School of Electrical Engineering, Computing and Mathematical Sciences

Various Financial Applications of Regime-Switching Jump-Diffusion Models

Yu Yang

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Declaration

To the best of my knowledge and belief, this thesis contains no material previously published by any other person except where due acknowledgment has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

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Yu Yang

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Abstract

Regime-switching jump-diffusion (RSJD) models, where a discontinuous Markov-modulated geometric Brownian motion is used to model the price process of the underlying asset, have attracted great attention from the research community recently due to their ability to capture the random market movements during both short-term and long-term periods. This research focuses on the applications of various RSJD models to two important financial problems, the mean-variance asset-liability management (MVALM) problem and the pricing of variance (volatility) swaps, where little work has been done to our knowledge.

We first investigate the MVALM problem under a basic RSJD model in which the investor invests in a market consisting of a riskless bond and a risky stock subject to an uncontrollable liability under a mean-variance preference. The problem is formulated as a non-cooperate game where the investor at each time point during the investment horizon is regarded as a different player aiming to optimize the objective functional. The objective functional combines the multi-objective of mean and variance via a regime-dependent risk aversion coefficient. By employing the stochastic dynamic programming techniques, we derive the regime-switching jump-diffusion version of the extended Hamilton-Jacobi-Bellman equations together with a verification theorem, based on which the Nash equilibrium control and equilibrium value function are obtained in terms of five systems of ordinary differential equations. Moreover, numerical and sensitivity analysis are carried out to examine the influence of the change of model parameters such as the transition rates and jump intensities.

Then we establish a RSJD model with Heston's stochastic volatility for the pricing of discretely-sampled variance swaps. A variance swap is in essence a forward contract which requires zero initial cost. Then under the risk-neutral probability measure, the pricing problem is reduced to calculating a series of conditional expectations. By applying the two-stage approach and the generalized Fourier transform method, the fair strike prices are obtained by solving a partial differential equation arising from the martingale property of the discounted value according to the Feynman-Kac theorem. The accuracy and efficiency of our solution is validated by the semi-Monte-Carlo simulation. Also, a counterpart pricing formula for a continuously-sampled variance swap is derived and compared with our discrete price to show the improvement of our solution. Furthermore, several numerical examples are presented where the influence of regime switching and jump diffusion is investigated.

Finally, we further include the Cox-Ingersoll-Ross stochastic interest rate in our Heston-RSJD model to study the pricing of both variance and volatility swaps with discrete sampling times. The dynamics of the underlying asset under an equivalent T-forward probability measure is first derived. Then by applying the risk-neutral pricing and characteristic function method, we obtain the fair delivery prices for different pre-specified calculating formulae of the realized variance and volatility. Similarly, the efficiency and accuracy of our solution is validated with a semi-Monte-Carlo simulation. Then we conduct numerical and sensitivity analysis to examine the effect of each of the factors considered in our hybrid model on the fair strike price, including regime switching, jump diffusion, stochastic interest rate and stochastic volatility.

List of Publications

The following papers were completed during the Ph.D. candidature:

1. Yu Yang, Yonghong Wu, and Benchawan Wiwatanapataphee. Time-Consistent Mean-Variance Asset-Liability Management in a Regime-Switching Jump-Diffusion Market. *Financial Markets and Portfolio Management*, 2020. doi: 10.1007/s11408-020-00360-6.

2. Yu Yang, Shican Liu, Yonghong Wu, and Benchawan Wiwatanapataphee. Pricing of Volatility Derivatives in a Heston-CIR Model with Markov-modulated Jump Diffusion. *Journal of Computational and Applied Mathematics*, *revise and resubmit*, 2020.

3. Shican Liu, Yu Yang, Yonghong Wu, and Benchawan Wiwatanapataphee. Variance Swap Pricing under a Markov-modulated Jump-Diffusion Model. *Discrete Dynamics in Nature and Society*, *under review*, 2020.

4. Shican Liu, Benchawan Wiwatanapataphee, Yonghong Wu, and Yu Yang. Variance Swap Pricing under Hybrid Jump Model. *East Asian Journal on Applied Mathematics*, 2020. doi: 10.4208/eajam.071119.010320

Statement of Contributions by Others

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- Pricing of Volatility Derivatives in a Heston-CIR Model with Markov-modulated Jump Diffusion. *Journal of Computational and Applied Mathematics*, *revise and resubmit*, 2020.

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- Variance Swap Pricing under a Markov-modulated Jump-Diffusion Model. *Discrete Dynamics in Nature and Society*, *under review*, 2020.

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- Variance Swap Pricing under Hybrid Jump Model. *East Asian Journal on Applied Mathematics*, 2020. doi: 10.4208/eajam.071119.010320

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Contents

CHAPTER 1

Introduction

1.1 Background

The well-known Black-Scholes-Merton (BSM) model [1, 2], though extensively adopted ever since it was established, exhibits many drawbacks due to its unrealistic assumptions such as Gaussian-distributed asset returns, a frictionless and complete market, and a known constant interest rate or volatility rate. Therefore numerous research has been made on relaxing some of the assumptions to build up more generalized and realistic financial models. Examples include Heston's stochastic volatility model [3, 4] where the volatility is assumed to follow a mean-reverting stochastic process, the jump-diffusion models [5, 6] where the path of the underlying asset's price process is discontinuous with jumps, and the stochastic interest rate models such as the Cox–Ingersoll–Ross (CIR) model [7] where the risk-free rate is no longer deterministic but follows a non-central chi-square process. However, the models mentioned above fail to reflect the long-term movements of the financial market affected by the structural changes of macroeconomic conditions, while the relationship between the financial market and the macroeconomic environment is well-documented in the empirical finance literature [8–11].

Markov regime-switching models capture the random market movements by including a finite-state continuous Markov chain which represents different market states. The model coefficients such as the appreciation rate, the risk-free rate, and the volatility rate are all modulated by the Markov chain to reflect the effect of the varying market state on the underlying assets' price processes. A typical and simple regime-switching market is divided into two states, the so-called "bullish state" and "bearish state". In 1989, Hamilton [12] first adopted the regime-switching model to investigate the stock return time series. The author maximized a log-likelihood function based on the transition probabilities between regimes and proved that the Markov-modulated model parameters are more realistic compared to the deterministic ones. After that, regime-switching models have been extensively applied to solving various financial problems such as option pricing [13–15], portfolio selection [16,17], optimal selling rules [18,19], and asset-liability management [20, 21]. However, in the literatures mentioned above, the price processes of the assets and liabilities are actually assumed to be continuous under a fixed market state, which in real world are often discontinuous and have jumps even under the given market state. Empirical studies showed that jump diffusion models better describe the price processes than the simple geometric Brownian motion models [5, 6, 22–24]. As a matter of fact, the Markov regime-switching model also describes jumps, but of a different type from the jumps depicted in jump-diffusion models. The general jumps may be caused by some unexpected financial events, which may have short-term and temporal effects on the prices of the assets and liabilities. On the other hand, the Markovian-type jump may result from the structural changes in the entire economic environment, which may affect the prices in the long run. Therefore, to formulate a suitable model for various financial problems over a time period of any length incorporating both types of jumps, it is reasonable to focus on the hybrid regime-switching jump-diffusion (RSJD) model. In fact, recently RSJD models have been applied to solve financial problems such as portfolio selection, asset-liability management and option pricing [25–28]. Moreover, a stochastic control problem in a RSJD market is investigated in [29] and a sufficient stochastic maximum principle has been developed .

Asset-liability management (ALM), often under the mean-variance (MV) criterion, has been one of the important topics in finance. In a typical ALM problem, the investor seeks to find the optimal investment strategy that maximizes the expected terminal surplus wealth while keeping the variance at the lowest. By taking into account the exogenous liability, the ALM model is actually one of the extensions of the portfolio selection problem that is initially investigated by Markowitz [30] and has attracted many researchers' interest ever since.

Volatility derivative products, mainly variance swaps and volatility swaps, have also drawn much attention from the practitioners and researchers. A variance (volatility) swap is not a swap in a traditional sense, but in essence a forward contract whose pay-off at expiry is determined by the difference between the realized variance (volatility) and a pre-set fixed delivery price. The realized variance (volatility) is usually calculated according to a pre-specified formula. Numerous research has been carried out on variance (volatility) swap pricing over the last decades.

1.2 Objectives of the Thesis

The main purpose of this dissertation is to establish various regime-switching jump-diffusion (RSJD) models and investigate the mean-variance asset-liability management (MVALM) problem and pricing of variance (volatility) swaps under the corresponding framework. Analytical solutions, either in closed-form or semi-closed-form, will be derived for each problem. Numerical analysis will be conducted to examine the effect of incorporating different realistic factors and the influence of the model parameters. More specifically, the objectives of this research include the following aspects,

- Establish a RSJD model for the MVALM problem, derive the extended Hamilton-Jacobi-Bellman (HJB) system and obtain the closed-form equilibrium control as well as the equilibrium value function by solving five systems of ordinary differential equations. Examine the influence of the change of model parameters such as the transition rates and the jump intensities with numerical examples.

- Develop a stochastic volatility model with Markov-modulated jump-diffusion for the pricing of a discretely-sampled variance swap, obtain a semi-closed-form pricing formula, derive a continuous counterpart and compare the two formulas together with a semi-Monte-Carlo simulation to examine the accuracy and efficiency of our solution. Conduct numerical analysis to examine the effect of regime switching and jump diffusion.

- Establish a Heston-CIR model with Markov-switching jump-diffusion for the valuation of both variance and volatility swaps defined on different pre-specified realized variance (volatility) calculating formulae, obtain an analytical solution for the fair delivery price for each case, verify the accuracy and efficiency of the solution with a semi-Monte-Carlo simulation and investigate the influence of each of the factors considered in our hybrid model via numerical examples and sensitivity analysis.

- Look for further improvements for our model to take into account more realistic factors, and derive more accurate and effective solutions for various financial problems.

1.3 Outline of the Thesis

This thesis consists of six chapters organized as follows.

In Chapter 1, a brief background introduction of the regime-switching jump-diffusion (RSJD) models and the two financial problems is presented together with the main objectives of this dissertation.

Chapter 2 provides necessary preliminary knowledge for the research and presents literature reviews for both the RSJD models and the mean-variance asset-liability management (MVALM) problem as well as the pricing of variance (volatility) swaps, including some basic concepts and methodologies.

In Chapter 3, a basic RSJD model is developed to investigate the MVALM problem under a game theoretic framework. The problem is considered as a non-cooperate game where an equilibrium control is defined. The RSJD version of the extended Hamilton-Jacobi-Bellman (HJB) system is derived together with a verification theorem, based on which the problem is reduced to solving five systems of ordinary differential equations. The equilibrium control and the equilibrium value function are finally obtained in closed form. Then we visualize the analytical solutions with some numerical examples and examine the effect of several model coefficients with sensitivity analysis.

In Chapter 4, we study the pricing of a discretely-sampled variance swap in a stochastic volatility model with Markov-modulated jump-diffusion. A semi-closed-form pricing formula is obtained by applying the generalized Fourier transform and the dimension-reducing two-stage approach. Moreover, to reduce the complexity of the computation, we first derive the solution based on a given realized path of the Markov chain, and finally obtain the fair delivery price conditional on various paths. The price of a counterpart with continuous sampling times is also derived and our solution is compared with the continuous counterpart as well as the result of a semi-Monte-Carlo simulation under a range of observation frequencies. The effects of regime switching and jump diffusion along with the influence of the model parameters are studied in the numerical analysis section.

