Algorithms and Multiplicative Thinking: Are Children Prisoners of Process?

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Multiplicative thinking is a critical component of mathematics which largely determines the extent to which people develop mathematical understanding beyond middle primary years. We contend that there are several major issues, one being that much teaching about multiplicative ideas is focussed on algorithms and procedures. An associated issue is the extent to which algorithms are taught without the necessary explicit connections to key mathematical ideas. This article explores the extent to which some primary students use the algorithm as a preferred choice of method and whether they can recognise and use alternative ways of calculating answers. We also consider the extent to which the students understand ideas that underpin algorithms. Our findings suggest that most students in the sample are 'prisoners to procedures and processes' irrespective of whether or not they understand the mathematics behind the algorithms.

Introduction

Multiplicative thinking is acknowledged as a key component of mathematics and underpins much of the mathematics needed beyond middle primary or elementary years. Researchers have found that children who do not develop multiplicative thinking are unable to understand higher level concepts such as fractions, proportional reasoning, and algebra (Clark & Kamii, 1996; Siemon, Breed, Dole, Izard, & Virgona, 2006). Hence it is important to understand what constitutes multiplicative thinking and to identify its key elements which can then be developed in a connected way across all school years.

Siemon et al. (2006) defined multiplicative thinking in the following terms:

- a capacity to work *flexibly* and efficiently with an extended range of numbers (e.g.,
- larger whole numbers, decimals, common fractions, ratio and percent)
- an ability to recognise and solve a *range* of problems involving multiplication or division including direct and indirect proportion
- the means to communicate this effectively in a *variety* of ways (e.g., materials, words, diagrams, symbolic expressions and written algorithms).

We have taken the liberty of highlighting three words in the above definition as they are particularly important to this paper. In order to work 'flexibly' with a 'range' of problems and communicate in a 'variety' of ways, we assert that there must be explicit teaching of the many connections within the broad idea of multiplicative thinking. Indeed, Anghileri (2006, p. 2) suggested that learning and practising procedures is not appropriate and that we must "focus on the links that demonstrate the logical structure underlying numbers and number operations". Anghileri (2000, p. 25) also noted that "children tend to use algorithms as 'mechanical' procedures [and] where they do not understand the procedures, they are unable to reconstruct the processes involved". She cited the work of Ruthven and Chaplin (1998) who refer to 'the improvisation of malgorithms' to describe children's inappropriate adaptations of the procedures.

In keeping with these ideas, Askew (2016) added a word of caution that models for multiplication like the empty number line and the array need to be carefully developed and not introduced as 'a topic' and discarded. Rather, if students use such models as 'tools for thinking', "learners working with more formal column methods of calculation will have a strong number sense [which] will support thinking about what to do" (Askew, 2016, p. 146). Anghileri (2006, p. 109) had some specific concerns about the use of formal written algorithms which she describes as focusing on the digits rather than on the values of the numbers which tends to lead to a mechanical approach devoid of real understanding. Anghileri also suggested that errors are more likely to result if children do not understand what they are doing and why they are doing it. She also notes that "Children make inappropriate adaptations of methods they learn as the try to reconstruct a procedure they can't remember or don't understand" (Anghileri, 2006, p. 109).

It seems that there is a strong case for the teaching of algorithms based on conceptual understanding rather than as a procedure. Our view is that multiplicative thinking is much more than the teaching of algorithms and should be about the structure of the mathematics that underpins processes and procedures.

In this article, we explore a number of issues related to primary students' understanding and use of algorithms, procedures, and rules.

- The extent to which primary school students unnecessarily use written algorithms as a preferred method of computation.
- The level of understanding of algorithms and procedures shown by primary students.
- Teacher understanding of algorithms and the mathematics that underpins them.

Background

Multiplicative Thinking and Procedural Teaching

We suggest that the teaching and learning of traditional algorithms for multiplication as a procedure is not necessarily supported by the explicit teaching of the mathematics that underpins the use of such algorithms. The discussion around the learning of procedures, or instrumental understanding, at the expense of relational understanding, is not new and can be traced at least to the work of Skemp. He described instrumental understanding as 'rules without reasons' which are generally devoid of understanding in a conceptual way, yet the learning of mathematics in this way is often equated with true understanding by its proponents (Skemp, 1976). In seeking to understand why this is so, Skemp referred to one of the reasons for the preponderance of instrumental teaching as being 'the backwash of examinations'. It seems reasonable to consider this in broader terms to include standardised and national testing, given the prevalence of it in the guise of the National Assessment of Educational Progress (NAEP) in the USA, National Assessment Plan Literacy and Numeracy (NAPLAN) in Australia, and what was previously called the Scholastic Aptitude Test (SAT) in the United Kingdom.

Skemp's comments about instrumental or procedural teaching were supported by Ross (2002, p. 420) in citing earlier work by Kamii (1989) about children in their first years of schooling demonstrating "significant gains in conceptual understanding of place value by . . . the learning of multidigit concepts and procedures as a conceptual problem-solving activity rather than as the transmission of established rules and procedures". More recently, Hartnett (2015) described a study into the use of methods other than traditional algorithms and noted that it was difficult to entice students to 'give up' the methods they had already learned. Hartnett (2015, p. 288) suggested that "One possible reason was that they were successful with the algorithms and predicted they would not be as successful with something that was new and different".

Further to this, and in support of the earlier work of Kamii (as cited by Ross, 2002), Pearn (2009) suggested that many children experience considerable difficulty in learning traditional algorithms. This may well be because they are taught as a procedure rather than from a conceptual standpoint which builds on the multiplicative property of place value, and the distributive property of multiplication, which in turn is based on place value partitioning. The importance of developing students' conceptual understanding is underlined by Young-Loveridge and Mills (2009) who suggested that contemporary mathematics instruction is aimed at helping students understand mathematical structure. It would follow that such understanding necessarily entails knowing the connections between ideas and the language associated with those ideas and connections. They went on to cite Mulligan and colleagues whose work suggests that the difference between high and low achievers in mathematics is related to the extent to which students understand mathematical structure and pattern. They also cited Baek (1998) about the central importance of students understanding the concept of multiplication. It is important to appreciate that their emphasis is on understanding, and not learning isolated facts and procedures. Young-Loveridge and Mills noted the complexity of multiplicative thinking and that the most effective representation is based on the array. "An area based representation can be used to show how the algorithm for multi-digit whole-number multiplication works" (p. 636). This is a point that we will explore when we report on our research in a later section.

Young-Loveridge and Mills noted how some students in their study who already were using written algorithms or the grid method had difficulty understanding how other more conceptually based models such as arrays could be useful. This is supported by Pesek and Kirschner's (2000) study which compared the outcomes for students taught through instrumental (procedural) instruction followed by relational (conceptual) instruction, against outcomes for students taught solely through relational instruction. They prefaced their report by noting that teachers in general "balance their professional obligation to teach for understanding against administrators' push for higher standardized test scores . . . and teach part of the time for meaning and part of the time for recall and procedural-skill development" (Pesek & Kirschner, 2000, p. 524). They found that students in the 'relational-only' group "used conceptual and flexible methods of constructing solutions" and that students in the instrumental instruction group prior to receiving relational instruction "achieved no more, and most probably less, conceptual understanding than students exposed only to the relational unit" (Pesek & Kirschner, 2000, p. 10). Their findings sit well with the work of Young-Loveridge and Mills (2009, p. 641) who noted that "once students have been trained to use written algorithms, it can be extremely difficult to then try to help them develop relational understanding" (p. 641).

