\ \phi'_B(\eta)\eta = \eta + (B - |\eta|)sign(\eta)\chi_{(-\infty,-B)\cap(B,\infty)}(\eta), \\ \phi_B^+(\eta) = \frac{1}{2}(\eta_+)^2 - \frac{1}{2}(B-\eta)^2\chi_{(B,\infty)}(\eta), \\ (\phi_B^+)'(\eta) = \eta_+ + (B-\eta)\chi_{(B,\infty)}(\eta), \\ \phi_B^-(\eta) = \frac{1}{2}(\eta_-)^2 - \frac{1}{2}(B+\eta)^2\chi_{(-\infty,-B)}(\eta), \\ (\phi_B^-)'(\eta) = \eta_- - (B+\eta)\chi_{(-\infty,-B)}(\eta). \end{cases} \tag{3.62}$$

Lemmas 3.15-3.17 can also be found in [9].

Lemma 3.18. *For almost all* $t > 0$, *if* $a \ge 0$, $u_0(x) \in H^1(\mathbb{R})$ *and* $|\frac{\partial u_0(x)}{\partial x}| < \infty$, *then*

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(\overline{(q_+)^2} - q_+^2\right)(t,x)dx \le \int_0^t\int_{-\infty}^{\infty} K(s,x)[\overline{q_+}(s,x) - q_+(s,x)]dsdx. \tag{3.63}$$

Proof. For any $T > 0$ $(0 < t < T)$, from Lemmas 3.8 and 3.16, we choose the constant B sufficiently large. Subtracting (3.53) from (3.57), and using the definition ϕ_B^+, we get

$$\frac{\partial}{\partial t}\Big(\overline{\phi_B^+(q)} - \phi_B^+(q)\Big) - k\frac{\partial}{\partial x}\Big[\overline{\phi_B^+(q)} - \phi_B^+(q)\Big] \leq -a\Big[\overline{q(\phi_B^+)'(q)} - q(\phi_B^+)'(q)\Big] + K(t,x)\Big(\overline{(\phi_B^+)'(q)} - (\phi_B^+)'(q)\Big). \quad (3.64)$$

Note that

$$-a\Big(\overline{q(\phi_B^+)'(q)} - q(\phi_B^+)'(q)\Big) \leq 0.$$

Let B be sufficiently large and $\Omega_B = \Big(\frac{1}{B-2c}, \infty\Big) \times \mathbb{R}$. In Ω_B, we have

$$\begin{cases} \phi_B^+ = \frac{1}{2}(q_+)^2, \quad (\phi_B^+)'(q) = q_+, \\ \overline{\phi_B^+(q)} = \frac{1}{2}\overline{(q_+)^2}, \quad \overline{(\phi_B^+)'(q)} = \overline{q_+}. \end{cases} \quad (3.65)$$

Integrating inequality (3.64) over $\Big(\frac{1}{B-2c}, t\Big) \times \mathbb{R}$, for almost all $t > \frac{1}{B-2c}$, yields

$$\frac{1}{2}\int_{-\infty}^{\infty}\Big(\overline{(q_+)^2} - q_+^2(t,x)\Big)dx \leq \lim_{B\to\infty}\int_{-\infty}^{\infty}\Big[\overline{\phi_B^+(q)}(\frac{1}{B-2c}, x) - \phi_B^+(q)(\frac{1}{B-2c}, x)\Big]dx + \int_{\frac{1}{\alpha(B-2c)}}^{t}\int_{-\infty}^{\infty} K(s,x)[\overline{q_+}(s,x) - q_+(s,x)]dsdx. \quad (3.66)$$

Using Lemma 3.16 and letting $B \to \infty$, we complete the proof. ■

Lemma 3.19. *If $a \geq 0$, $t > 0$, $u_0(x) \in H^1(\mathbb{R})$ and $|\frac{\partial u_0(x)}{\partial x}| < \infty$, then*

$$\int_{-\infty}^{\infty}\Big(\overline{\phi_B^-(q)} - \phi_B^-(q)\Big)(t,x)dx \leq \int_0^t\int_{-\infty}^{\infty} K(t,x)\Big(\overline{(\phi_B^-)'(q)} - (\phi_B^-)'(q)\Big)dsdx, \quad (3.67)$$

where B is sufficiently large.

Proof. Subtracting (3.53) from (3.57) and making use of the entropy ϕ_B^-, we derive that

$$\frac{\partial}{\partial t}\Big(\overline{\phi_B^-(q)} - \phi_B^-(q)\Big) - k\frac{\partial}{\partial x}\Big[\overline{\phi_B^-(q)} - \phi_B^-(q)\Big]$$
$$\leq -a\Big[\overline{q(\phi_B^-)'(q)} - q(\phi_B^-)'(q)\Big] + K(t,x)\Big(\overline{(\phi_B^-)'(q)} - (\phi_B^-)'(q)\Big). \quad (3.68)$$

Similar to the proof of Lemma 3.18, integrating inequality (3.68) directly completes the proof. ■

Lemma 3.20. *Let the assumptions in Theorem 3.2 hold. Then*

$$\overline{q^2} = q^2 \quad \textit{almost everywhere in} \quad [0,\infty)\times(-\infty,\infty). \quad (3.69)$$

Proof. Employing Lemmas 3.18 and 3.19 leads to

$$\int_{-\infty}^{\infty}\left(\frac{1}{2}\Big[\overline{(q_+)^2} - (q_+)^2\Big] + \Big[\overline{\phi_B^-} - \phi_B^-\Big]\right)(t,x)dx$$
$$\leq \int_0^t\int_{-\infty}^{\infty} K(s,x)\left(\Big[\overline{q_+} - q_+\Big] + \Big[\overline{(\phi_B^-)'(q)} - (\phi_B^-)'(q)\Big]\right)dsdx. \quad (3.70)$$

Using Lemma 3.17, we have

$$\overline{\phi_B^-(q)} - \phi_B^-(q) = \frac{1}{2}\Big(\overline{(q_-)^2} - (q_-)^2\Big) + \frac{1}{2}(B+q)^2\chi_{(-\infty,-B)}(q)$$
$$-\frac{1}{2}\overline{(B+q)^2\chi_{(-\infty,-B)}(q)}. \quad (3.71)$$

Applying Lemma 3.9, we can choose a constant $C > 0$ depending only on $\| u_0 \|_{H^1(\mathbb{R})}$ to satisfy

$$\| K(t,x) \|_{L^\infty([0,\infty)\times\mathbb{R})} \leq C. \quad (3.72)$$

Making use of Lemma 3.17 and (3.54) yields

$$\begin{cases} \overline{q_+} + \overline{(\phi_B^-)'(q)} = q - \overline{(B+q)\chi_{(-\infty,-B)}(q)}, \\ q_+ + (\phi_B^-)'(q) = q - (B+q)\chi_{(-\infty,-B)}. \end{cases} \tag{3.73}$$

Employing the convexity of the map $\eta \to \eta_+ + (\phi_B^-)'(\eta)$, we obtain that

$$\begin{aligned} 0 &\leq [\overline{q_+} - q_+] + [\overline{(\phi_B^-)'(q)} - (\phi_B^-)'(q)] \\ &= (B+q)\chi_{(-\infty,-B)} - \overline{(B+q)\chi_{(-\infty,-B)}(q)}. \end{aligned} \tag{3.74}$$

Then, from (3.70), (3.71) and (3.74), we have

$$\begin{aligned} 0 \leq &\int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\overline{(q_+)^2} - (q_+)^2\right) + \left(\overline{\phi_B^-} - \phi_B^-\right)\right](t,x)dx \\ &\to 0 \quad \text{as} \quad B \to \infty. \end{aligned} \tag{3.75}$$

Letting $B \to \infty$, for any $t > 0$, we obtain

$$0 \leq \int_{-\infty}^{\infty} \left(\overline{q^2} - q^2\right)(t,x)dx = 0. \tag{3.76}$$

We conclude from (3.76) that (3.69) holds. The proof is completed. ■

Proof of Theorem 3.2. Applying Lemma 3.9, (3.12) and (3.15), we conclude that (i) and (ii) in Definition 3.1 hold.

To complete the proof of (iii) in Definition 3.1, we utilize Lemma 3.20 to obtain

$$q_\varepsilon \to q \quad \text{in} \quad L^2_{loc}([0,\infty) \times \mathbb{R}). \tag{3.77}$$

From (3.37), (3.77) and Lemma 3.10, we conclude that $u(t,x)$ is a global weak solution to problem (3.3). From Lemmas 3.7 and 3.8, we derive that the inequalities (3.7) and (3.8) are valid. The proof is finished.

3.5 L^1 local stability

Firstly, we state some notations which will be used in this section.

Set $W_T = [0,T] \times \mathbb{R}$ for every $T > 0$. We use notation $C_0^\infty(W_T)$ to represent all C^∞ functions which have compact support in the domain W_T. Assume that the function $\phi_0(\varsigma)$ belongs to $C_0^\infty(\mathbb{R})$ such that

$$\begin{cases} \phi_0(\varsigma) \geq 0, \\ \phi_0(\varsigma) = 0, \quad \text{if} \quad |\varsigma| \geq 1, \\ \int_{-\infty}^{\infty} \phi_0(\varsigma) d\varsigma = 1. \end{cases}$$

For any real number $h > 0$, setting $\phi_h(\varsigma) = \frac{\phi_0(h^{-1}\varsigma)}{h}$, we know $\phi_h(\varsigma) \in C_0^\infty(\mathbb{R})$ and

$$\begin{cases} \phi_h(\varsigma) \geq 0, \\ \phi_h(\varsigma) = 0 \quad \text{if} \quad |\varsigma| \geq h, \\ |\phi_h(\varsigma)| \leq \frac{c}{h}, \\ \int_{-\infty}^{\infty} \phi_h(\varsigma) d\varsigma = 1, \end{cases} \tag{3.78}$$

where $c > 0$ represents a constant.

For locally integrable function $\lambda(x)$, $x \in (-\infty, \infty)$, its approximation function is defined by

$$\lambda^h(x) = \frac{1}{h} \int_{-\infty}^{\infty} \phi(\frac{x-y}{h}) \lambda(y) dy, \qquad h > 0.$$

If x_0 is a Lebesgue point of $\lambda(x)$, we have

$$\lim_{h\to 0}\frac{1}{h}\int_{x_0-h}^{x_0+h}|\lambda(x)-\lambda(x_0)|dx=0$$

and

$$\lim_{h\to 0}\lambda^h(x_0)=\lambda(x_0).$$

Therefore, we have that if $h\to 0$, then $\lambda^h(x)\to\lambda(x)$ almost everywhere.

If $\parallel u\parallel_{L^\infty(\mathbb{R})}<\infty$, we choose $N_0>\sup_{t\in[0,\infty)}\parallel u\parallel_{L^\infty(\mathbb{R})}$ and $M>N_0$. We define the cone $\daleth$ as

$$\daleth=\Big\{(t,x)\quad\Big|\quad |x|\leq M-N_0t,\quad 0<t<T_0=\min(T,MN_0^{-1})\Big\}.$$

Suppose that S_τ denotes the cross section of $\daleth$ when $t=\tau,\tau\in[0,T_0]$. For a real number $r>0$, we set $G_r=\Big\{x:\Big||x|\leq r\Big\}$.

Lemma 3.21.([53]). *If $\epsilon\in(0,\min[r,T])$, $h\in(0,\epsilon)$ and $g(t,x)$ is measurable and bounded in the domain $[0,T]\times G_r$, for the function*

$$V_h=\frac{1}{h^2}\iiiint_D|g(t,x)-g(\tau,y)|dxdtdyd\tau,\tag{3.79}$$

where

$$D=\Big\{(t,x,\tau,y):\Big|\quad|\frac{t-\tau}{2}|\leq h,\epsilon\leq\frac{t+\tau}{2}\leq T-\epsilon,|\frac{x-y}{2}|\leq h,|\frac{x+y}{2}|\leq r-\epsilon\Big\},$$

then $\lim_{h\to 0}V_h=0$.

Lemma 3.22.([53]). *Assume that $|\frac{\partial E(u)}{\partial u}|$ is bounded for $-\infty<u<\infty$. Then*

$$|sign(u_1-u_2)(E(u_1)-E(u_2))|\leq L|u_1-u_2|,\tag{3.80}$$

where $L > 0$ is the Lipschitz constant.

Applying operator $\Lambda^{-2} = (1 - \frac{\partial^2}{\partial x^2})^{-1}$ on Eq.(3.1), we have

$$u_t - ku_x + au - a\Lambda^{-2}u + \Lambda^{-2}\Big((b+k)u + \frac{1}{p+1}u^{p+1}\Big)_x = 0. \tag{3.81}$$

From (3.5) and (3.6), we derive that $\| u \|_{L^\infty} < \infty$ if $u_0 \in H^1(\mathbb{R})$. For conciseness, we write

$$F_u(t,x) = au - a\Lambda^{-2}u + \Lambda^{-2}\Big((b+k)u + \frac{1}{p+1}u^{p+1}\Big)_x$$

.

Lemma 3.23. *If $u_0(x) \in H^1(\mathbb{R})$, then*

$$\| F_u \|_{L^\infty(\mathbb{R})} < C, \tag{3.82}$$

where the constant $C > 0$ depends on the coefficients of Eq.(3.1) and the norm $\| u_0 \|_{H^1(\mathbb{R})}$.

Proof: By calculation, we have

$$\int_{-\infty}^{\infty} e^{-|x-y|}dy = 2.$$

Using $u_0(x) \in H^1(\mathbb{R})$, (3.5) and (3.6), we obtain that $\| u \|_{L^\infty} < \infty$ and $\| \Lambda^{-2}u \|_{L^\infty} < \infty$.

If the function $G(t,x) \in L^{\infty}(\mathbb{R})$ or $G(t,x) \in L^{p_0}(\mathbb{R})$ $(1 \le p_0 < \infty)$, we have

$$\begin{aligned}
\left|\Lambda^{-2}\partial_x G(t,x)\right| &= \left|\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\frac{\partial G(t,y)}{\partial y}dy\right| \\
&= \left|\frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}G(t,y)dy - \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}G(t,y)dy\right| \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}|G(t,y)|dy \\
&\le \| G \|_{L^{\infty}},
\end{aligned} \tag{3.83}$$

which derives that

$$\left|\Lambda^{-2}\Big((k+b)u + \frac{1}{p+1}u^{p+1}\Big)_x\right| \le C.$$

The proof is completed. ■

Lemma 3.24. *Suppose that $u(t,x)$ is a strong solution of Eq.(3.1). For an arbitrary constant γ, if $\mu(t,x) \in C_0^{\infty}(W_T)$, then*

$$\iint_{W_T} \Big\{|u-\gamma|\mu_t - k|u-\gamma|\mu_x - sign(u-\gamma)F_u(t,x)\mu\Big\}dxdt = 0. \tag{3.84}$$

Proof: Assume that $L(u)$ is a smooth function if $u \in (-\infty, \infty)$. We employ $L'(u)\mu(t,x)$ to multiply Eq.(3.81) and integrate by parts, for any constant γ, we then obtain

$$\int_{-\infty}^{\infty}[\int_{\gamma}^{u} L'(z)dz]\mu_x dx = -\int_{-\infty}^{\infty} \mu L'(u)u_x dx$$

and

$$\iint_{W_T} \Big[L(u)\mu_t - [\int_{\gamma}^{u} kL'(z)dz]\mu_x - L'(u)F_u(t,x)\mu\Big]dxdt = 0. \tag{3.85}$$

Let $L^h(u)$ be an approximation of $|u-\gamma|$. Replacing $L(u)$ by $L^h(u)$ in (3.85) and letting $h \to 0$ produce the desired result. The proof is completed. ■

We should address here that the approach to prove Lemma 3.24 can be found in [53].

Lemma 3.25. *Suppose that Eq.(3.1) has two strong solutions $u_1(t,x)$ and $u_2(t,x)$ associated with $u_{i,0} = u_i(0,x) \in L^1(\mathbb{R}) \cap H^1(\mathbb{R}), i = 1,2$. If $\mu \in C_0^\infty(W_T)$, then*

$$\begin{aligned} &|\int_{-\infty}^{\infty} sign(u_1-u_2)\Big[F_{u_1}(t,x) - F_{u_2}(t,x)\Big]\mu dx| \\ &\le C\int_{-\infty}^{\infty} |u_1-u_2|dx, \end{aligned} \tag{3.86}$$

where $C > 0$ depends on $\| u_{1,0} \|_{H^1(\mathbb{R})}$, $\| u_{2,0} \|_{H^1(\mathbb{R})}$ and $\mu(t,x)$.

Proof: Applying the property of Λ^{-2} and using inequality (3.83) produce

$$\begin{aligned} &\Big|\Lambda^{-2}[u_1(t,x) - u_2(t,x)]_x\Big| \\ &\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\Big|u_1(t,y) - u_2(t,y)\Big|dy \\ &\le \| u_1 - u_2 \|_{L^1(\mathbb{R})} . \end{aligned} \tag{3.87}$$

Using $\| u_1 \|_{L^\infty} \le \| u_{1,0} \|_{H^1(\mathbb{R})}$, $\| u_2 \|_{L^\infty} \le \| u_{2,0} \|_{H^1(\mathbb{R})}$ (see (3.6)) and inequality (3.83) gives rise to

$$\begin{aligned} &\Big|\Lambda^{-2}[u_1^{p+1}(t,x) - u_2^{p+1}(t,x)]_x\Big| \\ &\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\Big|u_1^{p+1}(t,y) - u_2^{p+1}(t,y)\Big|dy \\ &\le c \| u_1 - u_2 \|_{L^1(\mathbb{R})}, \end{aligned} \tag{3.88}$$

where c relies on $\| u_{i,0} \|_{H^1(\mathbb{R})}, i = 1,2$. The proof is completed by using (3.87) and (3.88). ■

Proof of the L^1 local stability

Using the methods in [53], we will prove Theorem 3.3.

For $T>0$, we write $W_T=[0,T]\times\mathbb{R}$. We choose $\mu(t,x)\in C_0^\infty(W_T)$ and $\mu(t,x)=0$ outside the region

$$\Omega_0=\{(t,x)\}=[\epsilon,T-2\epsilon]\times G_{r-2\epsilon},\quad 0<2\epsilon\le\min(T,r). \tag{3.89}$$

Let

$$J=\mu(\frac{t+\tau}{2},\frac{x+y}{2})\phi_h(\frac{t-\tau}{2})\phi_h(\frac{x-y}{2})=\mu(\cdots)\rho_h(*), \tag{3.90}$$

where $(*)=(\frac{t-\tau}{2},\frac{x-y}{2}),\quad(\cdots)=(\frac{t+\tau}{2},\frac{x+y}{2})$ and ϕ_h is defined in (3.78).

By calculation, we get

$$\begin{cases} J_t+J_\tau=\mu_t(\cdots)\rho_h(*),\\ J_x+J_y=\mu_x(\cdots)\rho_h(*). \end{cases} \tag{3.91}$$

Choosing $u_1=u_1(t,x)$ and $\gamma=u_2(\tau,y)$ for a fixed point (τ,y), and using the definition of $\mu(t,x)$ and Lemma 3.24, we obtain

$$\begin{aligned}\iiiint\limits_{W_T\times W_T}&\Big\{|u_1(t,x)-u_2(\tau,y)|J_t-k|u_1(t,x)-u_2(\tau,y)|J_x\\ &-sign(u_1(t,x)-u_2(\tau,y))F_{u_1}(t,x)J\Big\}dxdtdyd\tau=0.\end{aligned} \tag{3.92}$$

In exactly the same way, we have

$$\begin{aligned}\iiiint\limits_{W_T\times W_T}&\Big\{|u_2(\tau,y)-u_1(t,x)|J_\tau-k|u_2(\tau,y)-u_1(t,x)|J_y\\ &-sign(u_2(\tau,y)-u_1(t,x))F_{u_2}(\tau,y)J\Big\}dxdtdyd\tau=0.\end{aligned} \tag{3.93}$$

Adding (3.92) and (3.93) together yields

$$\begin{aligned}
0 \le & \iiiint\limits_{W_T\times W_T} \Big\{|u_1(t,x)-u_2(\tau,y)|(J_t+J_\tau) \\
& \qquad -k|u_1(t,x)-u_2(\tau,y)|(J_x+J_y)\Big\}dxdtdyd\tau \\
& +\Big|\iiiint\limits_{W_T\times W_T} sign(u_1(t,x)-u_2(\tau,y))(F_{u_1}(t,x)-F_{u_2}(\tau,y))Jdxdtdyd\tau\Big| \\
& = \iiiint\limits_{W_T\times W_T} K_1 dxdtdyd\tau + |\iiiint\limits_{W_T\times W_T} K_2 dxdtdyd\tau| \\
& = X_1+X_2. \qquad (3.94)
\end{aligned}$$

We shall prove that the inequality

$$\begin{aligned}
0 \le & \iint\limits_{W_T} \Big\{|u_1(t,x)-u_2(t,x)|\mu_t - k|u_1(t,x)-u_2(t,x)|\mu_x\Big\}dxdt \\
& +\Big|\iint\limits_{W_T} sign(u_1(t,x)-u_2(t,x))[F_{u_1}(t,x)-F_{u_2}(t,x)]\mu dxdt\Big| \qquad (3.95)
\end{aligned}$$

holds.

In fact, the integrand of X_1 in (3.94)(namely K_1) can be expressed by the form

$$I_h = I(t,x,\tau,y,u_1(t,x),u_2(\tau,y))\rho_h(*). \qquad (3.96)$$

Noticing the choice of J, we obtain $I_h = 0$ outside the region

$$\begin{aligned}
& \{(t,x;\tau,y)\} \\
& = \Big\{\epsilon \le \frac{t+\tau}{2} \le T-2\epsilon, \frac{|t-\tau|}{2} \le h, \frac{|x+y|}{2} \le r-2\epsilon, \frac{|x-y|}{2} \le h\Big\}. \qquad (3.97)
\end{aligned}$$

Thus, we have

$$\begin{aligned}\iiiint\limits_{W_T\times W_T} I_h dxdtdyd\tau &= \iiiint\limits_{W_T\times W_T} \Big[I(t,x,\tau,y,u_1(t,x),u_2(\tau,y)) \\ &\qquad -I(t,x,t,x,u_1(t,x),u_2(t,x))\Big]\rho_h(*)dxdtdyd\tau \\ &\quad + \iiiint\limits_{W_T\times W_T} I(t,x,t,x,u_1(t,x),u_2(t,x))\rho_h(*)dxdtdyd\tau \\ &= I_{11}(h) + I_{12}. \end{aligned} \tag{3.98}$$

Using $|\rho(*)| \le \frac{c}{h^2}$, we have

$$|I_{11}(h)| \le c\left[h + \frac{1}{h^2}\iiiint\limits_{D} |u_1(t,x) - u_2(\tau,y)|dxdtdyd\tau\right], \tag{3.99}$$

where D is defined in Lemma 3.21, $c > 0$ does not depend on the parameter h. From Lemma 3.21, we know that $I_{11}(h) \to 0$ if $h \to 0$.

Letting $t = \alpha_1$, $\frac{t-\tau}{2} = \alpha_2, x = \beta_1$, $\frac{x-y}{2} = \beta_2$, we obtain

$$\int_{-h}^{h}\int_{-\infty}^{\infty} \rho_h(\alpha_2,\beta_2)d\alpha_2 d\beta_2 = 1 \tag{3.100}$$

and

$$\begin{aligned} I_{12} &= 4\iint\limits_{W_T} \Big[I_h(\alpha_1,\beta_1,\alpha_1,\beta_1,u_1(\alpha_1,\beta_1),u_2(\alpha_1,\beta_1)) \\ &\qquad \times\Big\{\int_{-h}^{h}\int_{-\infty}^{\infty} \rho_h(\alpha_2,\beta_2)d\alpha_2 d\beta_2\Big\}\Big]d\beta_1 d\alpha_1 \\ &= 4\iint\limits_{W_T} I(t,x,t,x,u_1(t,x),u_2(t,x))dxdt. \end{aligned} \tag{3.101}$$

From (3.98) to (3.101), we have

$$\lim_{h\to 0}\iiiint\limits_{W_T\times W_T} I_h dxdtdyd\tau = 4\iint\limits_{W_T} I(t,x,t,x,u_1(t,x),u_2(t,x))dxdt. \tag{3.102}$$

Since

$$K_2 = sign(u_1(t,x)-u_2(\tau,y))(F_{u_1}(t,x)-F_{u_2}(\tau,y))\mu(\cdots)\rho_h(*) \tag{3.103}$$

and

$$\begin{aligned}
&\iiiint\limits_{W_T\times W_T} K_2 dxdtdyd\tau \\
&= \iiiint\limits_{W_T\times W_T} \Big(K_2(t,x,\tau,y)-K_2(t,x,t,x)\Big)dxdtdyd\tau \\
&\qquad + \iiiint\limits_{W_T\times W_T} K_2(t,x,t,x)dxdtdyd\tau \\
&= I_{21}(h)+I_{22},
\end{aligned} \tag{3.104}$$

we derive that

$$|I_{21}(h)| \le c\left(h+\frac{1}{h^2}\iiiint\limits_{D} |F_{u_1}(t,x)-F_{u_2}(\tau,y)|dxdtdyd\tau\right), \tag{3.105}$$

where D is defined in Lemma 3.21.

Applying Lemmas 3.21 and 3.22, we acquire $I_{21}(h)\to 0$ when $h\to 0$. Using

(3.100) gives rise to

$$
\begin{aligned}
I_{22} = 2^2 \iint\limits_{W_T} \Bigg[& K_2(\alpha_1, \beta_1, \alpha_1, \beta_1, u_1(\alpha_1, \beta_1), u_2(\alpha_1, \beta_1)) \\
& \times \Big\{ \int_{-h}^{h} \int_{-\infty}^{\infty} \rho_h(\alpha_2, \beta_2) d\alpha_2 d\beta_2 \Big\} \Bigg] d\alpha_1 d\beta_1 \\
= 2^2 \iint\limits_{W_T} & K_2(t, x, t, x, u_1(t, x), u_2(t, x)) dx dt \\
= 4 \iint\limits_{W_T} & sign(u_1(t, x) - u_2(t, x))(F_{u_1}(t, x) - F_{u_2}(t, x))\mu(t, x) dx dt.
\end{aligned}
\tag{3.106}
$$

From (3.102) and (3.106), we have proved that (3.95) is valid.

Set

$$
W(t) = \int_{-\infty}^{\infty} |u_1(t, x) - u_2(t, x)| dx. \tag{3.107}
$$

We define

$$
\upsilon_h(\sigma) = \int_{-\infty}^{\sigma} \phi_h(\xi) d\xi \qquad \Big(\upsilon_h'(\sigma) = \phi_h(\sigma) \geq 0\Big). \tag{3.108}
$$

If $r_1 < r_2$, $r_1 \in (0, T_0)$ and $r_2 \in (0, T_0)$, we set

$$
\mu = [\upsilon_h(t - r_1) - \upsilon_h(t - r_2)]\pi_\varepsilon(t, x), \quad h < min(r_1, T_0 - r_2), \tag{3.109}
$$

where

$$
\pi(t, x) = \pi_\varepsilon(t, x) = 1 - \upsilon_h(|x| + N_0 t - M + \varepsilon), \quad \varepsilon > 0. \tag{3.110}
$$

Letting ε be sufficiently small, we derive that $\mu(t, x) = 0$ outside the region

Ω_0 and $\pi(t,x) = 0$ outside the area $\daleth$. In the region $(t,x) \in \daleth$, we have

$$0 = \pi_t + N_0|\pi_x| \geq \pi_t + N_0\pi_x. \tag{3.111}$$

Using the inequality (3.95), and (3.108)-(3.111), we derive that

$$\begin{aligned} 0 \leq & \iint\limits_{W_{T_0}} \Big\{[\phi_h(t-r_1) - \phi_h(t-r_2)]\pi_\varepsilon|u_1(t,x) - u_2(t,x)|\Big\}dxdt \\ & + \Big|\iint\limits_{W_{T_0}} \Big[\upsilon_h(t-r_1) - \upsilon_h(t-r_2)\Big]\Psi(t,x)\pi(t,x)dxdt\Big|, \end{aligned} \tag{3.112}$$

where $\Psi(t,x) = [F_{u_1}(t,x) - F_{u_2}(t,x)]sign[u_1(t,x) - u_2(t,x)]$.

Using (3.112) results in

$$\begin{aligned} 0 \leq & \int_0^{T_0}\int_{-\infty}^{\infty} \Big\{[\phi_h(t-r_1) - \phi_h(t-r_2)]\pi_\varepsilon|u_1(t,x) - u_2(t,x)|\Big\}dxdt \\ & + \int_0^{T_0} \Big(\upsilon_h(t-r_1) - \upsilon_h(t-r_2)\Big)\Big|\int_{-\infty}^{\infty} \Psi(t,x)\pi(t,x)dx\Big|dt, \end{aligned} \tag{3.113}$$

which together with Lemma 3.25 produces

$$\begin{aligned} 0 \leq & \int_0^{T_0}\int_{-\infty}^{\infty} \Big\{[\phi_h(t-r_1) - \phi_h(t-r_2)]\pi_\varepsilon|u_1(t,x) - u_2(t,x)|\Big\}dxdt \\ & + c\int_0^{T_0} \Big(\upsilon_h(t-r_1) - \upsilon_h(t-r_2)\Big)\int_{-\infty}^{\infty} |u_1 - u_2|dxdt, \end{aligned} \tag{3.114}$$

where $c > 0$ does not depend on h.

Letting $\varepsilon \to 0$ in (3.114) and $M \to \infty$, we get

$$0 \leq \int_0^{T_0} \Big\{[\phi_h(t-r_1) - \phi_h(t-r_2)] \int_{-\infty}^{\infty} |u_1(t,x) - u_2(t,x)|dx\Big\}dt$$
$$+c\int_0^{T_0} \Big(\upsilon_h(t-r_1) - \upsilon_h(t-r_2)\Big)\Big(\int_{-\infty}^{\infty} |u_1 - u_2|dx\Big)dt. \tag{3.115}$$

If $h \leq \min(r_1, T_0 - r_1)$, using the definition of ϕ_h produces

$$\Big|\int_0^{T_0} \phi_h(t-r_1)W(t)dt - W(r_1)\Big| = \Big|\int_0^{T_0} \phi_h(t-r_1)\Big[W(t) - W(r_1)\Big]dt\Big|$$
$$\leq \frac{c}{h}\int_{r_1-h}^{r_1+h} |W(t) - W(r_1)|dt \to 0 \quad \text{as} \quad h \to 0, \tag{3.116}$$

where $c > 0$ does not depend on h.