Chapter 5 investigates the valuation of both variance swaps and volatility swaps with discrete sampling times under a Heston-CIR model with Markov-modulated jump diffusion. The model under a risk-neutral T-forward measure is first established via a change of *numéraire*. Applying the characteristic function method, we obtain the fair delivery prices for a variance (volatility) swap based on different pre-specified realized variance (volatility) calculating formulae. A semi-Monte-Carlo simulation is carried out to validate the solution. Finally, numerical examples and sensitivity analysis are conducted to examine and compare the effect of each of the factors considered in our hybrid model on the fair strike price.

Chapter 6 presents a summary of the main work and findings of the research and gives further research directions for improving the models established and extending the applications.

CHAPTER 2

Literature Review and Preliminaries

2.1 General Overview

The research focuses on the applications of various regime-switching jump-diffusion (RSJD) models on mean-variance asset-liability management (MVALM) and the pricing of variance (volatility) swaps. In this chapter, we first present the preliminary knowledge required for the research, including the Markov chain as well as the jump diffusion processes which are the building blocks for the RSJD models, stochastic dynamic programming technique which is an extensively adopted technique to solve the MVALM problem, and the risk-neutral pricing as well as the generalized Fourier transform which provide the basic logic and technique for the pricing of variance (volatility) swaps. Then we give a brief literature review of the RSJD models in finance and introduce four types of RSJD that we will develop in the following chapters. Finally a review of the research progress of MVALM and the pricing of variance (volatility) swaps is given, including the computing algorithms.

The rest of this chapter is organized as follows. Section 2.2 presents the mathematical and financial preliminaries, including the relevant concepts and techniques for the research. The development of the financial RSJD models is reviewed in Section 2.3 with an introduction of four types of RSJD models. Section 2.4 gives a review of the research progress in MVALM, including the algorithm for solving MVALM under a game-theoretical framework. Section 2.5 reviews the valuation approaches for variance (volatility) swaps. Finally a concluding remark is presented in Section 2.6.

2.2 Preliminaries

In this section, we present some mathematical and financial preliminaries on the regime-switching jump-diffusion (RSJD) models as well as the methodologies used to solve the mean-variance asset-liability management (MVALM) problem and the pricing of variance (volatility) swaps.

2.2.1 Markov chain

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space where $\mathbb{P}$ denotes the real-world probability measure. Consider a continuous-time financial market and a finite time horizon $[0, T]$. Under a regime-switching model, we assume that the market regime is divided into n different states modelled by an observable finite-state continuous-time Markov chain $\{\alpha(t), t \in [0, T]\}$ whose value can be selected from the state space $S = \{s_1, \dots, s_n\}$. The definition of a continuous-time Markov chain is given as follows,

Definition 2.2.1. *A stochastic process $\alpha(t)$ is called a* ***continuous-time Markov chain*** *on a specific time horizon $[0, T]$ if it takes values in a finite or countably infinite space S and satisfies the following*

$$\mathbb{P}\Big[\alpha(t_{m+1}) = j \mid \alpha(t_m) = i, \alpha(t_{m-1}) = i_{m-1} \dots, \alpha(t_0) = i_0] = \mathbb{P}[\alpha(t_{m+1}) = j \mid \alpha(t_m) = i\Big],$$
$$\forall j, i, i_{m-1}, \dots, i_0 \in S, \text{ and } \forall T > t_{m+1} > t_m > t_{m-1} > \dots > t_0 > 0. \tag{2.2.1}$$

Moreover, the Markov chain is said to be ***time-homogeneous*** *if*

$$\mathbb{P}\Big[\alpha(t+m) = j \mid \alpha(t) = i\Big] = \mathbb{P}\Big[\alpha(m) = j \mid \alpha(0) = i\Big] = p_{ij}(m), \forall i, j \in S, \tag{2.2.2}$$

where $p_{ij}(m)$ specifies the transition probability from state i to state j.

To incorporate the Markov chain into the stochastic calculus analysis, we can derive the following semi-martingale dynamics for $\alpha(t)$ according to [31],

$$\alpha(t) = \alpha(0) + \int_0^t Q(u)'\alpha(u)du + M(t), \tag{2.2.3}$$

where $M(t), t \in [0, T]$ is a $\mathbb{R}^n$-valued martingale increment process , and $Q(t) = (q_{ij})_{n \times n}$ is the transition matrix (generator) of $\alpha(t)$, where q_{ij} denotes the transition rate from state i to state j. Q' denotes the transpose of the generator.

The generator Q plays an essential role in the regime-switching models with the following properties:

- For $\forall i, j = 1, \ldots, n$, the transition rates satisfy $\sum_{j=1}^{n} q_{ij} = 0$. Moreover, we have $q_{ij} > 0$ $(j \neq i)$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$.
- The relationship between the stationary transition probabilities and the transition rates is given by,

$$\begin{cases} \mathbb{P}\Big[\alpha(t+h) = i \mid \alpha(t) = i\Big] = p_{ii}(t) = 1 - q_i h + \circ(h), \\ \mathbb{P}\Big[\alpha(t+h) = j \mid \alpha(t) = i\Big] = p_{ij}(t) = q_{ij} h + \circ(h), \end{cases} \tag{2.2.4}$$

where $j \neq i$ and $q_i = \sum_{j\neq i} q_{ij}$.

2.2.2 Jump diffusion processes

The jump diffusion (JD) processes refer to Brownian motions (the "diffusion" part) with jumps in the paths of the processes whose dynamics generally take the following form:

$$dS(t) = \Big[\mu(t, S(t))dt + \sigma(t, S(t))dW(t) + d\sum_{k=1}^{N(t)} Z_k\Big] S(t). \tag{2.2.5}$$

Here the jump part is represented by a compound Poisson process, $\sum_{k=1}^{N(t)} Z_k$.

We first give the following definition of a Poisson process, which is a pure jump process and the building block of the JD models.

Definition 2.2.2. *Let* $T_m = \sum_{k=1}^{m} \tau_k$, $m \in \mathbb{R}$, *where* $\{\tau_i, i = 1, 2, \ldots\}$ *is a sequence of independent exponential random variables with the parameter* λ, *the* ***Poisson process*** $N(t)$ *with parameter* λ *is defined as follows,*

$$N(t) = \sum_{m\geq 1} 1_{t \geq T_m}. \tag{2.2.6}$$

The Poisson process is essentially a counting process that counts the number of jumps occuring before time t with the sizes of all jumps being equal to one unit. The random variable τ_i in Definition 2.2.2 denotes the units of time that the i-th jump takes after the $(i-1)$-th jump. Thus $T_m = \sum_{k=1}^{m} \tau_k$ denotes the arrival time of the m-th jump. $1_{t\geq T_m}$ is a delta function that equals to 1 when $t \geq T_m$.

However, the assumption of one unit size for all jumps is unrealistic, therefore the compound Poisson process with random jump sizes is more extensively used in JD models instead.

Definition 2.2.3. *Let $N(t)$ be a Poisson process with parameter λ, and $\{Z_k\}_{k\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) random variables. Then the* ***compound Poisson process*** *is defined as*

$$Y(t) = \sum_{k=1}^{N(t)} Z_k, \quad t \geq 0. \tag{2.2.7}$$

Note that the jumps in $Y(t)$ and $N(t)$ occur at the same time but with different sizes.

Now we present some significant properties of the jump process.

- **Martingale property** Let $N(t)$ and $Y(t)$ be the Poisson process and the compound Poisson process defined as Definition 2.2.2 and Definition 2.2.3 respectively. Then the compensated Poisson process $\tilde{N}(t)$ and the compensated compound Poisson process $\tilde{Y}(t)$ are given as follows and are both martingales.

$$\begin{aligned} \tilde{N}(t) &= N(t) - \lambda t, \\ \tilde{Y}(t) &= Y(t) - \mathbb{E}[Z_k]\lambda t, \end{aligned} \tag{2.2.8}$$

  where $\mathbb{E}[\cdot]$ denotes the expectation operator. This is a very useful property for risk-neutral pricing under the JD model.

- **Characteristic function** The characteristic function plays an essential role in the analysis of jump process. Generally it is difficult to have the closed-form distribution function of the jump process. Alternatively, the characteristic function is often less complicated and given, based on which we could obtain the associated integral of expectation of the jump diffusion part. Usually the characteristic function of a random variable $Y(t)$ is defined by

$$\Phi_Y(w) = \mathbb{E}[e^{iwY}]. \tag{2.2.9}$$

  The two most widely adopted characteristic functions of the jump process are the Merton-type one and Kou-type one given as follows,

$$\begin{aligned} \text{Merton: } \Phi_z(w) &= e^{j\mu w - \frac{w^2}{2}\sigma^2}, \\ \text{Kou: } \Phi_z(w) &= \frac{p\lambda_1}{\lambda_1 - jw} + \frac{(1-p)\lambda_2}{\lambda_2 + jw}. \end{aligned} \tag{2.2.10}$$

2.2.3 Stochastic dynamic programming for MVALM

The mean-variance asset-liability management (MVALM) is usually formulated as an optimal control problem which can be solved by employing the stochastic dynamic programming (DP) method. Originally proposed by Richard Bellman [32], the basic idea of DP is to break the complicated optimal control problem into a family of sub-problems and obtain the optimal solution recursively based on the relationship between the sub-problems that is established via a so-called Hamilton-Jacobi-Bellman (HJB) equation.

Now we briefly review the algorithm of deriving the optimal strategy for a continuous-time MVALM problem by using the stochastic DP technique.

Generally the MVALM problem is formulated as follows,

$$(P_1) \begin{cases} \min\limits_{u(\cdot)} & J(u(\cdot)) := Var[Z(T)], \\ s.t. & \mathbb{E}[Z(T)] = d, \quad u(\cdot) \in U, \end{cases} \tag{2.2.11}$$

where U denotes the set of all admissible strategies and the wealth surplus process $Z(t)$, in the simplest case, evolves according to the following stochastic differential equation (SDE):

$$dZ(t) = \mu(t, Z(t), u(t))dt + \sigma(t, Z(t), u(t))dW(t), \tag{2.2.12}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are the appreciation rate and the volatility rate respectively.

Step 1: Transform the constrained problem to an unconstrained one.

The constraint on $\mathbb{E}[Z(T)]$ can be incorporated into the objective function using the Lagrange multiplier method. For $w \in \mathbb{R}$, the constrained problem (P_1) satisfies the following dual equation:

$$\begin{aligned} \min_{u(\cdot)} J(u(\cdot)) := \min_{u(\cdot)} Var[Z(T)] & \\ &= \max_{w\in\mathbb{R}} \min_{u(\cdot)} \mathbb{E}\Big[(Z(T)-d)^2\Big] + 2w\Big(\mathbb{E}[Z(T)] - d\Big) \\ &= \max_{w\in\mathbb{R}} \min_{u(\cdot)} \mathbb{E}\Big[(Z(T)+w-d)^2\Big] - w^2. \end{aligned} \tag{2.2.13}$$

Step 2: Develop the HJB equation.