Warren and English also discussed mathematical structure and described it as being about many ideas, including properties of quantitative relationships and properties of operations, such as commutativity. They noted the need for students to understand such important ideas as they underpin not only their immediate work with algorithms, but also their later work with algebra. While it is often assumed that students are familiar with such structures and concepts when they begin to learn about algebra, Warren and English said that this is not the case. They made a strong statement that "we argue that the overriding emphasis on computational procedures at the expense of exploring relationships is largely responsible for children's limited understanding of mathematical structure" (Warren &

English, 2000, p. 625). Again, this resonates with the work of Pesek and Kirschner and Young-Loveridge and Mills in pointing out the impact of teacher content knowledge and pedagogical knowledge on the way that teachers teach and consequently, on how their students learn.

There is quite a deal of common ground here. Young-Loveridge and Mills (2009) pointed out that children have difficulty relating the array to solving multiplication problems once they have been taught how to use algorithms. Pesek and Kirschner (2000) observed that students in the instrumental-relational group suffered cognitive interference, in which their previously entrenched instrumental ways of thinking intruded on their ability to cope with new ideas. Warren and English (2000) found that students who are over-reliant on algorithms and procedures were disadvantaged in terms of their understanding of mathematical structure. These points relate well with Devlin's (2008) comments about how difficult it is for students to 'unlearn' particular practices once they become entrenched. Askew (2016, p. 64) made a similar point in discussing methods of doing division and says, ". . . whichever model is introduced first, children are likely to accept that as 'the' model, rather than a conditional model" (2016, p. 64).

Mathematical Underpinnings of Algorithms

Algorithms are used to calculate answers to calculations in addition, subtraction, multiplication, and division. Haylock (2010, pp. 51-52) described an algorithm as "a step-by-step process for obtaining the solution to a mathematical problem, or the result of a calculation . . . [or] . . . the formal paper-and-pencil methods that we might use for doing calculations, which, if the procedures are followed correctly, will always lead to the required result". This paper focuses only on the use of the multiplication algorithm.

Numerous researchers and mathematics educators have described the mathematics that underpin the use of written algorithms for multiplication and division. Davis presented a concise summary of how important the array is as a base for understanding multiplication and related ideas, describing it as a flexible and powerful representation of multiplication. He stated that "An algorithm for multi-digit whole-number multiplication can be reformatted in a grid, which can connect the standard algorithm to area. This area-based interpretation can be extended to multiplication of decimal fractions, common fractions, algebraic expressions, and other continuous values" (Davis, 2008, p. 88). Davis has situated the array as a vital tool for understanding not only the written multiplication algorithm but also higher order ideas such as algebraic expressions.

While the array provides a model for understanding algorithms, basic number facts extended by powers of ten, and the distributive property represent specific aspects of mathematical understanding that facilitate the use of the algorithm. Jazby and Pearn (2015, p. 288) suggested that one of the reasons for the teaching of algorithms is that they are viewed as "cognitive aids which enable a multiplication problem to be broken up into a series of less cognitively demanding subroutines". However, they identified a problem which sits at the heart of the issue of procedural teaching. Jazby and Pearn used 34 × 26 as an example and state that when it comes to multiply the 30 by the 6, a student can 'suspend place value' and carry out the calculation as 3 × 6. They noted that "If an algorithm suspends place value, then successful use of the algorithm requires some cognitive work which recognises that this 3 × 6 is in fact 3 tens × 6 (and is therefore 18 tens rather than 18)" (Jazby & Pearn, 2015, p. 288). This issue can be taken further to include the distributive property of multiplication in that when multiplying 34 by 26, students often multiply the 30 by the 20 and the 4 by the 6, omitting the other two (30 × 6 and 4 × 20).

This latter problem can be ameliorated by using the array as a model for the distributive property.

A short article by Goutard, originally written in 1962, was recently re-published in the journal Mathematics Teaching. It shows the attempts of a group of young students aged between 7 and 9 years to solve two division examples: 235 divided by 5 and 402 divided by 6. The students had not been taught how to do division examples, let alone through the use of an algorithm. However, the thinking that they displayed was impressive with the following being two examples.

- For 235 divided by 5, Francois (8 years) offered this: "I said: how many 5's in 100, there are 20; then in 200 I must have 40, and in 30 there are 6, and in 5 there is 1. Then I add: 40 plus 6 plus 1 = 47".
- For 402 divided by 6, Pierre (9 years) offered this: "I said how many 6's in 200? 30, remainder 20. In 400? 60, remainder 40. And I said: 40 + 2 = 42. How many 6's in 42? 7. 60 + 7 = 67".

We believe that Goutard makes a valuable point. The students concerned had not been taught an algorithm for calculating answers to division examples yet they were perfectly capable of thinking of and executing a mathematically sound method. The mathematics underpinning their thinking is sound, as indeed it should be before students are taught how to use algorithms.

Making Effective Computational Choices

The capacity of children to make effective choices about methods of computation is strongly related to the issues already raised here. Results from research by Swan (2004, p. 30) suggested that, in order to help students develop and use flexible and appropriate strategies, teachers could "assist students to reflect on their computation strategy choice . . . place more emphasis on understanding numbers and number properties [and] encourage discussion about computation choice". Swan noted that students' choices of strategy were evenly spread between mental, written and calculator based strategies which indicates that there is clearly a need for more than instruction regarding written algorithms. Goutard's short article discussed above implies that students who have a conceptual understanding of the mechanics of an operation (in this case division) are well placed to learn how to use an algorithm at a later time, if indeed they need to do so. This is supported by Gravemeijer and van Galen (2003) who said that teaching an algorithm without understanding results in isolated and unconnected knowledge which is only useful to students in familiar situations and which they find difficult to apply more widely.

Van de Walle, Karp and Bay-Williams (2013) compared the merits of 'student invented strategies' with standard written algorithms and arrived at similar conclusions to Anghileri, as described earlier. They noted that when students use their own invented strategies, their work is generally based on understanding the numbers rather than manipulating digits and that they have a more flexible repertoire of strategies for mental computation. As well, student invented strategies help students to develop number sense, represent the process of actually 'doing' mathematics, and are often faster and more accurate than standard algorithms.

Contemporary curriculum documents also provide plenty of support for teaching students to use a choice of computation strategies. The Australian Curriculum: Mathematics states that at Year 7, students should be involved in "justifying choices of written, mental, or calculator strategies for solving specific problems including those involving larger numbers" (Australian Curriculum Assessment and Reporting Authority, 2017, ACMNA157). The Common Core State Standards for Mathematics states the following as part of the Year 4 standard: “Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products” (NGA Centre, 2010, p. 27). The United Kingdom National Curriculum in Mathematics includes many statements about different methods including the following for Lower Key Stage 2 that students should “become increasingly fluent with whole numbers and the four operations, including number facts and the concept of place value . . . [to] . . . ensure that pupils develop efficient written and mental methods” (Department of Education, 2014, p. 17). The year/age levels cited here are merely provided as examples of typical statements made in curriculum documents. In summary, national curriculum documents in various countries, supported by research suggest that computational choice is important.