We define

$$B(r_1) = \int_0^{T_0} \upsilon_h(t-r_1)W(t)dt = \int_0^{T_0}\int_{-\infty}^{t-r_1} \phi_h(\sigma)d\sigma W(t)dt. \tag{3.117}$$

Following the similar proof of (3.116), we derive that

$$B'(r_1) = -\int_0^{T_0} \phi_h(t-r_1)W(t)dt \to -W(r_1) \quad \text{as} \quad h \to 0, \tag{3.118}$$

from which we derive

$$B(r_1) \to B(0) - \int_0^{r_1} W(\xi)d\xi \quad \text{as} \quad h \to 0. \tag{3.119}$$

Similarly, we get

$$B(r_2) \to B(0) - \int_0^{r_2} W(\xi)d\xi \quad \text{as} \quad h \to 0. \tag{3.120}$$

Using (3.119) and (3.120) derives

$$B(r_1) - B(r_2) \to \int_{r_1}^{r_2} W(\xi)d\xi \quad \text{as} \quad h \to 0. \tag{3.121}$$

Letting $r_1 \to 0$, $r_2 \to t$ $(0 \leq t \leq T_0)$, and using (3.116),(3.117) and (3.121), we obtain

$$W(t) \leq W(0) + c\int_0^t W(\tau)d\tau. \tag{3.122}$$

Employing (3.107), (3.122) and the Gronwall inequality, we acquire the desired result and finish the proof of Theorem 3.3. ■

CHAPTER 4

Local strong and weak solutions to an integrable equation with quadratic nonlinearities

An integrable non-evolutionary partial differential equation, which has quadratic nonlinearities and possesses quasi-local higher symmetries, is studied in this chapter. We prove that there exists a unique local strong solution to the equation in the space $C([0,T);H^s)\cap C^1([0,T);H^{s-1}(\mathbb{R}))$ in the case $s>\frac{3}{2}$. We give a sufficient and necessary condition of the blow-up solutions. For the case of the Sobolev index $1\le s\le \frac{3}{2}$, we prove that the equation has local weak solutions in $H^s(\mathbb{R})$ under suitable assumptions.

4.1 General

Novikov [87] classified and generalized the Camassa-Holm-type equations which possess integrability and quasi-local higher symmetries. One of the models in [87] is in the form

$$u_t-\alpha^2u_{txx}+4uu_x-2\alpha u_x^2-2\alpha uu_{xx}=6\alpha^2u_xu_{xx}+2\alpha^2uu_{xxx}, \tag{4.1}$$

where $\alpha \neq 0$ is a constant.

Motivated by the works presented in [59,66] and the desire to find the dynamical properties of Eq.(4.1), which has quadratic nonlinearity, we aim to investigate Eq.(4.1) in this chapter. We prove that Eq.(4.1) has a unique local strong solution belonging to the space $C([0,T);H^s)\cap C^1([0,T);H^{s-1}(\mathbb{R}))$ in the case $s>\frac{3}{2}$ and find a sufficient and necessary condition for the blow-up solution. For the case of the Sobolev index $1\leq s\leq \frac{3}{2}$, we show that Eq.(4.1) has local weak solutions in $H^s(\mathbb{R})$. As far as we know, these dynamical properties have not been found for Eq.(4.1) in the literature.

This chapter is structured as follows. We present the main conclusions in section 4.2. The existence and uniqueness of the local strong solution are established in section 4.3. In section 4.4, the results about the blow-up solution and local weak solutions are proved.

4.2 Main results

For Eq.(4.1), its Cauchy proplem is written by the form

$$\begin{cases} u_t-\alpha^2u_{txx}+4uu_x-2\alpha u_x^2-2\alpha uu_{xx}=6\alpha^2u_xu_{xx}+2\alpha^2uu_{xxx}, \\ u(0,x)=u_0(x). \end{cases} \tag{4.2}$$

By applying the operator $\Lambda_1^{-2}=(1-\alpha^2\frac{\partial^2}{\partial x^2})^{-1}$ on both sides of (4.2) through some derivation, problem (4.2) becomes the following problem

$$\begin{cases} u_t+2uu_x=\Lambda_1^{-2}[\frac{1}{\alpha}u^2-2uu_x]-\frac{1}{\alpha}u^2, \\ u(0,x)=u_0(x). \end{cases} \tag{4.3}$$

The main conclusions of this chapter are stated in following theorems.

Theorem 4.1. *Let $s > \frac{3}{2}$ and $u_0(x) \in H^s$. Then, problem (4.2) or (4.3) has a unique solution in the space $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$, where $T > 0$ depends on the initial value $\| u_0 \|_{H^s}$.*

Theorem 4.2. *Assume $s > \frac{3}{2}$, $u_0(x) \in H^s$. Suppose that $u(t,x)$ is a solution of problem (4.2) or (4.3). Then, $\lim\limits_{t \to T} \| u \|_{H^s} = \infty$ if and only if $\| u_x \|_{L^\infty(\mathbb{R})} = \infty$.*

For ε satisfying $0 < \varepsilon < \frac{1}{4}$, using Theorem 4.1, we derive that there is a unique solution $u_\varepsilon(t,x)$ in $C^\infty([0,T_\varepsilon); H^\infty(\mathbb{R}))$ for the following system

$$\begin{cases} u_t - \alpha^2 u_{txx} + 4uu_x - 2\alpha u_x^2 - 2\alpha uu_{xx} = 6\alpha^2 u_x u_{xx} + 2\alpha^2 uu_{xxx} \\ u(0,x) = u_{\varepsilon,0}(x). \end{cases} \tag{4.4}$$

Here we illustrate that T_ε may be dependent on the parameter ε and $u_{\varepsilon,0}(x)$ is defined in (3.11). Under certain assumptions, however, we can find two positive constants $c > 0$ and $T > 0$, which do not depend on the parameter ε, to ensure that the space derivative of the solution $u_\varepsilon(t,x)$ to system (4.4) satisfies $\| u_{\varepsilon x} \|_{L^\infty} \leq c$ if $(t,x) \in [0,T) \times (-\infty, \infty)$. We employ this result to derive the existence of local weak solutions to problem (4.2).

Theorem 4.3. *Assume $s \in [1, \frac{3}{2}]$, $\| u_{0x} \|_{L^\infty} < \infty$ and $u_0(x) \in H^s$. Then solution $u_\varepsilon(t,x)$ of (4.4) satisfies*

$$\| u_{\varepsilon x} \|_{L^\infty} \leq c, \quad \text{for } t \in [0,T),\ x \in (-\infty, \infty),$$

where $T > 0$ and $c > 0$ do not depend on the parameter ε.

Theorem 4.4. *Let $u_0(x) \in H^s$, $1 \leq s \leq \frac{3}{2}$, $\| u_{0x} \|_{L^\infty} < \infty$. Then, for suitably small $T > 0$, there exists at least one local weak solutions for problem (4.2) in $L^2([0,T], H^s(\mathbb{R}))$. Moreover, $u_x \in L^\infty([0,T] \times \mathbb{R})$.*

4.3 Local strong solutions

We recall several statements for the following quasi-linear abstract operator equation

$$\frac{dw}{dt} + E(w)w = J(w), \qquad t \geq 0 \qquad \text{and} \qquad w(0) = w_0, \qquad (4.5)$$

where $E(w)$ is an operator. Assume that W and V are the Hilbert spaces. Let V be densely and continuously embedded in W. Let $P : V \to W$ be a topological isomorphism. Assume that the space $L(V, W)$ denotes all bounded linear operators from V to W. In the case $W = V$, we set $L(W, W) = L(W)$. We write out assumptions $(a_1), (a_2)$ and (a_3) in which the constants $\gamma_1, \gamma_2, \gamma_3$ and γ_4 depend only on the norms $\max\{\| \cdot \|_W, \| \cdot \|_V\}$.

(a_1). $E(y) \in L(V, W)$ for $y \in W$ with

$$\| (E(y) - E(z))v \|_W \leq \gamma_1 \| v \|_V \| y - z \|_W, \qquad v, y, z \in V,$$

and $E(y) \in G(W, 1, \beta)$ (namely, $E(y)$ is quasi-m-accretive) is uniformly bounded for bounded sets of V.

(a_2). For operator P, $PE(y)P^{-1} = E(y) + A(y)$, where $A(y) \in L(W)$ is uniformly bounded on bounded sets in V, it holds that

$$\| (A(y) - A(z))v \|_W \leq \gamma_2 \| y - z \|_V \| v \|_V, \quad v \in W, \quad y, z \in V.$$

(a_3). $J : V \to W$ is bounded for bounded sets of V, and

$$\| J(z) - J(y) \|_V \leq \gamma_3 \| z - y \|_V, \qquad z, y \in V,$$

$$\| J(z) - J(y) \|_W \leq \gamma_4 \| z - y \|_W, \qquad z, y \in W.$$

Kato Theorem [54, 55] *If $(a_1), (a_2)$ and (a_3) hold, letting $w_0 \in V$, then there exists a maximal $T > 0$ relying only on $\| w_0 \|_V$ to ensure that problem (4.5) has a unique solution w and*

$$w = w(\cdot, w_0) \in C([0,T); V) \cap C^1([0,T); W).$$

In addition, the mapping $w_0 \to w(., v_0)$ from V to the space $C([0,T); V) \cap C^1([0,T); W)$ is continuous.

Let $E(u) = 2u\partial_x$, $V = H^s(\mathbb{R}), W = H^{s-1}(\mathbb{R})$, $\Lambda_1 = (1 - \alpha^2\partial_x^2)^{\frac{1}{2}}$,

$$J(u) = \Lambda_1^{-2}[\frac{1}{\alpha}u^2 - 2uu_x] - \frac{1}{\alpha}u^2$$

and $P = \Lambda^s$. We know that the operator P is an isomorphism of H^s onto $H^{s-1}(\mathbb{R})$ (see [104]). For the aim to give the proof of Theorem 4.1, it suffices to show that $E(u)$ and $J(u)$ satisfy $(a_1) - (a_3)$.

Lemma 4.5 [104]. *The operator $E(u) = 2u\partial_x \in G(H^{s-1}(\mathbb{R}), 1, \beta)$ if $s > \frac{3}{2}$, $u \in H^s(\mathbb{R})$.*

Lemma 4.6 [104]. *Assume $u \in H^s(\mathbb{R})$ and $s > \frac{3}{2}$. If $E(u) = 2u\partial_x$, for each $u \in H^s(\mathbb{R})$, then $E(u) \in L(H^s(\mathbb{R}), H^{s-1}(\mathbb{R}))$ and*

$$\| (E(z) - E(u))v \|_{H^{s-1}(\mathbb{R})} \leq \gamma_1 \| z - u \|_{H^{s-1}(\mathbb{R})} \| v \|_{H^s(\mathbb{R})}, \quad u, z, v \in H^s(\mathbb{R}). \tag{4.6}$$

Lemma 4.7 [104]. *Let $s > \frac{3}{2}$, $v \in H^{s-1}(\mathbb{R})$ and $u, z \in H^s(\mathbb{R})$. Then $I(u) = [\Lambda^s, bu^2\partial_x]\Lambda^{-s}$ belongs to $L(H^{s-1})$ if*

$$\| (I(u) - I(z))v \|_{H^{s-1}} \leq \gamma_2 \| u - z \|_{H^s} \| v \|_{H^{s-1}} . \tag{4.7}$$

Lemma 4.8 [54]. *Assume that λ_1 and λ_2 are real numbers satisfying $-\lambda_1 < \lambda_2 \leq \lambda_1$. Then*

$$\| uv \|_{H^{\lambda_2}} \leq c \| u \|_{H^{\lambda_1}} \| v \|_{H^{\lambda_2}}, \qquad if \quad \lambda_1 > \frac{1}{2},$$
$$\| uv \|_{H^{\lambda_1+\lambda_2-1/2}} \leq c \| u \|_{H^{\lambda_1}} \| v \|_{H^{\lambda_2}}, \qquad if \quad \lambda_1 < \frac{1}{2}.$$

Lemma 4.9. *Let $u, z \in H^s$, $s > \frac{3}{2}$ and $J(u) = \Lambda_1^{-2}[\frac{1}{\alpha}u^2 - 2uu_x] - \frac{1}{\alpha}u^2$. Then $J(u)$ satisfies*

$$\| J(u) - J(z) \|_{H^s} \leq \gamma_3 \| u - z \|_{H^s}, \tag{4.8}$$
$$\| J(u) - J(z) \|_{H^{s-1}} \leq \gamma_4 \| u - z \|_{H^{s-1}} . \tag{4.9}$$

Proof. Noting the multiplying properties of $H^s(\mathbb{R})$ $(s > \frac{1}{2})$ and using the equivalency of operators Λ_1^{-2} and Λ^{-2} give rise to

$$\begin{aligned}
\| J(u) - J(z) \|_{H^s(\mathbb{R})} &\leq c\Big[\| \Lambda_1^{-2}u^2 - \Lambda_1^{-2}z^2 \|_{H^s(\mathbb{R})} + \| (u^2)_x - (z^2)_x \|_{H^{s-2}(\mathbb{R})} \\
&\qquad + \| u^2 - z^2 \|_{H^{s-2}(\mathbb{R})} \Big] \\
&\leq c\Big[\| u^2 - z^2 \|_{H^s(\mathbb{R})} + \| u^2 - z^2 \|_{H^{s-1}(\mathbb{R})} \Big] \\
&\leq c(\| (u-z)(u+z) \|_{H^s(\mathbb{R})}) \\
&\leq \gamma_3 \| u - z \|_{H^s(\mathbb{R})},
\end{aligned} \tag{4.10}$$

from which we complete the proof of (4.8).

Since $s-1>\frac{1}{2}$, applying Lemma 4.8, we get

$$\begin{aligned}
&\| \Lambda_1^{-2}[(u^2)_x-(z^2)_x] \|_{H^{s-1}} \\
&\qquad \le c \| u^2-z^2 \|_{H^{s-1}} \\
&\qquad \le \gamma_4 \| u-z \|_{H^{s-1}} \qquad (4.11)
\end{aligned}$$

and

$$\begin{aligned}
&\| \Lambda_1^{-2}[u^2-z^2]+u^2-z^2 \|_{H^{s-1}} \\
&\qquad \le c \| u^2-z^2 \|_{H^{s-1}} \\
&\qquad \le c \| u-z \|_{H^{s-1}} \ . \qquad (4.12)
\end{aligned}$$

Applying (4.11) and (4.12) yields

$$\| J(u)-J(z) \|_{H^{s-1}(\mathbb{R})} \le \gamma_4 \| u-z \|_{H^{s-1}(\mathbb{R})} \ . \qquad (4.13)$$

Using (4.10) and (4.13) finishes the proof of Lemma 4.9. ■

Proof of Theorem 4.1. Employing Lemmas 4.5, 4.6, 4.7 and 4.9 together with the Kato Theorem, we get that there is a unique solution $u(t,x)$ satisfying

$$u(t,x) \in C([0,T);H^s(\mathbb{R})) \cap C^1([0,T);H^{s-1}(\mathbb{R}))$$

for problem (4.2) or system (4.3). The proof is completed. ■

4.4 Local weak solutions

Employing the decay property of the Sobolev $H^s(\mathbb{R}), s > \frac{3}{2}$ and the first equation of system (4.2) yields the following identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(u^2+\alpha^2 u_x^2)dx = -\alpha^2\int_{\mathbb{R}} u_x^3 dx - 2\int_{\mathbb{R}} u u_x^2 dx,$$

which results in the conservation law

$$\begin{aligned}\int_{\mathbb{R}}\big(u^2+\alpha^2 u_x^2\big)dx + 2\alpha^2\int_0^t\int_{\mathbb{R}} u_x^3 dxdt + 4\int_0^t\int_{\mathbb{R}} u u_x^2 dxdt \\ = \int_{\mathbb{R}}(u_0^2+\alpha^2 u_{0x}^2)dx. \qquad (4.14)\end{aligned}$$

Lemma 4.10. ([54]). *Assume the constant $\gamma \geq 0$. If $v_1, v_2 \in H^\gamma \cap L^\infty$, then*

$$\| v_1 v_2 \|_{H^\gamma} \leq c\big(\| v_1 \|_{L^\infty} \| v_2 \|_{H^\gamma} + \| v_1 \|_{H^\gamma} \| v_2 \|_{L^\infty} \big),$$

where the constant $c > 0$ only depends on γ.

Lemma 4.11. ([54]). *Let the constant $\gamma > 0$. If $v_1 \in H^\gamma \cap W^{1,\infty}$ and $v_2 \in H^{\gamma-1} \cap L^\infty$, then*

$$\| [\Lambda_0^\gamma, v_1] v_2 \|_{L^2} \leq c\big(\| \partial_x v_1 \|_{L^\infty} \| \Lambda_0^{\gamma-1} v_2 \|_{L^2} + \| \Lambda_0^\gamma v_1 \|_{L^2} \| v_2 \|_{L^\infty} \big),$$

where $\Lambda_0 = (1-\partial_x^2)^{\frac{1}{2}}$ and the constant $c > 0$ only depends on γ.

Lemma 4.12. *Assume that $s > 1$ and the initial value $u_0(x)$ belongs to $H^s(\mathbb{R})$. If $u_\varepsilon(t,x)$ is a solution of (4.4), still denoted by $u(t,x)$, then*

$$\| u \|_{L^\infty(\mathbb{R})} \leq \| u \|_{H^1(\mathbb{R})} \leq c_0 \| u_0 \|_{H^1(\mathbb{R})} e^{c_0\int_0^t \|u_x\|_{L^\infty(\mathbb{R})} d\tau}, \qquad (4.15)$$

where $c_0 > 0$ is a constant depending only on α.

Assume $r \in (0, s-1]$. Then there is a positive constant c, depending only on α, such that

$$\int_{\mathbb{R}} (\Lambda^{r+1} u)^2 dx \leq \int_{\mathbb{R}} [(\Lambda^{r+1} u_0)^2] dx + c \int_0^t \Big(1 + \| u \|_{L^\infty} + \| u_x \|_{L^\infty}\Big) \| u \|^2_{H^{r+1}} d\tau. \tag{4.16}$$

Let $r \in [0, s-1]$. Then there is a constant c, depending only on α, such that

$$\| u_t \|_{H^r} \leq c \Big[\| u \|_{H^1} + \| u \|_{H^r} + \| u \|_{H^{r+1}} \Big] \| u \|_{L^\infty} . \tag{4.17}$$

Proof. Using

$$|2uu_x| \leq (u^2 + u_x^2),$$

$$\Big| \int_{\mathbb{R}} u_x^3 dx \Big| \leq \| u_x \|_{L^\infty} \| u \|^2_{H^1(\mathbb{R})},$$

the Gronwall inequality and (4.14) derives (4.15).

Using the Parseval equality and $\partial_x^2 = 1 - \Lambda^2$ gives rise to

$$\int_{\mathbb{R}} \Lambda^r u \Lambda^r \partial_x^3 (u^2) dx = -2 \int_{\mathbb{R}} (\Lambda^{r+1} u) \Lambda^{r+1} (uu_x) dx + 2 \int_{\mathbb{R}} (\Lambda^r u) \Lambda^r (uu_x) dx. \tag{4.18}$$

We know that the identity

$$2uu_{xxx} = \partial_x^3 (u^2) - 4uu_{xx} - 2u_x u_{xx} \tag{4.19}$$

holds.

For a real number $r \in (0, s-1]$, we apply $(\Lambda^r u)\Lambda^r$ to multiply the first equation of (4.4). Subsequently, integrating the obtained equation by parts about variable x, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big[(\Lambda^r u)^2+\alpha^2(\Lambda^r u_x)^2\Big]dx \\
&\quad\le c\Bigg(\Big|\int_{\mathbb{R}}(\Lambda^r u)\Lambda^r(2\alpha u_x^2+2\alpha u u_{xx}-4\alpha^2 u u_{xx})dx\Big| \\
&\qquad\quad+\Big|\int_{\mathbb{R}}(\Lambda^{r+1}u)\Lambda^{r+1}(uu_x)dx\Big|+\Big|\int_{\mathbb{R}}\Lambda^r u\Lambda^r(uu_x)dx\Big| \\
&\qquad+\Big|\int_{\mathbb{R}}\Lambda^r u\Lambda^r(u_x u_{xx})dx\Big|\Bigg) \\
&\quad= K_1+K_2+K_3+K_4, \qquad (4.20)
\end{aligned}
$$

where c may depend on α.

Making use of Lemmas 4.10 and 4.11, and the Cauchy-Schwartz inequality, we get

$$
\begin{aligned}
K_2&=\Big|\int_{\mathbb{R}}(\Lambda^{r+1}u)\Lambda^{r+1}(uu_x)dx \\
&=\int_{\mathbb{R}}(\Lambda^{r+1}u)\Big[\Lambda^{r+1}(uu_x)-u\Lambda^{r+1}u_x\Big]dx \\
&\quad+\int_{\mathbb{R}}(\Lambda^{r+1}u)u\Lambda^{r+1}u_x dx
\end{aligned}
$$

and

$$
\begin{aligned}
|K_2|&\le c\parallel u\parallel_{H^{r+1}}\Big(\parallel u\parallel_{H^r}\parallel u_x\parallel_{L^\infty}+\parallel u\parallel_{H^{r+1}}\parallel u\parallel_{L^\infty}\Big) \\
&\qquad\qquad+\parallel\Lambda^{r+1}u\parallel_{L^2}\parallel u_x\parallel_{L^\infty} \\
&\le c\parallel u\parallel_{H^{r+1}}^2\Big(1+\parallel u\parallel_{L^\infty}+\parallel u_x\parallel_{L^\infty}\Big). \qquad (4.21)
\end{aligned}
$$

Using Lemma 4.10 yields

$$
\begin{aligned}
|K_3| &\le c \parallel u \parallel_{H^r} \parallel uu_x \parallel_{H^r} \\
&\le c \parallel u \parallel_{H^r} \big(\parallel u_x \parallel_{H^r} \parallel u \parallel_{L^\infty} + \parallel u \parallel_{H^r} \parallel u_x \parallel_{L^\infty} \big) \\
&\le c \parallel u \parallel_{H^{r+1}}^2 \big(\parallel u \parallel_{L^\infty} + \parallel u_x \parallel_{L^\infty} \big).
\end{aligned} \tag{4.22}
$$

Applying Lemma 4.10 and the Cauchy-Schwartz inequality gives rise to

$$
\begin{aligned}
\Big| \int_R (\Lambda^r u_x)\Lambda^r(u_x^2)dx \Big| &\le \parallel \Lambda^r u_x \parallel_{L^2} \parallel \Lambda^r(u_x^2) \parallel_{L^2} \\
&\le c\big(\parallel u_x \parallel_{L^\infty} \parallel u_x \parallel_{H^r} \big) \parallel u \parallel_{H^{r+1}} \\
&\le c \parallel u_x \parallel_{L^\infty} \parallel u \parallel_{H^{r+1}}^2 .
\end{aligned} \tag{4.23}
$$

Noticing identity $u_x u_{xx} = \frac{1}{2}(u_x^2)_x$, we obtain

$$
\begin{aligned}
&|K_4| = \Big| \int_R (\Lambda^r u)\Lambda^r(u_x u_{xx})dx \Big| \\
&\le \frac{1}{2} \Big| \int_{\mathbb{R}} \Lambda^r u_x \Lambda^r(u_x^2)dx \Big| \\
&\le c \parallel u \parallel_{H^{r+1}}^2 \parallel u_x \parallel_{L^\infty} .
\end{aligned} \tag{4.24}
$$

For the term K_1, we have

$$
|K_1| \le c \parallel u \parallel_{H^{r+1}}^2 (1 + \parallel u \parallel_{L^\infty} + \parallel u_x \parallel_{L^\infty}). \tag{4.25}
$$

From (4.20)-(4.25), we get that inequality (4.16) holds.

Now we estimate u_t. Employing $(1 - \alpha^2 \partial_x^2)^{-1}$ and system (4.2), we have

$$
u_t = \Lambda_1^{-2}[\frac{1}{\alpha}u^2 - 2uu_x] - \frac{1}{\alpha}u^2 - 2uu_x. \tag{4.26}
$$

Applying $(\Lambda^r u_t)\Lambda^r$ on Eq.(4.26) and noticing the equivalence of two operators

$(1-\alpha^2\partial_x^2)$ and $(1-\partial_x^2)$, for $r\in[0,s-1]$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}(\Lambda^r u_t)^2dx\\
&\quad\le c\Bigg(\Big|\int_{\mathbb{R}}(\Lambda^r u_t)\Lambda^{r-2}(\frac{1}{\alpha}u^2-2uu_x)dx\Big|+\Big|\frac{1}{\alpha}\int_{\mathbb{R}}(\Lambda^r u_t)(\Lambda^r u^2)dx\Big|\\
&\qquad\quad+\Big|\int_{\mathbb{R}}(\Lambda^r u_t)(\Lambda^r(uu_x)dx\Big|\Bigg)\\
&\quad\le c\parallel u_t\parallel_{H^r(\mathbb{R})}\big(\parallel u\parallel_{H^1(\mathbb{R})}+\parallel u\parallel_{H^r(\mathbb{R})}+\parallel u\parallel_{H^{r+1}(\mathbb{R})}\big)\parallel u\parallel_{L^\infty(\mathbb{R})}.
\end{aligned}\tag{4.27}$$

The proof is completed. ■

Lemma 4.13. [7, 59]. *If* $u_{\varepsilon,0}=\psi_\varepsilon\star u_0$, $0<\varepsilon<\frac{1}{4}$, $u_0\in H^s(\mathbb{R})$ *and* $s>0$, *then*

$$\begin{aligned}
&\parallel\frac{u_{\varepsilon,0}}{\partial x}\parallel_{L^\infty}\le c_0\parallel u_{0x}\parallel_{L^\infty},\\
&\parallel u_{\varepsilon,0}\parallel_{H^r}\le c_0,\quad \textit{if}\quad q\le s,\\
&\parallel u_{\varepsilon,0}\parallel_{H^r}\le c_0\varepsilon^{\frac{s-q}{4}},\quad \textit{if}\ q>s,\\
&\parallel u_{\varepsilon,0}-u_0\parallel_{H^r}\le c_0\varepsilon^{\frac{s-q}{4}},\quad \textit{if}\ q\le s,\\
&\parallel u_{\varepsilon,0}-u_0\parallel_{H^s}=o(1),
\end{aligned}$$

where ψ_ε *is defined in section 3.3 and the positive constant* c_0 *does not depend on* ε.

Proof of Theorem 4.2. Using inequalities (4.15), (4.16) and taking $q=s-1$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty}(\Lambda^s u)^2dx\le&\int_{-\infty}^{\infty}[(\Lambda^s u_0)^2]dx\\
&+c\int_0^t\parallel u\parallel_{H^s}^2\Big(1+\parallel u\parallel_{L^\infty}+\parallel u_x\parallel_{L^\infty}\Big)d\tau,
\end{aligned}\tag{4.28}$$

which results in

$$\| u \|_{H^s(\mathbb{R})} \leq \| u_0 \|_{H^s(\mathbb{R})} \, e^{c\int_0^t \left(1+\|u\|_{L^\infty}+\|u_x\|_{L^\infty}\right)d\tau}. \tag{4.29}$$

On the other hand, for $s > \frac{3}{2}$, we have

$$\| u_x \|_{L^\infty(\mathbb{R})} \leq c \| u \|_{H^s(\mathbb{R})} . \tag{4.30}$$

Employing (4.30) and Lemma 4.12, we immediately finish the proof. ■

Proof of Theorem 4.3. For simplicity, we use notation $u = u_\varepsilon$. Differentiating the first equation of (4.4) about variable x gives rise to

$$u_{tx} + 2u_x^2 + 2uu_{xx} = \Lambda_1^{-2}\Big[\frac{1}{\alpha}(u^2 - 2uu_x)\Big]_x - \frac{1}{\alpha}(u^2)_x. \tag{4.31}$$

Assuming that p is an integer, we have

$$\int_{\mathbb{R}} u(u_x)^{2p+1}u_{xx}dx = -\int_{\mathbb{R}}(u_x)^{2p+3}dx - (2p+1)\int_{\mathbb{R}} u(u_x)^{2p+1}u_{xx}dx, \tag{4.32}$$

which results in

$$\int_{\mathbb{R}} u(u_x)^{2p+1}u_{xx}dx = -\frac{1}{2p+2}\int_{\mathbb{R}}(u_x)^{2p+3}dx. \tag{4.33}$$

We use $(u_x)^{2p+1}$ to multiply (4.31) and integrate the obtained equation about variable x to get the identity

$$\begin{aligned}&\frac{1}{2p+2}\frac{d}{dt}\Big(\int_{\mathbb{R}}(u_x)^{2p+2}dx\Big) + 2\int_{\mathbb{R}}(u_x)^{2p+3}dx + 2\int_{\mathbb{R}} u(u_x)^{2p+1}u_{xx}dx\\ &= \int_{\mathbb{R}}\Big(\Lambda_1^{-2}\Big[\frac{1}{\alpha}(u^2 - 2uu_x)\Big]_x - \frac{1}{\alpha}(u^2)_x\Big)(u_x)^{2p+1}dx.\end{aligned} \tag{4.34}$$

Using (4.33) and (4.34) yields

$$\frac{1}{2p+2}\frac{d}{dt}\int_{\mathbb{R}}(u_x)^{2p+2}dx+\frac{2p+1}{p+1}\int_{\mathbb{R}}(u_x)^{2p+3}dx+2\int_{\mathbb{R}}u(u_x)^{2p+1}u_{xx}dx$$
$$=\int_{\mathbb{R}}\Big(\Lambda_1^{-2}[\frac{1}{\alpha}(u^2-2uu_x)\Big]_x-\frac{1}{\alpha}(u^2)_x\Big)(u_x)^{2p+1}dx. \tag{4.35}$$

Since $\frac{1}{2p+2}+\frac{2p+1}{2p+2}=1$, using Hölder's inequality and (4.35) gives rise to

$$\frac{1}{2p+2}\frac{d}{dt}\int_{\mathbb{R}}(u_x)^{2p+2}dx$$
$$\leq\Big\{\Big(\int_{\mathbb{R}}|J_1|^{2p+2}dx\Big)^{\frac{1}{2p+2}}\Big(\int_{\mathbb{R}}|u_x|^{2p+2dx}\Big)^{\frac{2p+1}{2p+2}}$$
$$+\frac{2p+1}{p+1}\parallel u_x\parallel_{L^\infty}\int_{\mathbb{R}}(u_x)^{2p+2}dx\Big\}, \tag{4.36}$$

where

$$J_1=\Lambda_1^{-2}[\frac{1}{\alpha}(u^2-2uu_x)\Big]_x-\frac{1}{\alpha}(u^2)_x.$$

From (4.36), we have

$$\frac{d}{dt}\Big(\int_{\mathbb{R}}(u_x)^{2p+2}dx\Big)^{\frac{1}{2p+2}}\leq\Big(\int_{\mathbb{R}}|J_1|^{2p+2}dx\Big)^{\frac{1}{2p+2}}$$
$$+\frac{2p+1}{p+1}\Big(\parallel u_x\parallel_{L^\infty}\Big)\Big(\int_{\mathbb{R}}(u_x)^{2p+2}dx\Big)^{\frac{1}{2p+2}}. \tag{4.37}$$

In fact, $\parallel G\parallel_{L^p}$ tends to $\parallel G\parallel_{L^\infty}$ if $p\to\infty$ for any function $G\in L^\infty\cap L^2$. Using (4.37) and letting $p\to\infty$, we obtain

$$\parallel u_x\parallel_{L^\infty}\leq\parallel u_{0x}\parallel_{L^\infty}+\int_0^t(\parallel J_1\parallel_{L^\infty(\mathbb{R})}+\parallel u_x\parallel_{L^\infty}^2)d\tau. \tag{4.38}$$

Making use of the multiplying property of the space $H^s(\mathbb{R})$, $s>\frac{1}{2}$, (4.16) and the Sobolev imbedding theorem, for sufficiently small $\delta > 0$, we have

$$\begin{aligned}\| J_1 \|_{L^\infty} &\le c \| J \|_{H^{\frac{1}{2}+\delta}} \\ &= \| \Lambda_1^{-2}\Big[\frac{1}{\alpha}(u^2 - 2uu_x)\Big]_x - \frac{1}{\alpha}(u^2)_x \|_{H^{\frac{1}{2}+\delta}} \\ &\le c(\| u_x \|_{L^\infty} \| u \|_{H^1(\mathbb{R})} + \| u \|^2_{H^1(\mathbb{R})}) \\ &\le c(1 + \| u_x \|_{L^\infty}) \exp\Big[c\int_0^t \| u_x \|_{L^\infty} d\tau\Big], \end{aligned} \tag{4.39}$$

where constant $c > 0$ is independent of ε.