Before deriving the HJB equation, we first define the value function $V(z,t)$ as

$$V(z,t) = \inf_{u(\cdot)} \mathbb{E}\Big[(Z(T)+w-d)^2 \mid Z(t)=z\Big]. \tag{2.2.14}$$

Then the HJB equation that the value function should satisfy can be derived by using the following controlled infinitesimal operator

$$\mathcal{A}^u = \frac{\partial}{\partial t} + \mu(t,Z(t),u(t))\frac{\partial}{\partial z} + \frac{1}{2}\sigma^2(t,Z(t),u(t))\frac{\partial^2}{\partial z^2}, \tag{2.2.15}$$

with the boundary condition:

$$V(z,T) = (z+w-d)^2. \tag{2.2.16}$$

Step 3: Derive the optimal strategy and the corresponding value function

Usually we would first guess the expression of the value function as follows:

$$V(z,t) = P(t)z^2 + M(t)z + N(t). \tag{2.2.17}$$

Then we substitute the *Ansatz* into the HJB equation and derive the optimal strategy $\hat{u}(\cdot)$ in terms of the coefficients. After some calculation we would obtain a system of ordinary differential equations (ODEs) with boundary conditions that the coefficients satisfy. Thus we can finally derive the closed-form expression of $\hat{u}(\cdot)$ by solving the associated ODEs and obtain the optimal value of the unconstrained problem as $V(z_0,0)$.

Step 4: Derive the efficient portfolio and formulate the efficient frontier for the original MVALM problem

Based on the results from **Step 3**, we can finally obtain the efficient strategy for the original problem according to (2.2.13) as well as the first-order condition with respect to the Lagrange multiplier w and formulate the efficient frontier $(Var[Z(T)], d)$.

2.2.4 Risk-neutral pricing

The valuation process of financial derivatives is generally carried out under the risk-neutral probability measure which is equivalent to the real-world probability measure, and this is what we call the risk-neutral pricing.

The basic logic of the risk-neutral pricing can be described as follows. Under the risk-neutral probability measure $\widetilde{\mathbb{P}}$, the discounted stock price $D(t)S(t)$ is a martingale, where $D(t) = e^{-\int_0^t r(u)du}$ is the discounting process with the instantaneous interest rate $r(t)$. Based on the martingale property of $D(t)S(t)$, the value of a derivative $V(t)$ can be calculated as the conditional expectation of the discounted terminal pay-off function at the current time point as follows,

$$\begin{aligned} V(t) &= \frac{1}{D(t)}\widetilde{\mathbb{E}}\Big[D(T)V(T) \mid \mathcal{F}(t)\Big] \\ &= \frac{1}{D(t)}\widetilde{\mathbb{E}}\Big[D(T)h[S(T)] \mid \mathcal{F}(t)\Big], \end{aligned} \tag{2.2.18}$$

where $V(T) = h(S(T))$ is the terminal pay-off and $h(x)$ is a Borel-measurable function. According to the **Feynman-Kac Theorem**, $V(t)$ satisfies a partial differential equation (PDE). Thus the pricing problem is reduced to solving the associated PDE.

Now we introduce two useful theorems for risk-neutral pricing , the **Girsanov Theorem** and the **Feynman-Kac Theorem**.

Theorem 2.2.1. *(**Girsanov Theorem** [33]) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space where $\mathbb{P}$ denotes the real-world probability measure and $\mathcal{F}(t), t \in [0, T]$ is a filtration generated by a Brownian motion $W(t)$ defined on this probability space, namely, $\mathcal{F}(t) = \sigma\{W(s) : s \leq t\}$. Let $\xi(t)$ be an adapted process. Define*

$$Z(t) = e^{-\int_0^t \xi(s)dW(s) - \frac{1}{2}\int_0^t \xi^2(s)ds}, \tag{2.2.19}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \xi(s)ds, \tag{2.2.20}$$

and assume that

$$\mathbb{E}\int_0^T \xi^2(s)Z^2(s)ds < \infty. \tag{2.2.21}$$

Set $Z = Z(T)$. Then we have $\mathbb{E}[Z] = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by the following definition

$$\widetilde{\mathbb{P}}(B) = \int_B Z(\omega)d\mathbb{P}(\omega), \text{ for } \forall B \in \mathcal{F}, \tag{2.2.22}$$

the process $\widetilde{W}(t)$ is a Brownian motion.

The Girsanov Theorem provides a way to construct the risk-neutral probability measure $\widetilde{\mathbb{P}}$ which can be easily proved by applying the *Itô*'s formula. The proof is omitted here. Readers are referred to [33, 34] if interested.

$Z = Z(T)$ is called the *Radon-Nikodým* derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ which is used to generate the risk-neutral probability measure and satisfies

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}. \tag{2.2.23}$$

Moreover, according to the definition of $\widetilde{\mathbb{P}}$ in (2.2.22), for a random variable X, the following relationship between the expectation under two measures holds,

$$\widetilde{\mathbb{E}}[X] = \int X d\widetilde{\mathbb{P}} = \int ZX d\mathbb{P} = \mathbb{E}[ZX]. \tag{2.2.24}$$

We present the following example for constructing the risk-neutral probability measure by applying Theorem 2.2.1.

Example 2.2.1. *For the underlying asset $S(t)$ with the following dynamics:*

$$dS(t) = \Big[\mu dt + \sigma dW(t)\Big] S(t), \tag{2.2.25}$$

where μ and σ are the appreciation rate and the volatility rate. Let $r(t)$ be the instantaneous interest rate. Then we can define $Z(t)$ and $\widetilde{W}(t)$ as

$$\begin{aligned} Z(t) &= \exp\Big[-\int_0^t (\mu - r(s)) dW(s) - \frac{1}{2}\int_0^t (\mu - r(s))^2 ds \Big], \\ d\widetilde{W}(t) &= dW(t) + \frac{(\mu - r(t))}{\sigma} dt, \end{aligned}$$

We can derive the dynamics of the discounted price process as follows,

$$\begin{aligned} d(D(t)S(t)) &= d\Big[e^{-\int_0^t r(u)du} S(t)\Big] = e^{-\int_0^t r(u)du}\Big[(\mu - r(t))S(t)dt + \sigma S(t) dW(t)\Big] \\ &= e^{-\int_0^t r(u)du} \sigma S(t) d\widetilde{W}(t). \end{aligned} \tag{2.2.26}$$

Then it is obviously that under the probability measure $\widetilde{\mathbb{P}}$ generated by the Radon-Nikodým derivative $Z = Z(T)$, $D(t)S(t)$ is a martingale.

Theorem 2.2.2. *(**Feynman-Kac Theorem** [35]) Assume that the stochastic* *process $Y(t)$ satisfies the following stochastic differential equation (SDE):*

$$dY(t) = \mu(t, Y(t))dt + \sigma(t, Y(t))dW(t). \tag{2.2.27}$$

For a Borel-measurable function $h(z)$ and a fixed time horizon $[0, T]$. Define the function

$$l(t, y) = \mathbb{E}^{t,y}h[Y(T)]. \tag{2.2.28}$$

where $\mathbb{E}^{t,y}[\cdot]$ denotes the expectation conditional on the event $\{Y(t) = y\}$ at the time t. Assume that $\mathbb{E}^{t,y}h[Y(T)] < \infty$. Then $l(t, y)$ satisfies the following partial differential equation

$$l_t(t, y) + \mu(t, y)l_y(t, y) + \frac{1}{2}\sigma^2(t, y)l_{yy}(t, y) = 0, \tag{2.2.29}$$

with the terminal condition

$$l(T, y) = h(y) \text{ for all } y. \tag{2.2.30}$$

The derivation of the PDE (2.2.29) is based on the following theorem and the martingale property of $l(t, Y(t))$.

Theorem 2.2.3. *(**Markov property** [36]) Let $Y(t), t \geq 0$ be a solution to the SDE in (2.2.27) with initial condition given at time 0. Then for $t \in [0, T]$, we have*

$$\mathbb{E}\Big[h(Y(T)) \mid \mathcal{F}(t)\Big] = l(t, Y(t)). \tag{2.2.31}$$

What the above theorem specifies is that the conditional expectation of the terminal condition $h(Y(T))$ given the information up to time t is a function of time t and the state process $Y(t)$. In this sense, we can replace $Y(t)$ with a dummy variable y and compute $l(t, y) = \mathbb{E}^{t,y}h[Y(T)]$ first and put the random variable $Y(t)$ back afterwards for further computation.

Moreover, since $l(t, Y(t))$ denotes the conditional expectation, it is easy to prove that $l(t, Y(t))$ satisfies the martingale property. Thus for any SDE defined as (2.2.27), we can derive the dynamics of $l(t, Y(t))$ according to *Itô*'s formula:

$$\begin{aligned} dl(t, Y(t)) &= l_t dt + l_y dY + \frac{1}{2}dY dY \\ &= [l_t + \mu l_y dt + \frac{1}{2}\sigma^2 l_{yy}]dt + \sigma l_y dW. \end{aligned} \tag{2.2.32}$$

Based on the martingale property, the diffusion part should equal to 0, leading to the PDE in (2.2.29).

2.2.5 Generalized Fourier transform

In subsection 2.2.4, we showed that the pricing problem of a derivative is reduced to solving the associated partial differential equation (PDE) derived according to the Feynman-Kac theorem. In this subsection, we briefly introduce one of the essential techniques in finance, the generalized Fourier transform method, which can be employed to solve the PDE with respect to the pricing problem.

Definition 2.2.4. *(**Generalized Fourier transform** [37]) Let $V(x)$ be the value of the derivative that satisfies the PDE derived by the Feynman-Kac theorem, which is a function of the current state of the underlying asset $X(t)$. Then the generalized Fourier transform of $V(x)$, denoted by $\mathcal{F}[V(x)]$, is given by*

$$\mathcal{F}[V(x)] = \int_{-\infty}^{+\infty} V(x)e^{-j\omega x}dx = U(\omega), \tag{2.2.33}$$

where $j = \sqrt{-1}$ and ω is the Fourier transform frequency.

Some useful properties of Fourier transform are given as follows,

(1) **Inverse Fourier transform** The inverse Fourier transform is defined as:

$$V(x) = \mathcal{F}^{-1}[U(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(\omega)e^{j\omega x}d\omega. \tag{2.2.34}$$

With the inverse Fourier transform, we can obtain the solution $V(x)$ to the original PDE based on the solution $U(\omega)$ we derived for the transformed-PDE.

(2) **Differentiation** With the Fourier transform, we can transform the partial derivative in the original PDE as follows,

$$\mathcal{F}\Big[\frac{\partial^n V(x)}{\partial X^n}\Big] = (j\omega)^n U(\omega). \tag{2.2.35}$$

Thus we can reduce the dimension of the associated PDE as well as the difficulty in computation.

(3) **Delta function** For a delta function $\delta_{x_0}(x)$ defined as

$$\delta_{x_0}(x) = \delta(x_0 - x) = \begin{cases} 1, x = x_0, \\ 0, x \neq x_0, \end{cases} \tag{2.2.36}$$

with the integration property $\int_R \delta_a F(t)dt = F(a)$, for $\forall a \in R$ and any integrable function $F(t)$, we have the corresponding Fourier transform:

$$\mathcal{F}[\delta_{x_0}(x)] = e^{-jx_0\omega}. \tag{2.2.37}$$

This can be useful for transforming the terminal conditions for the pricing of variance swaps.

(4) **Characteristic function** The characteristic function of a random variable X is closely related to the Fourier transform and can be given by

$$\Phi_X(\omega) = \mathbb{E}[e^{j\omega X}] = \int_R e^{j\omega x} p(x)dx, \tag{2.2.38}$$

where Φ_X denotes the characteristic function of X and $p(x)$ is the probability density function of X. We can see that Φ_X is essentially a complex conjugate of the Fourier transform of $p(x)$.