Methodology

Overview

This article reports on a study that developed from a larger and on-going study into multiplicative thinking of children from 9 to 11 years of age. The original study has been conducted for over three years in Western Australian primary schools and has gathered data from over 1000 children in eight schools. The research question for the original study is ‘What is the capacity of students in Years 4, 5, and 6 to think multiplicatively?’ This is clearly a broad question but it is still applicable to this current study and encompasses more specific questions which are the focus of work reported upon here. Specifically, we are seeking to answer the following:

- What level of understanding and use of written algorithms and mathematical procedures is demonstrated by 10 and 11 year old students?
- To what extent might teacher knowledge, pedagogies, and other factors have an impact on students’ understanding and use of written algorithms?

The main focus here is on Research Question 1 but it was deemed to be important to consider Research Question 2 due to its relationship with Research Question 1.Data related to RQ2 is not comprehensive but is included to indicate the need for further research. Two data gathering instruments – a written Multiplicative Thinking Quiz, and a semi-structured interview – have been developed and refined during that time and are used in this current study involving two primary school classes at one school in the south-west of the United Kingdom. The broad expectations are that children in the United Kingdom, in Year 6 – that is by the end of Key Stage 2 – will develop the ability to multiply four digit by two digit numbers, and divide four digit numbers by two digit numbers, both with formal written methods for long multiplication and division *(Department of Education, 2014, p. 39).*

Data Gathering Phase 1

The sample size (n) for Phase 1 of the current study was 40, consisting of 14 Year 5 and 26 Year 6 students with approximately equal numbers of male and female students. The Year 5 group does not include the whole class but rather only those students who were available for an interview. The quiz was administered to both classes on the same day. In the Multiplicative Thinking Quiz (MTQ), students were asked a total of 18 questions which covered many aspects of multiplicative thinking, including identification of equal groups, use of the array, and the identification and generation of factors and multiples. Other questions targeted the identification and understanding of the commutative and distributive properties, inverse relationship between multiplication and division, and extension of number facts by powers of ten. Finally, another group of questions focused on the understanding of what happened to numbers when multiplied and divided by powers of ten, and the identification of the relative magnitude of numbers in terms of the 'times bigger' relationship. Due to the scope of this article, data from the responses to the multiplicative thinking quiz is not included.

All students were interviewed within four days of completing the written quiz. To begin, they were asked questions to probe their responses to some of the quiz questions. The main interview questions specifically required them to explain or show how they worked out a range of calculations. (e.g., What is the answer to 23 × 400? Please show and/or explain how you worked it out). We deliberately chose not to ask students if they could 'work out the answer in their heads' because that was implicit in the instruction to 'explain and/or show'. We wanted to find out if students employed a learned procedure or 'rule' as the first choice, if they used the standard, or other, written multiplication algorithm, and the extent to which they understood what was happening at a conceptual or procedural level. The use of the interview questions in this way was purposeful in that we wanted to find out the extent to which an algorithm was used, whether it was used 'as a convenience' and/or in situations where it might not have been the best choice of computation method. We were also interested to see if there was evidence of an understanding of underpinning mathematics such as extension of basic number facts with powers of ten, partitioning, and the distributive property. We have chosen data generated from three of the interview questions as they provide a good indication of children's understanding, and the use of algorithms and procedures. The interview questions relevant to this discussion are as follows:

- Please explain and/or show me how you would work out the answer to 23 × 400. What is the answer to 23 × 4?
- Please explain and/or show me how you would work out the answer to 29 × 37.
- Please explain and/or show me how you would work out the answer to 200 ÷ 13.

Data Gathering Phase 2

Three months after the initial interview, the complete Year 5 cohort (n=28) was interviewed. The Year 6 cohort was not available at the time due to commitments associated with the national testing program. This time we were particularly interested to see whether students opted to use standard algorithms or were able to recognise a mental computation strategy that would shorten the calculation. Therefore a different set of questions was created. The Phase 2 interview questions relevant to this discussion are as follows:

- Please explain and/or show me how you would work out the answer to 25 × 17 (Were students able to recognize that 25 × 4 = 100 would be an appropriate strategy?)
- Please explain and/or show me how you would work out the answer to 19 × 0.5 (Were students able to recognize that 0.5 is equivalent to a half and then divide 19 by 2?)
- Please explain and/or show me how you would work out the answer to 4999 ÷ 25 (Were students able to recognize that 4999 could be rounded to 5000, which could then be easily divided by 25?)

- If 14 × 9 = 126, what is the answer to 18 × 14? (Could students recognize that 18 is double nine, and then double the answer?)

The Phase 2 data gathering also included a one hour interview with each of the class teachers of the Year 5 and Year 6 students, as well as the school's mathematics and numeracy coordinator. Various anonymous work samples from the Phase 1 and Phase 2 interviews were presented for discussion and teachers were asked questions such as "How would you expect students to respond to this question? Here is a sample of a typical response – is that what you would have expected students to do?" Other questions were more specific in terms of the mathematics involved and included questions such as "How do you teach partitioning (or the use of the array . . . or the distributive property)? How important do you think partitioning is (or the use of the array . . . or the distributive property)? These data were collected as we wanted to understand the levels of teacher knowledge and their understanding about algorithms and the underpinning mathematics, as well as the way in which they are taught. The Phase 2 in-depth teacher interview was designed to provide data about those ideas.

Analysis of Data

Data from the quiz were entered manually onto an Excel spreadsheet using a code of '1' for a correct/appropriate response and '0' for an incorrect/inappropriate response. A set of guidelines outlining what constituted acceptable responses for each question was developed by the researchers to aid in the recording process. Results for each question were then aggregated using the 'sum' tool in Microsoft Excel. The interviews were audio recorded and manually transcribed. We believe that manual transcription allows for richer data to be generated, and allows the researcher to pick up nuances and emphases made by interviewees, especially when the transcription is enhanced by anecdotal notes made during the interviews. During and after transcription, emergent themes were developed and frequency of responses noted for later consideration.

Results and Discussion

Phase 1: Student Interview Data

Data from three interview questions listed above are now discussed in turn.

- Question 1. Please explain and/or show me how you would work out the answer to 23 × 400. What is the answer to 23 × 4? [The second question was asked immediately following the student response to the first question].

Students used a variety of methods to provide the answers for the two questions. These are summarised in Table 1. Samples of student responses showing the various methods are included after Table 1.

Table 1
Student responses to the question 23 × 400

Student response	% n=40
Used a standard written algorithm	65
Worked out 23 × 4 and 'added two zeros'	17
Worked out 23 × 4 and said they would 'move the digits two places to the right'	8

Used a grid method	5
Worked out the answer by adding two partial products	5

It is significant that all but two of the students used an algorithm or grid method to calculate the answer to one or both of 23 × 400 and 23 × 4. In fact, 70% of the students used a written algorithm to show the answer for 23 × 4 after they had worked out the answer to 23 × 400. Only 30% of the students worked out the answer to 23 × 4 using mental computation, or indicated that they had already worked it out from the previous example (23 × 400).

Student Joe's sample typifies the way in which the majority of students used the standard vertical algorithm. Almost all of the Year Six students reversed the order of the numbers and wrote the 400 on top. None of the Year Five students did that. When the Year 6 students were asked why they did so, students' responses included "Because it's easier with the big number on the top" (Craig), "Because I'm used to doing the bigger number on top" (Cassie), and some students could give no reason for doing so. Aaron's sample is included specifically to show his error. He multiplied the 0 in each case by the bottom number (23) and said that he put a zero on the second line "because it is being multiplied by 20, not 2". His error seems to result from his poor alignment of the digits. When asked if he could do it in a different way without doing the algorithm, he said he would do 23 × 4 and then would "add the zeros". Both Joe's and Aaron's samples are included in Figure 1.