It follows from (4.38), (4.39) and Lemma 4.12 that

$$\begin{aligned}&\| u_x \|_{L^\infty} \le \| u_{0x} \|_{L^\infty} \\ &+c\int_0^t \Big[\exp\Big[c\int_0^\tau \| u_x \| d\xi\Big]\Big(1 + \| u_x \|_{L^\infty}\Big) + 1 + \| u_x \|^2_{L^\infty}\Big] d\tau. \end{aligned} \tag{4.40}$$

Making use of the contraction mapping principle, we conclude that there is a $T > 0$ to ensure that the following equation has a unique continuous solution $f \in C[0, T]$

$$\begin{aligned}&\| f \|_{L^\infty} = \| u_{0x} \|_{L^\infty} \\ &+c\int_0^t \Big[1 + \| f \|^2_{L^\infty} + (1 + \| f \|_{L^\infty}) \exp\Big(c\int_0^\tau \| f \|_{L^\infty} d\xi\Big)\Big] d\tau. \end{aligned} \tag{4.41}$$

The conclusion at page 51 in [66] guarantees that we can find a constant $T > 0$, which does not depend on ε, such that

$$\| u_x \|_{L^\infty} \le f(t), \quad \text{if} \quad t \in [0, T].$$

As the function $f(t)$ is bounded, we immediately obtain the desired result. ■

Proof of Theorem 4.4.

Letting $r_1 \in (0, s-1]$, $r \in (0, s]$, $t \in [0, T)$, using Lemmas 4.12, 4.13 and

Theorem 4.3, and using the notation $u_\varepsilon = u$, we have the inequalities

$$\| u_\varepsilon \|_{H^r} \leq c \tag{4.42}$$

and

$$\| u_{\varepsilon t} \|_{H^{r_1}} \leq \| u_\varepsilon \|_{L^\infty} (\| u_\varepsilon \|_{H^1} + \| u_\varepsilon \|_{H^{r_1}} + \| u_\varepsilon \|_{H^{r_1+1}}) \leq c. \tag{4.43}$$

Applying Aubin's compactness theorem, (4.42) and (4.43), we can find a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_j}\}$, $\varepsilon_j \to \infty$ if $j \to \infty$, to ensure that $\{u_{\varepsilon_j}\}$ converges weakly to $u(t,x)$ in $L^2([0,T], H^s(\mathbb{R}))$ and $\{u_{\varepsilon_j t}\}$ converges weakly to $u_t(t,x)$ in $L^2([0,T], H^{s-1}(\mathbb{R}))$.

Furthermore, for an arbitrary constant $C_1 > 0$, we know that $\{u_{\varepsilon_j}\}$ converges strongly to $u(t,x)$ in $L^2([0,T], H^r(-C_1, C_1))$ and $\{u_{\varepsilon_j t}\}$ converges strongly to $u_t(t,x)$ in $L^2([0,T], H^{r_1}(-C_1, C_1))$.

For every $g(t,x) \in C_0^\infty([0,T] \times \mathbb{R})$, using (4.2) and integration by parts, we get

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} u_t g dx dt + \int_0^T \int_{\mathbb{R}} (u_t g_{xx}) dx dt \\ &= \int_0^T \int_{\mathbb{R}} \Big(-4uu_x + 2\alpha u_x^2 + 2\alpha u u_{xx} + 6\alpha^2 u_x u_{xx} + 2\alpha^2 u u_{xxx} \Big) g dx dt \\ &= \int_0^T \int_{\mathbb{R}} \Big(2u^2 g_x + \alpha u^2 g_{xx} - \alpha^2 u^2 g_{xxx} \Big) dx dt. \end{aligned} \tag{4.44}$$

The proof is completed. ■

CHAPTER 5

Local weak and strong solutions to a nonlinear Camassa-Holm type equation with cubic nonlinearities

In this chapter, we study a Camassa-Holm type model with quasi-local higher symmetries and cubic nonlinearity. Approximation techniques are employed to establish the well-posedness of local strong solutions for the equation in $H^s(\mathbb{R})$ associated with index $s > \frac{3}{2}$. For initial value $u_0(x) \in H^{s_0}(\mathbb{R})$ $(1 \leq s_0 \leq \frac{3}{2})$, we prove that the equation has local weak solutions.

5.1 General

Novikov [87] derived many models which are called the generalizations of the Camassa-Holm type models. One of the generalized models with cubic nonlinearities takes the form

$$u_t - \alpha^2 u_{txx} = (1 + \alpha \frac{\partial}{\partial x})(\alpha u^2 u_{xx} + \alpha u u_x^2 - 2u^2 u_x), \tag{5.1}$$

where $\alpha \neq 0$ (see Theorem 5 in [87]).

Eq.(5.1) is related to the Camassa-Holm type equations. Intensive research has been made to investigate the classical Camassa-Holm model and its generalizations as reviewed in section 2.2.

Motivated by the work in Li and Olver [66] in which the uniqueness and existence of local strong solutions and the existence of local weak solutions for an integral non-linear dispersive wave equation are considered, the purpose of this chapter is to investigate Eq.(5.1). Imposing some assumptions on the initial data and the coefficient α in Eq.(5.1), several dynamical properties such as the existence of local weak solutions, the uniqueness and existence of local strong solutions have been found.

In section 5.2, we prove that Eq.(5.1) has local weak solutions if we impose certain restrictions on the initial value. In section 5.3, local strong solution to Eq.(5.1) is considered.

For conciseness in this chapter, we let c represent an arbitrary positive constant independent of the parameter ε unless we clearly state.

5.2 Local weak solutions

By calculation, we get the Cauchy problem for Eq.(5.1) as follows

$$\begin{cases} u_t - \alpha^2 u_{txx} = (1+\alpha\frac{\partial}{\partial x})(\alpha u^2 u_{xx} + \alpha u u_x^2 - 2u^2 u_x) \\ = -\frac{2}{3}(u^3)_x - \frac{\alpha}{3}(u^3)_{xx} - \alpha u u_x^2 + \frac{\alpha^2}{3}(u^3)_{xxx} - \alpha^2 (u u_x^2)_x, \\ u(0,x) = u_0(x). \end{cases} \tag{5.2}$$

We summarize the result about the local weak solutions for problem (5.2) below.

Theorem 5.1. *Let* $1 \le s \le \frac{3}{2}$, $|\alpha| > \frac{\sqrt{2}}{2}$, $u_0(x) \in H^s(\mathbb{R})$ *and* $\| u_{0x} \|_{L^\infty} < \infty$. *There is a* $T > 0$ *guaranteeing that system (5.2) admits at least one weak solution* $u(t,x) \in L^2([0,T], H^s(\mathbb{R}))$. *Moreover,* $u_x \in L^\infty([0,T]\times\mathbb{R})$.

To prove Theorem 5.1, the following approximation problem needs to be handled

$$\begin{cases} u_t - \alpha^2 u_{txx} + \varepsilon u_{txxxx} \\ = -\frac{2}{3}(u^3)_x - \frac{\alpha}{3}(u^3)_{xx} - \alpha u u_x^2 + \frac{\alpha^2}{3}(u^3)_{xxx} - \alpha^2 (u u_x^2)_x, \\ u(0,x) = u_0(x), \end{cases} \tag{5.3}$$

where $\mid \alpha \mid > \frac{\sqrt{2}}{2}$ and $0 < \varepsilon < \min(\frac{1}{4}, \alpha^2 - \frac{1}{2})$. It is clear that the solution of system (5.3) depends on the parameter ε. For conciseness, we write $u(t,x) = u_\varepsilon(t,x)$.

Before proving Theorem 5.1, several Lemmas are required.

Lemma 5.2. *Assume $s > \frac{3}{2}$ and $u_0(x) \in H^s(\mathbb{R})$. Then problem (5.3) has a unique solution $u(t,x) = u_\varepsilon(t,x) \in C([0,T]; H^s(\mathbb{R}))$ where $T > 0$ depends on $\parallel u_0 \parallel_{H^s(\mathbb{R})}$.*

Proof. Defining the inverse operator $X = (1 - \alpha^2 \partial_x^2 + \varepsilon \partial_x^4)^{-1}$, we get that the operator $X : H^s \to H^{s+4}$ is linear and bounded. Let

$$P_u(t,x) = -\frac{2}{3}(u^3)_x - \frac{\alpha}{3}(u^3)_{xx} - \alpha u u_x^2 + \frac{\alpha^2}{3}(u^3)_{xxx} - \alpha^2 (u u_x^2)_x. \tag{5.4}$$

We utilize the operator X to multiply the first equation of system (5.3). Subsequently, integrating the obtained equation about variable t gives rise to

$$u(t,x) = u_0(x) + \int_0^t X P_u(\tau, x) d\tau, \quad t \in [0,T). \tag{5.5}$$

We discuss the operator

$$Au(t,x) = u_0(x) + \int_0^t X P_u(\tau, x) d\tau.$$

We let u and v be the elements in the closed ball $K_{M_0}(0)$ where M_0 is the radius of the ball and $K_{M_0}(0) \subset C([0,T]; H^s(\mathbb{R}))$. Employing the multiplying property of the space $H^\delta(\mathbb{R})$ ($\delta > \frac{1}{2}$), we obtain the estimate

$$
\begin{aligned}
&\| \int_0^t XP_u(t,x)dt - \int_0^t XP_v(t,x)dt \|_{H^s} \\
&\le cT\Big[\sup_{0\le t\le T} \| (u^3)_x - (v^3)_x \|_{H^{s-4}} + \sup_{0\le t\le T} \| (u^3)_{xx} - (v^3)_{xx} \|_{H^{s-4}} \\
&\quad + \sup_{0\le t\le T} \| uu_x^2 - vv_x^2 \|_{H^{s-4}} + \sup_{0\le t\le T} \| (u^3)_{xxx} - (v^3)_{xxx} \|_{H^{s-4}} \\
&\quad\quad + \sup_{0\le t\le T} \| (uu_x^2)_x - (vv_x^2)_x \|_{H^{s-4}} \Big] \\
&\le C_0 T\Big[\sup_{0\le t\le T} \| u^3 - v^3 \|_{H^s} + \sup_{0\le t\le T} \| u^3 - v^3 \|_{H^{s-2}} \\
&\quad + \sup_{0\le t\le T} \| uu_x^2 - vv_x^2 \|_{H^{s-4}} + \sup_{0\le t\le T} \| u^3 - v^3 \|_{H^{s-1}} \\
&\quad\quad + \sup_{0\le t\le T} \| uu_x^2 - vv_x^2 \|_{H^{s-3}} \Big],
\end{aligned}
\tag{5.6}
$$

where the constant $C_0 > 0$ may depend on the parameter ε.

Since

$$
\begin{aligned}
&\| uu_x^2 - vv_x^2 \|_{H^{s-4}} \le c \| uu_x^2 - vv_x^2 \|_{H^{s-3}} \\
&\le c \| uu_x^2 - vu_x^2 + vu_x^2 - vv_x^2 \|_{H^{s-3}} \\
&\le cM_0^2 \| u - v \|_{H^s},
\end{aligned}
\tag{5.7}
$$

from (5.6) and (5.7), we get

$$
\| Au - Av \|_{H^s} \le C_2 M_0^2 T \| u - v \|_{H^s} . \tag{5.8}
$$

Letting T be small enough to satisfy $C_2 M_0^2 T < 1$, we conclude that the operator A is contractive. Making use of (5.8) yields

$$
\| Au \|_{H^s} \le \| u_0 \|_{H^s} + C_2 M_0^2 T \| u \|_{H^s} . \tag{5.9}
$$

We choose $T > 0$ enough small to ensure that $C_2 M_0^3 T + \| u_0 \|_{H^s} < M_0$. Then we recognize that the operator A maps $K_{M_0}(0)$ to itself. The contractive mapping principle guarantees that the operator A has a unique fixed point $u(t,x)$ in $K_{M_0}(0)$. Thus, we obtain the desired result and finish the proof. ■

For the index $s > 4$, we have

$$\int_{\mathbb{R}} u(u^3)_x dx = 0, \quad \int_{\mathbb{R}} u^2 u_x u_{xx} dx = -\int_{\mathbb{R}} u u_x^3 dx \tag{5.10}$$

and

$$\int_{\mathbb{R}} u \partial_x^3 (u^3) dx = 3 \int_{\mathbb{R}} u^2 u_x u_{xx} dx, \quad \int_{\mathbb{R}} u(u u_x^2)_x dx = -\int_{\mathbb{R}} u u_x^3 dx. \tag{5.11}$$

Using the first equation of problem (5.3), we derive that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (u^2 + \alpha^2 u_x^2 + \varepsilon u_{xx}^2) dx = 0, \tag{5.12}$$

which results in the following conservation law

$$\begin{aligned} &\int_{\mathbb{R}} (u^2 + \alpha^2 u_x^2 + \varepsilon u_{xx}^2) dx \\ &= \int_{\mathbb{R}} (u_0^2 + \alpha^2 u_{0x}^2 + \varepsilon u_{0xx}^2) dx. \end{aligned} \tag{5.13}$$

Lemma 5.3. *If $s \geq 4$, $u_0(x) \in H^s(\mathbb{R})$ and $u(t,x)$ satisfies (5.3), then*

$$\| u \|_{H^1} \leq \int_{\mathbb{R}} (u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2) dx. \tag{5.14}$$

If $0 < r \leq s-1$, *there exists a constant* $c > 0$ *such that*

$$\int_{-\infty}^{\infty} (\Lambda^{r+1}u)^2 dx \leq \int_{-\infty}^{\infty} [(\Lambda^{r+1}u_0)^2 + \varepsilon(\Lambda^r u_{0xx})^2] dx$$
$$+c\int_0^t (\| u \|_{H^r}^2 + \| u \|_{H^{r+1}}^2)\big(\| u \|_{L^\infty} \| u_x \|_{L^\infty} + \| u_x \|_{L^\infty}^2 \big) d\tau. \quad (5.15)$$

If $|\alpha| > \frac{\sqrt{2}}{2}$ *and* $0 \leq r \leq s-1$, *there exists a constant* $c > 0$ *such that*

$$(2\alpha^2 - 1 - 2\varepsilon) \| u_t \|_{H^r} \leq c \| u \|_{H^1} \| u \|_{H^{r+1}} \| u \|_{L^\infty} . \quad (5.16)$$

Proof. Applying (5.13), we can easily derive (5.14)*.

For the case $r \in (0, s-1]$, we apply $(\Lambda^r u)\Lambda^r$ to multiply the first equation of (5.3) and subsequently use integration by parts to the obtained equation. Then, we acquire

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R \Big[(\Lambda^r u)^2 + \alpha^2(\Lambda^r u_x)^2 + \varepsilon(\Lambda^r u_{xx})^2\Big] dx \\
&= -\frac{2}{3}\int_{\mathbb{R}} \Lambda^r u \Lambda^r (u^3)_x dx - \frac{\alpha}{3}\int_{\mathbb{R}} \Lambda^r u \Lambda^r (u^2)_{xx} dx - \alpha \int_{\mathbb{R}} \Lambda^r u \Lambda^r (u u_x^2) dx \\
&\quad + \frac{\alpha^2}{3}\int_{\mathbb{R}} \Lambda^r u \Lambda^r \partial_x^3 (u^3) dx - \alpha^2 \int_{\mathbb{R}} \Lambda^r u \Lambda^r (u u_x^2)_x dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5. \qquad (5.17)
\end{aligned}$$

Recalling the operator $\partial_x^2 = -\Lambda^2 + 1$ and combining the Parseval equality, it holds that

$$\int_{\mathbb{R}} \Lambda^r u \Lambda^r \partial_x^3 (u^3) dx = -3\int_{\mathbb{R}} (\Lambda^{r+1}u)\Lambda^{r+1}(u^2 u_x) dx + 3\int_{\mathbb{R}} (\Lambda^r u)\Lambda^r (u^2 u_x) dx$$

*Lemmas 4.10 and 4.11 are still used in this chapter.

and

$$\int_{\mathbb{R}} \Lambda^r u \Lambda^r \partial_x^2(u^3)dx = -\int_{\mathbb{R}} (\Lambda^{r+1}u)\Lambda^{r+1}(u^3)dx + \int_{\mathbb{R}} (\Lambda^r u)\Lambda^r(u^3)dx.$$

Using Lemmas 4.11, we get

$$\begin{aligned}&\int_R (\Lambda^r u)\Lambda^r(u^2u_x)dx\\ &\qquad = \int_{\mathbb{R}} (\Lambda^r u)[\Lambda^r(u^2u_x) - u^2\Lambda^r u_x]dx + \int_{\mathbb{R}} (\Lambda^r u)u^2\Lambda^r u_x dx.\end{aligned}$$

Employing Lemma 4.10 and the Cauchy-Schwartz inequality yields

$$\begin{aligned}&\Big|\int_R (\Lambda^r u)\Lambda^r(u^2u_x)dx\Big|\\ &\quad \le c \parallel u \parallel_{H^r} \Big(\parallel u \parallel_{H^{r-1}} \parallel u \parallel_{L^\infty} \parallel u_x \parallel_{L^\infty} + \parallel u \parallel_{H^r} \parallel u \parallel_{L^\infty} \parallel u_x \parallel_{L^\infty} \Big)\\ &\qquad + \parallel u_x \parallel_{L^\infty} \parallel u \parallel_{L^\infty} \parallel \Lambda^r u \parallel_{L^2}^2\\ &\quad \le c \parallel u \parallel_{H^r}^2 \parallel u_x \parallel_{L^\infty} \parallel u \parallel_{L^\infty} . \end{aligned} \tag{5.18}$$

Similar to the proof of (5.18), we derive that

$$|\int_R (\Lambda^{r+1}u)\Lambda^{r+1}(u^2u_x)dx| \le c \parallel u \parallel_{L^\infty} \parallel u_x \parallel_{L^\infty} \parallel u \parallel_{H^{r+1}}^2 . \tag{5.19}$$

From (5.18) and (5.19), we have

$$|I_4| \le c(\parallel u \parallel_{H^r}^2 + \parallel u \parallel_{H^{r+1}}^2) \parallel u \parallel_{L^\infty} \parallel u_x \parallel_{L^\infty} . \tag{5.20}$$

For I_5, making use of Lemma 4.10 and the Cauchy-Schwartz inequality derives that

$$\begin{aligned}|I_5| &\leq \Big| \int_R (\Lambda^r u_x)\Lambda^r(uu_x^2)dx\Big| \\ &\leq \| \Lambda^r u_x \|_{L^2} \| \Lambda^r(uu_x^2) \|_{L^2} \\ &\leq c \| u \|_{H^{r+1}} \Big(\| u_x \|_{H^r} \| uu_x \|_{L^\infty} + \| u_x \|_{L^\infty} \| uu_x \|_{H^r} \Big) \\ &\leq c \| u \|^2_{H^{r+1}} \big(\| u \|_{L^\infty} \| u_x \|_{L^\infty} + \| u_x \|^2_{L^\infty} \big). \end{aligned} \tag{5.21}$$

Similarly, for terms I_1, I_2 and I_3, we have

$$|I_1| + |I_2| + |I_3| \leq c(\| u \|^2_{H^r} + \| u \|^2_{H^{r+1}}) \| u \|_{L^\infty} \| u_x \|_{L^\infty} . \tag{5.22}$$

Applying (5.20)-(5.22), we derive the inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \big[(\Lambda^r u)^2 + \alpha^2(\Lambda^r u_x)^2 + \varepsilon(\Lambda^r u_{xx})^2\big]dx \\ &\quad \leq c(\| u \|^2_{H^r} + \| u \|^2_{H^{r+1}})\big(\| u \|_{L^\infty} \| u_x \|_{L^\infty} + \| u_x \|^2_{L^\infty} \big), \end{aligned} \tag{5.23}$$

where $c > 0$ depends on α. We complete the proof of (5.15).

Noticing $\partial_x^2 = 1 - \Lambda^2$ and applying Λ^{-2} to multiply the first equation of system (5.3), we have

$$\begin{aligned}(\alpha^2 - \varepsilon)u_t - \varepsilon u_{txx} = \Lambda^{-2}\Bigg[&- (1 - \alpha^2 + \varepsilon)u_t - \frac{2}{3}(u^3)_x - \frac{\alpha}{3}(u^3)_{xx} \\ &-\alpha uu_x^2 + \frac{\alpha^2}{3}(u^3)_{xxx} - \alpha^2(uu_x^2)_x\Bigg]. \end{aligned} \tag{5.24}$$

Applying $(\Lambda^r u_t)\Lambda^r$ on both sides of (5.24) gives rise to

$$(\alpha^2-\varepsilon)\int_{\mathbb{R}}(\Lambda^r u_t)^2dx+\varepsilon\int_{\mathbb{R}}(\Lambda^r u_{xt})^2dx$$
$$=\int_{\mathbb{R}}(\Lambda^r u_t)\Lambda^r(1-\partial_x^2)^{-1}\Big[-(1-\alpha^2+\varepsilon)u_t-\frac{2}{3}(u^3)_x-\frac{\alpha}{3}(u^3)_{xx}$$
$$-\alpha u u_x^2+\frac{\alpha^2}{3}(u^3)_{xxx}-\alpha^2(uu_x^2)_x\Big]dx. \tag{5.25}$$

Now we estimate the term

$$\int_{\mathbb{R}}(\Lambda^r u_t)\Lambda^{r-2}\Big[-(1-\alpha^2+\varepsilon)u_t-\frac{2}{3}(u^3)_x-\frac{\alpha}{3}(u^3)_{xx}$$
$$-\alpha u u_x^2+\frac{\alpha^2}{3}(u^3)_{xxx}-\alpha^2(uu_x^2)_x\Big]dx. \tag{5.26}$$

In fact, it holds that

$$|\int_{\mathbb{R}}(\Lambda^r u_t)\Lambda^{r-2}\Big[-(1-\alpha^2+\varepsilon)u_t-\frac{2}{3}(u^3)_x-\frac{\alpha}{3}(u^3)_{xx}-\alpha u u_x^2\Big]dx|$$
$$\leq|1-\alpha^2+\varepsilon|\parallel u_t\parallel_{H^r}^2+c\parallel u_t\parallel_{H^r}\parallel u\parallel_{H^r}\parallel u\parallel_{L^\infty}^2 \tag{5.27}$$

and

$$|\int_{\mathbb{R}}(\Lambda^r u_t)(1-\partial_x^2)^{-1}\Lambda^r(uu_x^2)_x dx|$$
$$\leq c\parallel u_t\parallel_{H^r}\left(\int_{\mathbb{R}}(1+\varsigma^2)^{r-1}\Big[\int_{\mathbb{R}}\big[\widehat{uu_x}(\varsigma-\eta)\widehat{u_x}(\eta)\big]d\eta\Big]^2d\varsigma\right)^{\frac{1}{2}}$$
$$\leq c\parallel u_t\parallel_{H^r}\parallel u\parallel_{H^{r+1}}\parallel u\parallel_{L^\infty}\parallel u\parallel_{H^1}\ . \tag{5.28}$$

We know the identity

$$\int(\Lambda^r u_t)(1-\partial_x^2)^{-1}\Lambda^r\partial_x^2(u^3)_x dx=-3\int(\Lambda^r u_t)\Lambda^r(u^2u_x)dx$$
$$+3\int(\Lambda^r u_t)(1-\partial_x^2)^{-1}\Lambda^r(u^2u_x)dx. \tag{5.29}$$

From Lemmas 4.10, 4.11, $\| u \|_{L^\infty} \leq \| u \|_{H^1}$ and

$$\| u^2 u_x \|_{H^r} \leq c \| (u^3)_x \|_{H^r} \leq c \| u \|_{L^\infty}^2 \| u \|_{H^{r+1}},$$

we have

$$\begin{aligned} & |\int (\Lambda^r u_t) \Lambda^r (u^2 u_x) dx| \\ & \leq c \| u_t \|_{H^r} \| u^2 u_x \|_{H^r} \\ & \leq c \| u_t \|_{H^r} \| u \|_{L^\infty} \| u \|_{H^1} \| u \|_{H^{r+1}} . \end{aligned} \tag{5.30}$$

Furthermore, we get

$$|\int (\Lambda^r u_t)(1-\partial_x^2)^{-1} \Lambda^r (u^2 u_x) dx| \leq c \| u_t \|_{H^r} \| u \|_{L^\infty} \| u \|_{H^1} \| u \|_{H^{r+1}} . \tag{5.31}$$

Noticing $|\alpha| > \frac{\sqrt{2}}{2}$ and choosing $\varepsilon > 0$ sufficiently small, from (5.25)-(5.31), we get that (5.16) holds. The proof is completed. ■

Assume that $\widehat{\phi_1}$ is the Fourier transform of ϕ_1 satisfying $\widehat{\phi_1} \in C_0^\infty(\mathbb{R})$, $\widehat{\phi_1(\eta)} \geq 0$, and $\widehat{\phi_1(\eta)} = 1$ for any arbitrary $\eta \in (-1, 1)$. For $s > 0$, assume that $u_0(x) \in H^s(\mathbb{R})$ and $u_{\varepsilon,0} = u_0 \star \phi_\varepsilon$ is the convolution u_0 and $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}} \phi_1(\varepsilon^{-\frac{1}{4}} x)$. Then $u_{\varepsilon,0}(x) \in C^\infty$. Applying Lemma 5.2, for the small parameter ε with $0 < \varepsilon < \min(\frac{1}{4}, \alpha^2 - \frac{1}{2})$, $|\alpha| > \frac{\sqrt{2}}{2}$, we get that the following Cauchy problem

$$\begin{cases} u_t - \alpha^2 u_{txx} + \varepsilon u_{txxxx} \\ = -\frac{2}{3}(u^3)_x - \frac{\alpha}{3}(u^3)_{xx} - \alpha u u_x^2 + \frac{\alpha^2}{3}(u^3)_{xxx} - \alpha^2 [u u_x^2]_x, \\ u(0,x) = u_{\varepsilon,0}(x) \end{cases} \tag{5.32}$$

possesses a unique solution $u_\varepsilon(t,x)$ belonging to $C^\infty([0,\infty); H^\beta(\mathbb{R}))$ in which $\beta \in [1, \infty)$.

Remark 5.4. *If $s \geq 1$, we have*

$$\| u_\varepsilon \|_{L^\infty} \leq c \| u_\varepsilon \|_{H^{\frac{1}{2}+}} \leq c \| u_\varepsilon \|_{H^1},$$

where $H^{\frac{1}{2}+} = H^{\frac{1}{2}+\delta}$, δ is sufficiently small, and

$$\| u_\varepsilon \|_{H^1}^2 \leq c \int_{\mathbb{R}} (u_\varepsilon^2 + u_{\varepsilon x}^2) dx.$$

Using Lemma 4.13, we get

$$\begin{aligned} \| u_\varepsilon \|_{L^\infty}^2 \leq c \| u_\varepsilon \|_{H^1}^2 &\leq c \int_{\mathbb{R}} \Big(u_{\varepsilon,0}^2 + (\frac{\partial u_{\varepsilon,0}}{\partial x})^2 + \varepsilon (\frac{\partial^2 u_{\varepsilon,0}}{\partial x^2})^2 \Big) dx \\ &\leq c(\| u_{\varepsilon,0} \|_{H^1}^2 + \varepsilon \| u_{\varepsilon,0} \|_{H^2}^2) \\ &\leq c(c + c\varepsilon \times \varepsilon^{\frac{s-2}{2}}) \\ &\leq c. \end{aligned} \tag{5.33}$$

Lemma 5.5. *Assume $s \in [1, \frac{3}{2}]$, $u_0(x) \in H^s(\mathbb{R})$, $\| u_{0x} \|_{L^\infty} < \infty$ and $|\alpha| > \frac{\sqrt{2}}{2}$. If u_ε is the solution of problem (5.32), then*

$$\| u_{\varepsilon x} \|_{L^\infty} \leq c, \quad t \in [0, T],$$

where both $c > 0$ and $T > 0$ are independent of ε.