2.3 Various RSJD Models

2.3.1 A general overview

Abrupt changes can take place in financial markets and may persist for a long period, affecting the performance of the markets. For example, during the financial crises, the financial assets' and derivatives' prices, interest rates, investors' behaviour and expectations change dramatically towards a negative direction and the phenomenon can last for years or decades. Therefore, this regime-switching property should be considered when we are developing models for investment strategies, pricing of financial assets and derivatives and other financial problems.

Regime-switching (RS) models have been widely used to capture the simultaneously changes of the regime-dependent model dynamics. The application of Markov RS models on financial problems can be traced back to the works by [38] and [39] where two separate regimes are considered for linear regressions. Then

Hamilton [12] proposed a very tractable approach to modelling regime switching and adopted the RS model to investigate the stock return time series, motivating the extensive successive research on the applications of RS models to other financial topics. A hidden Markov chain is used to model the switches between market states in the pricing of American options in [13] where the regimes are not observable. Elliott et al. [15] developed a random Esscher transform to determine a risk neutral measure in a RS market, which has been widely adopted by research on option or derivative pricing with regime switching. Most recently, Muhammad et al. [40] compared the regime-switching GARCH model with a single-regime counterpart in predicting the Value-at-Risk (VaR) in the precious metals markets and proved the efficiency and accuracy of the RSGARCH model. Novel approaches are developed in [41] for option pricing under regime switching where the path-dependence side effects are dealt with by novel and intuitive risk-neutral measures.

As we mentioned in Chapter 1, generally the price processes in the RS models under each given regime are still considered stochastic processes with continuity. While in the real world, even if the business cycle has entered a somewhat stable stage, there could still be unexpected events affecting the performance of the financial markets, motivating the applications of jump-diffusion models in finance. In 1976, Merton [5] pointed out that geometric Brownian motions with continuous sample paths can only model the "normal" vibrations of the stock prices, which may arise from the imbalance between supply and demand. While there still exists "abnormal" vibrations due to new information which can cause a substantial effect on the stock price. And this component of price changes is modelled by a "jump" process, whose prototype is a "Poisson-driven process". In the work of [42], an explicit option pricing formula is obtained via Fourier transform where the stock price is modelled by a Lévy process consisting of the continuous-time diffusion and a jump process. A double exponential jump diffusion model is considered for the pricing of American options and popular path-dependent options in [43]. In 2018, Marianito et al. [44] developed closed-form pricing formulas for European options whose underlying stock pays dividends and is modelled by a jump-diffusion process.

The regime-switching jump-diffusion (RSJD) models, or the Markov-modulated jump-diffusion models, which capture both the effect of the changes of the macroeconomic environment and the abnormal events, have attracted much attention from the researchers in recent years.

In 2012, Zhang et al. [29] established a sufficient stochastic maximum principle for the stochastic control problem with RSJD and presented financial applications to problems such as the mean-variance portfolio selection. While Azevedo et al. [45] developed the dynamic programming technique for the optimal control problem under RSJD framework and established the corresponding Hamilton-Jacobi-Bellman equation. They provided detailed proof of Bellman's optimality principle and presented an application of their results to a consumption-investment problem in such a RSJD market. The pricing of contingent claims under RSJD models is investigated in [46] where explicit pricing formula and a multinomial approach is obtained via a backward induction scheme. Bo et al. [47] priced the variance swaps in a RSJD market and obtained a dynamic optimal investment strategy for the variance swaps in closed form via dynamic programming methods. The optimal dividend payment strategy for an insurance company with a RSJD surplus is studied in [48]. Mollapourasl et al. [49] investigated localized kernel-based approximation for pricing both European and American options in a RSJD market by formulating free or fixed boundary problems.

In the following subsection, we will briefly introduce various types of RSJD models that we will develop in this thesis.

2.3.2 RSJD models

Consider a finite-state continuous-time observable Markov chain $\alpha(t)$ with the corresponding state space $E = \{e_1, \ldots, e_n\}$, where $e_i = (0, \ldots, 1, \ldots, 0)' \in \mathbb{R}^n$ is a n-dimensional canonical unit vector. Here M' denotes the transpose of a vector or a matrix M. Assume that there is only one risky asset in the financial market denoted by $S(t)$, then the dynamics of the price process of $S(t)$ under various RSJD models are formulated as follows,

Basic regime-switching model with independent jump diffusion

$$\begin{cases} dS(t) = S(t)\Big[\mu_{\alpha(t)}(t)dt + \sigma_{\alpha(t)}(t)dW(t) + d\sum_{m=1}^{M(t)} J_m\Big], \\ t \in [0, T], \\ S(0) = S_0 > 0, \end{cases} \tag{2.3.1}$$

where $\mu_{\alpha(t)}(t)$ and $\sigma_{\alpha(t)}(t)$ are the Markov-modulated appreciation rate and volatility rate of the risky asset $S(t)$ respectively. Specifically, $\mu_{\alpha(t)}(t)) =< \mu(t), \alpha(t) >$, $\sigma_{\alpha(t)}(t) =< \sigma, \alpha(t) >$, where $\mu(t) = (\mu_1(t), \ldots, \mu_n(t))$, $\sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t))$ and $< \cdot, \cdot >$ denotes the inner product in $\mathbb{R}^n$. For $i = 1, \ldots, n$, $\mu_i(t)$ denotes the appreciation rate under the i-th market regime, and $\sigma_i(t)$ is defined similarly. Thus the switching regime represented by the Markov chain is incorporated into our model. Moreover, $W(t)$ is a standard Brownian motion. J_m is the size of the m-th jump. $M(t)$ is a counting process that represents the number of jumps up to time t, which is essentially a Poisson process with intensity λ_m. Thus the compound Poisson process $\sum_{m=1}^{M(t)} J_m$ represents the cumulative amount of jumps in the time interval $[0, T]$. S_0 denotes the initial stock price.

Under this model, the jump process is assumed to be independent of the switching regime and is not modulated by the Markov chain. This can be interpreted as the situation where the unexpected event is irrelevant to the structural changes of the macroeconomic environment. Alternatively, in other cases, some of the abnormal vibrations may be closely linked to the macroeconomic condition. Thus we would formulate the following model,

Basic regime-switching model with Markov-modulated jump diffusion

$$\begin{cases} dS(t) = S(t)\Big[\mu_{\alpha(t)}(t)dt + \sigma_{\alpha(t)}(t)dW(t) + \displaystyle\int_R J_{\alpha(t)}(t,z)\tilde{N}_{\alpha(t)}(dz,dt)\Big], \\ t \in [0,T], \\ S(0) = S_0 > 0, \end{cases} \tag{2.3.2}$$

where $J_{\alpha(t)}(t, z)$ is the Markov-modulated generalized form of the jump size z. $\tilde{N}_{\alpha(t)}(dz, dt)$ is a compensated Poisson random measure which is also dependent on the market regime and can be given by:

$$\tilde{N}_{\alpha(t)}(dz, dt) = N_{\alpha(t)}(dz, dt) - \lambda_{\alpha(t)} v_{\alpha(t)}(dz)\eta(dt), \tag{2.3.3}$$

where $v_{\alpha(t)}(dz)$ denotes the jump size distribution and $\lambda_{\alpha(t)}$ is the jump intensity. $N_{\alpha(t)}(dz, dt)$ is the Markov-modulated Poisson Random measure and $\eta(dt)$ is a generalized form of dt. All of the parameters are defined similarly to $\mu_{\alpha(t)}(t)$ and $\sigma_{\alpha(t)}(t)$ as an inner product with the Markov chain. Under this model, the jump intensity and jump size along with its distribution takes different values or forms under different market regime.

Heston-RSJD model

Previous empirical studies reported that the volatility of a stock moves stochastically over time rather than stays deterministic [50–52]. Thus plenty of stochastic volatility (SV) models have been developed and applied in finance to capture the features of the implied volatility such as the volatility smile. In

1993, Heston [3] proposed the most famous SV model where the volatility rate of the stock is assumed to be a stochastic mean-reverting process and derived a closed-form pricing formula for options under this model. Heston's model exhibits superior performance over the Black-Scholes model and have been extensively adopted over decades.

We take into account the SV and formulate the Heston-RSJD model as follows,

$$\begin{cases} dS(t) = \Big[\mu_{\alpha(t)}(t)dt + \sigma(y(t))dW_s + \int_R J_{\alpha(t)}(t,z)\tilde{N}_{\alpha(t)}(dz,dt)\Big]S(t), \\ dy(t) = k(\theta_{\alpha(t)} - y(t))dt + \sigma_y\sqrt{y(t)}dW_y, \end{cases} \tag{2.3.4}$$

where $\sigma(y(t))$ is a generalized form of the stochastic volatility process $y(t)$ that satisfies the second stochastic differential equation. k is corresponding to the speed of mean reversion adjustment, $\theta_{\alpha(t)} =< \theta, \alpha(t) >$, $\theta = (\theta_1, \dots, \theta_n)'$ denotes the long run average of the volatility rate and σ_v is the so-called volatility of volatility. W_s and W_y are two standard Brownian motions, which are usually assumed to be correlated by a constant correlation coefficient ρ.

Heston-CIR model with Markov-modulated jump diffusion

In plenty of literatures [53–55], the stochastic volatility is often investigated together with the stochastic interest rate. The consideration for the stochastic property of the interest rate results from the fact that the maturities for many contingent claims contracts can be as long as five years or beyond when the interest rate cannot be taken as a constant or deterministic function. The most well-known stochastic interest rate models are the Cox–Ingersoll–Ross (CIR), the Hull-white and the Vasicek model. Here we formulate the following RSJD model combining both Heston's stochastic volatility and the stochastic interest rate that follows a CIR process,

$$\begin{cases} dS(t) = \Big[\mu_{\alpha(t)}(t)dt + \sigma(y(t))dW_s(t) + \int_{\mathbb{R}} J^s_{\alpha(t)}(t,z^s)\tilde{N}^s_{\alpha(t)}(dt,dz^s)\Big]S(t), \\ dy(t) = k(\theta_{\alpha(t)} - y(t))dt + \sigma_v\sqrt{y(t)}dW_y(t) + \int_{\mathbb{R}} J^y_{\alpha(t)}(t,z^y)\tilde{N}^y_{\alpha(t)}(dt,dz^y), \\ dr(t) = b(\gamma_{\alpha(t)} - r(t))dt + \eta\sqrt{r(t)}dW_r(t), \end{cases} \tag{2.3.5}$$

where $J^s_{\alpha(t)}(\cdot)$ and $J^y_{\alpha(t)}(\cdot)$ are the generalized jump sizes with respect to the stock price and the volatility process respectively. $\tilde{N}^s_{\alpha(t)}(dt,dz^s)$ and $\tilde{N}^y_{\alpha(t)}(dt,dz^y)$

denote the Markov-modulated compensated Poisson random measures. b and $\gamma_{\alpha(t)}$ indicate the speed and the mean of the reverting adjustment. η denotes the volatility of the instantaneous interest rate process.

2.4 Mean-Variance Asset-Liability Management

The Nobel-Prize-winning work of Harry Markowitz [30] first investigated the portfolio selection problem under the mean-variance (M-V) criterion, which was referred to as the "expected returns-variance of returns" rule, and undoubtedly laid the foundation of the modern portfolio theory and even the modern mathematical finance theory. Markowtiz's mean-variance portfolio selection aims to optimize the allocation of wealth among the risky assets to best handle the trade-off between the overall return and risk of the portfolio where the risk that comes from the uncertainty in the future return is measured by variance. In Markowitz's work, the optimal strategy of a single-period portfolio selection problem is derived, and some significant concepts in the portfolio theory such as the efficient frontier and expected return are introduced.