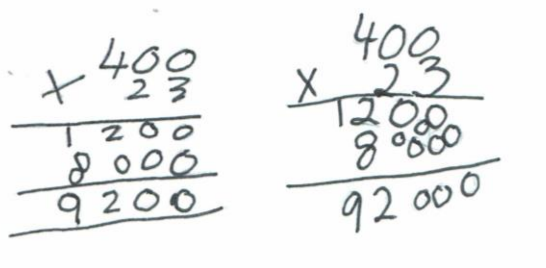


Figure 1: *Work samples from Joe and Aaron respectively.*

Figure 2 shows samples from Students Cathy, Greg, and Ellie respectively. Cathy wrote 23 × 4 as a vertical algorithm and then wrote the answer of 9200 – "I did that and then added zeros back on". Greg worked out the answer to 23 × 4 and said "It's a hundred times bigger so I move it two places . . . I fill in the bouncers".

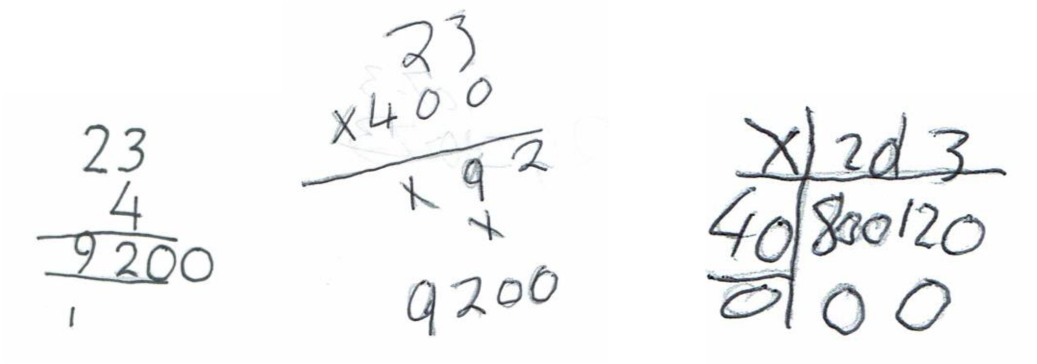


Figure 2: *Work samples from Cathy, Greg, and Ellie*

This is a term used by some of the students to explain multiplication by powers of ten. Greg explained it by saying, “The bouncers are there to stop the normal numbers [92] from getting back into the tens and ones places”. Ellie attempted to use the grid method, apparently without understanding completely what she was doing in terms of partitioning the two numbers.

Figure 3 shows Ruby’s use of a partial product method as well as Guy’s method of doubling 23, then doubling again to derive the answer of 92. It is interesting to note that Guy actually wrote algorithms to calculate what one would reasonably expect a Year Five student to do mentally. Freya’s sample shows how 70% of the students used a vertical algorithm to calculate the answer for 23 × 4.

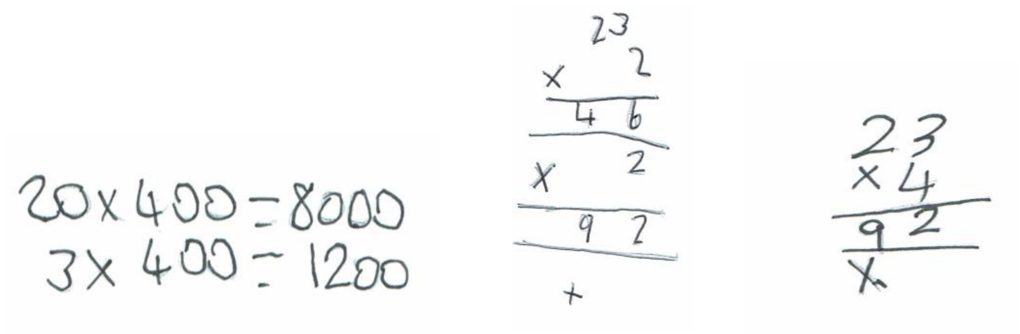


Figure 3: *Work samples from Ruby, Guy and Freya*

Whilst there were some errors made in the calculation of answers most students arrived at the correct answers through the use of a vertical algorithm. When asked, they were generally able to articulate why they had adopted a particular approach as this comment from David shows. “First I’d do 23 × 4 . . . I’d partition it, so 23 × 4 is 92 and this bit means it is times 100, so you add the two zeros because it is 100 times bigger” (Student David, Year Six). Some other students also used the term ‘partition’ which may indicate that they had been taught about it. Also, Conor displayed a measure of understanding with his comment. [He immediately wrote the example as 2300 × 4]. “I take the hundred times off that one and put it there, because I like to do multiplication sums by one digit” (Student Conor, Year Six).

Notwithstanding numerous comments like those of David and Conor, it is important to note that every student apart from two used a vertical algorithm for either one or both of 23 × 400 and 23 × 4. Ruby used a partial product strategy, and Guy used a ‘double-double’ strategy (see Figure 3) and even then, Guy still set out his work as a vertical algorithm.

- Question 2. Please explain and/or show me how you would work out the answer to 29 × 37

As was the case with the previous question, there was some variety of methods used including the standard vertical algorithm, the grid method, and a partial product strategy, all of which are related in being based on an understanding of partitioning. These are summarised in Table 2.

In general, students were well able to articulate what they were doing when using the vertical algorithm. For example, David (Year Six) explained the placement of the zero in the second line “because this one is actually multiplied by tens”. Similarly, Chris (Year Six) was asked about the same thing and responded with “Because that 2 [in the 29] is not a normal 2, it’s a 20, so it must be ten times bigger”. It was interesting to note that students used a variety of terms to refer to the algorithm, some calling it ‘long multiplication’, others ‘column multiplication’, and a number of others calling it ‘column addition’. This is an observation that could be followed up with further interviewing. Students appear to have identified a particular aspect of the process, possibly as an aid for remembering how they should calculate such examples. The fact that a number of them identified the process as ‘column addition’ may be of concern and the term is specifically mentioned in the school’s calculation policy at Year 5.

Table 2
Summary of responses to Question 2, 29 × 37

Student responses to 29 × 37	% n=40
Correctly used the standard vertical algorithm	63
Correctly used a partial product model	10
Correctly used a grid method	5
Made a methodological/computational error when using the vertical algorithm	10
Made a methodological/computational error when using partial product	2
Did not attempt the example	10

Another observation of particular interest was that every student who used the vertical algorithm set out the entire framework <u>before</u> commencing any calculation, as shown in Figure 4. When asked about this, Student David responded in this way. “Well, there are two numbers [indicated the 29 on the second line] so there will be two lines of working, and I put the answer underneath in those two lines. The zero goes there because this column is multiplying by tens”. His comments indicate a measure of understanding of the mathematics underpinning the process, that is, that the answer in the second line is obtained by 37 times 2 times 10. This exemplifies the earlier comments attributed to Jazby and Pearn (2015) about ‘suspending place value’ when using algorithms. When asked why he wrote the larger number on top, David replied, “Because that’s what I usually do”. Figure 4 also contains a sample from Student Ben, who did not reverse the order of the numbers, and Max, who used a grid method. All students who used the vertical algorithm as shown in Student Ben’s sample wrote the ‘carried numbers’ in the same way as did Ben.