Proof. For simplicity, we use notation $u = u_\varepsilon(t, x)$ and differentiate Eq.(5.24) about x. Then

$$(\alpha^2-\varepsilon)u_{tx}-\varepsilon u_{txxx}=\frac{\alpha}{3}(u^3)_x-\frac{\alpha^2}{3}(u^3)_{xx}+\alpha^2(uu_x^2)+\frac{2-\alpha^2}{3}u^3$$
$$+\Lambda^{-2}\Big[-(1-\alpha^2+\varepsilon)u_{tx}-\frac{2-\alpha^2}{3}u^3-\frac{\alpha}{3}(u^3)_x-\alpha(uu_x^2)_x-\alpha^2uu_x^2\Big]. \tag{5.34}$$

Set

$$G=\Lambda^{-2}\Big[-(1-\alpha^2+\varepsilon)u_{tx}-\frac{2-\alpha^2}{3}u^3-\frac{\alpha}{3}(u^3)_x-\alpha(uu_x^2)_x-\alpha^2uu_x^2\Big]. \tag{5.35}$$

Letting p be an integer, using the identity

$$\int_{\mathbb{R}}u^2u_x^{2p+1}u_{xx}dx=-\frac{1}{p+1}\int_{\mathbb{R}}uu_x^{2p+3}dx \tag{5.36}$$

and multiplying (5.34) by $(u_x)^{2p+1}$, we have

$$\frac{\alpha^2-\varepsilon}{2p+2}\frac{d}{dt}\int_{\mathbb{R}}(u_x)^{2p+2}dx-\varepsilon\int_{\mathbb{R}}(u_x)^{2p+1}u_{xxxt}dx+\alpha^2\frac{p}{p+1}\int_{\mathbb{R}}u(u_x)^{2p+3}dx$$
$$=\int_{\mathbb{R}}(u_x)^{2p+1}\Big[\frac{2-\alpha^2}{3}u^3+\frac{\alpha}{3}(u^3)_x\Big]dx+\int_{\mathbb{R}}(u_x)^{2p+1}\times Gdx. \tag{5.37}$$

Making use of Hölder's inequality produces the inequality

$$\Big(\frac{\alpha^2-\varepsilon}{2p+2}\Big)\frac{d}{dt}\int_{-\infty}^{\infty}(u_x)^{2p+2}dx$$
$$\leq\Big\{\varepsilon\Big(\int_{\mathbb{R}}|u_{txxx}|^{2p+2}dx\Big)^{\frac{1}{2p+2}}+\int_{\mathbb{R}}\Big|\frac{2-\alpha^2}{3}u^3+\frac{\alpha}{3}(u^3)_x\Big|dx$$
$$+\Big(\int_{\mathbb{R}}|G|^{2p+2}dx\Big)^{\frac{1}{2p+2}}\Big\}\Big(\int_{\mathbb{R}}|u_x|^{2p+2}dx\Big)$$
$$+\frac{p}{p+1}\alpha^2\parallel u\parallel_{L^\infty}\parallel u_x\parallel_{L^\infty}\int_{\mathbb{R}}|u_x|^{2p+1}dx, \tag{5.38}$$

which is equivalent to

$$\begin{aligned}
&\Big(\alpha^2-\varepsilon\Big)\frac{d}{dt}\Big(\int_{-\infty}^{\infty}(u_x)^{2p+2}dx\Big)^{\frac{1}{2p+2}}\\
&\leq \varepsilon\Big(\int_{\mathbb{R}}|u_{txxx}|^{2p+2}dx\Big)^{\frac{1}{2p+2}}+\Big(\int_{\mathbb{R}}|\frac{2-\alpha^2}{3}u^3+\frac{\alpha}{3}(u^3)_x|^{2p+2}dx\Big)^{\frac{1}{2p+2}}\\
&+\Big(\int_{\mathbb{R}}|G|^{2p+2}dx\Big)^{\frac{1}{2p+2}}+\frac{p}{p+1}\alpha^2\parallel u\parallel_{L^\infty}\parallel u_x\parallel_{L^\infty}\Big(\int_{\mathbb{R}}|u_x|^{2p+1}dx\Big)^{\frac{1}{2p+2}}.
\end{aligned} \tag{5.39}$$

We know $\parallel h\parallel_{L^p}\to\parallel h\parallel_{L^\infty}$ $(p\to\infty)$ if $h\in L^\infty\cap L^2$. From (5.39) and sending $p\to\infty$, we derive that

$$\begin{aligned}
&(\alpha^2-\varepsilon)\parallel u_x\parallel_{L^\infty}\leq(\alpha^2-\varepsilon)\parallel u_{0x}\parallel_{L^\infty}\\
&\quad+c\int_0^t\Big[\varepsilon\parallel u_{txxx}\parallel_{L^\infty}+\parallel u\parallel_{L^\infty}^2\parallel u_x\parallel_{L^\infty}\\
&\quad+\parallel u\parallel_{L^\infty}^3+\parallel G\parallel_{L^\infty}+\parallel u\parallel_{L^\infty}\parallel u_x\parallel_{L^\infty}^2\Big]d\tau.
\end{aligned} \tag{5.40}$$

For sufficiently small $\rho>0$, we have

$$\begin{aligned}
&\parallel G\parallel_{L^\infty}\leq c\parallel G\parallel_{H^{\frac{1}{2}+\rho}}\\
&=\parallel\Lambda^{-2}\Big[-(1-\alpha^2+\varepsilon)u_{tx}-\frac{2-\alpha^2}{3}u^3-\frac{\alpha}{3}(u^3)_x\\
&\qquad-\alpha(uu_x^2)_x-\alpha^2uu_x^2\Big]\parallel_{H^{\frac{1}{2}+\rho}}\\
&\leq c(\parallel u_t\parallel_{L^2(\mathbb{R})}+\parallel u\parallel_{H^1(\mathbb{R})}^3+\parallel u\parallel_{H^1(\mathbb{R})}^2\parallel u_x\parallel_{L^\infty}).
\end{aligned} \tag{5.41}$$

Using (5.16), (5.33) and (5.41), we derive that

$$\int_0^t\parallel G\parallel_{L^\infty}d\tau\leq c\Big[1+\int_0^t\Big(1+\parallel u_x\parallel_{L^\infty}\Big)d\tau\Big]. \tag{5.42}$$

Besides, for any fixed $r_1 \in (\frac{1}{2}, 1)$, using the Sobolev embedding lemma, we have

$$\| u_{txxx} \|_{L^\infty} \leq C_{r_1} \| u_{txxx} \|_{H^{r_1}} \leq C_{r_1} \| u_t \|_{H^{r_1+3}}, \tag{5.43}$$

where C_{r_1} only depends on r_1. From (5.15), (5.33) and (5.43), we get

$$\| u_{txxx} \|_{L^\infty} \leq c \| u \|_{H^{r_1+4}} \Big(1 + \| u_x \|_{L^\infty} + \| u_x \|_{L^\infty}^2 \Big). \tag{5.44}$$

Letting $r = r_1 + 3$, $u = u_\varepsilon$ and utilizing the Gronwall's inequality to (5.15), from (5.33), we derive that

$$\begin{aligned} \| u \|_{H^{r_1+4}}^2 \leq & \Big(\int_{\mathbb{R}} (\Lambda^{r_1+4} u_0)^2 + \varepsilon (\Lambda^{r_1+3} u_{0xx})^2 \Big) \\ & \times e^{[c \int_0^t (1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2) d\tau]}. \end{aligned} \tag{5.45}$$

From Lemma 5.3 and (5.45), we obtain

$$\begin{aligned} \| u_{txxx} \|_{L^\infty} \leq & c \varepsilon^{\frac{s - r_1 - 4}{4}} (1 + \| u_x \|_{L^\infty}) \\ & \times e^{[c \int_0^t (1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2) d\tau]}. \end{aligned} \tag{5.46}$$

For sufficiently small $\varepsilon > 0$, from (5.40),(5.42) and (5.46), we get

$$\begin{aligned} \| u_x \|_{L^\infty} \leq & \| u_{0x} \|_{L^\infty} \\ & + c \int_0^t \Big[\varepsilon^{\frac{s - r_1}{4}} (1 + \| u_x \|_{L^\infty}) e^{[c \int_0^\tau (1 + \|u_x\|_{L^\infty}^2) d\varrho]} \\ & + 1 + \| u_x \|_{L^\infty}^2 \Big] d\tau. \end{aligned} \tag{5.47}$$

Making use of the contractive mapping principle, we get that there must exist

$T > 0$ to guarantee that the following equation

$$\begin{aligned}\| K \|_{L^\infty} = &\| u_{0x} \|_{L^\infty} \\ &+c\int_0^t \Big[\varepsilon^{\frac{s-r_1}{4}}(1+\| K \|_{L^\infty})e^{[c\int_0^\tau(1+\|K\|^2_{L^\infty})d\varrho]} \\ +1+&\| K \|^2_{L^\infty}\Big]d\tau\end{aligned}$$

admits a unique solution $K(t) \in C[0,T]$. We cite the results from [66] and derive that there must have two constants $c > 0$ and $T > 0$, which do not depend on ε and $\| u_x(t,x) \|_{L^\infty} \le K(t) \le c$ for every $t \in [0,T]$. The proof is completed. ■

Proof of Theorem 5.1.

Assume that u_ε is the solution of problem (5.32). Making use of Lemmas 5.3 and 5.5, (5.15), (5.16), Gronwall's inequality and the notation $u_\varepsilon = u$, we have the conclusions

$$\| u_\varepsilon \|_{H^r} \le \| u_\varepsilon \|_{H^{r+1}} \le ce^{c\int_0^t(1+\|u_x\|_{L^\infty}+\|u_x\|^2_{L^\infty})d\tau} \le c \tag{5.48}$$

and

$$\| u_{\varepsilon t} \|_{H^{r_1}} \le c \| u \|_{H^1} \| u \|_{H^{r_1+1}} \| u_x \|_{L^\infty} \le c, \tag{5.49}$$

where $t \in [0,T)$ and $r_1 \in (0, s-1], r \in (0,s]$.

Applying Aubin's compactness theorem, (5.48) and (5.49), we derive that there exists a subsequence of $\{u_\varepsilon\}$, represented by $\{u_{\varepsilon_j}\}$, $\varepsilon_j \to 0$ when $j \to \infty$, to ensure that $\{u_{\varepsilon_j}\}$ converges weakly to $u(t,x)$ in $L^2([0,T], H^s(\mathbb{R}))$ and its time derivative $\{u_{\varepsilon_j t}\}$ converges weakly to a function $u_t(t,x)$ in $L^2([0,T], H^{s-1}(\mathbb{R}))$. In addition, for every constant $C_1 > 0$, we know that $\{u_{\varepsilon_j}\}$ converges strongly to $u(t,x)$ in $L^2([0,T], H^r(-C_1, C_1))$ and $\{u_{\varepsilon_j t}\}$ converges strongly to $u_t(t,x)$ in

$L^2([0,T],H^{r_1}(-C_1,C_1))$.

Using the conclusion in Lemma 5.5, we obtain that $\{u_{\varepsilon_j x}\}(\varepsilon_j \to 0)$ is bounded in $L^\infty(\mathbb{R})$. For any $r_1 \in [0, s-1)$, it is derived that the sequences $\{u_{\varepsilon_j x}\}$ and $\{u^2_{\varepsilon_j x}\}$ converge weakly to u_x and u_x^2 in $L^2[0,T], H^{r_1}(-C_1,C_1)$, respectively. Using integration by parts, from problem (5.3), we have

$$\int_0^T\int_{\mathbb{R}}(u_t g)dxdt - \int_0^T\int_{\mathbb{R}} u_t g_{xx})dxdt = \int_0^T\int_{\mathbb{R}}\Big[-\frac{2}{3}u^3 g_x - \frac{\alpha}{3}u^3 g_x \\ -\alpha u u_x^2 g - \frac{\alpha^2}{3}u^3 g_{xxx} + \alpha^2 u u_x^2 g_x\Big]dxdt,$$

where $g \in C_0^\infty$ and $u(0,x) = u_0(x)$. Since $\{u_{\varepsilon_j x}\}$ is a bounded sequence in $L^\infty([0,T]\times\mathbb{R})$ and the Banach space $L^1([0,T]\times\mathbb{R})$ is separable, we conclude that there must exist a subsequence of $\{u_{\varepsilon_j x}\}$, still represented by $\{u_{\varepsilon_j x}\}$, weakly star converges to a function $U(t,x)$ in $L^\infty([0,T]\times\mathbb{R})$. Using the fact that $\{u_{\varepsilon_j x}\}$ converges weakly to u_x in the space $L^2([0,T]\times\mathbb{R})$, we obtain $u_x = U(x)$ almost everywhere. Furthermore, we acquire $u_x \in L^\infty([0,T]\times\mathbb{R})$. The proof is completed. ∎

5.3 Local strong solutions

Here we use the ideas presented in [66] to handle the well-posedness for Eq.(5.1). We give the conclusion.

Theorem 5.6. *Let $|\alpha| > \frac{\sqrt{2}}{2}$, $s > \frac{3}{2}$ and $u_0(x) \in H^s(\mathbb{R})$. Then there exists a $T > 0$, which depends on $\| u_0 \|_{H^s}$, to guarantee that (5.2) has a unique solution u satisfying*

$$u(t,x) \in C([0,T];H^s(\mathbb{R})) \cap C^1([0,T];H^{s-1}(\mathbb{R})).$$

Lemma 5.7 ([66]). *If functions v_1 and v_2 are in $H^{r+1} \cap \{\| \frac{\partial v_1(x)}{\partial x} \|_{L^\infty} < \infty\}$, then*

$$| \int_{\mathbb{R}} \Lambda^r v_1 \Lambda^r (v_1 v_2)_x dx | \leq \begin{cases} c_q \| v_1 \|_{H^r}^2 \| v_2 \|_{H^{r+1}}, \quad r \in (\frac{1}{2}, 1], \\ c_q \Big(\| v_1 \|_{H^r} \| v_1 \|_{L_\infty} \| v_2 \|_{H^{r+1}} \\ + \| \frac{\partial v_2}{\partial x} \|_{L_\infty} \| v_1 \|_{H^r}^2 \\ + \| v_1 \|_{H^r} \| \frac{\partial v_1}{\partial x} \|_{L_\infty} \| v_2 \|_{H^r} \Big), \\ \quad r \in (0, \infty). \end{cases} \tag{5.50}$$

Lemma 5.8 ([66]). *Assume $s > \frac{3}{2}$, $u_0 \in H^s(\mathbb{R})$ and let the solution $u_\varepsilon(t,x)$ satisfy system (5.32). Then it holds that*

$$\| u_\varepsilon \|_{H^s} \leq M e^{ct}, \tag{5.51}$$

$$\| u_\varepsilon \|_{H^{s+k_1}} \leq \varepsilon^{-\frac{k_1}{4}} M e^{ct}, \quad k_1 > 0, \tag{5.52}$$

$$\| u_{\varepsilon t} \|_{H^{s+k_1}} \leq \varepsilon^{-\frac{(k_1+1)}{4}} M e^{ct}, \quad k_1 > -1, \tag{5.53}$$

in which the constants $M > 0$ and $c > 0$ do not depend on ε, $t \in [0,T)$, and ε is small enough.

Lemma 5.9 ([59]). *If $u, v \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, $w = u - v$, $r > \frac{1}{2}$, then the inequality*

$$| \int_{\mathbb{R}} \Lambda^s w \Lambda^s (u^3 - v^3)_x dx \big| \leq c(\| w \|_{H^s} \| w \|_{H^r} \| v \|_{H^{s+1}} + \| w \|_{H^s}^2)$$

holds.

Lemma 5.10 ([59]). *If $s > \frac{3}{2}$ and $\frac{1}{2} < r < \min\{1, s-1\}$, for functions f, w,* *the two inequalities*

$$| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}(wf)_x dx | \leq c \parallel w \parallel_{H^r}^2 \parallel f \parallel_{H^r}, \tag{5.54}$$

$$| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}(w_x f_x)_x dx | \leq c \parallel w \parallel_{H^r}^2 \parallel f \parallel_{H^s} \tag{5.55}$$

hold.

Now, we shall demonstrate that the solution u_ε of problem (5.32) is a Cauchy sequence. Assume that u_ε and u_δ are two solutions of problem (5.32), corresponding to the initial values $u_{\varepsilon,0}$ and $u_{\delta,0}$, respectively. Let $w_0(x) = u_{\varepsilon,0}(x) - u_{\delta,0}(x)$. We assume $0 < \varepsilon < \delta < \min(\frac{1}{4}, \alpha^2 - \frac{1}{2})$, $|\alpha| > \frac{\sqrt{2}}{2}$ and set $w = u_\varepsilon - u_\delta$. Using (5.24) and $\partial_x^2 = 1 - \Lambda^2$, we have

$$\begin{cases} (\alpha^2 - \varepsilon) w_t - \varepsilon w_{txx} + (\delta - \varepsilon)(u_{\delta t} + u_{\delta txx}) \\ \qquad = \frac{\alpha}{3}(u_\varepsilon^3 - u_\delta^3) - \frac{\alpha^2}{3}[(u_\varepsilon^3)_x - (u_\delta^3)_x] \\ +\Lambda^{-2}\Big[- (1 - \alpha^2 + \varepsilon) w_t + (\delta - \varepsilon) u_{\delta t} - \frac{2-\alpha^2}{3}(u_\varepsilon^3 - u_\delta^3)_x - \frac{\alpha}{3}(u_\varepsilon^3 - v_\delta^3) \\ -\alpha(u_\varepsilon u_{\varepsilon x}^2 - u_\delta u_{\delta x}^2) - \alpha^2(u_\varepsilon u_{\varepsilon x}^2 - u_\delta u_{\delta x}^2)_x \Big], \\ w(0, x) = w_0(x). \end{cases} \tag{5.56}$$

Lemma 5.11. *Let $|\alpha| > \frac{\sqrt{2}}{2}$, $s > \frac{3}{2}$ and $u_0(x) \in H^s(\mathbb{R})$. Let $u_\varepsilon(t, x)$ be the solution of problem (5.32). Then, there must exist a $T > 0$ to guarantee that $u_\varepsilon(t, x)$ is a Cauchy sequence in the space*

$$C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R})).$$

Proof. Letting r satisfy $\frac{1}{2} < r < min(1, s-1)$, we apply $\Lambda^r w \Lambda^r$ to multiply the first equation of problem (5.56). Subsequently, for the obtained equation, using integration by parts about variable x, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}[(\alpha^2-\varepsilon)(\Lambda^r w)^2+\varepsilon(\Lambda^r w_x)^2]dx \\
&=(\varepsilon-\delta)\int_{\mathbb{R}}(\Lambda^r w)[(\Lambda^r u_{\delta t})+(\Lambda^r u_{\delta xxt})]dx-(1-\alpha^2+\varepsilon)\int_{\mathbb{R}}\Lambda^r w\Lambda^{r-2}w_t dx \\
&+(\delta-\varepsilon)\int_{\mathbb{R}}\Lambda^r w\Lambda^{r-2}u_{\delta t}dx \\
&-\frac{\alpha^2}{3}\int_{\mathbb{R}}(\Lambda^r w)\Lambda^r(u_\varepsilon^3-u_\delta^3)_x dx+\frac{\alpha}{3}\int_{\mathbb{R}}\Lambda^r w\Lambda^r(u_\varepsilon^3-u_\delta^3)dx \\
&+\int_{\mathbb{R}}(\Lambda^r w)\Lambda^{r-2}\Big[-\frac{2-\alpha^2}{3}(u_\varepsilon^3-u_\delta^3)_x \\
&-\frac{\alpha}{3}(u_\varepsilon^3-v_\delta^3)-\alpha(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)-\alpha^2(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)_x\Big]dx. \qquad (5.57)
\end{aligned}$$

Applying the Schwarz inequality to (5.57) yields

$$\begin{aligned}
&\frac{d}{dt}\int[(\alpha^2-\varepsilon)(\Lambda^r w)^2+\varepsilon(\Lambda^r w_x)^2]dx \\
&\leq c\Big\{\parallel \Lambda^r w\parallel_{L^2}\Big[(\delta-\varepsilon)\big(\parallel \Lambda^r u_{\delta t}\parallel_{L^2}+\parallel \Lambda^r u_{\delta xxt}\parallel_{L^2}\big) \\
&+|1-\alpha^2+\varepsilon|\parallel \Lambda^{r-2}w_t\parallel_{L^2}+(\delta-\varepsilon)\parallel \Lambda^{r-2}u_{\delta t}\parallel_{L^2}\Big] \\
&+\Big|\int_{\mathbb{R}}\Lambda^r w\Lambda^r(u_\varepsilon^3-u_\delta^3)_x dx\Big|+\Big|\int_{\mathbb{R}}\Lambda^r w\Lambda^r(u_\varepsilon^3-u_\delta^3)dx\Big| \\
&+\Big|\int_{\mathbb{R}}\Lambda^r w\Lambda^{r-2}\Big[-\frac{2-\alpha^2}{3}(u_\varepsilon^3-u_\delta^3)_x \\
&-\frac{\alpha}{3}(u_\varepsilon^3-v_\delta^3)-\alpha(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)-\alpha^2(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)_x\Big]dx\Big|\Big\}. \qquad (5.58)
\end{aligned}$$

Making use of the first inequality in Lemma 5.7 gives rise to

$$\begin{aligned}
&\Big|\int_{\mathbb{R}}\Lambda^r w\Lambda^r(u_\varepsilon^3-u_\delta^3)_x dx\Big| \\
&=\Big|\int_{\mathbb{R}}\Lambda^r w\Lambda^r(wg)_x dx\Big| \\
&\leq c\parallel w\parallel_{H^r}^2\parallel g\parallel_{H^{r+1}} \qquad (5.59)
\end{aligned}$$

and

$$
\begin{aligned}
&|\int_{\mathbb{R}} \Lambda^r w \Lambda^r (u_\varepsilon^3 - u_\delta^3) dx| \\
&= |\int_{\mathbb{R}} \Lambda^r w \Lambda^r (wg) dx| \\
&\leq c \parallel w \parallel_{H^r}^2 \parallel g \parallel_{H^r},
\end{aligned} \tag{5.60}
$$

where $g = \sum_{i=0}^{3} u_\varepsilon^{3-i} u_\delta^i$. Furthermore, we have

$$
\begin{aligned}
&|\int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2} (u_\varepsilon^3 - u_\delta^3)_x dx| \\
&= |\int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2} (wg)_x dx| \\
&\leq c \parallel g \parallel_{H^r} \parallel w \parallel_{H^r}^2
\end{aligned} \tag{5.61}
$$

and

$$
\begin{aligned}
&|\int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2} (u_\varepsilon^3 - u_\delta^3) dx| \\
&= |\int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2} (wg) dx| \\
&\leq c \parallel g \parallel_{H^r} \parallel w \parallel_{H^r}^2 \ .
\end{aligned} \tag{5.62}
$$

We know the identity

$$
(u_\varepsilon u_{\varepsilon x}^2 - u_\delta u_{\delta x}^2)_x = [w u_{\varepsilon x}^2 + u_\delta w_x (u_{\varepsilon x} + u_{\delta x})]_x. \tag{5.63}
$$

Using Lemma 5.10 and the multiplying property of $H^{s_0}(\mathbb{R})$ with $s_0 > \frac{1}{2}$, we get

$$
\begin{aligned}
&| \int_{\mathbb{R}} \Lambda^r w \Lambda^r (1-\alpha^2\partial_x^2)^{-1}[u_\delta(u_{\varepsilon x}+u_{\delta x})\partial_x w]_x dx \;| \\
&\leq c \;| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}[u_\delta(u_{\varepsilon x}+u_{\delta x})\partial_x w]_x dx \;| \\
&\leq c \parallel w \parallel_{H^r}^2 \parallel u_\varepsilon \parallel_{H^s}^2,
\end{aligned} \tag{5.64}
$$

$$
\begin{aligned}
&| \int_{\mathbb{R}} \Lambda^r w \Lambda^r (1-\alpha^2\partial_x^2)^{-1}[w u_{\varepsilon x}^2]_x dx \;| \\
&\leq c \;| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}[w u_{\varepsilon x}^2]_x dx \;| \\
&\leq c \parallel w \parallel_{H^r}^2 \parallel u_\varepsilon \parallel_{H^s}^2 \;.
\end{aligned} \tag{5.65}
$$

Similarly, we have

$$
\begin{aligned}
&| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}(u_\varepsilon u_{\varepsilon x}^2 - u_\delta u_{\delta x}^2) dx \;| \\
&\leq c \;| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}(u_\varepsilon u_{\varepsilon x}^2 - u_\delta u_{\delta x}^2) dx \;| \\
&\leq c \parallel w \parallel_{H^r}^2 \parallel u_\varepsilon \parallel_{H^s}^2 \;.
\end{aligned} \tag{5.66}
$$

We derive the inequality

$$
\begin{aligned}
&| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}(u_\varepsilon^3 - u_\delta^3)_x dx| \\
&= | \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}(wg)_x dx| \\
&\leq c \parallel g \parallel_{H^r} \parallel w \parallel_{H^r}^2,
\end{aligned} \tag{5.67}
$$

where $g = \sum_{i=0}^{3} u_\varepsilon^{3-i} u_\delta^i$. Making use of the multiplying property of H^r $(r > \frac{1}{2})$, $r+1 < s$ and Lemma 5.8, we get

$$
\parallel g \parallel_{H^{r+1}} \leq c, \quad \text{if} \quad t \in (0, \widetilde{T}].
$$

From Lemma 5.8 and (5.58)-(5.67), we get that there must exist a constant c depending on $\widetilde{T}$, $t \in [0, \widetilde{T})$, to ensure that the inequality

$$\frac{d}{dt}\int_{\mathbb{R}}[(\alpha^2-\varepsilon)(\Lambda^r w)^2+\varepsilon(\Lambda^r w_x)^2]dx \leq c(\delta^\gamma \parallel w \parallel_{H^r} + \parallel w \parallel_{H^r}^2) \tag{5.68}$$

holds. In (5.68), we let $\gamma = \frac{1+s-r}{4}$ if $s < 3+r$; $\gamma = 1$ if $s \geq 3+r$. From (5.68), we obtain

$$\begin{aligned}\parallel w \parallel_{H^r}^2 &= \int_{\mathbb{R}}(\Lambda^r w)^2 dx \\ &\leq c_0 \int_{\mathbb{R}}[(\alpha^2-\varepsilon)(\Lambda^r w)^2+\varepsilon(\Lambda^r w)^2]dx \\ &\leq c_0 \int_{\mathbb{R}}[(\Lambda^r w_0)^2+\varepsilon(\Lambda^r w_{0x})^2]dx + c_0\int_0^t(\delta^\gamma \parallel w \parallel_{H^r} + \parallel w \parallel_{H^r}^2)d\tau,\end{aligned} \tag{5.69}$$

where c_0 depends on α.

Applying the Gronswall inequality and Lemma 5.8 results in

$$\parallel u \parallel_{H^r} \leq c\delta^{\frac{s-r}{4}}e^{ct}+\delta^\gamma(e^{ct}-1) \tag{5.70}$$

for every $t \in [0, \widetilde{T})$.

We now utilize $\Lambda^s w \Lambda^s$ to multiply the first equation of (5.56). For the obtained equation, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}[(\alpha^2-\varepsilon)(\Lambda^s w)^2+\varepsilon(\Lambda^s w_x)^2]dx\\
&=(\varepsilon-\delta)\int_{\mathbb{R}}(\Lambda^s w)[(\Lambda^s u_{\delta t})+(\Lambda^s u_{\delta xxt})]dx-(1-\alpha^2+\varepsilon)\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-2}w_t dx\\
&+(\delta-\varepsilon)\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-2}u_{\delta t}dx\\
&-\frac{\alpha^2}{3}\int_{\mathbb{R}}(\Lambda^s w)\Lambda^s(u_\varepsilon^3-u_\delta^3)_x dx+\frac{\alpha}{3}\int_{\mathbb{R}}\Lambda^s w\Lambda^s(u_\varepsilon^3-u_\delta^3)dx\\
&+\int_{\mathbb{R}}(\Lambda^s w)\Lambda^{s-2}\Big[-\frac{2-\alpha^2}{3}(u_\varepsilon^3-u_\delta^3)_x\\
&-\frac{\alpha}{3}(u_\varepsilon^3-v_\delta^3)-\alpha(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)-\alpha^2(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)_x\Big]dx. \qquad (5.71)
\end{aligned}
$$

Using the Schwarz inequality for (5.71) yields

$$
\begin{aligned}
&\frac{d}{dt}\int[(\alpha^2-\varepsilon)(\Lambda^s w)^2+\varepsilon(\Lambda^s w_x)^2]dx\\
&\le c\Big\{\parallel\Lambda^s w\parallel_{L^2}\Big[(\delta-\varepsilon)\big(\parallel\Lambda^s u_{\delta t}\parallel_{L^2}+\parallel\Lambda^s u_{\delta xxt}\parallel_{L^2}\big)\\
&+|1-\alpha^2+\varepsilon|\parallel\Lambda^{s-2}w_t\parallel_{L^2}+(\delta-\varepsilon)\parallel\Lambda^{s-2}u_{\delta t}\parallel_{L^2}\Big]\\
&+\Big|\int_{\mathbb{R}}\Lambda^s w\Lambda^s(u_\varepsilon^3-u_\delta^3)_x dx\Big|+\Big|\int_{\mathbb{R}}\Lambda^s w\Lambda^s(u_\varepsilon^3-u_\delta^3)dx\Big|\\
&+\Big|\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-2}\Big[-\frac{2-\alpha^2}{3}(u_\varepsilon^3-u_\delta^3)_x\\
&-\frac{\alpha}{3}(u_\varepsilon^3-v_\delta^3)-\alpha(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)-\alpha^2(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)_x\Big]dx\Big|\Big\}. \qquad (5.72)
\end{aligned}
$$

Using Lemma 5.9 gives rise to

$$
\Big|\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-2}(u_\varepsilon^3-u_\delta^3)_x dx\Big|\le c_3\parallel g\parallel_{H^s}\parallel w\parallel_{H^s}^2, \qquad (5.73)
$$

where $g=\sum_{i=0}^{3}u_\varepsilon^i u_\delta^{3-i}$.

Applying Lemma 5.9 derives that

$$\begin{aligned}&|\int_{\mathbb{R}}\Lambda^s w\Lambda^s(u_\varepsilon^3-u_\delta^3)_x dx|\\&\le c(\| w \|_{H^s}\| w \|_{H^r}\| u_\delta \|_{H^{s+1}}+\| w \|_{H^s}^2).\end{aligned} \tag{5.74}$$

For $s>\frac{3}{2}$, the algebra property of $H^{s_0}(\mathbb{R})$ ($s_0>\frac{1}{2}$) and the Cauchy-Schwartz inequality are used to derive that

$$\begin{aligned}&|\int_{\mathbb{R}}\Lambda^s w\Lambda^s(1-\alpha^2\partial_x^2)^{-1}[u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2]_x dx|\\&\le c\| \Lambda^s w \|_{L^2}\| \Lambda^{s-2}[u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2]_x \|_{L^2}\\&\le c\| w \|_{H^s}\| \partial_x(u_\varepsilon)\partial_x w \|_{H^{s-1}}\\&\le c\| u_\varepsilon \|_{H^s}\| w \|_{H^s}^2,\end{aligned} \tag{5.75}$$

$$\begin{aligned}&|\int_{\mathbb{R}}\Lambda^s w\Lambda^s(1-\alpha^2\partial_x^2)^{-1}[u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2]dx|\\&\le c\| \Lambda^s w \|_{L^2}\| \Lambda^{s-2}[u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2] \|_{L^2}\\&\le c\| w \|_{H^s}\| \partial_x(u_\varepsilon)\partial_x w \|_{H^{s-1}}\\&\le c\| u_\varepsilon \|_{H^s}\| w \|_{H^s}^2\end{aligned} \tag{5.76}$$

and

$$\begin{aligned}&|\int_{\mathbb{R}}\Lambda^s w\Lambda^s(1-\alpha^2\partial_x^2)^{-1}\Big[-\frac{1}{3}(u_\varepsilon^3-u_\delta^3)_x-\frac{1}{3\alpha}(u_\varepsilon^3-u_\delta^3)\Big]dx\ |\\&\le c\| u_\varepsilon \|_{H^s}\| w \|_{H^s}^2\ .\end{aligned} \tag{5.77}$$

Applying the bounded property of $\| u_\varepsilon \|_{H^s}$ and $\| u_\delta \|_{H^s}$ (see Lemma 5.8 and

(5.33)), it follows from (5.70)-(5.77), Lemmas 5.3 and 5.5 that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}[(\alpha^2-\varepsilon)(\Lambda^s w)^2+\varepsilon(\Lambda^s w_x)^2]dx\\
&\leq 2\delta(\| u_{\delta t} \|_{H^s}+\| u_{\delta xxt} \|_{H^s}+\| \Lambda^{s-2}w_t \|_{L^2}+\| \Lambda^{s-2}u_{\delta t} \|)\| w \|_{H^s}\\
&\quad +c(\| w \|^2_{H^s}+\| w \|_{H^r}\| w \|_{H^s}\| u_\delta \|_{H^{s+1}})\\
&\leq c(\delta^{\gamma_1}\| w \|_{H^s}+\| w \|^2_{H^s}),
\end{aligned} \tag{5.78}$$

where $\gamma_1=\min(\frac{1}{4},\frac{s-r-1}{4})>0$ and c does not depend on δ and ε.