However, the milestone work exhibits some limitations. For example, it only considers a single-period setting and it assumes a frictionless complete market. Therefore, efforts have been made by a vast number of researchers to employ and extend the mean-variance framework. One breakthrough research was made by Li and Ng in [56] where an embedding technique was first adopted to obtain the optimal strategy for a multi-period M-V portfolio selection problem which had been difficult to solve as a result of the non-separability of the M-V objective function. In the same year, Zhou and Li [57] investigated the continuous-time version of the same problem by also employing the embedding technique. In the work of Lim and Zhou [58], the model coefficients are assumed to be random instead of deterministic. The M-V portfolio selection problem is reduced to solving a backward stochastic differential equation (BSDE). Bielecki et al. [59] further considered a bankruptcy prohibition constraint based on the M-V portfolio selection model with random parameters. The investor's wealth of the portfolio is required to remain above zero at all times. A decomposition approach is adopted to obtain the constrained optimal control and efficient frontier. Most recently, Wang and Wei [60] established a continuous-time M-V portfolio selection model where the parameters are both random and modulated by a Markov chain. A new system of mean-field BSDEs driven by the Markov chain are formulated to obtain the analytical expressions of the optimal strategy. More research progress on the M-V portfolio selection can be found in [61–66].

As a matter of fact, the mean-variance asset-liability management (MVALM) is among the extensions of the Markowitz's mean-variance portfolio selection model, taking into account the exogenous liability faced by the investors. The major concern in a MVALM problem becomes the terminal surplus wealth, defined as the wealth of the portfolio subtracted by the uncontrollable liability. Investors seek an optimal investing strategy to minimize the variance of the terminal surplus given a fixed level of expected terminal surplus wealth. The first ALM model under the M-V criterion was investigated by Sharpe and Tint [67] in a static setting. Sequentially, Keel and Müller [68] provided proof that the efficient frontier (EF) would shift due to the effect of liabilities. Then a multi-period MVALM model was first considered in the paper by Leippold et al. [69] where the explicit expressions of the optimal control and the EF are derived by applying the embedding technique and a geometric approach. In 2006, Chiu and Li [70] established a continuous-time MVALM framework where geometric Brownian motions are used to model the price processes and the optimal strategy are obtained by employing the linear-quadratic method. The impact of incorporating liability and the optimal funding ratio are also investigated in their paper.

Following Leippold and Chiu, a great deal of research has been made to improve the MVALM models both in a multi-period and continuous-time setting by relaxing some of the assumptions and considering more realistic constraints. We here briefly review some of the MVALM literatures incorporating realistic factors such as endogenous liabilities, uncertain exit time, bankruptcy control, a constant elasticity variance process, stochastic interest rate, stochastic volatility and transaction costs.

Though the liability is often considered uncontrollable, some of the investors may be able to manage and optimize their liabilities, which may be the securities they issued to raise funds. Thus Leippold et al. [71] first considered endogenous liabilities and studied the strategy that optimizes simultaneously the assets and liabilities in a multi-period setting. It is showed in their paper that the efficient frontier could be decomposed to an orthogonal set of basis returns. After that, a continuous-time efficient strategy for the MVALM problem with endogenous liabilities is derived in closed form by Yao et al. [72]. Yao et al. [73] further incorporated uncertain exit time and Markov jumps to extend their model. In real world, the investment process can be interrupted by unexpected events, leading to investors exiting the market before the pre-specified terminal time. In the work of Yi et al. [74], by assuming that the uncertain exit time follows a given distribution, the problem is reduced to the familiar one with certain investment horizon. Cui et al. [75] established a mean-field formulation for the uncertain-exit-time MVALM problem where two-dimensional state variables are involved to study the influence of an uncertain exit time. Due to the consideration of liability, there is a higher probability that the investor may go bankruptcy and have to borrow money for investment. In the paper of Li et al. [76], the investor intends to control the probability of bankruptcy during the investment period. By constructing and solving a Lagrangian problem and a corresponding auxiliary problem through a looping algorithm that keeps running as long as the current wealth is above zero, they obtained the optimal solution when the algorithm stops. The real financial market is not frictionless so investors cannot trade as often as they wish without transaction costs. Thus Zhou et al. [77] considered the MVALM model in a financial market with quadratic transaction costs and obtained both pre-committed and time-consistent strategies in closed form.

Apart from the constraints set on the financial markets, researchers are also dedicated to establishing more realistic and generalized models for the price processes of the financial assets. Zhang and Chen [78] considered a MVALM problem under constant elasticity of variance (CEV) processes where the efficient portfolio are obtained via two linear backward stochastic differential equations (BSDEs) with unbounded stochastic coefficients. They provided proof of the existence and uniqueness of the complicated BSDEs and established solutions under special cases. The interest rate is assumed to follow the Hull-White process in the paper by Pan and Xiao [79] where inflation risk also exists and the investor can invest in a zero-coupon bond and an inflation-indexed bond besides the stocks and cash. Pan et al. [80] employed the Heston's stochastic volatility model to investigate a continuous-time MVALM and presented a sensitivity analysis of the parameters with respect to the volatility process on the efficient frontier. The stochastic volatility model is also adopted in the work of Li et al. [81] where a derivative is allowed to be traded besides a bond and a stock and they derived the explicit expressions of the optimal strategies and the EF via two backward stochastic differential equations.

However, we have so far neglected a significant problem which has drawn more and more attractions these years, the time inconsistency property of the MVALM problem. The time inconsistency problem arises from the non-linearity of the MV objective functional, making it difficult to apply the dynamic programming techniques since the Bellman optimality principle does not hold anymore. Therefore the optimal strategy that maximizes the objective functional at initial time does not necessarily optimize the ones at latter time. Basically there are three approaches to handle the time-inconsistency problem. Firstly, the investor adopts different optimal strategies each day that maximizes the M-V objective functional at that time. Secondly, as in all the literatures mentioned above, only the optimal strategy obtained at the initial time is investigated and derived, regardless of the fact that it may not be optimal for the objective functionals in the future and it is assumed that the investor would hold on to the allocation rule which is the so-called pre-committed strategy. The third way is to take the time consistency more seriously and investigate the problem in a game theoretic framework. The game theoretic approach that addresses general time inconsistency via Nash equilibrium was first investigated by Scrotz [82] and Pollak [83] who considered a deterministic Ramsey problem. Following them, consecutive work is provided in [84], [85], [86] and [87]. Especially, Björk and Murgoci [87] provided the first study of a general Markovian framework where a general controlled Markov process and a general objective functional are considered. The authors generalized many previously models, presented a verification theorem and derived an extended HJB equation which plays an essential role in the game theoretic approach to the time inconsistent problems. From then on, a vast number of papers adopted this game theoretic framework. For example, Björk et al. [88] considered a risk aversion coefficient which is dependent on the current wealth condition in the MV portfolio selection problem and derived a time-consistent equilibrium control. Li et al. [89] derived the time-consistent reinsurance-investment strategy for an insurer and a reinsurer under the M-V criterion under the constant elasticity of variance (CEV) model. Wei and Wang [90] investigated a MVALM problem with random coefficients and obtained a time-consistent equilibrium strategy in a linear feedback form of the surplus and the liability. Zhang et al. [91] derived the extended HJB equations for a MVALM problem with state-dependent risk aversion and obtained the time-consistent equilibrium control through the verification theorem.

Now we move on to the applications of RSJD models on the MVALM problem. Regime switching (R-S) was originally considered under the M-V framework by Zhou and Yin [?] in a continuous-time portfolio selection problem where they showed that unexpectedly the results are similar to the non-regime-switching counterparts when the interest rate is not regime-dependent. Chen et al. [20] employed maximum principle to investigate a continuous-time MVALM problem with R-S and obtained the optimal feedback control via four systems of ordinary differential equations. Three years later, Chen and Yang [92] established a multi-period R-S model and compared the results with and without liabilities. Inspired by the game theoretic framework of [87], Wei et al. [21] derived the equilibrium control and value function for a regime-switching MVALM via the extended HJB equations and verification theorem. Along another line, Zeng and Li [93] compared the benchmark and M-V criteria of the ALM problem under a jump-diffusion model where the price processes of the assets and the liabilities are modelled by an exponential Lévy process and a Lévy process respectively. Zeng and Li [94] further extended their jump diffusion model under a time-consistent framework and derived both investment and reinsurance strategies for a mean-variance investor.

The regime-switching jump-diffusion (RSJD) model was investigated by Zhang et al. [29] where a sufficient stochastic maximum principle is established and applied to the M-V portfolio selection. Wu [25] used a Markov-modulated Lévy process and a Brownian motion to model the price process of the stock and investors' stochastic cash flow respectively in a M-V portfolio selection problem. Then Yu [26] formulated a more generalized Markov-switching jump-diffusion framework of the MVALM problem and derived the feasibility condition as well as the efficient strategy by employing the stochastic linear-quadratic technique.

However, so far the literatures above regarding RSJD models only focused on the pre-committed strategy of the M-V portfolio selection or ALM problem. In this thesis, we aim to establish a basic RSJD model for a MVALM problem under a game theoretic framework and derive the corresponding Nash equilibrium control and value function, which is one of our major contributions.

In Section 2.2.3 we reviewed the algorithm of deriving the efficient strategy and efficient frontier by using the stochastic dynamic programming, which is on a pre-committed basis. Now we briefly introduce the algorithm of deriving the time-consistent equilibrium strategy by also applying the stochastic dynamic programming technique. For more details, the readers can refer to [86, 87]

Step 1: Formulate the objective functional.

Under the game theoretical framework, the objective functional is generally formulated as follows,

$$\begin{aligned} J(t,z,\pi(\cdot)) &= \mathbb{E}_{t,z}[Z^{\pi}(T)] - \frac{\gamma(\cdot)}{2} Var_{t,z}[Z^{\pi}(T)] \\ &= \mathbb{E}_{t,z}[Z^{\pi}(T)] - \frac{\gamma(\cdot)}{2}\Big\{\mathbb{E}_{t,z}[Z^{\pi}(T)^2] - \mathbb{E}^2_{t,z}[Z^{\pi}(T)]\Big\} \\ &= \mathbb{E}_{t,z}[F(z,Z^{\pi}(T))] + G(z,\mathbb{E}_{t,z,}[Z^{\pi}(T)]), \end{aligned} \tag{2.4.1}$$

where $\mathbb{E}_{t,z}[\cdot]$ and $Var_{t,z}[\cdot]$ are the expectation and variance conditional on the current surplus condition $\{Z(t) = z\}$ respectively. $\gamma(\cdot)$ denotes the risk aversion coefficient of the investor, which can be state-dependent as $\gamma(z)$ or regime-dependent as $\gamma(j)$ with j representing the j-th market state. $F(z,x) = x - \frac{\gamma(\cdot)}{2}x^2$, $G(z,x) = \frac{\gamma(\cdot)}{2}x^2$.

Step 2: Define the equilibrium control law and the equilibrium value function

The problem is considered as a non-cooperate game where the investor at each time point $t \in [0,T]$ is treated as a different player. The equilibrium control $\hat{\pi}$ is defined as such a strategy that if for all players $k > t$, $\hat{\pi}$ is the optimal strategy they choose, then it is also the optimal strategy for player t.