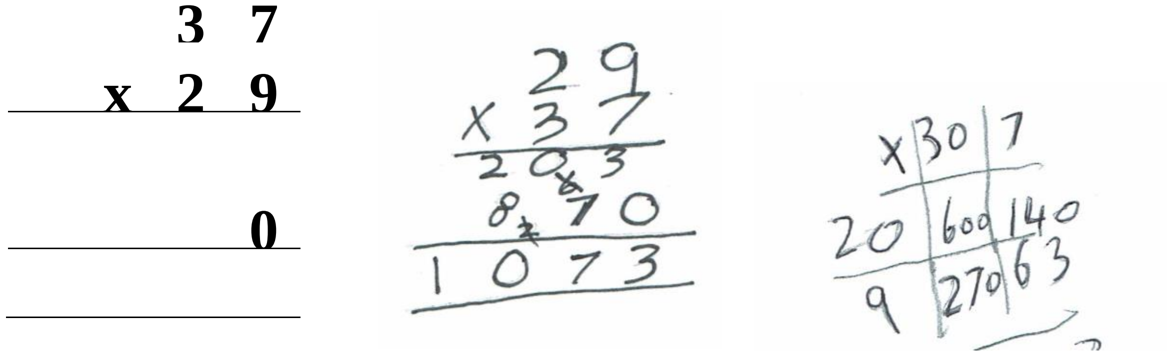


Figure 4: *Typical setting out for algorithm, and samples from Ben and Max*

- Question 3. Please explain and/or show me how you would work out the answer to 200 ÷ 13

All students who attempted this example used a division algorithm which they called the ‘bus stop method’. One student described the reason for this as “It is like a bus shelter that covers the numbers you are dividing into”. All but one of the students began by writing the algorithm and then immediately wrote answers to ‘the thirteen times table’ at the side. No student in the sample attempted to work out the answer in any way other than using the algorithm. A summary of responses to the division question is provided in Table 3.

Table 3
Summary of responses to 200 ÷ 13

Student response	% n=40
Correctly used the ‘bus stop’ written algorithm	55
Unable to begin to work out the answer	33
Inappropriately reversed the numbers to divide 13 by 200	7
Inappropriately partitioned the divisor 13	5

Two samples are shown in Figure 5. Ian’s sample indicates a typical solution provided by students but also contains a second aspect. Having seen other students use the algorithm in the same way as Ian, it was decided to follow up the 200 ÷ 13 example and ask Ian how he would solve 200 ÷ 3. He used the same procedure as for solving 200 ÷ 13 when it would be reasonable to expect that he would not need an algorithm but would work it out mentally. It would have been of interest to have asked other students to also solve 200 ÷ 3 to see they used an algorithm in the same way as did Ian. Alice’s sample indicates how a number of students exhaustively wrote down all of the multiples of 13 even though they were not required. It is also worthy of note that Alice worked out successive multiples after 65 by using an algorithm!

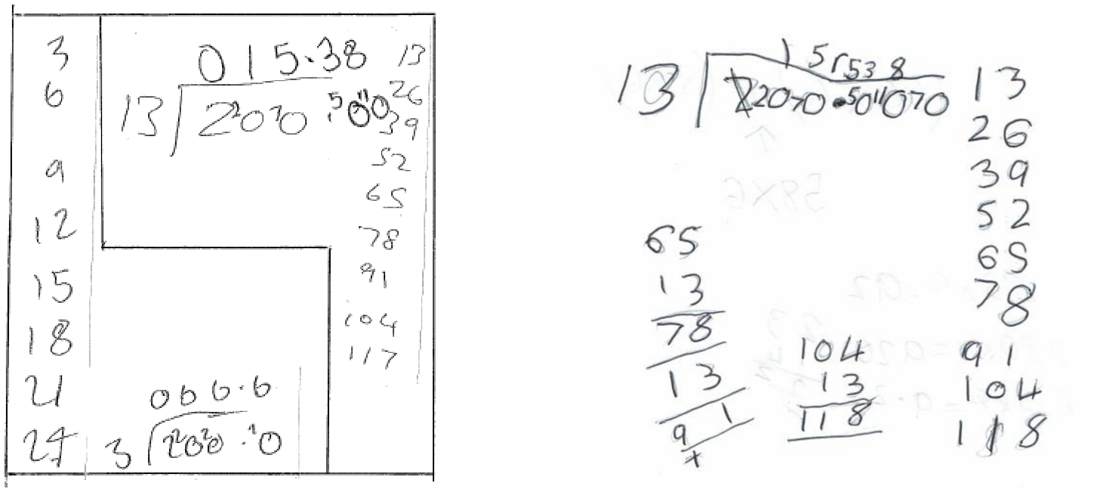


Figure 5: Samples from Student Ian and Student Alice

The students appear to be instinctively using the ‘bus stop’ algorithm yet the example could have been easily done mentally by using a ‘chunking’ method. That is, 200 could have been ‘chunked’ into 130 + 70, so that ten lots of 13 and five lots of 13 respectively could be found, giving an answer of 15 with 5 remaining. When this was put to a number of the students, they acknowledged it and said that they had used it before but they preferred to use the ‘bus stop’ method. When asked how they would ‘chunk 200’ if dividing by 14, 15, or 16, they were able to explain it well. However, not one student chose to ‘chunk’ 200, and instead defaulted to the ‘bus stop’ method. This is interesting when considered in the context of the school’s calculation policy. At Years 3 and 4, students are encouraged to use ‘informal chunking’ of dividend but in late Year 4 and in Year 5, they are introduced to the ‘bus stop’ method for ‘formal short division’. Students are seemingly calculating in line with policy expectations.

Phase 2: Student Interview Data

As noted earlier, the complete Year Five cohort was interviewed with different questions designed to see if the students defaulted to the use of a standard algorithm or whether they were able to identify a different approach that would shorten the calculation. Each question is dealt with in turn.

- Question 1. Please explain and/or show me how you would work out the answer to 25 × 17 (Were students able to recognize that 25 × 4 = 100 would be an appropriate strategy?)

Only two students saw the mental strategy based on 25 × 4 = 100 and worked through the stage of 25 × 16 = 400, with one more 25 to make 425. The rest (93%) used a standard vertical algorithm or ‘partial product’ model and of them, 61% arrived at the correct answer and 32% made an error. Figure 6 contains samples from Grace (showing a correct use of the algorithm) and Hugh (showing an incorrect use of the ‘partial product’ model).

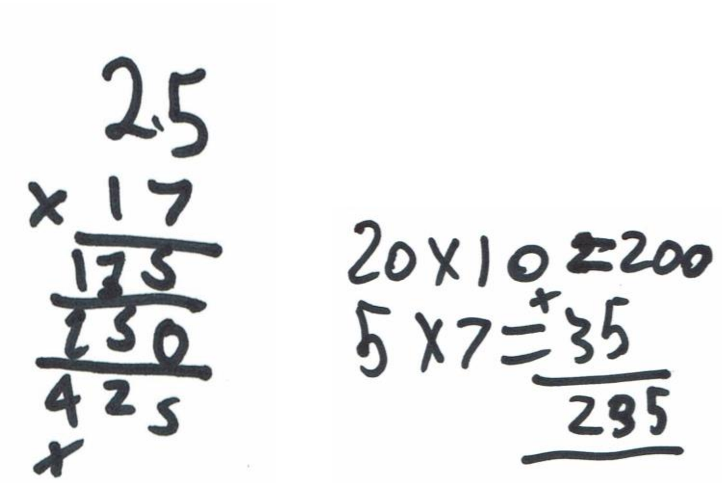


Figure 6: *Samples from Students Grace and Hugh*

Once they had arrived at an answer, students who had used the algorithm were prompted about the mental strategy based on 25 × 4 = 100. One student, Daniel, quickly saw the shorter mental strategy as this exchange indicates.