Integrating (5.78) with respect to t, we get

$$\begin{aligned}
\| w \|^2_{H^s}&\leq c\int_{\mathbb{R}}[(\alpha^2-\varepsilon)(\Lambda^s w)^2+\varepsilon(\Lambda^s w_x)^2]dx\\
&\leq c\int_{\mathbb{R}}[(\Lambda^s w_0)^2+\varepsilon(\Lambda^s w_{0x})^2]dx+c(\delta^{\gamma_1}\| w \|_{H^s}+\| w \|^2_{H^s}).
\end{aligned} \tag{5.79}$$

Using the Gronwall inequality and (5.79) gives rise to

$$\begin{aligned}
\| w \|_{H^s}&\leq (c\int_{\mathbb{R}}[(\Lambda^s w_0)^2+\varepsilon(\Lambda^s w_{0x})^2]dx)^{\frac{1}{2}}e^{ct}+c\delta^{\gamma_1}(e^{ct}-1)\\
&\leq c_1(\| w_0 \|_{H^s}+\delta^{\frac{3}{4}})e^{ct}+c_1\delta^{\gamma_1}(e^{ct}-1),
\end{aligned} \tag{5.80}$$

where the constant $c_1>0$ is independent of ε and δ. Therefore, from Lemma 4.13 and (5.80), we obtain

$$\| w \|_{H^s}\to 0, \quad \text{as} \quad \delta\to 0, \quad \varepsilon\to 0. \tag{5.81}$$

Now, we deal with the convergence of $\{u_{\varepsilon t}\}$. Applying $\Lambda^{s-1}w_t\Lambda^{s-1}$ on the first equation of (5.56) and integrating the obtained equation about x, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}[(\alpha^2-\varepsilon)(\Lambda^{s-1}w)^2+\varepsilon(\Lambda^{s-1}w_x)^2]dx\\
&=(\varepsilon-\delta)\int_{\mathbb{R}}(\Lambda^{s-1}w)[(\Lambda^s u_{\delta t})+(\Lambda^{s-1}u_{\delta xxt})]dx\\
&\quad-(1-\alpha^2+\varepsilon)\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-3}w_t dx+(\delta-\varepsilon)\int_{\mathbb{R}}\Lambda^{s-1}w\Lambda^{s-3}u_{\delta t}dx\\
&-\frac{\alpha^2}{3}\int_{\mathbb{R}}(\Lambda^{s-1}w)\Lambda^{s-1}(u_\varepsilon^3-u_\delta^3)_x dx+\frac{\alpha}{3}\int_{\mathbb{R}}\Lambda^{s-1}w\Lambda^{s-1}(u_\varepsilon^3-u_\delta^3)dx\\
&+\int_{\mathbb{R}}(\Lambda^{s-1}w)\Lambda^{s-3}\Big[-\frac{2-\alpha^2}{3}(u_\varepsilon^3-u_\delta^3)_x\\
&-\frac{\alpha}{3}(u_\varepsilon^3-v_\delta^3)-\alpha(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)-\alpha^2(u_\varepsilon u_{\varepsilon x}^2-u_\delta u_{\delta x}^2)_x\Big]dx. \qquad (5.82)
\end{aligned}$$

Using Lemma 5.8 and the Schwartz inequality, by the similar proof of (5.80), we get that there must exist a constant $c>0$, depending on $\widetilde{T}$ and α, to ensure that the inequality

$$\begin{aligned}
(\alpha^2-\varepsilon)\parallel w_t\parallel_{H^{s-1}}^2 &\leq c(\delta^{\frac{1}{2}}+\parallel w\parallel_{H^s}+\parallel w\parallel_{s-1})\parallel w_t\parallel_{H^{s-1}}\\
&+|1-\alpha^2+\varepsilon|\parallel w_t\parallel_{H^{s-1}}^2 \qquad (5.83)
\end{aligned}$$

holds.

Therefore, we have

$$\begin{aligned}
\parallel w_t\parallel_{H^{s-1}}^2 &\leq c\parallel w_t\parallel_{H^{s-1}}^2\\
&\leq c(\delta^{\frac{1}{2}}+\parallel w\parallel_{H^s}+\parallel w\parallel_{H^{s-1}})\parallel w_t\parallel_{H^{s-1}},
\end{aligned}$$

from which we obtain

$$\parallel w_t\parallel_{H^{s-1}}\leq c(\delta^{\frac{1}{2}}+\parallel w\parallel_{H^s}+\parallel w\parallel_{H^{s-1}}). \qquad (5.84)$$

From (5.81) and (5.84), we get that $w_t\to 0$ as both $\varepsilon\to 0$ and $\delta\to 0$ in

$H^{s-1}(\mathbb{R})$. We conclude that u_ε is a Cauchy sequence in the space

$$C([0,T);H^s(\mathbb{R}))\cap C([0,T);H^{s-1}(\mathbb{R})).$$

The proof of Lemma 5.11 is finished. ∎

Proof of Theorem 5.6. Now we consider the system

$$\begin{cases} (\alpha^2-\varepsilon)u_t-\varepsilon u_{txx}=\Lambda^{-2}\Bigg[-(1-\alpha^2+\varepsilon)u_t-\frac{2}{3}(u^3)_x-\frac{\alpha}{3}(u^3)_{xx} \\ -\alpha uu_x^2+\frac{\alpha^2}{3}(u^3)_{xxx}-\alpha^2(uu_x^2)_x\Bigg] \\ u(0,x)=u_{\varepsilon,0}(x). \end{cases} \tag{5.85}$$

Suppose that the sequence u_ε has the limit $u(t,x)$. Letting $\varepsilon\to 0$ in problem (5.85) and using Lemma 5.11, we get that $u(t,x)$ satisfies

$$\begin{cases} \alpha^2u_t=\Lambda^{-2}\Bigg[-(1-\alpha^2)u_t-\frac{2}{3}(u^3)_x-\frac{\alpha}{3}(u^3)_{xx} \\ -\alpha uu_x^2+\frac{\alpha^2}{3}(u^3)_{xxx}-\alpha^2(uu_x^2)_x\Bigg], \\ u(0,x)=u_{\varepsilon,0}(x), \end{cases} \tag{5.86}$$

which means that we have proven the existence of local strong solutions.

Suppose that $v(t,x)$ and $u(t,x)$, belonging to $C([0,T);H^s(\mathbb{R}))$, are two solutions of (5.86) which have the same initial value $u_0(0,x)$. Setting $w=u-v$, we get that

$$\begin{cases} w_t=\frac{1}{\alpha^2}\Lambda^{-2}\Big[-\frac{1}{3}(u^3-v^3)_x-\frac{1}{3\alpha}(u^3-v^3) \\ -\alpha(uu_x^2-vv_x^2)-\alpha(uu_x^2-vv_x^2)_x\Big]-\frac{1}{3}\partial_x(u^3-v^3)+\frac{1}{3\alpha}(u^3-v^3), \\ w(x,0)=0. \end{cases} \tag{5.87}$$

For the case $\frac{1}{2}<r<min\{1,s-1\}$, we apply $\Lambda^r w\Lambda^r$ on the first equation of

(5.87), and then integrate the obtained equation about x to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \parallel w \parallel_{H^r}^2 = &\frac{1}{\alpha^2}\int_{\mathbb{R}} (\Lambda^r w)\Lambda^r (1-\partial_x^2)^{-1}\Big[-\frac{1}{3}(u^3-v^3)_x - \frac{1}{3\alpha}(u^3-v^3) \\
&-\alpha(uu_x^2 - vv_x^2) - \alpha(uu_x^2 - vv_x^2)_x\Big]dx \\
&+\int_{\mathbb{R}} \Lambda^r w)\Lambda^r \Big[-\frac{1}{3}\partial_x(u^3-v^3) + \frac{1}{3\alpha}(u^3-v^3)\Big]dx. \qquad (5.88)
\end{aligned}$$

Similar to the estimates derived in Lemma 5.11, we obtain

$$\frac{d}{dt} \parallel w \parallel_{H^r}^2 \leq \widetilde{c} \parallel w \parallel_{H^r}^2 . \qquad (5.89)$$

Using the Gronwall inequality gives rise to

$$\parallel w \parallel_{H^r} = 0, \quad \text{for} \quad t \in [0, \widetilde{T}). \qquad (5.90)$$

The uniqueness is proved. Up to now, we obtain the desired result and finish the proof. ■

CHAPTER 6

Global weak solutions to a nonlinear Camassa-Holm type equation with quartic nonlinearities

In this chapter, we investigate global weak solutions to a nonlinear Camassa-Holm type equation with quartic nonlinearities. For the initial value $u_0(x) \in H^1(\mathbb{R})$ and $\| \frac{\partial u_0}{\partial x} \|_{L^\infty(\mathbb{R})} < \infty$, it is shown that the nonlinear equation has at least one global weak solution in $C([0,\infty)\times\mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$. We do not assume the initial value to satisfy the sign condition. Namely, our assumption is weaker than the sign condition. In previous works, the sign condition about the initial data is required to prove the existence of global weak solutions of the Camassa-Holm-type equations such as the Degasperis-Procesi equation and many generalized Camassa-Holm equations(see [36–38, 76–78]). The key elements for the proof in this chapter include establishing a space-time higher integrability estimate and a super bound estimate on the first order spatial derivative of the solution.

6.1 General

The following Camassa-Holm-type equation

$$u_t - u_{txx} + (a+1)u^k u_x = au^{k-1}u_x u_{xx} + u^k u_{xxx}, \quad a \in \mathbb{R}, \quad \text{integer} \quad k \geq 1 \tag{6.1}$$

was proposed by Grayshan and Himonas [41]. If $(a, k) = (2, 1)$, Eq.(6.1) is the standard Camassa-Holm equation (CH) [19]. If $(a, k) = (3, 1)$, Eq.(6.1) reduces to the Degasperis-Procesi equation (DP) [23]. Both the CH and DP models are integrable and possess quadratic nonlinearities and peaked solutions [13]. When $(a, k) = (3, 2)$, Eq.(6.1) is turned into the Novikov equation, which is integrable with cubic nonlinearities [87]. In this chapter, we discuss global weak solutions of (6.1) in the case $(a, k) = (4, 3)$. Namely, we investigate the following Camassa-Holm-type equation

$$u_t - u_{txx} + 5u^3 u_x = 4u^2 u_x u_{xx} + u^3 u_{xxx}. \tag{6.2}$$

Here we mention that without the sign condition on the initial value, the existence of global weak solutions to Eq.(6.2) in $C([0,\infty)\times\mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$ has not been established yet (see recent work in [40, 76]).

The approaches in [101] will be employed to prove our main result. After we prove the higher-integrability estimate (see Lemmas 6.3) and the one-sided super bound estimate (see Lemma 6.5), considering the derivative $q_\varepsilon = \frac{\partial u_\varepsilon(t,x)}{\partial x}$ (see (6.9)), which is only weakly compact, we will prove that q_ε converges strongly. Namely, we will show that this weak convergence is equivalent to strong convergence.

The structure of this chapter is as follows. We provide the main result in section 6.2. Several lemmas about the viscous approximation problem are presented in section 6.3. In section 6.4, we prove strong compactness of the derivative of the solutions for the approximation problem and give the proof of our main result.

6.2 Main results

We write the Cauchy problem for Eq.(6.2) in the form

$$\begin{cases} u_t - u_{txx} + 5u^3 u_x = 4u^2 u_x u_{xx} + u^3 u_{xxx}, \\ u(0,x) = u_0(x). \end{cases} \tag{6.3}$$

Applying the operator $\Lambda^{-2} = (1 - \frac{\partial^2}{\partial x^2})^{-1}$ on problem (6.3), we obtain

$$\begin{cases} u_t + u^3 u_x + \frac{\partial H}{\partial x} = 0, \\ \frac{\partial H}{\partial x} = \Lambda^{-2}[4u^3 u_x + 3\partial_x (u^2 u_x^2) - u^2 u_x u_{xx}], \\ u(0,x) = u_0(x). \end{cases} \tag{6.4}$$

In this chapter, we use the definition of global weak solutions similar to that of Definition 3.1 in chapter 3. Namely, it is defined as the solution $u(t,x)$ that satisfies (6.3) or (6.4) in the sense of distribution and the conditions (i) and (ii) in Definition 3.1 hold. Then, the main conclusion of this chapter is given below.

Theorem 6.1. *Let $u_0(x) \in H^1(\mathbb{R})$ and $\| u_{0x} \|_{L^\infty(\mathbb{R})} < \infty$. Then, there exists at least one global weak solution $u(t,x)$ for problem (6.3) or (6.4) in the sense of distribution. In addition, the following results (a) and (b) hold.*

(a). For any time $T > 0$, there exists a positive constant $C = C(\| u_0 \|_{H^1(\mathbb{R})}, \| u_{0x} \|_{L^\infty(\mathbb{R})})$ to ensure that the one-sided L^∞ estimate

$$\frac{\partial u(t,x)}{\partial x} \le C(1+t), \qquad \text{for } t \in [0,T),\ x \in \mathbb{R} \tag{6.5}$$

holds.

(b). For any $T > 0$, there must exist a positive constant $C = C(\| u_0 \|_{H^1(\mathbb{R})}, \| u_{0x} \|_{L^\infty(\mathbb{R})})$ to guarantee that the space-time-higher integrability inequality

$$\int_{\mathbb{R}} \Big| \frac{\partial u(t,x)}{\partial x} \Big|^6 dx \leq C(1+T)e^{CT}, \quad t \in [0,T) \tag{6.6}$$

holds.

6.3 Viscous approximations

Assume $\psi(x)$ be defined in (3.10). we know that $\psi \in C^\infty$ has compact set $0 \leq x \leq 1$. Set the smooth function $\psi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\psi(\varepsilon^{-\frac{1}{4}}x)$ associated with $0 < \varepsilon < \frac{1}{4}$. Let

$$u_{\varepsilon,0} = \int_{\mathbb{R}} \psi_\varepsilon(x - \varsigma) u_0(\varsigma) d\varsigma = \psi_\varepsilon \star u_0,$$

which has the property $u_{\varepsilon,0} \in C^\infty$ for any $u_0 \in H^s, s > 0$ and

$$\| u_{\varepsilon,0} \|_{H^1(\mathbb{R})} \leq \| u_0 \|_{H^1(\mathbb{R})}, \quad \text{and} \quad u_{\varepsilon,0} \to u_0 \quad \text{in} \quad H^1(\mathbb{R}), \text{as} \quad \varepsilon \to 0. \tag{6.7}$$

Consider the following viscous approximation problem

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon^3 \frac{\partial u_\varepsilon}{\partial x} + \frac{\partial H_\varepsilon}{\partial x} = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), \end{cases} \tag{6.8}$$

where

$$\frac{\partial H_\varepsilon}{\partial x} = \Lambda^{-2}\Big[4u_\varepsilon^3 \frac{\partial u_\varepsilon}{\partial x} + 3\partial_x\Big(u_\varepsilon^2 (\frac{\partial u_\varepsilon}{\partial x})^2\Big) - u_\varepsilon^2 \frac{\partial u_\varepsilon}{\partial x}\frac{\partial^2 u_\varepsilon}{\partial x^2}\Big].$$

In order to establish the existence of global weak solutions for (6.4), we will investigate the compactness of a sequence of smooth differentiable functions $\{u_\varepsilon\}_{\varepsilon>0}$. Specifically, we will handle the factor $\frac{\partial u_\varepsilon(t,x)}{\partial x}$, which plays a key role in proving the existence of global weak solutions.

Letting $q_\varepsilon(t,x) = \frac{\partial u_\varepsilon(t,x)}{\partial x}$ and differentiating the first equation of (6.8) about variable x yield

$$\frac{\partial q_\varepsilon}{\partial t} + u_\varepsilon^3 \frac{\partial q_\varepsilon}{\partial x} + \frac{1}{2} u_\varepsilon^2 q_\varepsilon^2 - \varepsilon \frac{\partial^2 q_\varepsilon}{\partial x^2} = u_\varepsilon^4 - \Lambda^{-2}\left(u_\varepsilon^4 + \frac{5}{2}(u_\varepsilon^2 q_\varepsilon^2) + (u_\varepsilon q_\varepsilon^3)_x \right). \quad (6.9)$$

In the following discussion of this chapter, we let

$$J_\varepsilon(t,x) = u_\varepsilon^4 - \Lambda^{-2}\left(u_\varepsilon^4 + \frac{5}{2}(u_\varepsilon^2 q_\varepsilon^2) + (u_\varepsilon q_\varepsilon^3)_x \right). \quad (6.10)$$

For problem (6.8), we have the well-posedness conclusion.

Lemma 6.2. *Provided that $u_0 \in H^1(\mathbb{R})$ and $\kappa \geq 2$. Then system (6.8) admits a unique solution $u_\varepsilon(t,x) \in C([0,\infty); H^\kappa(\mathbb{R}))$. Moreover, the following identity holds*

$$\int_{\mathbb{R}} \Big(u_\varepsilon^2 + (\frac{\partial u_\varepsilon}{\partial x})^2 \Big) dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} \Big((\frac{\partial u_\varepsilon}{\partial x})^2 + (\frac{\partial^2 u_\varepsilon}{\partial x^2})^2 \Big)(s,x) dx ds = \| u_{\varepsilon,0} \|_{H^1(\mathbb{R})}^2, \quad (6.11)$$

which is equivalent to

$$\| u_\varepsilon(t,\cdot) \|_{H^1(\mathbb{R})}^2 + 2\varepsilon \int_0^t \| \frac{\partial u_\varepsilon}{\partial x}(s,\cdot) \|_{H^1(\mathbb{R})}^2 ds = \| u_{\varepsilon,0} \|_{H^1(\mathbb{R})}^2 . \quad (6.12)$$

Proof. For every $\kappa \geq 2$ and $u_0 \in H^1(\mathbb{R})$, we obtain $u_{\varepsilon,0} \in C([0,\infty); H^\infty(\mathbb{R}))$. Employing theorem 2.3 in [10], we get that system (6.8) has a unique solution $u_\varepsilon \in C([0,\infty); H^\kappa(\mathbb{R}))$.

According to system (6.8), we have

$$\frac{\partial u_\varepsilon}{\partial t} - \frac{\partial^3 u_\varepsilon}{\partial t x^2} + 5u_\varepsilon^3 \frac{\partial u_\varepsilon}{\partial x} = 4u_\varepsilon^2 \frac{\partial u_\varepsilon}{\partial x} \frac{\partial^2 u_\varepsilon}{\partial x^2} + u_\varepsilon^3 \frac{\partial^3 u_\varepsilon}{\partial x^3} + \varepsilon \Big(\frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{\partial^4 u_\varepsilon}{\partial x^4} \Big). \quad (6.13)$$

Taking $\kappa = 5$, we obtain

$$u_\varepsilon(t, \pm\infty) = \frac{\partial u_\varepsilon(t, \pm\infty)}{\partial x} = \frac{\partial^2 u_\varepsilon(t, \pm\infty)}{\partial x^2} = \frac{\partial^3 u_\varepsilon(t, \pm\infty)}{\partial x^3} = 0. \tag{6.14}$$

Using (6.13) and (6.14), we obtain the identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Big(u_\varepsilon^2 + (\frac{\partial u_\varepsilon}{\partial x})^2\Big)dx + \varepsilon\int_{\mathbb{R}}\Big((\frac{\partial u_\varepsilon}{\partial x})^2 + (\frac{\partial^2 u_\varepsilon}{\partial x^2})^2\Big)dx = 0. \tag{6.15}$$

The proof is completed. $\square$

Using (6.7) and Lemma 6.2 gives rise to

$$\| u_\varepsilon \|_{L^\infty(\mathbb{R})} \le c \| u_\varepsilon \|_{H^1(\mathbb{R})} \le c \| u_{\varepsilon,0} \|_{H^1(\mathbb{R})} \le c \| u_0 \|_{H^1(\mathbb{R})} \le c, \tag{6.16}$$

where c does not depend on the parameter ε.

Note that

$$\begin{aligned}\Lambda^{-2}[g(x)]_x &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\frac{\partial g(y)}{\partial y}dy \\ &= \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\frac{\partial g(y)}{\partial y}dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}\frac{\partial g(y)}{\partial y}dy \\ &= -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}g(y)dy + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-y}g(y)dy,\end{aligned}$$

from which we acquire

$$|\Lambda^{-2}[g(x)]_x| \le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}|g(y)|dy. \tag{6.17}$$

Lemma 6.3. *Assume* $\| \frac{\partial u_0}{\partial x} \|_{L^\infty} < \infty$, $0 < \varepsilon < \frac{1}{4}$, $0 < t < T$, *and let* $u_\varepsilon(t, x)$ *be a solution of (6.8). Then, there exists a positive constant* $c = c(\| u_0 \|_{H^1(\mathbb{R})}, \|$

$\frac{\partial u_0}{\partial x}\|_{L^\infty})$ *such that*

$$\int_{\mathbb{R}} \Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^6 dxdt \leq c(1+T)e^{cT} \tag{6.18}$$

and

$$\varepsilon \int_0^T \int_{\mathbb{R}} \Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^4 \Big(\frac{\partial^2 u_\varepsilon}{\partial x^2}\Big)^2 dxdt \leq c(1+T)e^{cT}, \tag{6.19}$$

where c does not depend on ε.

Proof. For conciseness, we write $u = u_\varepsilon$. Differentiating the first equation of problem (6.8) about x, we have

$$u_{tx} + \frac{1}{2}u^2u_x^2 + u^3u_{xx} - \varepsilon u_{xxx} = u^4 - \Lambda^{-2}\left[u^4 + \frac{5}{2}u^2u_x^2 + \partial_x(uu_x^3)\right]. \tag{6.20}$$

Applying (6.14) and the identity

$$\int_{\mathbb{R}} u_x^5 u^3 u_{xx} dx = \int_{\mathbb{R}} u_x^5 u^3 du_x = -\int_{\mathbb{R}} u_x[5u_x^4 u_{xx} u^3 + 3u^2 u_x^6] dx$$

yields

$$6\int_{\mathbb{R}} u^3 u_x^5 u_{xx} dx = -3\int_{\mathbb{R}} u^2 u_x^7 dx,$$

from which we have

$$\int_{\mathbb{R}} (\frac{1}{2}u^2u_x^2 + u^3u_{xx})u_x^5 dx = 0. \tag{6.21}$$

Multiplying (6.20) by u_x^5 and using (6.21), we obtain

$$\frac{1}{6}\frac{d}{dt}\int_{\mathbb{R}} u_x^6 dx + 5\varepsilon \int_{\mathbb{R}} u_x^4 u_{xx}^2 dx$$
$$= \int_{\mathbb{R}} u^4 u_x^5 dx - \int_{\mathbb{R}} u_x^5 \Lambda^{-2}[u^4 + \frac{5}{2}u^2 u_x^2 + (uu_x^3)_x] dx. \qquad (6.22)$$

Applying the Hölder inequality gives rise to

$$\int_{\mathbb{R}} \mid u^4 u_x^5 \mid dx \le \Big(\int_{\mathbb{R}} u_x^6 dx\Big)^{\frac{5}{6}} \Big(\int_{\mathbb{R}} u^{24} dx\Big)^{\frac{1}{6}}$$
$$\le \| u \|_{L^\infty}^{\frac{11}{3}} \Big(\int_{\mathbb{R}} u_x^6 dx\Big)^{\frac{5}{6}} \Big(\int_{\mathbb{R}} u^2 dx\Big)^{\frac{1}{6}}$$
$$\le c\Big(1 + \int_{\mathbb{R}} u_x^6 dx\Big), \qquad (6.23)$$

in which we have used inequality (6.16). Using (6.16) and the Hölder inequality again, we have

$$\int_{\mathbb{R}} |u_x|^4 dx \le \left(\int_{\mathbb{R}} u_x^6 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} u_x^2 dx\right)^{\frac{1}{2}} \le c(1 + \int_{\mathbb{R}} |u_x|^6 dx), \qquad (6.24)$$

$$\int_{\mathbb{R}} |u_x|^3 dx \le \left(\int_{\mathbb{R}} u_x^4 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} u_x^2 dx\right)^{\frac{1}{2}}$$
$$\le c\left(\int_{\mathbb{R}} u_x^6 dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} u_x^2 dx\right)^{\frac{1}{2}+\frac{1}{4}}$$
$$\le c\left(\int_{\mathbb{R}} u_x^6 dx\right)^{\frac{1}{4}}$$
$$\le c(1 + \int_{\mathbb{R}} |u_x|^6 dx) \qquad (6.25)$$

and

$$
\begin{aligned}
\int_{\mathbb{R}} |u_x|^5 dx &\le \left(\int_{\mathbb{R}} u_x^6 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} u_x^4 dx \right)^{\frac{1}{2}} \\
&\le c\left(\int_{\mathbb{R}} u_x^6 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} u_x^6 dx \right)^{\frac{1}{4}} \\
&\le c\left(\int_{\mathbb{R}} u_x^6 dx \right)^{\frac{3}{4}}. \qquad (6.26)
\end{aligned}
$$

We then have

$$
\begin{aligned}
\mid \Lambda^{-2}[u^4 + \frac{5}{2}u^2u_x^2] \mid =\mid \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}(u^4 + \frac{5}{2}u^2u_y^2)dy \mid \\
\le c(1 + \int_{\mathbb{R}} u_y^2 dy) \le c. \qquad (6.27)
\end{aligned}
$$

Using integration by parts yields

$$
\begin{aligned}
\mid \Lambda^{-2}[uu_x^3]_x \mid &=\mid \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\frac{\partial(uu_y^3)}{\partial y}dy \mid \\
&=\mid \frac{1}{2}e^{x}\int_x^{\infty} e^{-y}\frac{\partial(uu_y^3)}{\partial y}dy + \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}\frac{\partial(uu_y^3)}{\partial y}dy \mid \\
&=\mid \frac{1}{2}e^{x}\int_x^{\infty} e^{-y}(uu_y^3)dy - \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{y}(uu_y^3)dy \mid \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}|uu_y^3|dy \\
&\le c\int_{-\infty}^{\infty} |u_x|^3 dx \\
&\le c\left(\int_{\mathbb{R}} u_x^6 dx \right)^{\frac{1}{4}}. \qquad (6.28)
\end{aligned}
$$

Using (6.26) and (6.28), we have

$$\begin{aligned}
&\int_{\mathbb{R}} |u_x^5 \Lambda^{-2}(uu_x^3)_x| dx \\
&\leq \int_{\mathbb{R}} |u_x^5||\Lambda^{-2}(uu_x^3)_x| dx \\
&\leq c\left(\int_{\mathbb{R}} u_x^6 dx\right)^{\frac{3}{4}}\left(\int_{\mathbb{R}} u_x^6 dx\right)^{\frac{1}{4}} = c\int_{\mathbb{R}} u_x^6 dx. \qquad (6.29)
\end{aligned}$$

Using (6.22)-(6.29) results in

$$\frac{d}{dt}\int_{\mathbb{R}} u_x^6 dx \leq c(1+\int_{\mathbb{R}} u_x^6 dx), \qquad (6.30)$$

from which we obtain

$$\begin{aligned}
\int_{\mathbb{R}} u_x^6 dx &\leq \Big(cT + \int_{\mathbb{R}} u_{0x}^6(x) dx\Big) e^{cT} \leq ce^{cT}(T + \| u_{0x} \|_{L^\infty}^2) \\
&\leq c(1+T)e^{cT}. \qquad (6.31)
\end{aligned}$$

Using (6.22) and (6.31) leads to (6.19). The proof is completed. □

Lemma 6.4. *Let $0 < t < T$. Then there exists a constant $C > 0$, which does not depend on ε but may depend on T, $\| u_0 \|_{H^1(\mathbb{R})}$ and $\| u_{0x} \|$, such that*

$$\| H_\varepsilon(t,\cdot) \|_{L^\infty(\mathbb{R})} \leq C, \qquad (6.32)$$

$$\| \frac{\partial H_\varepsilon(t,\cdot)}{\partial x} \|_{L^\infty(\mathbb{R})} \leq C, \qquad (6.33)$$

$$\| \frac{\partial H_\varepsilon(t,\cdot)}{\partial x} \|_{L^1(\mathbb{R})} \leq C, \qquad (6.34)$$

$$\| \frac{\partial H_\varepsilon(t,\cdot)}{\partial x} \|_{L^2(\mathbb{R})} \leq C \qquad (6.35)$$

and

$$\| J_\varepsilon(t,\cdot) \|_{L^\infty(\mathbb{R})} \le C, \tag{6.36}$$

$$\| J_\varepsilon(t,\cdot) \|_{L^1(\mathbb{R})} \le C, \tag{6.37}$$

$$\| J_\varepsilon(t,\cdot) \|_{L^2(\mathbb{R})} \le C, \tag{6.38}$$

where $H_\varepsilon(t,x)$ and $J_\varepsilon(t,x)$ are defined in (6.8) and (6.10).

Proof. We write $u(t,x) = u_\varepsilon(t,x)$ concisely. Namely, we write

$$H_\varepsilon(t,x) = \Lambda^{-2}[u^4 + 3u^2u_x^2 - \frac{1}{2}\int_{-\infty}^{x} u^2(u_\xi^2)_\xi d\xi], \tag{6.39}$$

and

$$\frac{\partial H_\varepsilon(t,x)}{\partial x} = \Lambda^{-2}[4u^3u_x + 3\partial_x(u^2u_x^2) - u^2u_xu_{xx}]. \tag{6.40}$$

Using (6.16) yields

$$\begin{aligned} \mid \Lambda^{-2}(u^4 + 3u^2u_x^2) \mid &= \mid \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\Big(u^4 + 3u^2u_y^2\Big)dy \mid \\ &\le C\Big(\int_{\mathbb{R}} e^{-|x-y|}u^2dy + \int_{\mathbb{R}} e^{-|x-y|}u_y^2dy\Big), \end{aligned} \tag{6.41}$$

which leads to

$$\int_{\mathbb{R}} \mid \Lambda^{-2}(u^3 + 2uu_x^2) \mid dx \le C, \quad |\Lambda^{-2}(u^3 + 2uu_x^2)|dx \le C. \tag{6.42}$$

Using Lemma 6.3 and (6.16), we have

$$\Lambda^{-2}[\int_{-\infty}^{x} u^2(u_\xi^2)_\xi d\xi] = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}[u^2u_x^2 - \int_{-\infty}^{x} 2uu_\xi^3 d\xi]dy. \tag{6.43}$$

Applying (6.18) and the Hölder inequality gives rise to

$$\left|\Lambda^{-2}[\int_{-\infty}^{x} u^2(u_\xi^2)_\xi d\xi]\right| \leq C. \tag{6.44}$$

Using (6.41) and (6.44), we prove that (6.32) holds.