Step 3: Derive the extended HJB equations and establish the verification theorem

The extended HJB equations are a system of PDEs that the value function satisfies. Based on the extended HJB equations and the verification theorem, the problem is reduced to solving a number of ODEs.

Step 4: Obtain the equilibrium control and value function by solving the associated ODEs derived from the extended HJB equations.

2.5 Variance (Volatility) Swaps Pricing

Variance (volatility) is often used as a measurement of the risk of investing in the underlying asset. The variance (volatility) changes sarcastically over the investment period, providing practitioners opportunities to speculate on the spread between the realized variance (volatility) and the implied variance (volatility), as well as the motivation to hedge against the variance risk. As a consequence, variance (volatility) related derivative products such as the volatility index (VIX) futures, the VIX options, variance swaps and volatility swaps have emerged and are widely traded over the counter or in the exchanges in recent years.

Among all the variance (volatility) related derivative products, variance (volatility) swaps have drawn much attention from the practitioners and researchers. A variance (volatility) swap is not a swap in a traditional sense, but a forward contract whose payoff at expiry is determined by the difference between the realized variance (volatility) and a pre-set fixed strike price. The realized variance (volatility) is usually calculated according to a pre-specified formula. For details regarding calculation of the realized variance (volatility), one can refer to [95–97]. A long position in a variance (volatility) swap generates profit if the realized variance exceeds the pre-set strike price.

Numerous research has been carried out on variance (volatility) swap pricing over the last decades. The valuation approaches can be categorized into two types: the non-parametric model-free approach and the parametric stochastic approach.

Early studies of variance (volatility) swaps mainly focused on the model-free approach. The main idea of the model-free pricing technique is to replicate the variance (volatility) swap with a portfolio composed of a call option, a put option and a forward, and follow the routine of calculating the VIX index (see [98]). It is less complicated and easier to apply since it does not involve any assumption of the specific form of the dynamics of the asset's price process. The replicating method was first proposed in [99] where a log contract is replicated with a static position in a series of options. Then in 1994, Neuberger [100] claimed that a variance swap with continuous sampling times could be replicated with a log contract and a dynamically adjusted position in the underlying asset under certain circumstances. Carr and Lee [101] developed robust replicating portfolios for volatility derivatives that trade on vanilla options and the underlying asset and obtained explicit pricing formulas for variance swaps and variance options under an independence condition. Two years later, Carr and Lee [102] extended their results to allow for a continuous semi-martingale process and obtained lower and upper bounds on the prices of variance options.

Although this model-free method gives the prices of variance (volatility) swaps as well as a hedging strategy, it still exhibits several drawbacks. First of all, it relies heavily on the assumption that the price process of the underlying asset follows a continuous path, which is empirically proved unrealistic. Secondly, the approach assumes continuously sampled variance (volatility) swaps, which is only an approximation for the real-world ones and in some situations might not be a good proxy. Last but not least, the replication strategy requires continuous exercise of the options, which is not operable.

As a consequence, to deal with more sophisticated and realistic variance (volatility) swaps, researchers developed the parametric stochastic approach which is based on the assumption that the underlying asset's price process is driven by a specific stochastic process. The stochastic approach can be further categorized into analytical methods and numerical methods. Analytical methods have been adopted in order to derive closed-form expressions of the fair strike prices.

Generally, the payoff function of a long position in a variance swap at expiry takes the form $V(T) = (\sigma_R^2 - K_{var}) \times G$, where σ_R^2 denotes the annualized realized variance, K_{var} is the strike price of the variance swap, and G denotes the notional amount of the swap in dollars per volatility point squared. One significant feature of the variance swap useful in the valuation process is that it requires zero initial cost since it is essentially a forward contract. Therefore, if we denote the value of a variance swap at time t as $V(t)$, which equals the expected present value of the payoff under the risk-neutral measurement of $\mathbb{Q}$, we can have the following initial value:

$$V(0) = \mathbb{E}_0^{\mathbb{Q}}\Big[e^{-\int_0^T r_t dt}(\sigma_R^2 - K_{var}) \times G\Big] = 0, \tag{2.5.1}$$

where r_t is the related interest rate and $\mathbb{E}_t^{\mathbb{Q}}$ denotes the conditional expectation at time t. Based on the above equation, the fair strike price can be obtained as

$$K_{var} = \mathbb{E}_0^{\mathbb{Q}}[\sigma_R^2]. \tag{2.5.2}$$

Usually σ_R^2 is calculated according to some pre-determined formula specified in the contract. Plenty of research on the analytical methods for pricing variance (volatility) swaps has been done based on (2.5.2).

Heston and Nandi [103] proposed analytical pricing formulae for volatility swaps as well as other volatility derivatives based on the GARCH volatility model. A hedging strategy that trades only on the underlying asset and a risk-free asset is also presented, compared to the hedging strategies on a continuum of options in previous literatures. Carr [104] developed the pricing formulae for derivatives on future realized quadratic variation where the underlying stock's price follows a Lévy process and illustrated substantial differences from the counterparts in a continuous setting. Elliott et al. [105] established a Markov-modulated stochastic volatility model for the pricing of volatility swaps and derived strike prices both from a probabilistic perspective and a differential equation perspective. Broadie and Jain [106] investigated the pricing of variance swaps via a partial differential equation approach and presented a hedging strategy for volatility derivatives using variance swaps based on the no-arbitrage relationship. Itkin and Carr [107] considered various stochastic time-changed Lévy models for the pricing of variance swaps and compared the pricing formulae obtained via a forward characteristic function approach and a log-contract approach.

However, the literatures mentioned above, mostly in early years, though taking into account more realistic factors, still relies on the continuously-sampled approximation for the realized variance. Even though the continuous approximation may provide reasonable estimates when the observation frequencies for the real-world variance (volatility) swaps are high, it can diverge drastically from the prices for variance (volatility) swaps with low sampling frequencies, leading to large relative errors.

As a result, most of the recent literatures have shifted attention directly to discretely sampled variance (volatility) swaps on yearly, quarterly, monthly and daily bases. Broadie [108] investigated the fair strike prices of discretely-sampled variance and volatility swaps under various stochastic models, including the jump-diffusion models as well as the stochastic volatility models, and compared the obtained pricing formulae with the continuously-sampled counterpart, concluding that the discrete price converges to the continuous price as the observation frequency approaches infinity. A variance swap with discrete sampling times is priced in [109] based on the Heston stochastic volatility model where an explicit solution with high efficiency and accuracy is derived by adopting the dimension reduction technique and using the generalized Fourier transform. A hybrid model is established in [110] incorporating both stochastic volatility and stochastic interest rate for the discretely-sampled variance swaps where the authors examined both the effect of the volatility and interest rate on the results. Most recently, Yan and Zhao [111] proposed a tractable approach to price volatility derivative with discrete sampling times under a general stochastic volatility model. In their work, the underlying volatility is assumed to comply with a beta prime distribution, which is flexible and consistent with the feature of the market data, and the jump diffusion process is also captured by extending the model through stochastic time changes.

However, it can be technically difficult to obtain analytical solutions most of the time which gives rise to various numerical methods as an alternative. In 2001, Little and Pant [112] assumed that the local volatility varies with both time and the stock price according to a known function, and studied the discretely-sampled variance swap via a finite difference method. They reduced the dimension of the pricing problem by introducing a two-stage approach, which greatly improved the efficiency and accuracy of their pricing formula. Windcliff et al. [113] established a general framework applicable for various stochastic models using the numerical partial integral differential equation (PIDE) methods and compared the effects of transaction costs, jump-diffusion and local volatility on the discretely-sampled variance swaps prices. A numerical partial differential equation (PDE) method is adopted in [108] to derive the fair volatility strikes with jumps and discrete sampling times where it is showed that discrete sampling exhibits little effect when the effect of jumps is significant.

Elliott et al. [105] originally considered regime-switching (R-S) models for pricing volatility swaps and presented a R-S Esscher transform under the Markov-modulted stochastic volatility model to determine the risk neutral measure. However, they only considered the continuously-sampled approximation. Therefore Elliott and Lian [114] extended their results to variance and volatility swaps with discrete sampling times by applying the characteristic function method. They presented a very useful idea to reduce the difficulty of calculation caused by the incorporation of the Markov chain, which is to determine the forward characteristic function based on a given realized path of the Markov chain and then consider the effect of changing paths on the conditional expectation. Shen and Siu [115] investigated a *Schöbel*-Zhu-Hull-White stochastic interest rate model modulated by a continuous-time observable Markov chain and illustrated the significant effect of regime switching as well as the instantaneous interest rate with numerical examples. Dilloo and Tangman [116] adopted a finite difference method for pricing variance swaps under R-S models as well as other stochastic models and validated the results with obtained analytical solutions under R-S models and Merton's models.

To our knowledge, little work has been done on the pricing of volatility derivatives under regime-switching jump-diffusion (RSJD) models, therefore we aim to establish a Heston-RSJD model and a Heston-CIR model with Markov-modulated jump diffusion for the pricing of variance swaps and volatility swaps in this thesis.

2.6 Concluding Remarks

In this chapter, we review some essential concepts for the regime-switching jump-diffusion (RSJD) models as well as methodologies for solving mean-variance asset-liability management (MVALM) and the pricing of variance (volatility) swaps. The RSJD model assumes jump diffusion processes whose parameters are dependent on a finite-state observable continuous-time Markov chain. Generally a jump diffusion process refers to a geometric Brownian motion with jumps described by a compound Poisson process. The Markov chain defined with a transition matrix whose elements stand for the transition rates between two regimes is used to represent the market states. The MVALM is usually formulated as a stochastic control problem which is generally solved by utilizing the stochastic dynamic programming technique. The optimal strategy can be obtained by deriving and solving the associated HJB equation or the extended Hamilton-Jacobi-Bellman (HJB) equations. The pricing of variance (volatility) swaps is carried out under the risk-neutral pricing framework where the value of a variance (volatility) swap is calculated as the conditional expectation of the discounted terminal pay-off function under the risk-neutral probability measure. According to the Feynman-Kac theorem, the discounted value function satisfies a partial differential equation (PDE) and can be solved by using the generalized Fourier transform method. Thus the pricing problem is reduced to solving the associated PDE.

In the upcoming chapters, various realistic RSJD models will be established for the MVALM as well as the pricing of variance (volatility) swaps and the problems will be solved by utilizing the methodologies introduced in this chapter.

CHAPTER 3

Time-Consistent Mean-Variance Asset-Liability Management in a Regime-Switching Jump-Diffusion Market

3.1 General Overview

In this chapter, we investigate the mean-variance asset-liability management (MVALM) problem under a regime-switching jump-diffusion (RSJD) model. The investor (a company) is faced with an exogenous liability while investing in the financial market consisting of a risk-less bond and a risky stock whose price follows a Markov-modulated jump diffusion process. To handle the time-inconsistency existing in the MVALM problem, we formulate this problem as an optimal stochastic control problem under a game theoretic framework. In this sense, our solution is time-consistent compared to the pre-committed strategy in previous literatures. By applying the stochastic dynamic programming techniques we introduce in Chapter 2, we obtain the Nash equilibrium control and equilibrium value function in closed form. The contributions of this chapter include the following aspects. Firstly, to our knowledge, this appears to be the first attempt to establish a RSJD model for the MVALM problem under a game-theoretic framework. Secondly, we derive the RSJD version of the extended Hamilton-Jacobi-Bellman (HJB) equations together with the verification theorem to obtain our solution. Also, we assume a regime-dependent risk aversion coefficient for the investor. Moreover, the jumps of the stock and liability are assumed to be related via a common Poisson process $N_0(t)$, which is more realistic. Finally, we obtain the equilibrium strategy and value function in terms of five systems of ordinary differential equations arising from the HJB equations and we examine the influence of regime switching and jump diffusion via numerical and sensitivity analysis.