D: I'd do 20 times 17 and 5 times 17.

INT [prompted]: How many 25's does it take to make 100?

D: Oh yeah! It would be four fours with is 16 . . . that's 400 plus the one 25 that's . . . Oh, I get it!

It could be argued that the use of an algorithm to solve 25 × 17 is justified and the school calculation policy certainly states that at Year 5 level, students should use a formal written method to multiply numbers of up to four digits by one and two digit numbers. However, it should be more about developing number sense given that the UK National Curriculum mentions the need for students to use efficient written <u>and</u> mental methods.

- Question 2. Please explain and/or show me how you would work out the answer to 19 × 0.5. (Were students able to recognize that 0.5 is equivalent to a half and then divide 19 by 2?)

The UK National Curriculum states that, at Year 4 level, students should recognise decimal equivalents for common fractions including a half. It seems reasonable to suggest that students might use that knowledge to employ a mental strategy rather than use an algorithm to solve 19 × 0.5. Three students immediately saw that 0.5 was equivalent to a half and quickly halved 19 to mentally calculate the answer of 9.5. One of the students was Daniel who also recognised the 'short' strategy for the previous example. 39% of the students correctly used a vertical algorithm and 42% made an error in doing so. There was a variety of ways in which the algorithm was set out but whichever way was used, it is striking to note the extent to which these students used an algorithm 'by default' instead of a relatively easy mental calculation. Three samples of algorithms producing a correct answer (from John, Ryan, and Ben) are presented in Figure 7 to show some of the variation in methods.

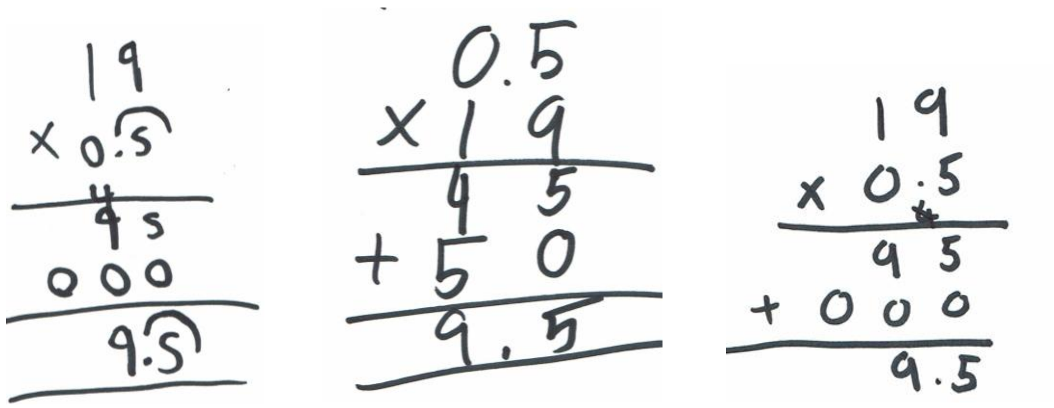


Figure 7: *Samples of algorithms from John, Ryan, and Ben*

It is worth noting that each of the three students has ignored place value convention by aligning the five (tenths) with the nine (ones) yet each has obtained the correct answer. John appears to have 'taken off the decimal point' and replaced it later, while Ryan seems to have obtained part products, albeit without the decimal point. These samples provoke the question about how much these students really understand about the procedure they have used. Given that so many students used an algorithm incorrectly, it is worth looking at some of the ways in which the algorithm was misused. Figure 8 contains samples from Sissy, Claire, and Louise respectively.

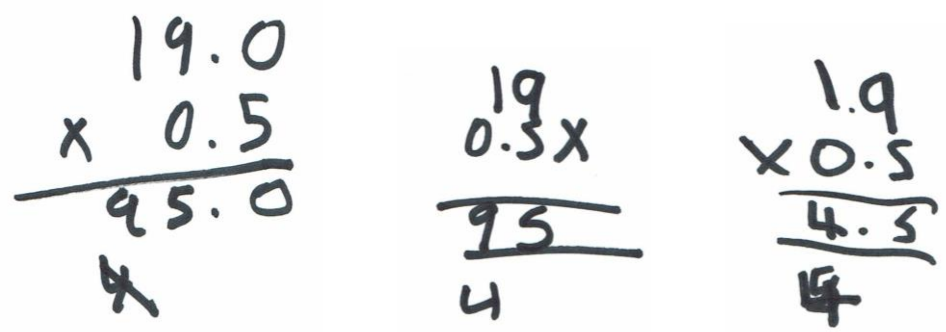


Figure 8: *Samples of algorithm use from Sissy, Claire, and Louise showing incorrect calculations*

A point of interest here is that none of the students whose samples appear in Figure 8 showed number sense to detect that their answers could not be correct. An answer of 95 is impossible but students did not identify that. Had they recognised that 0.5 was equivalent to a half, an answer of 4.5 would also be seen to be incorrect. The students clearly have a preference for using the written algorithm and may have developed a disproportionate level of trust in it, perhaps almost to the extent of assuming that the algorithm will provide them with a correct answer.

- Question 3. Please explain and/or show me how you would work out the answer to 4999 ÷ 25. (Were students able to recognize that 4999 could be rounded to 5000, which could then be easily divided by 25?)

As was the case with the division example (200 ÷ 13) in Phase 1, all students who attempted this example used the 'bus stop' method with 21% of them attempting to work out the answer by rounding the 4999 to 5000. However, only one student who did that arrived at the correct answer. Of the 79% who divided 4999 by 25, 61% made an error and only 18% worked out the correct answer.

There are several points needing discussion here. First, Daniel rounded the 4999 to 5000 and obtained an answer of 200. He knew that he had to take something away from the answer to compensate for the rounding but he subtracted 25 for an incorrect answer of 175. Similarly, Joe arrived at the answer of 200 and said, "I'll take one off so it's 199". He ignored the remainder. While these students saw that the number could be rounded to make it easier to divide, they were unsure what to do to complete the calculation. This suggests that they are unaccustomed to working with methods other than the standard 'bus stop' method, a view that is enhanced by the number of students who did so for the 4999 ÷ 25 example and the 200 ÷ 13 example in Phase 1.

Second, the majority of students who attempted the example made an error but none identified it through the use of number sense, even though some of the answers were clearly incorrect. Four students attempted to partition the divisor and do two separate calculations before adding the part answers together. It is hard to understand their basis for doing that but they appear to have faith in the algorithm and have applied some knowledge of partitioning in the mistaken belief that it is a legitimate way to do the calculation. Once again, a lack of number sense comes to the fore. Samples are contained in Figures 9 and 10.

The first sample from Hugh shows that he partitioned the 25 into 2 and 5, ignoring the place value, then divided the result from the first calculation by 5 before somehow arriving at a clearly incorrect answer. Sissy did something similar but divided both the 2 and the 5 into 4999 before adding the partial answers together (using an algorithm!).