From (6.17), we have

$$\begin{aligned} \mid \Lambda^{-2}[(u^4)_x] \mid &\leq \int_{\mathbb{R}} e^{-|x-y|}|u|^4 dy \\ &\leq \parallel u \parallel_{L^\infty}^2 \int_{\mathbb{R}} u^2 dx, \end{aligned} \tag{6.45}$$

$$\mid \Lambda^{-2}[3(u^2u_x^2)_x] \leq \frac{3}{2}\int_{\mathbb{R}} e^{-|x-y|}|u^2u_y^2|dy \tag{6.46}$$

and

$$\begin{aligned} \mid \Lambda^{-2}(u^2u_xu_{xx}) \mid &= \mid \frac{1}{4}e^{-x}\int_{-\infty}^{x} e^y u^2(u_y^2)_y dy + \frac{1}{4}e^x \int_x^\infty e^{-y}u^2(u_y^2)_y dy \mid \\ &= \mid -\frac{1}{4}e^{-x}\int_{-\infty}^{x} u_y^2[e^y u^2 + 2uu_y e^y]dy \\ &\quad -\frac{1}{4}e^x \int_x^\infty u_y^2 e^{-y}[-u^2 + 2uu_y]dy \mid \\ &\leq C\int_{\mathbb{R}} e^{-|x-y|}\Big(|u^2u_y^2| + |uu_y^3|\Big)dy \\ &\leq C\int_{\mathbb{R}} e^{-|x-y|}\Big(|u_y^2| + |u_y^3|\Big)dy. \end{aligned} \tag{6.47}$$

From (6.45) to (6.47), using (6.25) and and the Tonelli Theorem, we get that (6.33) and (6.34) hold.

Applying (6.33) and (6.34) yields

$$\parallel \frac{\partial H_\varepsilon(t,.)}{\partial x} \parallel_{L^2(\mathbb{R})}^2 \leq \parallel \frac{\partial H_\varepsilon(t,.)}{\partial x} \parallel_{L^\infty} \parallel \frac{\partial H_\varepsilon(t,.)}{\partial x} \parallel_{L^1(\mathbb{R})} \leq C. \tag{6.48}$$

Now we prove (6.36)-(6.38). Using (6.16) gives rise to

$$\| u^4 \|_{L^\infty} \leq C, \quad \int_{\mathbb{R}} u^4 dx \leq C \tag{6.49}$$

and

$$| \Lambda^{-2}\Big[u^4 + \frac{5}{2}u^2q^2\Big] | \leq | \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\Big[u^4 + \frac{5}{2}u^2q^2\Big]dy | \leq C, \tag{6.50}$$

where $C > 0$ does not depend on the parameter ε.

Employing (6.16) and (6.17) yields

$$| \Lambda^{-2}[uq^3]_x | \leq \int_{-\infty}^{\infty} e^{-|x-y|}|uq|^3 dy \leq C \int_{\mathbb{R}} |q|^3 dx. \tag{6.51}$$

From (6.10), we have

$$\frac{\partial J_\varepsilon(t,x)}{\partial x} = 4u^3q + uq^3 - \Lambda^{-2}\Big[(u^4 + \frac{5}{2}u^2q^2)_x + uq^3\Big]. \tag{6.52}$$

From (6.16) and (6.18), we have

$$\begin{aligned} &\int_{\mathbb{R}} |u^3q|dx \leq (\int_{\mathbb{R}} |u^6|dx)^{\frac{1}{2}}(\int_{\mathbb{R}} q^2dx)^{\frac{1}{2}} \leq C, \\ &\int_{\mathbb{R}} |uq|^3dx \leq \Big(\int_{\mathbb{R}} u^2dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}} q^6dx\Big)^{\frac{1}{2}} \leq C, \end{aligned} \tag{6.53}$$

$$
\begin{aligned}
&|\Lambda^{-2}\Big[(u^4+\frac{5}{2}u^2q^2)_x+uq^3\Big]| \\
&=\Big|\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{y}\frac{\partial(u^4+\frac{5}{2}u^2q^2)}{\partial y}dy+\frac{1}{2}e^{x}\int_{x}^{\infty}e^{-y}\frac{\partial(u^4+\frac{5}{2}u^2q^2)}{\partial y}dy \\
&\qquad\qquad +\frac{1}{2}\int_{-\infty}^{\infty}e^{-|x-y|}uq^3dy\Big| \\
&=\Big|-\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{y}(u^4+\frac{5}{2}u^2q^2)dy+\frac{1}{2}e^{x}\int_{x}^{\infty}e^{-y}(u^4+\frac{5}{2}u^2q^2)dy \\
&\qquad\qquad +\frac{1}{2}\int_{-\infty}^{\infty}e^{-|x-y|}uq^3dy\Big| \\
&\le\int_{-\infty}^{\infty}e^{-|x-y|}|u^4+\frac{5}{2}u^2q^2|dy+\frac{C}{2}\int_{-\infty}^{\infty}e^{-|x-y|}|q|^3dy. \qquad (6.54)
\end{aligned}
$$

Applying (6.16),(6.25), (6.52)-(6.54) and the Tonelli theorem, we get that (6.36) and (6.37) hold. The inequality (6.38) is directly derived by (6.36) and (6.37). The proof is completed. □

Lemma 6.5. *Suppose $u_0(x)\in H^1(\mathbb{R})$ and $\| u_{0x} \|_{L^\infty(\mathbb{R})}<\infty$. For every $T>0$, then the following one-sided L^∞ estimate holds*

$$
\frac{\partial u_\varepsilon(t,x)}{\partial x}\le C(1+t), \qquad \text{for } t\in[0,T),\ x\in\mathbb{R}, \qquad (6.55)
$$

in which the positive constant C only depends on $\| u_0 \|_{H^1(\mathbb{R})}$, $\| u_{0x} \|_{L^\infty(\mathbb{R})}$ and T.

Proof. From Lemma 6.4, we obtain $|J_\varepsilon|\le C$ and

$$
\frac{\partial q_\varepsilon}{\partial t}+u_\varepsilon^3\frac{\partial q_\varepsilon}{\partial x}+\frac{1}{2}u_\varepsilon^2q_\varepsilon^2-\varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2}=J_\varepsilon(t,x)\le C. \qquad (6.56)
$$

Assume that $h=h(t)$ is the solution of

$$
\frac{dh}{dt}+(u_\varepsilon^*)^2h^2=C, \quad t>0, \quad h(0)=\| \frac{\partial u_{\varepsilon,0}}{\partial x} \|_{L^\infty}. \qquad (6.57)
$$

Letting u_ε^* be the value of $u_\varepsilon(t,x)$ when $\sup_{x\in R} q_\varepsilon(t,x) = h(t)$, we get that $h = h(t)$ is a supersolution of the parabolic equation (6.56) with $u_{\varepsilon,0}(x)$. Using the comparison principle for parabolic equations, we obtain

$$q_\varepsilon(t,x) \le h(t). \tag{6.58}$$

Let $K(t) = Ct$. Consider that $\frac{dK(t)}{dt} + (u_\varepsilon^*)^2K^2(t) - C = (u_\varepsilon^*)^2(Ct)^2 > 0$ for any $t > 0$. Applying the comparison principle of ordinary differential equations, we know $h(t) \le K(t) = Ct + \| \frac{\partial u_{\varepsilon,0}}{\partial x} \|_{L^\infty}$ for all $t > 0$. Thus, we obtain the desired result. □

Lemma 6.6. *Suppose $u_0(x) \in H^1(\mathbb{R})$ and $\| u_{0x} \|_{L^\infty(\mathbb{R})} < \infty$. For the solution $u_\varepsilon(t,x)$ of problem (6.8), then there exists a subsequence $\varepsilon_i \to 0$ as $i \to \infty$ and a function $u \in L^\infty([0,\infty); H^1(\mathbb{R})) \cap H^1([0,T]\times\mathbb{R})$, such that*

$$u_{\varepsilon_i} \rightharpoonup u \quad in \quad H^1([0,T]\times\mathbb{R}), \quad for\ every \quad T \ge 0, \tag{6.59}$$

$$u_{\varepsilon_i} \to u \quad in \quad L^\infty_{loc}([0,\infty)\times\mathbb{R}). \tag{6.60}$$

Proof. We have

$$\frac{\partial u_\varepsilon}{\partial t} + u^3\frac{\partial u_\varepsilon}{\partial x} + \frac{\partial H_\varepsilon}{\partial x} = \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2}.$$

For every fixed $T > 0$, Applying Lemmas 6.2 and 6.4 yields

$$\| \frac{\partial u_\varepsilon}{\partial t} \|_{L^2(\mathbb{R})}, \quad \| \frac{\partial u_\varepsilon}{\partial t} \|_{L^2([0,T]\times\mathbb{R})} \le C_0(1 + \sqrt{\varepsilon} \| u_0 \|_{H^1(\mathbb{R})}), \tag{6.61}$$

where C_0 depends on $\| u_{0x} \|_{L^\infty(\mathbb{R})}$, $\| u_0 \|_{H^1(\mathbb{R})}$ and T. Consequently, we get that the solution $\{u_\varepsilon\}$ is uniformly bounded in the space $L^\infty([0,\infty); H^1(\mathbb{R})) \cap H^1([0,T]\times\mathbb{R})$ and (6.59) holds.

Note that, for each $0 \le s, t \le T$,

$$\| u_\varepsilon(t,.) - u_\varepsilon(s,.) \|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \Big(\int_s^t \frac{\partial u_\varepsilon}{\partial t}(\tau, x) d\tau \Big)^2 dx$$
$$\le |t-s| \int_{\mathbb{R}} \int_0^T \Big(\frac{\partial u_\varepsilon}{\partial t}(\tau, x) \Big)^2 d\tau dx. \quad (6.62)$$

Therefore, $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty([0,T]; H^1(\mathbb{R}))$ and $H^1(\mathbb{R}) \subset L^\infty_{loc} \subset L^2_{loc}(\mathbb{R})$. Applying the conclusions in [9], we get that (6.60) is valid (also see [101]). □

Lemma 6.7. *Suppose* $\| u_{0x} \|_{L^\infty(\mathbb{R})} < \infty$ *and* $u_0(x) \in H^1(\mathbb{R})$. *Then, the sequence* $J_\varepsilon(t,x)$ *is uniformly bounded in* $W^{1,1}_{loc}([0,\infty) \times \mathbb{R})$. *Moreover, there exists a sequence* $\varepsilon_i \to 0$ *if* $i \to \infty$ *and a function* $J \in L^\infty([0,\infty); W^{1,\infty}(\mathbb{R}))$ *such that*

$$J_{\varepsilon_i} \to J \quad \textit{strongly in} \quad L^p_{loc}([0,T) \times \mathbb{R}), \quad 1 < p < \infty. \quad (6.63)$$

Proof. Applying notations $u = u_\varepsilon(t,x)$ and $q = q_\varepsilon$ for simplicity, we acquire

$$\frac{\partial J_\varepsilon}{\partial t} = 4u^3 u_t - \Lambda^{-2}\Big[4u^3u_t + 5uu_tq^2 + 5u^2qq_t + \partial_x(u_tq^3 + 3uq^2q_t)\Big]$$
$$= 4u^3u_t - \Lambda^{-2}\Big[(4u^3 + 5uq^2)u_t\Big]$$
$$-\Lambda^{-2}\Big[5u^2q(J_\varepsilon - u^3q_x - \frac{1}{2}u^2q^2 + \varepsilon q_{xx})\Big]$$
$$-\Lambda^{-2}\Big(\partial_x\Big[u_tq^3 + 3uq^2q_t\Big]\Big)$$
$$= 4u^3u_t + Q_1 + Q_2 + Q_3. \quad (6.64)$$

Applying Lemma 6.6 and (6.16) yields

$$\int_{\mathbb{R}} |u^3u_t| dx \le (\int_{\mathbb{R}} u^6 dx)^{\frac{1}{2}} (\int_{\mathbb{R}} u_t^2 dx)^{\frac{1}{2}}$$
$$\le \| u \|^2_{L^\infty(\mathbb{R})} (\int_{\mathbb{R}} u^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}} u_t^2 dx)^{\frac{1}{2}} \le C.$$

Using (6.16), (6.24) and Lemma 6.6 gives rise to

$$\begin{aligned}\int_{\mathbb{R}} |Q_1| dxdt &\le \int_0^t \Big[\int_{\mathbb{R}} e^{-|x-y|} \Big(\int_{\mathbb{R}} (4u^3 + 5uq^2)^2 dy \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} u_t^2 dy \Big)^{\frac{1}{2}} dx \Big] dt \\ &\le CT. \end{aligned} \tag{6.65}$$

For Q_2, we have

$$\begin{aligned} |Q_2| &\le \Bigg(\int_{\mathbb{R}} e^{-|x-y|} \Big| 5u^2 q J_\varepsilon - \frac{5}{2} u^4 q^3 \Big| dy \\ &\quad + \Big| \int_{\mathbb{R}} e^{-|x-y|} 5u^5 q q_y dy \Big| + \varepsilon \Big| \int_{\mathbb{R}} e^{-|x-y|} 5u^2 q q_{yy} dy \Big| \Bigg) \\ &\le C \Bigg(\int_{\mathbb{R}} e^{-|x-y|} \Big| 5u^2 q J_\varepsilon - \frac{5}{2} u^4 q^3 \Big| dy \\ &\quad + \frac{1}{2} \Big| \int_{\mathbb{R}} e^{-|x-y|} q^2 \Big[25u^4 q + sign(x-y) u^5 \Big] dy \Big| \\ &\quad + \varepsilon \Big| \int_{\mathbb{R}} e^{-|x-y|} q_y \Big[10uq^2 + 5u^2 q_y + sign(x-y) 5u^2 q \Big] dy \Big| \Bigg) \end{aligned} \tag{6.66}$$

Using the Schwartz inequality, (6.16), (6.25) Lemmas 6.2 and 6.3, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} |Q_2| dxdt &\le C \Big(1 + T \parallel u_0 \parallel_{H^1(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} |q|^3 dydt \Big) \\ &\quad + \varepsilon C \int_0^t \int_{\mathbb{R}} \Big(|qq_y| + |q^2 q_y| + |q_y|^2 \Big) dxdt \\ &\le C \Big(1 + T \parallel u_0 \parallel_{H^1(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} |q|^3 dydt \Big) \\ &\quad + \varepsilon C \int_0^t \int_{\mathbb{R}} \Big(|q|^2 + q^4 + 3|q_y|^2 \Big) dxdt \\ &\le C(1+T). \end{aligned} \tag{6.67}$$

For Q_3, we have

$$\begin{aligned}
Q_3 &= \Lambda^{-2}\Big[u_t q^3 + 3uq^2 q_t\Big]_x \\
&= \Lambda^{-2}\Big[u_t q^3 + 3uq^2\Big(J_\varepsilon - u^3 q_x - \frac{1}{2}u^2 q^2 + \varepsilon q_{xx}\Big)\Big]_x \\
&= \Lambda^{-2}\Big[u_t q^3 + 3uq^2 J_\varepsilon - \frac{3}{2}u^3 q^4\Big]_x - 3\Lambda^{-2}\Big[u^4 q^2 q_x\Big]_x + 3\varepsilon\Lambda^{-2}\Big[uq^2 q_{xx}\Big]_x \\
&= I_1 + I_2 + I_3 \qquad (6.68)
\end{aligned}$$

Using (6.17), we get

$$\begin{aligned}
|I_1| &= |\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\Big[u_t q^3 + 3uq^2 J_\varepsilon - \frac{3}{2}u^3 q^4\Big]_y dy| \\
&\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}\Big|u_t q^3 + 3uq^2 J_\varepsilon - \frac{3}{2}u^3 q^4\Big| dy, \qquad (6.69)
\end{aligned}$$

$$\begin{aligned}
I_2 &= -3\Lambda^{-2}\Big[u^4 q^2 q_x\Big]_x = -\Lambda^{-2}\Big[(u^4 q^3)_x - (u^4)_x q^3\Big]_x \\
&= -\Lambda^{-2}(1-\Lambda^2)(u^4 q^3) + \Lambda^{-2}\Big[4u^3 q^4\Big]_x \\
&= -\Lambda^{-2}(u^4 q^3) + 4\Lambda^{-2}\Big[u^3 q^4\Big]_x + u^4 q^3 \qquad (6.70)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= 3\varepsilon\Lambda^{-2}\Big[(uq^2 q_x)_x - (uq^2)_x q_x\Big]_x \\
&= 3\varepsilon\Lambda^{-2}(1-\Lambda^2)[uq^2 q_x] - 3\varepsilon\Lambda^{-2}\Big[(q^3 + 2uqq_x)q_x\Big]_x \\
&= -3\varepsilon(uq^2 q_x) + 3\varepsilon\Lambda^{-2}[uq^2 q_x] - 3\varepsilon\Lambda^{-2}\Big[(q^3 + 2uqq_x)q_x\Big]_x. \qquad (6.71)
\end{aligned}$$

Using (6.24), (6.25) and (6.61), we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} |I_1| dxdt \\
&\leq C \int_0^t \left(1 + \Big(\int_{-\infty}^{\infty} (u_t)^2 dy \Big)^{\frac{1}{2}} \Big(\int_{-\infty}^{\infty} q^6 dy \Big)^{\frac{1}{2}} + C \Big(\int_{-\infty}^{\infty} q^4 dy \Big) \right) dt \\
&\leq C(1+T). \qquad (6.72)
\end{aligned}$$

Applying inequalities (6.16), (6.17), (6.24), (6.25) and the Tonelli theorem yields

$$\int_0^t \int_{\mathbb{R}} |I_2| dxdt \leq C \int_0^t \left[1 + \int_{\mathbb{R}} |q|^3 dx + \int_{\mathbb{R}} |q|^4 dx \right] dt \leq C(1+T) \quad (6.73)$$

Using Lemmas 6.2 and 6.3, we have

$$\begin{aligned}
\varepsilon \int_0^t \int_{\mathbb{R}} |(uq^2 q_x)| dxdt &\leq \varepsilon C \int_0^t \int_{\mathbb{R}} |q^2 q_x| dxdt \\
&\leq C\varepsilon \int_0^t \int_{\mathbb{R}} (q^4 + q_x^2) dxdt \\
&\leq C \qquad (6.74)
\end{aligned}$$

Using (6.17) gives rise to

$$\varepsilon |\Lambda^{-2} \Big[(q^3 + 2uqq_x) q_x \Big]_x | \leq \varepsilon \int_{\mathbb{R}} e^{-|x-y|} \Big| (q^3 + 2uqq_y) q_y \Big| dy.$$

Note that

$$|qq_y^2| \leq c[(qq_y)^2 + q_y^2] \leq c[q_y^2 + (q^2 q_y)^2 + q_y^2] \leq c[2q_y^2 + q^4 q_y^2].$$

Using Lemmas 6.2 and 6.3, we have

$$\begin{aligned}
&\varepsilon\int_0^t\int_{\mathbb{R}}|\Lambda^{-2}\Big[(q^3+2uqq_x)q_x\Big]_x|dxdt\\
&\quad\leq \varepsilon C\int_0^t\left(\int_{\mathbb{R}}\Big|(q^3+2uqq_y)q_y\Big|dy\int_{\mathbb{R}}e^{-|x-y|}dx\right)dt\\
&\quad\leq \varepsilon C\int_0^t\int_{\mathbb{R}}\Big|(q^3+2uqq_y)q_y\Big|dydt\\
&\quad\leq \varepsilon C\int_0^t\int_{\mathbb{R}}\Big(\big|qq^2q_y\big|+\big|qq_y^2|\big|\Big)dydt\\
&\quad\leq \varepsilon C\int_0^t\int_{\mathbb{R}}\Big(q^6+q_y^2+q^4q_y^2\Big)dydt\\
&\quad\leq C. \qquad (6.75)
\end{aligned}$$

From (6.69)-(6.75), we obtain

$$\int_0^t\int_{\mathbb{R}}|Q_3|dxdt\leq C(1+T). \qquad (6.76)$$

From (6.64)-(6.68), (6.73)-(6.74) and (6.76), we derive that $\frac{\partial J_\varepsilon}{\partial t}$ is uniformly bounded in $L^1_{loc}([0,\infty)\times\mathbb{R})$. Then we get that J_ε is bounded in $W^{1,1}_{loc}([0.\infty)\times\mathbb{R})$ which together with Lemma 6.4 results in (6.63) (see [94]). □

Throughout this chapter, for $1<r<3$, the overbars denote weak limits in the space $L^r([0,\infty)\times\mathbb{R})$. Let $\Omega_+=[0,\infty)\times\mathbb{R}$.

Lemma 6.8. *There exists a subsequence $\varepsilon_i\to 0$ if $i\to\infty$ and two functions $q\in L^p_{loc}(\Omega_+)$, $\overline{q^2}\in L^r_{loc}(\Omega_+)$ such that, for every $1<p<6$ and $1<r<3$, the following conclusions*

$$q_{\varepsilon_i}\rightharpoonup q \quad in \quad L^p_{loc}(\Omega_+),\quad q_{\varepsilon_i}\overset{\star}{\rightharpoonup} q\quad in\quad L^\infty_{loc}([0,\infty);L^2(\mathbb{R})), \qquad (6.77)$$

$$q^2_{\varepsilon_i}\rightharpoonup\overline{q^2}\quad in\quad L^r_{loc}(\Omega_+) \qquad (6.78)$$

hold. Moreover,

$$q^2(t,x) \le \overline{q^2}(t,x) \quad \textit{almost everywhere in} \quad (t,x) \in \Omega_+, \tag{6.79}$$

and

$$\frac{\partial u}{\partial x} = q \quad \textit{in the sense of distributions on the domain} \quad \Omega_+. \tag{6.80}$$

Proof. Using Lemmas 6.2 and 6.3 leads to (6.77) and (6.78). Applying the weak convergence in (6.78), we have inequality (6.79). The definition of q_ε, Lemma 6.6 and (6.77) directly derives (6.80). The proof is completed. □

For notational convenience and simplicity, we use $\{u_\varepsilon\}_{\varepsilon>0}$, $\{q_\varepsilon\}_{\varepsilon>0}$ and $\{J_\varepsilon\}_{\varepsilon>0}$ to replace the sequence $\{u_{\varepsilon_i}\}_{i\in N}$, $\{q_{\varepsilon_i}\}_{i\in N}$ and $\{J_{\varepsilon_i}\}_{i\in N}$ (where N denotes all the nature numbers), respectively.

For every convex function $\phi \in C^1(\mathbb{R})$ associated with ϕ' bounded, Lipschitz continuous on $\mathbb{R}$, using Lemma 6.8 yields

$$\phi(q_\varepsilon) \rightharpoonup \overline{\phi(q)} \text{ in } L^p_{loc}(\Omega_+), \quad 1 < p < 6, \tag{6.81}$$

$$\phi(q_\varepsilon) \overset{\star}{\rightharpoonup} \overline{\phi(q)} \text{ in } L^\infty_{loc}([0,\infty); L^2(\mathbb{R})). \tag{6.82}$$

Multiplying the equation (6.9) by $\phi'(q_\varepsilon)$ gives rise to

$$\begin{aligned} &\frac{\partial}{\partial t}\phi(q_\varepsilon) + \frac{1}{3}\frac{\partial}{\partial x}(u_\varepsilon^3\phi(q_\varepsilon)) - \varepsilon\frac{\partial^2\phi(q_\varepsilon)}{\partial x^2} + \varepsilon\phi''(q_\varepsilon)\Big(\frac{\partial q_\varepsilon}{\partial x}\Big)^2 \\ &\qquad = u_\varepsilon^2 q_\varepsilon\phi(q_\varepsilon) - \frac{1}{2}u_\varepsilon^2\phi'(q_\varepsilon)q_\varepsilon^2 - \frac{2}{3}u_\varepsilon^3\frac{\partial\phi(q)}{\partial x} + J_\varepsilon\phi'(q_\varepsilon). \end{aligned} \tag{6.83}$$

Lemma 6.9. *Assume that all the assumptions in Theorem 6.1 hold. Then for* *every convex $\phi \in C^1(\mathbb{R})$ associated with ϕ' bounded, Lipschitz continuous on $\mathbb{R}$, the following inequality*

$$\overline{\frac{\partial \phi(q)}{\partial t}} + \frac{1}{3}\frac{\partial}{\partial x}\left(u^3\overline{\phi(q)}\right) \le u^2\overline{q\phi(q)} - \frac{1}{2}u^2\overline{\phi'(q)q^2} - \frac{2}{3}u^3\overline{\frac{\partial \phi(q)}{\partial x}} + J\overline{\phi'(q)} \quad (6.84)$$

holds in the sense of distributions on the domain Ω_+. Here $\overline{q\phi(q)}$ and $\overline{\phi'(q)q^2}$ represent the weak limits of $q_\varepsilon\phi(q_\varepsilon)$ and $q_\varepsilon^2\phi'(q_\varepsilon)$ in $L^r_{loc}(\Omega_+)$, $1 < r < 3$, respectively.

Proof. Applying Lemmas 6.6-6.8, the convexity of ϕ and taking the limits for $\varepsilon \to 0$ in (6.83), we obtain (6.84). □

Remark 6.10. *From (6.77) and (6.78), we have*

$$q = q_+ + q_- = \overline{q_+} + \overline{q_-}, \quad q^2 = (q_+)^2 + (q_-)^2, \quad \overline{q^2} = \overline{(q_+)^2} + \overline{(q_-)^2} \qquad (6.85)$$

almost everywhere in Ω_+, where $\eta_+ := \eta_{\chi[0,+\infty)}(\eta), \eta_- := \eta_{\chi(-\infty,0]}(\eta)$ for $\eta \in \mathbb{R}$. Using Lemma 6.5 leads to

$$q_\varepsilon(t,x), \quad q(t,x) \le C(1+t), \quad 0 < t < T, \quad x \in \mathbb{R}, \qquad (6.86)$$

where $C > 0$ does not depend on the parameter ε.

Lemma 6.11. *Assume that all the assumptions in Theorem 6.1 hold. In the sense of distributions on Ω_+, then*

$$\frac{\partial q}{\partial t} + \frac{1}{3}\frac{\partial}{\partial x}(u^3 q) = \frac{1}{2}u^2\overline{q^2} - \frac{2}{3}u^3\frac{\partial q}{\partial x} + J. \qquad (6.87)$$

Proof. Using (6.9), Lemmas 6.6-6.8, and taking limits for $\varepsilon \to 0$ in (6.9), we get that (6.87) holds. □

A generalized formulation of (6.87) is presented in the lemma below.

Lemma 6.12. *Assume that all the assumptions in Theorem 6.1 hold. Then for every* $\phi \in C^1(\mathbb{R})$ *with* $\phi' \in L^\infty(\mathbb{R})$ *and an arbitrary* $T > 0$*, the following identity*

$$\frac{\partial \phi(q)}{\partial t} + \frac{1}{3}\frac{\partial}{\partial x}(u^3\phi(q)) = u^2 q\phi(q) - u^2q^2\phi'(q) + \frac{1}{2}u^2\overline{q^2}\phi'(q) - \frac{2}{3}u^3\frac{\partial \phi(q)}{\partial x} + J\phi'(q) \quad (6.88)$$

holds in the sense of distributions on $[0,T) \times \mathbb{R}$.

Proof. Assume that $\{w_\gamma\}_\gamma$ is a family of mollifiers defined in $\mathbb{R}$. Set $q_\gamma(t,x) := (q(t,.)\star w_\gamma)(x)$, where the notation $\star$ represents the convolution about the variable x. Multiplying (6.87) by $\phi'(q_\gamma)$ gives rise to

$$\frac{\partial \phi(q_\gamma)}{\partial t} = \phi'(q_\gamma)\frac{\partial q_\gamma}{\partial t} = \phi'(q_\gamma)\Bigg[-\frac{1}{3}\frac{\partial}{\partial x}(u^3 q)\star w_\gamma + \frac{1}{2}u^2\overline{q^2}\star w_\gamma - \frac{2}{3}u^3\frac{\partial \phi(q)}{\partial x}\star w_\gamma + J\star w_\gamma\Bigg]. \quad (6.89)$$

Using the boundedness of ϕ, ϕ' and letting $\gamma \to 0$ in (6.89), we have

$$\frac{\partial \phi(q)}{\partial t} + \frac{1}{3}u^3\frac{\partial \phi q}{\partial x} = -u^2q^2\phi'(q) + \frac{1}{2}u^2\overline{q^2}\phi'(q) - \frac{2}{3}u^3\frac{\partial \phi(q)}{\partial x} + J\phi'(q), \quad (6.90)$$

which leads to

$$\frac{\partial \phi(q)}{\partial t} + \frac{1}{3}\frac{(\partial u^3\phi q)}{\partial x} = u^2q\phi(q) - u^2q^2\phi'(q) + \frac{1}{2}u^2\overline{q^2}\phi'(q) - \frac{2}{3}u^3\frac{\partial \phi(q)}{\partial x} + J\phi'(q).$$

The proof is completed. □

6.4 The proof of main results

Using the methods in [101] or [9] or the techniques in Chapter 3, we shall show that the weak convergence of q_ε in (6.78) is equal to strong convergence. Subsequently, we establish the existence of global weak solutions for problem (6.3).

In this section, we will use Lemmas 3.15,3.16,3.17 in chapter 3 to prove our results. All the notations used are the same as those in Lemmas 3.15-3.17.

Lemma 6.13. *Assume $u_0 \in H^1(\mathbb{R})$ and $\| u_{0x} \|_{L^\infty} < \infty$. Then for almost all $t > 0$, the inequality*

$$\frac{1}{2}\int_{\mathbb{R}}\Big(\overline{(q_+)^2}-q_+^2\Big)(t,x)dx \le -\frac{3}{2}\int_0^t\int_{\mathbb{R}} u^3\Big(\overline{\frac{\partial\phi_B^+(q)}{\partial x}}-\frac{\partial\phi_B^+(q)}{\partial x}\Big)dxds + \int_0^t\int_{\mathbb{R}} J(s,x)[\overline{q_+}(s,x)-q_+(s,x)]dxds \quad (6.91)$$

holds.