The rest of this chapter is organized as follows. The basic model setup and the game theoretic framework are formulated in Section 3.2. The extended HJB system of equations and the verification theorem are derived and presented in Section 3.3. Then we solve the extended HJB equations for the time-consistent control based on a suitable *Ansatz* in Section 3.4. In Section 3.5, a numerical and sensitivity analysis illustrating the effect of different model parameters on the equilibrium control and value function is conducted. Finally, we conclude this chapter in Section 3.6.

3.2 Model Formulation

This chapter fixes a complete probability space $(\Omega, \mathscr{F}, P)$ with the real-world probability measure P and the filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ that satisfies the usual conditions, namely, it is right continuous and contains all P-null sets. Here $\mathscr{F}_t$ contains the information available up to time t based on which all the decisions are made. $[0, T]$ represents the finite time horizon where $T \in (0, \infty)$. $W_1(t)$ and $W_2(t)$ are two correlated standard Brownian motions with the correlation coefficient denoted by ρ. Then $Cov[W_1(t), W_2(t)] = \rho t$. We only consider cases where $\rho^2 < 1$ in this paper. $\alpha(t)$ is a stationary continuous-time Markov chain process defined on the probability space taking discrete values from the state space $\mathcal{M} = \{1, 2, \ldots, d\}, d \in \mathbb{N}^+$, and the dynamics of the market state are modelled by $\alpha(t)$, which means the market states are divided into d regimes. The generator (which is also called the Q-matrix) of the Markov chain is defined as $Q = (q_{jk})_{d\times d}$ where q_{jk} is the constant transition intensity from state j to state k. Note that $\sum_{j=1}^{d} q_{jk} = 0$ and $q_{jk} > 0$ for $j \neq k$, so we have $q_{jj} = -\sum_{j\neq k}^{d} q_{jk} < 0$. Moreover, we give three mutually independent Poisson processes $N_1(t), N_2(t)$ and $N_0(t)$ defined on the probability space, with intensities $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_0 > 0$ respectively. Furthermore, we assume that the Brownian motions $\{W_i(t), i = 1, 2, t \geq 0\}$, Poisson processes $\{N_i(t), i = 0, 1, 2, t \geq 0\}$ and the Markov chain $\{\alpha(t), t \geq 0\}$ are well defined and adapted processes that are mutually independent.

3.2.1 Financial market

For simplicity, we consider a continuous-time financial market where one riskless bond and one risky stock are traded continuously over the time horizon

$[0, T]$ with no transaction cost. Moreover, these assets are assumed to be infinitely divisible, which means any fractional units of assets can be traded.

The price process of the bond is supposed to be subject to the following ordinary differential equation (ODE):

$$\begin{cases} \mathrm{d}P_0(t) = r(t)P_0(t)\,\mathrm{d}t, \\ P_0(0) = p_0 > 0, \end{cases} \tag{3.2.1}$$

where $r(t)$ is the riskless interest rate. p_0 denotes the initial price of the bond at $t = 0$.

The price process of the stock is supposed to satisfy the following Markov-modulated jump-diffusion stochastic differential equation (SDE):

$$\begin{cases} \mathrm{d}P_1(t) = P_1(t)\Big[b(t, \alpha(t))\,\mathrm{d}t + \sigma(t, \alpha(t))\,\mathrm{d}W_1(t) + \mathrm{d}\sum_{n=1}^{M_1(t)} Q_n\Big], \\ t \in [0, T], \\ P_1(0) = p_1 > 0, \end{cases} \tag{3.2.2}$$

where $b(t, j)$ and $\sigma(t, j)$ are the appreciation rate and volatility rate of the price process respectively for every $j \in \mathcal{M}$. We assume that $b(t, j) > r(t)$ for all $(t, j) \in [0, T] \times \mathcal{M}$. Q_n is the size of the n-th jump, where $\{Q_n, n = 1, 2, \dots\}$ is a sequence of independent and identically distributed random variables. We assume that $Q_n > -1$ for $\forall n = 1, 2, \dots$ to ensure that the asset price always takes positive values. Moreover, Q_n has a finite mean $\mathbb{E}(Q_n) = \mu_1$ and a second moment $\mathbb{E}[(Q_n)^2] = \sigma_1^2$. $M_1(t) = N_1(t) + N_0(t)$ is a Poisson process with intensity $\lambda_1 + \lambda_0$ due to the independency of $N_1(t)$ and $N_0(t)$. Thus $\sum_{n=1}^{M_1(t)} Q_n$ is a compound Poisson process representing the total amount of jumps of the stock price up to time $t \in [0, T]$.

3.2.2 Liability process

Consider a company with initial liability l_0. The uncontrollable liability process $L(t)$ is modelled by the following stochastic differential equation (SDE):

$$\begin{cases} \mathrm{d}L(t) = \mu(t, \alpha(t))\,\mathrm{d}t + \widetilde{\sigma}(t, \alpha(t))\,\mathrm{d}W_2(t) + \mathrm{d}\sum_{n=1}^{M_2(t)} R_n, \\ t \in [0, T], \\ L(0) = l_0 > 0, \end{cases} \tag{3.2.3}$$

where $\mu(t,j)$ and $\widetilde{\sigma}(t,j)$ are the appreciation rate and the volatility rate of the liability process respectively for each $j \in \mathcal{M}$. $\{R_n, n = 1, 2, \dots\}$ is a sequence of independent and identically distributed non-negative random variables with a common distribution. Similarly, R_n has a finite mean $\mathbb{E}(R_n) = \mu_2$ as well as a second moment $\mathbb{E}[(R_n)^2] = \sigma_2^2$. $M_2(t) = N_2(t) + N_0(t)$ is a Poisson process with intensity $\lambda_2 + \lambda_0$. Thus $\sum_{n=1}^{M_2(t)} R_n$ is a compound Poisson process representing the cumulative amount of jumps of the liability process up to time $t \in [0, T]$. Note that the liability can be negative here.

3.2.3 Wealth process

Suppose that the company is allocating all of its wealth between the bond and the stock with the initial wealth z_0. Suppose that the market state at initial time is $\alpha(0) = j_0$. Let $Z(t)$ denote the wealth surplus of the company at time $t \geq 0$. We assume the continuity of the operation of the company even if the wealth surplus falls negative. Furthermore, we assume that there is no other income or consumption during the investment process. Let $\pi(t) = \pi(t, Z(t), \alpha(t))$ denote the total amount of money invested in the risky stock at time t. $\pi(t, Z(t), \alpha(t)) \in [0,T] \times \mathbb{R} \times \mathcal{M}$ here is a feedback control law and the investment strategy based on the condition $\{Z(t) = z, \alpha(t) = j\}$. Then after deducing the liability, the amount invested in the riskless bond, denoted by $\pi_0(t)$, satisfies $\pi_0(t) = Z(t) - \pi(t)$.

Hence we can deduce the dynamics of $Z(t)$ from (3.2.1)-(3.2.3) as follows,

$$
\begin{cases}
\mathrm{d}Z(t) = \Big[r(t)Z(t) + \pi(t)(b(t,\alpha(t)) - r(t)) - \mu(t,\alpha(t))\Big]\,\mathrm{d}t \\
\qquad + \Big[\pi(t)\sigma(t,\alpha(t))\,\mathrm{d}W_1(t) - \widetilde{\sigma}(t,\alpha(t))\,\mathrm{d}W_2(t)\Big] \\
\qquad + \pi(t)\,\mathrm{d}\displaystyle\sum_{n=1}^{M_1(t)} Q_n - \mathrm{d}\sum_{n=1}^{M_2(t)} R_n, \quad t \in [0,T], \\
Z(0) = z_0.
\end{cases}
\tag{3.2.4}
$$

Compared to the existing models developed in literatures such as [88], [70], [61], the parameters in our model, including the appreciation rates and volatility rates of the risky asset's price and liability process, are modulated by a continuous-time Markov chain $\alpha(t)$ which represents different market regimes by taking corresponding values. For example, let $\tilde{b}(t) = (b_1(t), \dots, b_d(t))$ denote the vector containing all possible appreciation rates under each market regime, then we have $b(t, \alpha(t)) = b_{\alpha(t)}(t)$, for $\alpha(t) \in \mathcal{M}$. Thus the parameters switches along with the Markov chain. In this way, our model is able to capture the long-term effect on the financial asset's return caused by the macroeconomic environment movements simulated by the Markov chain. This is a meaningful and attractive feature since the relationship between the financial market and the macroeconomic environment is well-documented in the empirical literature [8–11].

The regime-switching feature is similar to the model in [21]. However, we further considered two correlated jumps in the asset's price process and liability process with the jump size Q_n and R_n respectively. Generally in the regime-switching model the price process of the risky asset is considered continuous under a given market regime, while in the real world even during a period of a somewhat stable macroeconomic environment, there can still be jumps caused by some unexpected events. Thus combining jump diffusion enables the regime-switching model to capture both the longer-term effect and the short-term effect on the investing activity resulting from market movements and individual events, which makes the model more realistic and meaningful. In fact, recently the regime-switching jump-diffusion models have been applied to solve various financial problems [25–28]. In some literatures, the jump diffusion part can also be regime-dependent via a Markov-modulated Poisson measure. In this chapter, we want to investigate the long-term and short-term effects caused by macroeconomic and microeconomic events more separately, and that is the main purpose of establishing such a hybrid model. Thus the jump diffusion is assumed to be independent of the Markov chain in our model.

Definition 3.2.1. *(Admissible strategy)* $\pi(\cdot) \in \mathbb{R}$ *is said to be an admissible strategy if it satisfies: a)* $\pi(\cdot)$ *is* $\mathscr{F}_t$*-adapted satisfying* $\int_0^T ||\pi(\cdot)||^2 dt < \infty$*. b)* Z^π *corresponding to the strategy* $\pi(\cdot)$ *is the unique solution to the SDE in (3.2.4). We let* Π *denote the set of all admissible strategies.*

3.2.4 Problem formulation in a game theoretic framework

For any fixed $(t, z, j, \pi(\cdot)) \in [0, T] \times \mathbb{R} \times \mathcal{M} \times \Pi$, the mean-variance objective (reward) functional is given as follows,

$$\begin{aligned} J(t, z, j, \pi(\cdot)) &:= \mathbb{E}_{t,z,j}[Z^\pi(T)] - \frac{\gamma(j)}{2} Var_{t,z,j}[Z^\pi(T)] \\ &= \mathbb{E}_{t,z,j}[Z^\pi(T)] - \frac{\gamma(j)}{2}\Big\{\mathbb{E}_{t,z,j}[Z^\pi(T)^2] - \mathbb{E}^2_{t,z,j}[Z^\pi(T)]\Big\}, \end{aligned} \tag{3.2.5}$$

where $\mathbb{E}_{t,z,j}[\cdot]$ and $Var_{t,z,j}[\cdot]$ are the conditional expectation and variance based on the condition $\{Z(t) = z, \alpha(t) = j\}$ respectively and $\gamma(j), j = 1, 2, \ldots, d$ is a pre-determined regime-dependent risk aversion coefficient of the investor (the company).