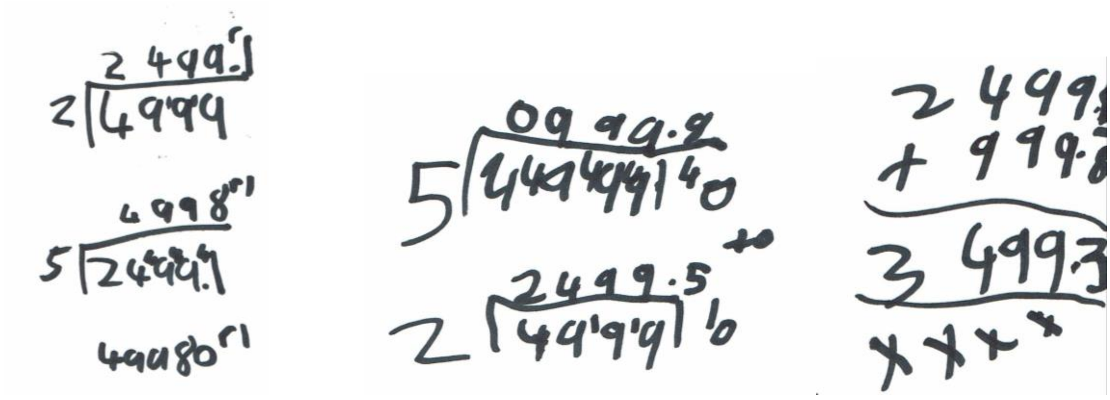
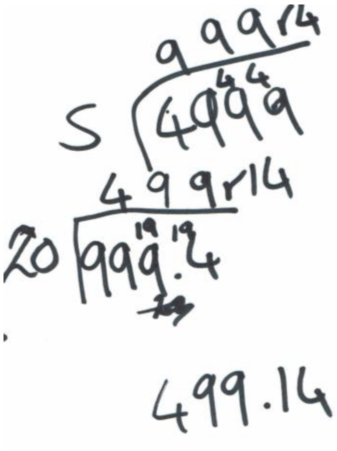


Figure 9: *Samples from Students Hugh and Sissy*

Grace's sample shows that she also partitioned the 25 and divided it into 4999 in two parts before ignoring one of the answers she had obtained. The other sample, from Max, shows that he rounded the 25 to 20, ignored the zero and divided by 2. He seems to be indicating that he needs to multiply something by ten.



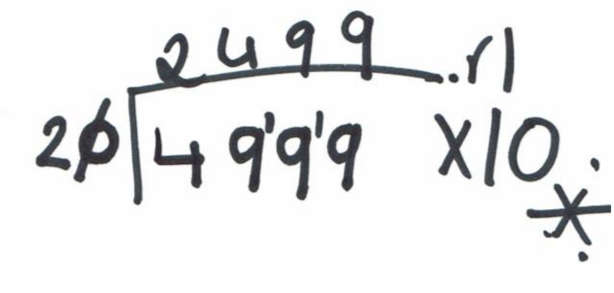


Figure 10: *Samples from Students Grace and Max.*

The samples in Figures 9 and 10 appear to suggest that the students have considerable faith in, but limited understanding of, the division algorithm, to the extent that number sense has not been employed to see that the answers are clearly incorrect and also to understand that divisors cannot be partitioned in that way. In seeking to understand why students might attempt to work out the answer in that way, we looked at the school's calculation policy which states that, at Year 3, students will partition to support division. It is possible that some students recalled that and decided that they might (inappropriately of course) partition the divisor.

- Question 4. If 14 × 9 = 126, what is the answer to 18 × 14?

We were interested to see if students recognised that they could simply double the answer for 14 × 9 to derive the answer for 18 ×14, or whether they would use a written algorithm. The majority of students (65%) obtained a correct answer for this question but a range of related strategies were used. A further 18% made an error, and another 17% did not attempt the question. There were three ways in which the answer was calculated. First, 29% used a vertical algorithm or partial product method, with 11% being correct and 18% incorrect. Second, a further 25% recognised the 'doubling link' and calculated the answer mentally. Third, it is interesting to note that a further 29% recognised that they could double 126 but still used a vertical algorithm to work out the answer for 126 x 2. There appears to be something driving many of these students to use a written algorithm for virtually any purpose, even when it is unnecessary to do so. The fact that the majority of students who used the vertical algorithm made a mistake suggests that an alternative, perhaps a mental strategy, may be better.

Phase 2: Teacher Interview Data

In order to find out the extent of mathematical content knowledge held by the teachers of these students, teachers were asked to comment on work samples generated from the MTQ and interview. A one hour interview was also conducted with each teacher. Teacher Martin was asked to respond to student work samples shown in Figure 11.

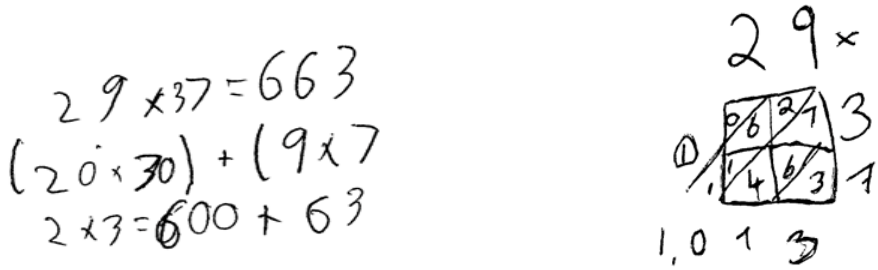


Figure 11: *Student work samples requiring comment from teachers.*

Martin's response to the first part of the sample was that the student didn't understand partitioning and that "I'd need to talk to them and demonstrate that just partitioning both numbers doesn't work". He made no mention of the distributive property and the fact that it had not been applied by the student in the sample. With regard to the second part of the sample, Martin commented that the student used a 'secure method' and 'understands partitioning'. However, the link between the lattice method and partitioning is at best tenuous. During the interview, Martin was asked whether he explicitly taught the distributive property. He said that he did not and proceeded to give examples of what he called 'missing number problems' such as 24 × ___ = 96 and 96 ÷ ___ = 24. Martin appeared to be confusing the distributive property with the inverse relationship between multiplication and division. Martin was also asked to comment on the samples contained in Figure 12 showing students' representations of 3 × 4.

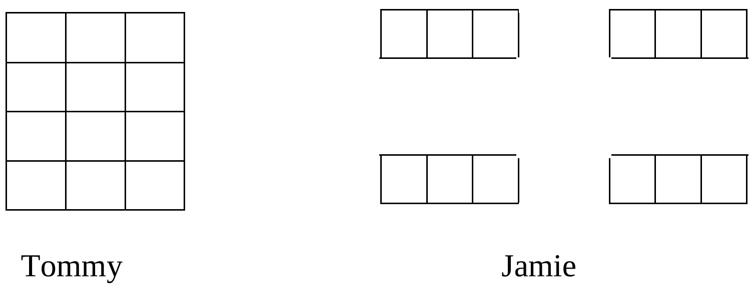


Figure 12: *Student work samples requiring comment from teachers.*

Martin commented that both students had used an array to represent the number fact whereas it is clear that Tommy did but Jamie showed it as separate groups. Also, he did not identify that neither student had in fact shown 3 × 4 but had represented 4 × 3. When asked about how he used the array in his general teaching, Martin responded with, "If I'm honest with you, hardly ever . . . perhaps it is something that I should be bringing in to my practice". Perhaps Martin's apparent lack of understanding, and use in teaching, of the array and distributive property has contributed to a lack of underpinning mathematical understanding in the student cohort represented here.