Proof. For any $T>0$ $t\in[0,T]$, we choose B sufficiently large to satisfy $B > C$ (see Lemma 6.5). Employing Lemmas 6.9 and 6.12, and the entropy ϕ_B^+ (see Lemma 3.17) leads to

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\overline{\phi_B^+(q)}-\phi_B^+(q)\Big)&+\frac{1}{3}\frac{\partial}{\partial x}\Big(u^3\Big[\,\overline{\phi_B^+(q)}-\phi_B^+(q)\Big]\Big)\\ &\le u^2\Big(\overline{q\phi_B^+(q)}-q\phi_B^+(q)\Big)-\frac{u^2}{2}\Big(\overline{q^2(\phi_B^+)'(q)}-q^2(\phi_B^+)'(q)\Big)\\ &-\frac{1}{2}u^2\Big(\overline{q^2}-q^2\Big)(\phi_B^+)'(q)-\frac{2}{3}u^3\Big(\overline{\frac{\partial\phi_B^+(q)}{\partial x}}-\frac{\partial\phi_B^+(q)}{\partial x}\Big)\\ &\quad+J(t,x)\Big(\overline{(\phi_B^+)'(q)}-(\phi_B^+)'(q)\Big). \end{aligned} \quad (6.92)$$

Since ϕ_B^+ is increasing, from (6.79), we have

$$-u^2(\overline{q^2}-q^2)(\phi_B^+)'(q)\le 0. \quad (6.93)$$

Applying Lemma 3.17 results in

$$
\begin{aligned}
q\phi_B^+(q) - \frac{1}{2}q^2(\phi_B^+)'(q) &= -\frac{B}{2}q(B-q)\chi_{(B,\infty)}(q),\\
\overline{q\phi_B^+(q)} - \frac{1}{2}\overline{q^2(\phi_B^+)'(q)} &= -\frac{B}{2}\overline{q(B-q)\chi_{(B,\infty)}(q)}.
\end{aligned}
\tag{6.94}
$$

Using Remark 6.10, (6.86) and Lemma 6.5, we can choose sufficiently large $B > 0$ to ensure $q < C(1+t) < B$. Let $\Upsilon_B = \Big(0, \frac{B}{C} - 1\Big) \times \mathbb{R}$. From (6.86), we have

$$
q\phi_B^+(q) - \frac{1}{2}q^2(\phi_B^+)'(q) = \overline{q\phi_B^+(q)} - \frac{1}{2}\overline{q^2(\phi_B^+)'(q)} = 0, \quad \text{in} \quad \Upsilon_B.
\tag{6.95}
$$

In $\Big(0, \frac{B}{C} - 1\Big) \times \mathbb{R}$, it holds that

$$
\begin{cases}
\phi_B^+ = \frac{1}{2}(q_+)^2, \quad (\phi_B^+)'(q) = q_+,\\
\overline{\phi_B^+(q)} = \frac{1}{2}\overline{(q_+)^2}, \quad \overline{(\phi_B^+)'(q)} = \overline{q_+}.
\end{cases}
\tag{6.96}
$$

Using (6.93)-(6.96), in the domain $\Big(0, \frac{B}{C}-1\Big)\times\mathbb{R}$, we have the following inequality

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big(\overline{\phi_B^+(q)} - \phi_B^+(q)\Big) + \frac{1}{3}\frac{\partial}{\partial x}\Big(u^3\Big[\,\overline{\phi_B^+(q)} - \phi_B^+(q)\Big]\Big)\\
&\le -\frac{2}{3}u^3\Big(\overline{\frac{\partial\phi_B^+(q)}{\partial x}} - \frac{\partial\phi_B^+(q)}{\partial x}\Big) + J(t,x)\Big(\overline{(\phi_B^+)'(q)} - (\phi_B^+)'(q)\Big).
\end{aligned}
\tag{6.97}
$$

For almost all $0 < t < \frac{B}{C} - 1$, integrating (6.97) over $(0,t)\times\mathbb{R}$ produces

$$
\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}}\Big(\overline{(q_+)^2} - q_+^2(t,x)\Big)dx \le{}& \lim_{t\to 0}\int_{\mathbb{R}}\Big[\overline{\phi_B^+(q)}(t,x) - \phi_B^+(q)(t,x)\Big]dx\\
&-\frac{2}{3}\int_0^t\int_{\mathbb{R}}u^3\Big(\overline{\frac{\partial\phi_B^+(q)}{\partial x}} - \frac{\partial\phi_B^+(q)}{\partial x}\Big)dxds\\
&+\int_0^t\int_{\mathbb{R}}J(s,x)[\overline{q_+}(s,x) - q_+(s,x)]dxds.
\end{aligned}
\tag{6.98}
$$

Letting $B\to\infty$ and using Lemma 3.16 complete the proof. □

Lemma 6.14. *Assume $u_0 \in H^1(\mathbb{R})$ and $\| u_{0x} \|_{L^\infty} < \infty$. For almost all $t > 0$, then*

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(\overline{\phi_B^-(q)} - \phi_B^-(q)\Big)(t,x)dx \\
&\le \frac{B^2}{2}\int_0^t \int_{\mathbb{R}} u^2\overline{(B+q)\chi_{(-\infty,-B)}(q)}dxds \\
&-\frac{B^2}{2}\int_0^t \int_{\mathbb{R}} u^2(B+q)\chi_{(-\infty,-B)}(q)dxds \\
&+B\int_0^t \int_{\mathbb{R}} u^2\Big[\overline{\phi_B^-(q)} - \phi_B^-(q)\Big]dxds \\
&+\frac{B}{2}\int_0^t \int_{\mathbb{R}} u^2\Big(\overline{q_+^2} - q_+^2\Big)dxds \\
&-\frac{2}{3}\int_0^t \int_{\mathbb{R}} u^3\Big(\overline{\frac{\partial \phi_B^-(q)}{\partial x}} - \frac{\partial \phi_B^-(q)}{\partial x}\Big)dxds \\
&+\int_0^t \int_{\mathbb{R}} J(t,x)\Big(\overline{(\phi_B^-)'(q)} - (\phi_B^-)'(q)\Big)dxds, \qquad (6.99)
\end{aligned}$$

where $B > 0$ is sufficiently large.

Proof. From Lemmas 6.9 and 6.12, choosing $B > 0$ suitably large and using entropy ϕ_B^-, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\overline{\phi_B^-(q)} - \phi_B^-(q)\Big) + \frac{1}{3}\frac{\partial}{\partial x}\Big(u^3\Big[\,\overline{\phi_B^-(q)} - \phi_B^-(q)\Big]\Big) \\
&\le u^2\Big(\overline{q\phi_B^-(q)} - q\phi_B^-(q)\Big) - \frac{u^2}{2}\Big(\overline{q^2(\phi_B^-)'(q)} - q^2(\phi_B^-)'(q)\Big) \\
&-\frac{1}{2}u^2\Big(\overline{q^2} - q^2\Big)(\phi_B^-)'(q) - \frac{2}{3}u^3\Big(\overline{\frac{\partial \phi_B^-(q)}{\partial x}} - \frac{\partial \phi_B^-(q)}{\partial x}\Big) \\
&+J(t,x)\Big(\overline{(\phi_B^-)'(q)} - (\phi_B^-)'(q)\Big). \qquad (6.100)
\end{aligned}$$

As $-B \le (\phi_B^-)' \le 0$ and $u^2 \ge 0$, we obtain

$$-\frac{u^2}{2}\Big(\overline{q^2} - q^2\Big)(\phi_B^-)'(q) \le \frac{u^2 B}{2}\Big(\overline{q^2} - q^2\Big), \qquad (6.101)$$

Using Remark 6.10 and Lemma 3.17 yields

$$u^2 q\phi_B^-(q) - \frac{u^2}{2}q^2(\phi_B^-)'(q) = -\frac{Bu^2}{2}q(B+q)\chi_{(-\infty,-B)}(q) \tag{6.102}$$
$$u^2\overline{q\phi_B^-(q)} - \frac{u^2}{2}\overline{q^2(\phi_B^-)'(q)} = -\frac{Bu^2}{2}\overline{q(B+q)\chi_{(-\infty,-B)}(q)}. \tag{6.103}$$

Applying (6.100), (6.101) and (6.103) leads to

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big(\overline{\phi_B^-(q)} - \phi_B^-(q)\Big) + \frac{1}{3}\frac{\partial}{\partial x}\Big(u^3\Big[\,\overline{\phi_B^-(q)} - \phi_B^-(q)\Big]\Big)\\
&\quad\le -\frac{1}{2}Bu^2\overline{q(B+q)\chi_{(-\infty,-B)}(q)} + \frac{1}{2}Bu^2 q(B+q)\chi_{(-\infty,-B)}(q)\\
&\quad+\frac{1}{2}Bu^2\Big(\overline{q^2} - q^2\Big) - \frac{2}{3}u^3\Big(\overline{\frac{\partial\phi_B^-(q)}{\partial x}} - \frac{\partial\phi_B^-(q)}{\partial x}\Big)\\
&\quad+J(t,x)\Big(\overline{(\phi_B^-)'(q)} - (\phi_B^-)'(q)\Big).
\end{aligned} \tag{6.104}$$

Integrating (6.104) over $(0,t)\times\mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\overline{\phi_B^-(q)} - \phi_B^-(q)\Big)(t,x)dx\\
&\quad\le -\frac{B}{2}\int_0^t\int_{\mathbb{R}} u^2\overline{q(B+q)\chi_{(-\infty,-B)}(q)}dxds\\
&\quad+\frac{B}{2}\int_0^t\int_{\mathbb{R}} u^2 q(B+q)\chi_{(-\infty,-B)}(q)dxds\\
&+\frac{B}{2}\int_0^t\int_{\mathbb{R}} u^2\Big(\overline{q^2} - q^2\Big)dxds - \frac{2}{3}\int_0^t\int_{\mathbb{R}} u^3\Big(\overline{\frac{\partial\phi_B^-(q)}{\partial x}} - \frac{\partial\phi_B^-(q)}{\partial x}\Big)dxds\\
&\quad+\int_0^t\int_{\mathbb{R}} J(t,x)\Big(\overline{(\phi_B^-)'(q)} - (\phi_B^-)'(q)\Big)dxds.
\end{aligned} \tag{6.105}$$

Using Lemma 3.17 yields

$$\begin{aligned}
\overline{\phi_B^-(q)} - \phi_B^-(q) &= \frac{1}{2}\Big(\overline{(q_-)^2} - (q_-)^2\Big) + \frac{1}{2}(B+q)^2\chi_{(-\infty,-B)}(q)\\
&-\frac{1}{2}\overline{(B+q)^2\chi_{(-\infty,-B)}(q)},
\end{aligned} \tag{6.106}$$

which results in

$$Bu^2\Big[\overline{\phi_B^-(q)}-\phi_B^-(q)\Big]=\frac{B}{2}u^2\Big(\overline{(q_-)^2}-(q_-)^2\Big)+\frac{B}{2}u^2(B+q)^2\chi_{(-\infty,-B)}(q)$$
$$-\frac{B}{2}u^2\overline{(B+q)^2\chi_{(-\infty,-B)}(q)}. \tag{6.107}$$

Making use of Remark 6.10, (6.105), (6.107), we acquire

$$\begin{aligned}
&\int_{\mathbb{R}}\Big(\overline{\phi_B^-(q)}-\phi_B^-(q)\Big)(t,x)dx\\
&\le -\frac{B}{2}\int_0^t\int_{\mathbb{R}}u^2\overline{q(B+q)\chi_{(-\infty,-B)}(q)}dxds\\
&+\frac{B}{2}\int_0^t\int_{\mathbb{R}}u^2q(B+q)\chi_{(-\infty,-B)}(q)dxds\\
&+B\int_0^t\int_{\mathbb{R}}u^2\Big[\overline{\phi_B^-(q)}-\phi_B^-(q)\Big]dxds\\
&+\frac{B}{2}\int_0^t\int_{\mathbb{R}}u^2\overline{(B+q)^2\chi_{(-\infty,-B)}(q)}dxds\\
&-\frac{B}{2}\int_0^t\int_{\mathbb{R}}u^2(B+q)^2\chi_{(-\infty,-B)}(q)dxds+\frac{B}{2}\int_0^t\int_{\mathbb{R}}u^2\Big(\overline{q_+^2}-q_+^2\Big)dxds\\
&-\frac{2}{3}\int_0^t\int_{\mathbb{R}}u^3\Big(\overline{\frac{\partial\phi_B^-(q)}{\partial x}}-\frac{\partial\phi_B^-(q)}{\partial x}\Big)dxds\\
&+\int_0^t\int_{\mathbb{R}}J(t,x)\Big(\overline{(\phi_B^-)'(q)}-(\phi_B^-)'(q)\Big)dxds.
\end{aligned}\tag{6.108}$$

Using the identity $B(B+q)^2-Bq(B+q)=B^2(B+q)$, we obtain (6.99). The proof is completed. □

Lemma 6.15. *Suppose that all the assumptions in Theorem 6.1 hold. Then*

$$\overline{q^2}=q^2 \qquad \textit{almost everywhere in} \quad [0,\infty)\times\mathbb{R}. \tag{6.109}$$

Proof. From Lemmas 6.13 and 6.14, we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\Big(\frac{1}{2}\Big[\overline{(q_+)^2}-(q_+)^2\Big]+\Big[\overline{\phi_B^-}-\phi_B^-\Big]\Big)(t,x)dx\\
&\le \frac{B^2}{2}\int_0^t\int_{\mathbb{R}}u^2\overline{(B+q)\chi_{(-\infty,-B)}(q)}dxds\\
&\qquad -\frac{B^2}{2}\int_0^t\int_{\mathbb{R}}u^2(B+q)\chi_{(-\infty,-B)}(q)dxds\\
&\qquad +B\int_0^t\int_{\mathbb{R}}u^2\Big[\overline{\phi_B^-(q)}-\phi_B^-(q)\Big]dxds\\
&\qquad +\frac{B}{2}\int_0^t\int_{\mathbb{R}}u^2\Big(\overline{q_+^2}-q_+^2\Big)dxds\\
&\qquad -\frac{2}{3}\int_0^t\int_{\mathbb{R}}u^3\Big(\overline{\frac{\partial\phi_B(q)}{\partial x}}-\frac{\partial\phi_B(q)}{\partial x}\Big)dxds\\
&+\int_0^t\int_{\mathbb{R}}J(s,x)\Bigg(\Big[\overline{q_+}-q_+\Big]+\Big[\overline{(\phi_B^-)'(q)}-(\phi_B^-)'(q)\Big]\Bigg)dxds. \qquad (6.110)
\end{aligned}
$$

Using Lemma 6.7, for $0<t<T$, there exists a constant $K_0>0$, relying only on $\| u_{0x} \|_{L^\infty}$, $\| u_0 \|_{H^1(\mathbb{R})}$, and T, such that

$$
\| J(t,x) \|_{L^\infty([0,T)\times\mathbb{R})}\le K_0. \qquad (6.111)
$$

Employing Remark 6.10 and Lemma 3.17 yields

$$
\begin{aligned}
&q_+ + (\phi_B^-)'(q) = q-(B+q)\chi_{(-\infty,-B)},\\
&\overline{q_+}+\overline{(\phi_B^-)'(q)} = q-\overline{(B+q)\chi_{(-\infty,-B)}(q)}. \qquad (6.112)
\end{aligned}
$$

Since the map $\eta\to\eta_+ + (\phi_B^-)'(\eta)$ is convex and concave, we obtain

$$
\begin{aligned}
0&\le[\overline{q_+}-q_+]+[\overline{(\phi_B^-)'(q)}-(\phi_B^-)'(q)]\\
&=(B+q)\chi_{(-\infty,-B)}-\overline{(B+q)\chi_{(-\infty,-B)}(q)}. \qquad (6.113)
\end{aligned}
$$

It follows from (6.110) that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\frac{1}{2} \Big[\overline{(q_+)^2} - (q_+)^2 \Big] + \Big[\overline{\phi_B^-} - \phi_B^- \Big] \right)(t,x)dx \\
&\quad \le B \int_0^t \int_{\mathbb{R}} u^2 \Big[\overline{\phi_B^-(q)} - \phi_B^-(q) \Big] dxds + \frac{B}{2} \int_0^t \int_{\mathbb{R}} u^2 \Big(\overline{q_+^2} - q_+^2 \Big) dxds \\
&\quad + \int_0^t \int_{\mathbb{R}} 2u^2 q \Big(\overline{\phi_B(q)} - \phi_B(q) \Big) dxds. \qquad (6.114)
\end{aligned}$$

Using Lemma 6.5, we get that there must exist a sufficiently large B such that $u^2 q \le CB$. Thus, from Lemma 3.17 and (6.114), we acquire

$$\begin{aligned}
0 \le &\int_{\mathbb{R}} \left(\frac{1}{2} \Big[\overline{(q_+)^2} - (q_+)^2 \Big] + \Big[\overline{\phi_B^-} - \phi_B^- \Big] \right)(t,x)dx \\
&\le CB \int_0^t \int_{\mathbb{R}} \left(\frac{1}{2} \Big[\overline{(q_+)^2} - (q_+)^2 \Big] + \Big[\overline{\phi_B^-} - \phi_B^- \Big] \right)(t,x)dxds. \qquad (6.115)
\end{aligned}$$

For each $t > 0$, the Gronwall inequality is used to produce

$$0 \le \int_{\mathbb{R}} \left(\frac{1}{2} \Big[\overline{(q_+)^2} - (q_+)^2 \Big] + \Big[\overline{\phi_B^-} - \phi_B^- \Big] \right)(t,x)dx = 0.$$

Using the Fatou lemma, (6.79) and Remark 6.10, we let $B \to \infty$ and obtain

$$0 \le \int_{\mathbb{R}} \Big(\overline{q^2} - q^2 \Big)(t,x)dx = 0, \quad \text{for} \quad t > 0, \qquad (6.116)$$

which completes the proof. □

Proof of Theorem 6.1. Applying (6.7), (6.11) and Lemma 6.6, we get that the conditions (i) and (ii) in Definition 3.1 hold. Using Lemma 6.15 gives rise to

$$q_\varepsilon \to q \quad \text{in} \quad L^2_{loc}([0,\infty) \times \mathbb{R}). \qquad (6.117)$$

From Lemmas 6.6 and 6.7, and (6.117), we get that u is a distributional solution to system (6.3). Using Lemmas 6.3 and 6.5, we obtain inequalities (6.5) and (6.6).

The proof of Theorem 6.1 is completed. □

CHAPTER 7

Wave breaking to a nonlinear shallow water wave equation

This chapter considers a nonlinear shallow water wave model including the Degasperis-Procesi equation. Several estimates, which are derived from the shallow water model itself, are established to discuss the wave breaking of its solutions.

7.1 General

Constantin and Lannes [13] derived the nonlinear equation

$$u_t + u_x + \frac{3}{2}\rho u u_x + \mu(\alpha u_{xxx} + \beta u_{txx}) = \rho\mu(\gamma u_x u_{xx} + \delta u u_{xxx}), \tag{7.1}$$

where the constants $\rho, \gamma, \delta, \alpha, \beta$ and μ are required to satisfy certain restrictions. Eq.(7.1) describes the propagation and motion of shallow water waves over a flat bed (see [13]). From [13], we know that Eq.(7.1) can be turned into the equation

$$u_t - u_{txx} + k_0 u_x + m u u_x = k_1 u_x u_{xx} + k_2 u u_{xxx}, \tag{7.2}$$

where k_0, k_1, k_2 and m are constants. If $m = 4, k_1 = 3, k_2 = 1$, Eq.(7.2) becomes the Degasperis-Procesi equation. For $m = 3, k_1 = 2, k_2 = 1$, Eq.(7.2) reduces to the Camassa-Holm equation (see [19]). When $k_0 = -1, m = \frac{3}{2}, k_1 = \frac{9}{2}, k_2 = \frac{3}{2}$, Eq.(7.2) is turned into the Fornberg-Whitham (FW) equation (see [43, 44]). The wave breaking of solutions and local strong solutions in $H^s(\mathbb{R})(s > \frac{3}{2})$ for the Fornberg-Whitham equation are discussed in [43, 44].

Motivated by the recent work in [44] where the wave breaking for the FW equation is investigated, we study the wave breaking for a special case of Eq.(7.2), which has the form

$$u_t - u_{txx} + ku_x + muu_x = 3u_x u_{xx} + uu_{xxx}, \tag{7.3}$$

where $m > 0$ and k are constants. As stated in [14], the wave breaking implies that the solution itself is bounded, but its slope remains unbounded as the time tends to a finite time. Assuming that the initial value $u_0(x) \in H^2(\mathbb{R})$, we shall discuss the wave breaking of Eq.(7.3).

For Eq.(7.3), Lai et al. [61] derive the conservation law

$$\int_{\mathbb{R}} \frac{1+\xi^2}{m+\xi^2} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{1+\xi^2}{m+\xi^2} |\hat{u_0}(\xi)|^2 d\xi \sim \| u_0 \|_{L^2(\mathbb{R})}, \tag{7.4}$$

where $u_0 = u(0, x) \in H^2(\mathbb{R})$.

Several Lemmas shall be given in section 7.2. The main results of this chapter and their proofs will be presented in section 7.3.

7.2 Lemmas

We write the Cauchy problem of Eq.(7.3) as follows

$$\begin{cases} u_t - u_{txx} + ku_x + muu_x = 3u_x u_{xx} + uu_{xxx}, \\ u(0, x) = u_0(x). \end{cases} \tag{7.5}$$

Multiplying the first equation of problem (7.5) by $\Lambda^{-2} = (1 - \frac{\partial^2}{\partial x^2})^{-1}$, we have

$$\begin{cases} u_t + uu_x = -k\Lambda^{-2}u_x - \frac{m-1}{2}\Lambda^{-2}(u^2)_x, \\ u(0,x) = u_0(x). \end{cases} \tag{7.6}$$

We shall address here that the operator Λ^{-2} has also been used in chapter 3. For the classical solutions $u(t,x) \in H^2(\mathbb{R}) \in C^1(\mathbb{R})$, we have $u(t, \pm\infty) = 0$.

Lemma 7.1 [61]. *For problem (7.5), if $m > 0$ and $u_0(x) \in H^2(\mathbb{R})$, it holds that*

$$\int_{\mathbb{R}} yv dx = \int_{\mathbb{R}} \frac{1+\xi^2}{m+\xi^2} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}} \frac{1+\xi^2}{m+\xi^2} |\hat{u_0}(\xi)|^2 d\xi \sim \| u_0 \|_{L^2(\mathbb{R})} \tag{7.7}$$

and

$$c_1 \| u_0 \|_{L^2(\mathbb{R})} \leq c_1 \| u \|_{L^2(\mathbb{R})} \leq c_2 \| u_0 \|_{L^2(\mathbb{R})},$$

where c_1, c_2 are constants depending only on m.

Lemma 7.2 [61]. *Let $s \geq 2, u_0 \in H^s(\mathbb{R})$. Then there must exist $T = T(u_0) > 0$ to guarantee that the problem (7.6) has a unique strong solution $u(t,x)$ and*

$$u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})).$$

Lemma 7.3 [61]. *Assume $s \geq 2, u_0 \in H^s(\mathbb{R})$. If T is the maximal existence time of solution to Eq.(7.3), then*

$$\| u(t,x) \|_{L^\infty} \leq \| u_0 \|_{L^\infty} + c_0 t \| u_0 \|^2_{L^2}, \quad \textit{for } t \in [0,T], \tag{7.8}$$

where c_0 only depends on m.

From this Lemma, we know that for all $t \in [0, \infty)$, inequality (7.8) is still valid.

Lemma 7.4. *If* $u(t,x) \in H^2(\mathbb{R})$, *then*

$$\int_{\mathbb{R}} u_x \Lambda^{-2} u dx = -\int_{\mathbb{R}} u \Lambda^{-2} u_x dx = 0, \quad 0 \leq \int_{\mathbb{R}} \Lambda^{-2} u^2 dx < \infty. \tag{7.9}$$

Proof. Letting $\Lambda^{-2} u = v$, we have

$$u = v - v_{xx},$$

which results in

$$\int_{\mathbb{R}} u \Lambda^{-2} u_x dx = \int_{\mathbb{R}} (v - v_{xx}) v_x dx = \int_{\mathbb{R}} v dv - \int_{\mathbb{R}} v_x dv_x = 0. \tag{7.10}$$

Applying

$$\begin{aligned} 0 \leq \int_{\mathbb{R}} & \Lambda^{-2} u^2 dx \\ &= \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-|x-y|} u^2(t,y) dy dx \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2(t,y) dy \int_{\mathbb{R}} e^{-|x-y|} dx \end{aligned} \tag{7.11}$$

and Lemma 7.1, we derive that

$$0 \leq \int_{\mathbb{R}} \Lambda^{-2} u^2 dx < \infty.$$

The proof is completed. ∎

Lemma 7.5. *If $u_0 \in H^2(\mathbb{R})$ and $u(t,x)$ satisfies problem (7.5), then*

$$|\Lambda^{-2}u^2| < c, \tag{7.12}$$

$$|\Lambda^{-2}\partial_x(u^2)| < c, \tag{7.13}$$

$$|\int_{\mathbb{R}} \Lambda^{-2}\partial_x(u^2)dx| < c, \tag{7.14}$$

$$|\int_{\mathbb{R}} u\Lambda^{-2}(u^2)_x dx| < c, \tag{7.15}$$

where constant $c > 0$ is independent of time t.

proof. Since

$$\Lambda^{-2}u^2 dx = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}u^2(t,y)dy \tag{7.16}$$

and

$$\Lambda^{-2}\partial_x(u^2) = -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^y u^2(t,y)dy \quad + \frac{e^x}{2}\int_x^{\infty} e^{-y}u^2(t,y)dy,$$

using Lemma 7.1, we get

$$\begin{aligned} |\Lambda^{-2}\partial_x(u^2)| &= |\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}sign(x-y)u^2(t,y)dy| \\ &\leq \frac{1}{2}\int_{\mathbb{R}} u^2(t,y)dy < \infty \end{aligned} \tag{7.17}$$

and

$$\begin{aligned} \int_{\mathbb{R}} |\Lambda^{-2}\partial_x(u^2)|dx &= \frac{1}{2}\int_{\mathbb{R}} |\int_{\mathbb{R}} e^{-|x-y|}sign(x-y)u^2(t,y)dy|dx \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2(t,y)dy \int_{-\infty}^{\infty} |e^{-|x-y|}sign(x-y)|dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}} u^2(t,y)dy \int_{-\infty}^{\infty} e^{-|x-y|}dx < \infty. \end{aligned} \tag{7.18}$$

Using (7.17) yields

$$\begin{aligned}
|\int_{\mathbb{R}} u\Lambda^{-2}\partial_x(u^2)dx| &\le \Big(\int_{\mathbb{R}} u^2dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}}[\Lambda^{-2}\partial_x(u^2)]^2dx\Big)^{\frac{1}{2}} \\
&\le c\Big(\int_{\mathbb{R}} u^2dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}}|\Lambda^{-2}\partial_x(u^2)|dx\Big)^{\frac{1}{2}} \\
&\le c. \qquad (7.19)
\end{aligned}$$

From (7.16)-(7.19), we complete the proof. ■

7.3 Blow-up criteria

If the maximal existence time $T > 0$ for problem (7.5) is finite and $u_0(x) \in H^2(\mathbb{R})$, from Lemmas 7.2 and 7.3, we get that

$$\sup_{(t,x)\in[0,T)\times\mathbb{R}} |u(t,x)| < \infty$$

and

$$\lim_{t\to T} \| u_x \|_{H^2(\mathbb{R})} = \infty.$$

We will prove that $\sup\limits_{(t,x)\in[0,T)\times\mathbb{R}} |u_x(t,x)| = \infty$, which means that the wave breaking of equation (7.3) occurs. For detailed discussions of wave breaking, the readers are referred to [14].

Theorem 7.6. Let $u_0(x) \in H^2(\mathbb{R})$. If the maximal existence time T is finite, then the solution of problem (7.5) or problem (7.6) blows up if and only if

$$\lim_{t\to T} \inf[u_x(t,x)] = -\infty. \qquad (7.20)$$

proof. It follows from Lemma 7.2 that there exists $u(t,x) \in C([0,T), H^2(\mathbb{R})) \cap C^1([0,T), H^1(\mathbb{R}))$. We shall use the classical energy estimates in our proof. We multiply the first equation of problem (7.6) by $u(t,x)$ and integrate the resultant equation on $\mathbb{R}$ to acquire

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2dx &= -\int_{\mathbb{R}} u^2u_xdx - k\int_{\mathbb{R}} u\Lambda^{-2}u_x - \frac{m-1}{2}\int_{\mathbb{R}} u\Lambda^{-2}(u^2)_xdx \\ &= -\frac{m-1}{2}\int_{\mathbb{R}} u\Lambda^{-2}(u^2)_xdx, \end{aligned} \tag{7.21}$$

in which we have used Lemma 7.4.