Teacher Solly was also asked to reflect on the work samples in Figure 11. He did not comment at all about the sample showing the lattice method and his only remark about the first part of the sample in Figure 11 was that it showed "Method gone wrong – needs to be modelled". He made no mention of partitioning or the distributive property. With regard to the samples in Figure 12, Solly said that, “Tommy and Jamie both understand that multiplication requires either 3 lots of 4 or 4 lots of 3. [I would] work on representing calculations in different ways to deepen understanding”. He made no mention that Tommy had used an array compared to Jamie’s separate group model. When asked about how he used the array in his teaching, the following exchange occurred:

INT: Do you use the array in your teaching?

S: Is that the . . . [pause]

INT: Rows and columns of dots or squares.

S: [further pause] No we don’t, not in Year 6, no. Definitely a hundred per cent, yeah.

Solly added that there was less use of materials and different representations as students moved from the early childhood years, and that the Mathematics Enhancement Program (MEP) lesson plans were ‘very prescribed and don’t allow for the opportunity to get these sorts of things out’. Solly was also asked about the samples shown in Figures 9 and 10 where students had inappropriately partitioned the divisor. His comment was, “It’s eye-opening to see how they’ve lost the ideas . . . number sense has gone out the window”.

Conclusions

The title to this article poses the question – Are children prisoners of process? To help answer that, we developed two research questions regarding students’ level of understanding and use of standard written algorithms, and the impact that teacher knowledge, pedagogies, and ‘other factors’ might have on the way in which algorithms are learned and used. One point we want to make here is that we have no objection per se to the teaching and use of algorithms such as those described in this paper. Our main concern is that a lot of students opt to use algorithms when there is no need to do so, and they are often used at the expense of mental computation strategies. The definition of multiplicative thinking (Siemon et al., 2006) cited earlier in the paper mentioned the need for students to ‘work flexibly’ and to communicate ‘in a variety of ways’. The evidence presented from this relatively small sample of students suggests that this ‘flexibility’ and ‘variety’ may be compromised somewhat by the preference for using algorithms.

Student use and understanding of algorithms and procedures

A number of students demonstrated sound understanding of the mathematics that underpins the multiplication algorithm and therefore how and why it works. For instance, students were able to say that the zero was placed in the second line, because they were multiplying by tens, not ones. Perhaps of more concern is the range of procedures adopted by a number of students for dealing with zeros and the decimal point when multiplying and dividing. Indeed, some students were able to explain that ‘adding a zero’ had the same effect as multiplying by ten, but this was not the case with all students. As well, the majority of students were able, with prompting, to describe how digits moved a place when a number was multiplied or divided by a power of ten. Unfortunately, there were also as many students who described the situation in terms of ‘moving the decimal point’ and ‘taking off the zero and adding them back on to your answer’.

We believe that there is sufficient evidence here to suggest that many students in this cohort are ‘prisoners of process’. As shown by data from the Phase 2 Interview, 70% of the students used a vertical written algorithm to show the answer for 23 × 4. It would be reasonable to expect that only a small proportion of students would need to use an algorithm to calculate the answer for an example like that. In fact, the National Curriculum (UK) states that Year 5 students should be able to "multiply and divide numbers mentally drawing upon known facts" (Department of Education, 2014, p. 32). That such a large proportion of this cohort saw it necessary to use an algorithm, and were generally unable to recognise and use alternative methods in the Phase 2 Interview, supports the work of Young-Loveridge and Mills (2009) and Hartnett (2015) who found that students find it difficult to change habits once they have been taught the use of algorithms.

It seems reasonable to suggest that the reliance on algorithms held by these students has undermined the development of their number sense, in many instances. Examples have been presented to show how there is a distinct lack of number sense in how many students approached some of the examples. For instance, it is reasonable to expect that Year 5 and 6 students would know that $25 \times 4 = 100$ and that 4999 can be rounded to 5000, yet only a small proportion of the cohort applied that knowledge to their work, instead 'defaulting' to the tried and familiar 'bus stop' algorithm. Similarly, very few students recognised 0.5 as a half in order to perform a simple mental calculation to halve 19 in the example 0.5×19. Again, even when a number of students did see that 14×18 could be solved by doubling the answer to 9×14, a considerable number of them used an algorithm to work out the answer when a simple mental calculation should have sufficed. Perhaps Freya's comment in response to 38×6 illuminates the situation. She proceeded to set out the vertical algorithm and was asked, "Can you work it out from 6×40 or do you have to work it out like that? [algorithm]?" She responded, "You can work it out from that but I'm doing this just to make sure and as a back-up". As noted above, we do not object to the use of algorithms per se, but we think that many of the students in this cohort may be 'prisoners of process'.

Another consequence of the lack of number sense is linked to students' mindset, which in turn may determine a student's capacity to make effective computational choices. Several students were asked if they could think of 'another way' to work out an answer. The response from Student John was quite typical – when asked about 25×17, he said "I could do 20×10 and 5×7 but that wouldn't give the right answer". Other students responded in a similar way that seems to imply that 'an alternative way' would only lead to an incorrect answer. Furthermore, when any of the students attempted to use an alternative method (such as rounding 4999 to 5000 in the division example), they seemed unsure of how to complete the calculation, suggesting that working with alternative methods, or even looking for them, might not be the norm for them. This seems to support Hartnett's (2015) findings that students were reluctant to try alternative methods because they generally had success with using algorithms. However, as we have seen here, that is not necessarily the case where there is a paucity of number sense. Another example of this is provided by Claire who had calculated an answer to 25×17 using an algorithm. She had made several calculation errors and arrived at an answer of 1400. When asked to check the second line of her working, she responded with, "I've left off a zero" and then promptly added a zero to her answer line to make it 14000. In all there were three computational errors in her work but she made the assumption that she could place 'the missing zero' in her answer and that it would be correct.

Teacher content knowledge, pedagogies, and other factors

In seeking to understand student's reliance on algorithms and an apparent lack of understanding of them, we must consider teacher knowledge as a factor. Although minimal data has been presented to answer Research Question 2 about teacher knowledge, it has highlighted the need for researchers to consider teacher knowledge in future studies. Evidence from the two teacher interviews seems to indicate that the teachers are unclear about some of key ideas that underpin the understanding of algorithms, namely the array, partitioning, and the distributive property. Alongside this, students have indicated a propensity for using algorithms as a first option and in many cases, have struggled to explain how and why they are used, nor identify errors made. Perhaps it is not surprising given two underlying and powerful factors. First, the school calculation policy emphasizes the use of written algorithms from Year 4 onwards. Second, there is a requirement for teachers to base their teaching on the MEP lesson plans, which are quite prescriptive and seem to be based on remembering facts and ideas rather than developing them in a conceptual way.

The case for using the array to develop an understanding of the distributive property, the grid method of multiplication, and subsequently, the standard multiplication algorithm, has been made earlier, citing the work of Davis (2008) and others. It seems reasonable to suggest that students' inability to understand the distributive property and the standard algorithm could be at least in part due to the fact that their teachers did not use the array in their teaching. It seems reasonable to also suggest that teaching of algorithms may be instrumental in nature, rather than relational.

In summary, we believe that it is important for students to understand the mathematics that underpin algorithms and procedures and that the mathematics should be explicitly taught to emphasize connections between ideas. Also, it is important for algorithms to be used as one way of calculating along with a wide range of mental and other strategies. If this is done, it might be possible for students to avoid becoming' prisoners of process'.

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