We differentiate the first equation of (7.6) about x and get

$$\begin{aligned} u_{tx} + (uu_x)_x &= -k\Lambda^{-2}u_{xx} - \frac{m-1}{2}\Lambda^{-2}(u^2)_{xx} \\ &= -k\Lambda^{-2}(1-\Lambda^2)u - \frac{m-1}{2}\Lambda^{-2}(1-\Lambda^2)u^2 \\ &= -k\Lambda^{-2}u + ku - \frac{m-1}{2}\Lambda^{-2}u^2 + \frac{m-1}{2}u^2. \end{aligned} \tag{7.22}$$

Multiplying Eq.(7.22) by u_x and using Lemma 7.4, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_x^2dx &= -\int_{\mathbb{R}} u_x(uu_x)_x - \frac{m-1}{2}\int_{\mathbb{R}} u_x\Lambda^{-2}u^2dx \\ &= -\frac{1}{2}\int_{\mathbb{R}} u_x^3dx + \frac{m-1}{2}\int_{\mathbb{R}} u\Lambda^{-2}(u^2)_xdx. \end{aligned} \tag{7.23}$$

Differentiating (7.22) again with respect to x, we get

$$\begin{aligned} u_{txx} = -(uu_x)_{xx} - k\Lambda^{-2}u_x + ku_x - \frac{m-1}{2}\Lambda^{-2}(u^2)_x \\ + \frac{m-1}{2}(u^2)_x. \end{aligned} \tag{7.24}$$

Using Lemma 7.4, we have

$$\int_{\mathbb{R}} u_{xx}\Lambda^{-2}u_xdx = -\int_{\mathbb{R}} u_x\Lambda^{-2}u_{xx}dx = -\int_{\mathbb{R}} u_x(\Lambda^{-2}u - u)dx = 0 \tag{7.25}$$

and

$$\begin{aligned}\int_{\mathbb{R}} u_{xx}\Lambda^{-2}(u^2)_x dx &= -\int_{\mathbb{R}} u_x\Lambda^{-2}(u^2)_{xx} dx \\ &= -\int_{\mathbb{R}} u_x(\Lambda^{-2}u^2 - u^2)dx = -\int_{\mathbb{R}} u_x(\Lambda^{-2}u^2)dx. \end{aligned} \tag{7.26}$$

We multiply (7.24) by u_{xx}, integrate the obtained equation over $\mathbb{R}$ and get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_{xx}^2 dx &= -\int_{\mathbb{R}} u_{xx}(uu_x)_{xx}dx - k\int_{\mathbb{R}} u_{xx}\Lambda^{-2}u_x dx \\ &\quad -\frac{m-1}{2}\int_{\mathbb{R}} u_{xx}\Lambda^{-2}(u^2)_x + \frac{m-1}{2}\int_{\mathbb{R}} u_{xx}(u^2)_x dx \\ &= -\frac{5}{2}\int_{\mathbb{R}} u_x u_{xx}^2 dx - \frac{m-1}{2}\int_{\mathbb{R}} u_x^3 dx + \frac{m-1}{2}\int_{\mathbb{R}} u_x\Lambda^{-2}u^2 dx \\ &= -\frac{5}{2}\int_{\mathbb{R}} u_x u_{xx}^2 dx - \frac{m-1}{2}\int_{\mathbb{R}} u_x^3 dx - \frac{m-1}{2}\int_{\mathbb{R}} u\Lambda^{-2}(u^2)_x dx. \end{aligned} \tag{7.27}$$

From (7.21), (7.23) and (7.27), we have

$$\begin{aligned}&\frac{1}{2}\left[\frac{d}{dt}\int_{\mathbb{R}} u^2 dx + \frac{d}{dt}\int_{\mathbb{R}} u_x^2 dx + \frac{d}{dt}\int_{\mathbb{R}} u_{xx}^2 dx\right] \\ &\quad = -\frac{5}{2}\int_{\mathbb{R}} u_x u_{xx}^2 dx - \frac{m}{2}\int_{\mathbb{R}} u_x^3 dx - \frac{m-1}{2}\int_{\mathbb{R}} u\Lambda^{-2}(u^2)_x dx. \end{aligned} \tag{7.28}$$

Assume that we can choose a constant $C > 0$ to satisfy

$$u_x(t,x) \geq -C, \quad t \in [0,T), \quad x \in \mathbb{R}. \tag{7.29}$$

From (7.28) and Lemma 7.5, we have

$$\begin{aligned}&\frac{1}{2}\left[\frac{d}{dt}\int_{\mathbb{R}} u^2 dx + \frac{d}{dt}\int_{\mathbb{R}} u_x^2 dx + \frac{d}{dt}\int_{\mathbb{R}} u_{xx}^2 dx\right] \\ &\quad \leq \max(\frac{5C}{2}, \frac{mC}{2})\Big(\int_{\mathbb{R}} u^2 dx + \int_{\mathbb{R}} u_x^2 dx + \int_{\mathbb{R}} u_{xx}^2 dx\Big) + c. \end{aligned} \tag{7.30}$$

Letting

$$E(t) = \int_{\mathbb{R}} \left(u^2 + u_x^2 + u_{xx}^2\right) dx \tag{7.31}$$

and using (7.30), we get

$$E(t) \leq \max(5C, mC) \int_0^t E(\tau) d\tau + 2ct + E(0), \tag{7.32}$$

from which together with the Gronwall inequality, we have

$$E(t) \leq (2ct + E(0)) e^{\max(5C, mC)t}, \tag{7.33}$$

which derives $u(t, x) \in H^2(\mathbb{R})$. Since $L^\infty(\mathbb{R}) \in H^1(\mathbb{R})$, we get $u_x(t, x) \in L^\infty(\mathbb{R})$. Thus we have shown that the bound of $u_x(t, x)$ from below for $(t, x) \in [0, T) \times \mathbb{R}$, which contradicts the assumption of the theorem. The proof is completed. ∎

Theorem 7.7. Assume $u_0(x) \in H^s(\mathbb{R})$ with $s \geq 2$. If $\lim\limits_{t \to T} \| u \|_{H^s(\mathbb{R})} = \infty$ and $T < \infty$, then

$$\int_0^T \| u_x(\tau, x) \|_{L^\infty} d\tau = \infty. \tag{7.34}$$

Proof. We write

$$u_t - u_{txx} + ku_x + muu_x = (uu_x)_{xx} = (1 - \Lambda^2)(uu_x). \tag{7.35}$$

Applying $(\Lambda^{s-1}u)\Lambda^{s-1}$ to both sides of (7.35) and using integration by parts, we derive the inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}[(\Lambda^{s-1}u)^2+(\Lambda^{s-1}u_x)^2]dx \\
&\qquad \le |k|\Big|\int_{\mathbb{R}}(\Lambda^{s-1}u)\Lambda^{s-1}(u_x)dx\Big| + |m-1|\Big|\int_{\mathbb{R}}(\Lambda^{s-1}u)\Lambda^{s-1}(uu_x)dx\Big| \\
&\qquad\quad +\Big|\int_{\mathbb{R}}\Lambda^{s-1}u\Lambda^{s+1}(uu_x)dx\Big| \\
&\le c\Bigg(\Big|\int_{\mathbb{R}}(\Lambda^{s-1}u)\Lambda^{s-1}(u_x)dx\Big| + \Big|\int_{\mathbb{R}}(\Lambda^{s-1}u)\Lambda^{s-1}(uu_x)dx\Big| \\
&\qquad\qquad +\Big|\int_{\mathbb{R}}\Lambda^{s}u\Lambda^{s}(uu_x)dx\Big|\Bigg) \\
&\qquad = c(K_1+K_2+K_3),
\end{aligned}
\tag{7.36}
$$

where c depends on k and m.

Using the Cauchy-Schwartz inequality results in the inequality

$$
|K_1| \le c \parallel u \parallel_{H^{s-1}} \parallel u \parallel_{H^s} . \tag{7.37}
$$

Applying Lemmas 4.10 and 4.11, we get

$$
\begin{aligned}
\Big|\int_{\mathbb{R}}(\Lambda^s u)\Lambda^s(uu_x)dx\Big| &= \Big|\int_{\mathbb{R}}(\Lambda^s u)[\Lambda^s(uu_x)-u\Lambda^s u_x]dx \\
&\qquad + \int_{\mathbb{R}}(\Lambda^s u)u\Lambda^s u_x dx\Big| \\
&\le c \parallel u \parallel_{H^s}\Big(\parallel u \parallel_{H^{s-1}}\parallel u_x \parallel_{L^\infty} + \parallel u \parallel_{H^s}\parallel u \parallel_{L^\infty}\Big) + \\
&\qquad + \parallel u_x \parallel_{L^\infty}\parallel \Lambda^s u \parallel_{L^2} \\
&\le c\Big(\parallel u \parallel_{L^\infty} + \parallel u_x \parallel_{L^\infty}\Big)\parallel u \parallel_{H^s}^2,
\end{aligned}
\tag{7.38}
$$

which results in

$$
|K_3| \le c\Big(\parallel u \parallel_{L^\infty} + \parallel u_x \parallel_{L^\infty}\Big)\parallel u \parallel_{H^s}^2 . \tag{7.39}
$$

Similarly, we get

$$|K_2| \leq c \parallel u \parallel_{H^{s-1}}^2 \Big(\parallel u \parallel_{L^\infty} + \parallel u_x \parallel_{L^\infty} \Big). \tag{7.40}$$

It follows from (7.37), (7.39) and (7.40) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}[(\Lambda^{s-1}u)^2 + (\Lambda^{s-1}u_x)^2]dx \\ &\quad \leq c(\parallel u \parallel_{H^{s-1}(\mathbb{R})}^2 + \parallel u \parallel_{H^s(\mathbb{R})}^2) \\ &\qquad \times \Big(1 + \parallel u \parallel_{L^\infty(\mathbb{R})} + \parallel u_x \parallel_{L^\infty(\mathbb{R})} \Big), \end{aligned} \tag{7.41}$$

where $c > 0$ is a constant. From (7.41), we get

$$\parallel u \parallel_{H^s} \leq c \parallel u_0 \parallel_{H^s} e^{\int_0^t (1 + \parallel u \parallel_{L^\infty} + \parallel u_x \parallel_{L^\infty})} dt. \tag{7.42}$$

If $\lim\limits_{t\to T} \parallel u \parallel_{H^s(\mathbb{R})} = \infty$ and $T < \infty$, using (7.42), we obtain

$$\int_0^T \parallel u_x(\tau, x) \parallel_{L^\infty} d\tau = \infty. \tag{7.43}$$

The proof is completed. ■

CHAPTER 8

Summary and future research

8.1 Summary

In this paper, a generalized Benjamin-Bona-Mahony-Burgers equation(GBBMB), a generalized Degasperis-Procesi equation, and three integrable non-evolutionary equations with quadratic, cubic and quartic nonlinearities, respectively, have been investigated. Various dynamical properties of the five nonlinear equations have been derived. Specifically, we obtain the following results.

(1). For a generalized Benjamin-Bona-Mahony-Burgers equation with third order spatial derivative, which takes the form

$$u_t - u_{txx} - au_{xx} + bu_x + u^p u_x + ku_{xxx} = 0, \quad a \geq 0,$$

we prove that for any $T > 0$, $\parallel u \parallel_{L^\infty(\mathbb{R})} \leq c \parallel u_0 \parallel_{H^1(\mathbb{R})}$, and derive the one-sided L^∞ estimate on the first order spatial derivative $\frac{\partial u(t,x)}{\partial x} \leq c_0(1 + e^{-at})$ and the estimate $\int_0^t \int_{-\infty}^{\infty} \left| \frac{\partial u(t,x)}{\partial x} \right|^4 dxdt \leq c_1 T e^{c_1 T}, \quad t \in [0, T]$, where c_0 and c_1 only depend on the coefficients a, b, p, k and the norm $\parallel u_0 \parallel_{H^1(\mathbb{R})}$. Making use of the method in Xin and Zhang [101], we prove that the GBBMB model has global weak solutions in $C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$. If the GBBMB equation exists strong solution and its initial data satisfy certain assumptions, the $L^1(\mathbb{R})$ local stability of solutions to the GBBMB equation is established by using Kruzkov's techniques of doubling the space variables.

(2). For an integrable non-evolutionary partial differential equation with quadratic nonlinearities and quasi-local higher symmetries, using the Kato theorem, we prove the existence and uniqueness of local strong solutions to the equation in the space $C([0,T);H^s(\mathbb{R}))\cap C^1([0,T);H^{s-1}(\mathbb{R}))$ where $s>\frac{3}{2}$. The blow-up condition is given under certain assumptions. If $1\le s\le \frac{3}{2}$, using Aubin's compactness theorem and various estimates of the solutions, we prove the existence of local weak solutions in $H^s(\mathbb{R})$.

(3). For an integrable non-evolutionary partial differential equation with cubic nonlinearities and quasi-local higher symmetries, imposing certain restrictions on the initial value and using Aubin's compactness theorem and various estimates of the solutions, we prove that the equation possesses local weak solutions in $L^2([0,T],H^s(\mathbb{R})), 1<s\le\frac{3}{2}$ in the sense of distribution. Constructing a Cauchy sequence of the solutions in $C([0,T);H^s(\mathbb{R}))\cap C^1([0,T);H^{s-1}(\mathbb{R}))$ with $s>\frac{3}{2}$, we show that the equation has a unique local strong solution.

(4). The existence of global weak solutions to the Cauchy problem for a nonlinear Camassa-Holm type equation with quartic nonlinearities is proved in $C([0,\infty)\times\mathbb{R})\cap L^\infty([0,\infty);H^1(R))$ by assuming that initial value satisfy $u_0(x)\in H^1(R)$ and $\| \frac{\partial u_0(x)}{\partial x} \|_{L^\infty(R)}<\infty$. We do not assume that the initial value satisfies the sign condition. Namely, our assumption is weaker than the sign condition. The key contributions in this chapter include establishing a space-time higher integrability estimate and a super bound estimate on the first order spatial derivative of the solution.

(5). For a generalized Degasperis-Procesi equation(GDP), we derive conditions for which the local strong solutions of the GDP equation blow up in finite time. The wave breaking of solutions to the equation is investigated.

8.2 Future research

For the GBBMB equation discussed in chapter 3, we only prove the existence of global weak solutions and the L^1 local stability if the initial data satisfy certain conditions. However, we do not prove the uniqueness of global weak solutions and the uniformly $L^p(\mathbb{R}) \quad (1 \leq p < \infty)$ stability of the solution. These problems remain to be investigated in our future works.

For the two integrable nonlinear partial differential equations we investigated in Chapters 4 and 5, we only establish the well-posedness of local strong solutions in $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ $(s > \frac{3}{2})$. Establishing conditions on the initial data to ensure the existence of global strong solution for the two equations in $C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))(s > \frac{3}{2})$ is a challenging problem. We shall consider this problem by adding certain assumptions on the initial value in future studies.

For the nonlinear Camassa-Holm-type equation with quartic nonlinearities in Chapter 6, we only prove the existence of global weak solutions in $C([0,\infty) \times \mathbb{R}) \cap L^\infty([0,\infty); H^1(\mathbb{R}))$. The uniqueness of the global weak solutions remains unexplored. Establishing conditions to ensure the uniqueness of the global weak solutions could be considered in further studies.

For the generalized Degasperis-Procesi equation discussed in Chapter 7, other blow-up criteria need to be found to further study the equation. Establishing suitable conditions to ensure the existence of the global solution in the Sobolev space also constitutes our future research.

Bibliography

[1] A.S. AbdelRady, E.S. Osman, M. Khalfallah: The homogeneous balance method and its application to the BBM equation. Appl. Math. Compt. **217**, 1385-1390 (2010).

[2] J. Angulo, M. Scialom, C. Banquet: The regularized Benjamin-one and BBM equations: Well-posedness and nonlinear stability. J. Differ. Equ. **251**, 4011-4036 (2011).

[3] A. Bressan, A. Constantin: Global conservative solutions of the Camassa-Holm equation. Arch. Rat. Mech. Anal. **183**, 215-239 (2007).

[4] A. Bressan, A. Constantin: Global dissipative solutions of the Camassa-Holm equation. Anal. Appl. (Singap). **5**, 1-27 (2007).

[5] T.B. Benjamin, J.L. Bona, J.J. Mahony: Model equation for long waves in nonlinear dispersive system. Philos. Trans. Roy. Soc. London Ser A. **272**, 47-78 (1972).

[6] C. Burtea: Long time existence results for bore-type initial data for BBM-Boussinesq systems. J. Differ. Equ. **261**, 4825-4860 (2016).

[7] J.L. Bona, R. Smith: The initial value problem for the Korteweg-de-Vries equation. Philos. Trans. Royal Soc. London Ser A **278**, 555-601 (1975).

[8] H.Q. Chen, W.J. Wang: Stability of the solitary wave solutions to a coupled BBM equations. J. Differ. Equ. **261**, 1604-1621 (2016).

[9] G.M. Coclite, H. Holden, K.H. Karlsen: Global weak solutions to a generalized hyperelastic-rod wave equation. SIAM J. Math. Anal. **37**, 1044-1069 (2005).

[10] G.M. Coclite, H. Holden, K.H. Karlsen: Well-posedness for a parabolic-elliptic system. Disc. Cont. Dyn. Syst. **13**, 659-682 (2005).

[11] G.M. Coclite, K.H. Karlsen, On the well-posedness of the Degasperis-Procesi equation. J. Funct. Anal. **223**, 60-91 (2006).

[12] A. Constantin, L. Molinet: Global weak solutions for a shallow water equation. Commun. Math. Phys. **211**, 45-61 (2000).

[13] A. Constantin, D. Lannes: The hydro-dynamical relevance of the Camassa-Holm and Degasperis-Procesi equations. Arch. Rat. Mech. Anal. **193**, 165-186 (2009).

[14] A. Constantin, J. Escher: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. **181**, 229-243 (1998).

[15] A. Constantin, W. Strauss: Stability of peakons. Comm. Pure Appl. Math. **53**, 603-610 (2000).

[16] A. Constantin, H.P. McKean: A shallow water equation on the circle. Comm. Pure Appl. Math. **52**, 949-982 (1999).

[17] A. Constantin, R.I. Ivanov: Dressing method for the Degasperis-Procesi equation. Stud. Appl. Math. **138**, 205-226 (2017).

[18] A. Constantin, T. Kappeler, B. Kolev, P. Topalov: On geodesic exponential maps of the Virasoro group. Ann. Global Anal. Geom. **31**, 155-180 (2007).

[19] R. Camassa, D. Holm: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661-1664 (1993).

[20] J. Chen: The stability of solutions for the generalized Degasperis-Procesi equation with variable coefficients. Math. Probl. Eng. Article ID 207427,8 pages. (2015).

[21] J. Chen, R. Li: The local well-posedness and stability to a nonlinear generalized Degasperis-Procesi equation. Bound. Value Probl. 2015:170, (2015).

[22] A. Degasperis, D. Holm, A. Hone: A new integral equation with peakon solutions. Theor. Math. Phys. **133**, 1461-1472 (2002).

[23] A. Degasperis, M. Procesi: Asymptotic integrability, in: Symmetry and Perturbation Theory (A. Degasperis and G. Gaeta, eds.). World Scientific, Singapore, 23-37 (1999).

[24] H.R. Dullin, G.A. Gottwald, D.D. Holm: Camassa-Holm, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves. Fluid Dynam. Research. **33**, 73-79 (2003).

[25] J. Escher, Y. Liu, Z.Y. Yin: Global weak solutions and blow-up structure for the Degasperis-Procesi equation. J. Func. Anal. **241**, 457-485 (2006).

[26] J. Escher, Z.Y. Yin: Initial boundary value problems for nonlinear dispersive wave equations. J. Funct. Anal. **256**, 479-508 (2009).

[27] J. Escher, Z.Y. Yin: Well-posedness, blow-up phenomena and global solutions for the b-equation. J. Reine Angew. Math. **624**, 51-80 (2008).

[28] Y. Fu, Y. Liu, C.Z. Qu: On the blow-up structure for the generalized periodic Camassa-Holm and Degasperis-Procesi equation. J. Func. Anal. **262**, 3125-3158 (2012).

[29] Y. Fu, C.Z. Qu: Well-posedness and wave breaking of the degenerate Novikov equation. J. Differ. Equa. **263**, 4634-4657 (2017).

[30] B. Fuchssteiner, A.S. Fokas: Symplectic structures, their Backlund transformations and hereditary symmetries. Phys. D. **4**, 821-831 (1981).

[31] Y.P. Fu, B.L. Guo: Time periodic solution of the viscous Camassa-Holm equation. J. Math. Anal. Appl. **313**, 311-321 (2006).

[32] Y. Gao, L. Li, J.G. Liu: A dispersive regularization for the modified Camassa-Holm equation. SIAM J. Math. Anal. **50**, 2807-2838 (2018).

[33] X.J. Gao, S.Y. Lai, H.J. Chen: The stability of solutions for the Fornberg-Whitham equation in $L^1(\mathbb{R})$ space. Bound. Value Probl. 2018:142, (2018).

[34] B.L. Guo, Z.R. Liu: Two new types of bounded waves of CH-γ equations. Science in China Ser. A **48**, 1618-1630 (2005).

[35] Y.X. Guo, Y.H. Wu, S.Y. Lai, L. Caccetta: The global weak solution for the shallow water wave model of moderate amplitude. Appl. Anal. **96**, 663-678 (2017).

[36] Y.X. Guo, Y.H. Wu, S.Y. Lai: On weak solutions to a shallow water wave model of moderate amplitude. Appl. Anal. **95**, 1806-1829 (2016).

[37] Y.X. Guo, S.Y. Lai: On the Cauchy problem for a periodic rotation-two-component Hunter-Saxton system. Appl. Anal. **98**, 2218-2238 (2019).

[38] Y.X. Guo, Y.H. Wu, S.Y. Lai: On existence and uniqueness of the global weak solution for a shallow water equation. Appl. Math. Comput. **218**, 11410-11420 (2012).

[39] Z.G. Guo: On an integrable Camassa-Holm type equation with cubic nonlinearity. Nonl. Anal.: Real Worl. Appl. **34**, 225-232 (2017).

[40] Z.G. Guo, X.G. Li, C. Yu: Some properties of solutions to the Camassa-Holm-type equation with higher order nonlinearities. J. Nonl. Sci. **28**, 1901-1914 (2018).

[41] K. Grayshan, A.A. Himonas: Equations with peakon traveling wave solutions. Adv. Dyn. Sys. Appl. **8**, 217-232 (2013).

[42] O. Glass: Controllability and asymptotic stabilization of the Camassa-Holm equation. J. Differ. Equ. **245**, 1584-1615 (2008).

[43] J. Holmes, R.C. Thompson: Well-posedness and continuity properties of the Fornberg-Whitham equation in Besov spaces. J. Differ. Equ. **263**, 4355-4381 (2017).

[44] S.V.Haziot: Wave breaking for the Fornberg-Whitham equation. J. Differ. Equ. **263**, 8178-8185 (2017).

[45] D.D. Holm, M.F. Staley: Wave structure and nonlinear balances in a family of evolutionary PDEs. SIAM J. Appl. Dyn. Syst. **2**, 323-380 (2003).

[46] D. Henry: Infinite propagation speed for the Degasperis-Procesi equation. J. Math. Anal. Appl. **311**, 755-759 (2005).

[47] A. Himonas, C. Holliman: The Cauchy problem for the Novikov equation. Nonlinearity. **25**, 449-479 (2012).

[48] A. Himonas, D. Mantzavinos: A b-family equation with peakon travelling waves. Proc. Amer. Math. Soci. **144**, 3797-3811 (2016).

[49] A. Himonas, C. Holliman, C. Kenig: Construction of 2-peakon solutions and ill-posedness for the Novikov equation. SIAM J. Math. Anal. **50**, 2968-3006 (2018).

[50] S. Hakkaev, K. Kirchev: Local well-posedness and orbital stability of solitary wave solutions for the generalized Camassa-Holm equation. Comm. PDE. **30**, 761-781 (2005).

[51] M.A. Johnson: On the stability of periodic solutions of the generalized Benjamin-Bona-Mahony equation. Physica D **239**, 1892-1908 (2010).

[52] R.S. Johnson: On solutions of the Camassa-Holm equation. Proc. R. Soc. Lon. **459**, 1687-1708 (2003).

[53] S.N. Kružkov: First order quaslinear equations in several independent variables. Math. USSR Sbornik **10**, 217-243 (1970).

[54] T. Kato, G. Ponce: Commutator estimates and the Euler and Navier-Stokes equations. Comm. Pure Appl. Math. **41**, 891-907 (1998).

[55] T. Kato, Quasi-linear equations of evolution with applications to partial differential equations, Spectral Theory and Differential Equations, Lecture Notes in Math. Springer Verlag Berlin. **448**, 25-70 (1975).

[56] J.J. Kang, Y.T. Guo, Y.B. Tang: Local well-posedness of generalized BBM equations with generalized damping on $1D$ torus. Bound. Value Probl. 2015:227, (2015).

[57] S. Kouranbaeva: The Camassa-Holm equation as a geodesic flow on the diffeomorphism group. J. Math. Phys. **40**, 857-868 (1999).

[58] S.Y. Lai, Y.H. Wu: The existence of global strong and weak solutions for the Novikov equation. J. Math. Anal. Appl. **399**, 682-691 (2013).

[59] S.Y. Lai, Y.H. Wu: The local well-posedness and existence of weak solutions for a generalized Camassa-Holm equation. J. Differ. Equ. **248**, 2038-2063 (2010).

[60] S.Y. Lai, Y.H. Wu: Global solutions and blow-up phenomena to a shallow water equation. J. Differ. Equ. **249**, 693-706 (2010).

[61] S.Y. Lai, H.B. Yan, H.J. Chen, Y. Wang, The stability of local strong solutions for a shallow water equation, J. Inequal. Appl. 2014:410, (2014).

[62] M.G. Li, Q.T. Zhang: Generic regularity of conservative solutions to Camassa-Holm-type equations. SIAM J. Math. Anal. **49**, 2920-2949 (2017).

[63] J. Lenells: Traveling wave solutions of the Camassa-Holm equation. J. Differ. Equ. **217**, 393-430 (2005).

[64] J. Lenells: Traveling wave solutions of the Degasperis-Procesi equation. J. Math. Anal. Appl. **306**, 72-82 (2005).

[65] J. Lenells: Conservation laws of the Camassa-Holm equation. J. Phys. A. **38**, 869-880 (2005).

[66] Y. Li, P. Olver: Well-posedness and blow-up solutions for an integ nonlinearly dispersive model wave equation. J. Differ. Equ. **162**, 27-63 (2000).

[67] L.C. Li: Long time behaviour for a class of low-regularity solutions of the Camassa-Holm equation. Commun. Math. Phys. **285**, 265-291 (2009).

[68] Z.W. Lin, Y. Liu: Stability of peakons for the Degasperis-Procesi equation. Comm. Pure. Appl. Math. **62**, 125-146 (2009).

[69] Y. Liu: Global existence and blow-up solutions for a nonlinear shallow water equation. Math. Ann. **335**, 717-735 (2006).

[70] Y. Liu, Z.Y. Yin: Global existence and blow-up phenomena for the Degasperis-Procesi equation. Commun. Math. Phys. **267**, 801-820 (2006).

[71] H. Lundmark, J. Szmigielski: Multi-peakon solutions of the Degasperis-Procesi equation. Inverse Prob. **19**, 1241-1245 (2003).

[72] Y. Matsuno: Multisoliton solutions of the Degasperis-Procesi equation and their peakon limit. Inverse Prob. **21**, 1553-1570 (2005).

[73] H.P. McKean: Fredholm determinants and the Camassa-Holm hierarchy. Comm. Pure Appl. Math. **56**, 638-680 (2003).

[74] G.A. Misiolek: Shallow water equation as a geodesic flow on the Bott-Virasoro group. J. Geom. Phys. **24**, 203-208 (1998).

[75] L. Molinet: A Liouville property with application to asymptotic stability for the Camassa-Holm equation. Arch. Rat. Mech. Anal. **230**, 185-230 (2018).

[76] Y.S. Mi, Y. Liu, B.L. Guo, T. Luo: The Cauchy problem for a generalized Camassa-Holm equation. J. Differ. Equ. **266**, 6739-6770 (2019).

[77] S. Ming, S.Y. Lai, Y.Q. Su: The Cauchy problem of a weakly dissipative shallow water equation. Appl. Anal. **98**, 1387-1402 (2019).

[78] S. Ming, H. Yang, Z.L. Chen: The global stabilization of the CH equation with a distributed feedback control. Bound. Value Probl. 2016:67, (2016).

[79] S. Ming, H. Yang, Z.L. Chen, S.Y. Lai: The properties of solutions to the dissipative 2-component Camassa-Holm system. Appl. Anal. **95**, 1165-1183, (2016).

[80] S. Ming, H. Yang, Z.L. Chen: The Cauchy problem for the liquid crystals system in the critical Besov space with negative index. Czechoslovak Math. J. **67**, 37-55, (2017).

[81] S. Ming, S.Y. Lai, Y.Q. Su: The optimal control problem with necessary condition for a viscous shallow water equation. Bound. Value Probl. 2018:71, (2018).

[82] O.G. Mustafa: A note on the Degasperis-Procesi equation. J. Nonlinear Math. Phys. **12**, 10-14 (2005).

[83] O.G. Mustafa: Existence and uniqueness of low regularity solutions for the Dullin-Gottwald-Holm equation. Commun. Math. Phys. **265**, 189-200 (2006).

[84] M. Mei: L_q decay rates of solutions for Benjamin-Bona-Mahony-Burgers Equations. J. Differ. Equ. **158**, 314-340 (1999).

[85] Y.S. Mi, C.L. Mu: On the Cauchy problem for the modified Novikov equation with peakon solutions. J. Differ. Equ. **254**, 961-982 (2013).

[86] L. Ni, Y. Zhou: Well-posedness and persistence properties for the Novikov equation. J. Differ. Equ. **250**, 3002-3021 (2011).

[87] V. Novikov: Generalizations of the Camassa-Holm equation. J. Phys. A: Math. Theor. **42**, 342002 (2009).

[88] E.A. Olson: Well posedness for a higher order modified Camassa-Holm equation. J. Differ. Equ. **246**, 4154-4172 (2009).

[89] F.G. Omer, A. Samil: Exact solutions of Benjamin-Bona-Mahony-Burgers-type nonlinear pseudo-parabolic equations. Bound. Value Probl. **2012**, 2012:144, (2012).

[90] D.H. Peregrine: Calculations of the developmrnt of an undular bore. J. Fluid Mech. **25**, 321-330 (1966).

[91] G. Rodriguez-Blanco: On the Cauchy problem for the Camassa-Holm equation. Nonl. Anal. **46**, 309-327 (2001).

[92] L. Rosier, B.Y. Zhang: Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain. J. Differ. Equ. **254**, 141-178 (2013).

[93] J.C. Saut, N. Tzetkov: Global well-posedness for the KP-BBM equations. Appl. Math. Res. Express **1**(1), 1-6 (2004).

[94] J. Simon: Compact sets in the space $L^p((0,T),B)$. Ann. Mat. Pure Appl. **146**, 65-96 (1987).

[95] X.Y. Tu, Y. Liu, C.L. Mu: Existence and uniqueness of the global conservative weak solutions to the rotation Camassa-Holm equation. J. Differ. Equa. **266**, 4864-4900 (2019).

[96] V.O. Vakhnenko, E.J. Parkes: Periodic and solitary-wave solutions of the Degasperis-Procesi equation. Chaos Solitons Fractals. **20**, 1059-1073 (2004).

[97] W. Walter: Differential and Integral Inequalities. Springer-Verlag, New York, (1970).

[98] A.M. Wazwaz: A class of nonlinear fourth order variant of a generalized Camassa-Holm equation with compact and noncompact solutions. Appl. Math. Comput. **165**, 485-501 (2005).

[99] S.Y. Wu, Z.Y. Yin: Global existence and blow-up phenomena for the weakly dissipative Camassa-Holm equation. J. Differ. Equ. **246**, 4309-4321 (2009).

[100] X. Wu: On the finite time singularities for a class of Degasperis-Procesi equations. Nonl. Anal. **44**, 1-17 (2018)

[101] Z. Xin, P. Zhang: On the weak solutions to a shallow water equation. Comm. Pure Appl. Math. **53**, 1411-1433 (2000).

[102] Z. Xin, P. Zhang: On the uniqueness and large time behavior of the weak solutions to a shallow water equation. Comm. PDE. **27**, 1815-1844 (2002).

[103] Z.Y. Yin: On the blow-up scenario for the generalized Camassa-Holm equation. Comm. PDE. **29**, 867-877 (2004).

[104] Z.Y. Yin: On the Cauchy problem for an intergrable equation with peakon solutions. Illinois J. of Math. **47**, 649-666 (2003).

[105] Z.Y. Yin: Global weak solutions for a new periodic integrable equation with peakon solutions. J. Funct. Anal. **212**, 182-194 (2004).

[106] W. Yan, Y.S. Li, Y,M. Zhang: The Cauchy problem for the integrable Novikov equation. J. Differ. Equ. **253**, 298-318 (2002).

[107] W. Yan, Y.S. Li, Y.M. Zhang: Global existence and blow-up phenomena for the weakly dissipative Novikov equation. Nonl. Anal. **75**, 2464-2473 (2012).

[108] L. Zhang, B. Liu: On the Cauchy problem for a class of shallow water wave equations with $(k+1)$-order nonlinearities. J. Math. Anal. Appl. **445**, 151-185 (2017).

Appendix 1. Statement of Candidate's Contributions to Joint-authored Publications

To Whom It May Concern,

I, Rui Li, made major contributions in the design of the research, development of the theories, derivation of the results and drafting of the paper entitled 'Global weak solutions to a generalized Benjamin-Bona-Mahony-Burgers equation', Acta Mathematica Scientia, vol.38(3), 915-925, 2018.

Rui Li

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Chong Lai

Yonghong Wu

Appendix 2. Statement of Candidate's Contributions to Joint-authored publications

To Whom It May Concern,

I, Rui Li, made major contributions in the design of the research, development of the theories, derivation of the results and drafting of the paper entitled 'The L^1 stability to a generalized Benjamin-Bona-Mahony-Burgers model', Dynamics of Continuous, Discrete and Impulsive Systems Series B, vol.26(2), 123-132, 2019.

Rui Li

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Chong Lai

Kexin Luo