For convenience, we rewrite $J(t,z,j,\pi(\cdot))$ as

$$J(t,z,j,\pi(\cdot)) = \mathbb{E}_{t,z,j}[F(j,Z^{\pi}(T))] + G(j,\mathbb{E}_{t,z,j}[Z^{\pi}(T)]), \tag{3.2.6}$$

where $F(j,x) = x - \frac{\gamma(j)}{2}x^2$, $G(j,x) = \frac{\gamma(j)}{2}x^2$.

Assumption 3.2.1. *All the parameters mentioned above, such as $r(t,j)$, $b(t,j)$ $\sigma(t,j)$ and $\gamma(j)$, are uniformly bounded. Among them, $\gamma(j)$, $\sigma(t,j)$, and $\widetilde{\sigma}(t,j)$ satisfy the non-degeneracy condition, namely, there exists a $C > 0$ such that $h(t,j) > C$ for any $t \in [0,T]$ and $j \in \mathcal{M}$, where $h(\cdot) = \gamma(\cdot), \sigma(\cdot), \widetilde{\sigma}(\cdot)$.*

As we have mentioned in Chapter 2, due to the non-linearity and lack of iterated-expectation property of the M-V objective functional, the MVALM problem is time inconsistent. Therefore, by applying the stochastic dynamic programming techniques similar to those in [87], we formulate this MVALM problem as a stochastic control problem under a game theoretic framework.

To be specific, we treat our problem as a non-cooperate game where the investor at each time point $t \in [0,T]$ is regarded as a different player that is to optimize its objective functional $J(t,z,j,\pi(\cdot))$ at time t. Then we aim to find such an equilibrium strategy π^* that if for all players $k > t$, π^* is the optimal strategy they choose, then it is also optimal for player t to choose the same strategy $\pi^*(\cdot)$. In this sense, we could reach a Nash equilibrium point by this time-consistent equilibrium strategy or equilibrium control. To summarize this idea, We give the definition of the equilibrium control (EC) as follows,

Definition 3.2.2. *(Equilibrium control) For any fixed initial condition $(t,z,j) \in [0,T] \times \mathbb{R} \times \mathcal{M}$, an admissible strategy $\pi^*(t,z,j) \in \Pi$ is said to be an equilibrium control if for any $\delta > 0$, one can define a new control law $\pi_\delta(k,y,j)$ by*

$$\pi_\delta(k,y,j) = \begin{cases} \pi(k,y,j), & for\ t \leq k < t+\delta, \\ \pi^*(k,y,j), & for\ t+\delta \leq k \leq T, \end{cases} \tag{3.2.7}$$

such that

$$\begin{aligned} &\lim_{\delta \to 0^+} \inf \frac{J(t,z,j,\pi^*(\cdot)) - J(t,z,j,\pi_\delta(\cdot))}{\delta} \geq 0, \\ &for\ \forall (t,z,j) \in [0,T] \times \mathbb{R} \times \mathcal{M}. \end{aligned} \tag{3.2.8}$$

To derive the extended HJB equation in the next section, we also define the corresponding equilibrium value function (EVF) as follows,

$$V(t,z,j) = J(t,z,j,\pi^*(\cdot)). \tag{3.2.9}$$

3.3 The Extended HJB Equations and the Verification Theorem

In this section, we present a regime-switching jump-diffusion (RSJD) version of the verification theorem and derive the associated extended HJB system of equations for the MVALM problem, based on which the equilibrium control and value function are derived in the next section.

First, if a random variable $Z(t)$ evolves according to (3.2.4), then for any fixed admissible control $\pi \in \Pi$ and any function $\Phi(t,z,j) \in C^{1,2}([0,T]\times\mathbb{R}\times\mathcal{M})$ where $C^{1,2}([0,T]\times\mathbb{R}\times\mathcal{M})$ denotes the space of the function $\Phi(t,z,j)$ such that $\Phi(t,z,j)$ and the derivatives $\Phi_t(t,z,j)$, $\Phi_z(t,z,j)$, $\Phi_{zz}(t,z,j)$ are continuous on $[0,T]\times\mathbb{R}\times\mathcal{M}$, the controlled infinitesimal generator is given by

$$\begin{aligned}\mathcal{A}^{\pi}\Phi(t,z,j) = {} & \Phi_t(t,z,j) + \Phi_z(t,z,j)\Big[r(t)z + \pi(t)(b(t,j)-r(t)) \\ & - \mu(t,j)\Big] + \frac{1}{2}\Phi_{zz}(t,z,j)\Big[\pi^2(t)\sigma^2(t,j) + \widetilde{\sigma}^2(t,j) \\ & - 2\rho\sigma(t,j)\widetilde{\sigma}(t,j)\pi(t)\Big] + \sum_{k=1}^{d} q_{jk} d\Phi(t,z,k) \\ & + \lambda_1 E\Big[\Phi(t,z+\pi(t)Q,j) - \Phi(t,z,j)\Big] \\ & + \lambda_2 E\Big[\Phi(t,z-R,j) - \Phi(t,z,j)\Big] \\ & + \lambda_0 E\Big[\Phi(t,z+\pi(t)Q-R,j) - \Phi(t,z,j)\Big].\end{aligned} \tag{3.3.1}$$

Based on our new model with the above defined controlled infinitesimal generator, we give the following definition of the regime-switching jump-diffusion version of the extended HJB equations by applying similar techniques to those in [87].

Definition 3.3.1. *For a Nash equilibrium problem with its equilibrium control and value function defined as Definition 3.2.2 and (3.2.9) respectively, and for any fixed $(t,z,j) \in [0,T]\times\mathbb{R}\times\mathcal{M}$, the extended HJB system of equations for the Nash equilibrium problem is defined as follows,*

the extended HJB system of equations for the Nash equilibrium problem is *defined as follows,*

$$
\begin{cases}
\sup_{\pi}\Big\{(\mathcal{A}^{\pi}V)(t,z,j)-(\mathcal{A}^{\pi}f)(t,z,j,j)+(\mathcal{A}^{\pi}f^{j})(t,z,j)\\
\quad-\mathcal{A}^{\pi}(G\circ g)(t,z,j)+(\mathcal{H}^{\pi}g)(t,z,j)\Big\}=0,\quad t\in[0,T],\\
\mathcal{A}^{\pi^*}f^{q}(t,z,j)=0,\quad t\in[0,T],\\
\mathcal{A}^{\pi^*}g(t,z,j)=0,\quad t\in[0,T],\\
V(T,z,j)=F(j,z)+G(j,z)=z,\\
f^{q}(T,z,j)=F(q,z),\\
g(T,z,j)=z,
\end{cases}
\tag{3.3.2}
$$

where

$$
\begin{aligned}
\pi^*=\arg\sup_{\pi}\Big\{&(\mathcal{A}^{\pi}V)(t,z,j)-(\mathcal{A}^{\pi}f)(t,z,j,j)\\
&+(\mathcal{A}^{\pi}f^{j})(t,z,j)-\mathcal{A}^{\pi}(G\circ g)(t,z,j)+(\mathcal{H}^{\pi}g)(t,z,j)\Big\},
\end{aligned}
\tag{3.3.3}
$$

$\mathcal{A}^{\pi}$ *is the infinitesimal generator defined in (3.3.1) and the definitions of* $f^{q}(\cdot)$, $(G\circ g)(\cdot)$ *and* $\mathcal{H}^{\pi}g(\cdot)$ *are given by*

$$
\begin{aligned}
&f^{q}(t,z,j)\triangleq f(t,z,j,q),\\
&(G\circ g)(t,z,j)\triangleq G(j,g((t,z,j)))=\frac{\gamma(j)}{2}g^{2}(t,z,j),\\
&\mathcal{H}^{\pi}g(t,z,j)\triangleq G_{g}(j,g(t,z,j))\cdot\mathcal{A}^{\pi}g(t,z,j).
\end{aligned}
\tag{3.3.4}
$$

Furthermore, $f^{q}(\cdot)$ *and* $g(\cdot)$ *have the following probabilistic representations:*

$$
\begin{aligned}
&f(t,z,j,q)=\mathbb{E}_{t,z,j}[F(q,Z^{\pi^*}(T))],\\
&g(t,z,j)=\mathbb{E}_{t,z,j}[Z^{\pi^*}(T)].
\end{aligned}
\tag{3.3.5}
$$

Next we present the verification theorem, based on which the MVALM problem is reduced to the process of solving the corresponding extended HJB equations.

Theorem 3.3.1. *(Verification Theorem) Assume* $V,g:[0,T]\times R\times\mathcal{M}\to R$, $f:[0,T]\times R\times\mathcal{M}\times\mathcal{M}\to R$, $\pi^*:[0,T]\times R\times\mathcal{M}\to R$ *are such functions that* (V,f,g) *is a solution to the extended HJB system (3.3.2) and the supremum can be realized by the control law* π^**. Then* π^* *is an equilibrium control with the corresponding equilibrium value function* V*. Moreover,* f *and* g *satisfy the probabilistic representations in (3.3.5).*

Proof. The proof is rather similar to that of Theorem 4.1 in [87], so we omit it here. □

From (3.2.6), (3.2.9) and (3.3.5), it is obvious that the following equation holds,

$$V(t,z,j) = f(t,z,j,j) + G(j, g(t,z,j)). \tag{3.3.6}$$

Then due to the linearity of the infinitesimal generator (3.3.1), the first equation in (3.3.2) can be simplified as follows,

$$\begin{aligned}
&\sup_{\pi}\Big\{(\mathcal{A}^{\pi}V)(t,z,j) - (\mathcal{A}^{\pi}f)(t,z,j,j) + (\mathcal{A}^{\pi}f^{j})(t,z,j) \\
&- \mathcal{A}^{\pi}(G\circ g)(t,z,j) + (\mathcal{H}^{\pi}g)(t,z,j)\Big\} \\
&= \sup_{\pi}\Big\{(\mathcal{A}^{\pi}f^{j})(t,z,j) + (\mathcal{H}^{\pi}g)(t,z,j)\Big\} = 0.
\end{aligned} \tag{3.3.7}$$

Therefore, substituting into the explicit expression of the infinitesimal generator (3.3.1), we can rewrite the extended HJB system of equations as follows,

$$\begin{aligned}
&\sup_{\pi}\Bigg\{f^{j}_{t}(t,z,j) + g_t(t,z,j)\gamma(j)g(t,z,j) \\
&+ \Big[f^{j}_{z}(t,z,j) + \gamma(j)g(t,z,j)g_z(t,z,j)\Big]\Big[r(t)z \\
&+ \pi(t)(b(t,j) - r(t)) - \mu(t,j)\Big] + \frac{1}{2}\Big[f^{j}_{zz}(t,z,j) \\
&+ \gamma(j)g(t,z,j)g_{zz}(t,z,j)\Big]\Big[\sigma^2(t,j)\pi^2(t) + \widetilde{\sigma}^2(t,j) \\
&- 2\rho\sigma(t,j)\widetilde{\sigma}(t,j)\pi(t)\Big] + \sum_{k=1}^{d} q_{jk}\Big[f^{j}(t,z,k) + \gamma(j)g(t,z,j)g(t,z,k)\Big] \\
&+ \lambda_1\mathbb{E}\Big[f^{j}(t,z+\pi(t)Q,j) + \gamma(j)g(t,z,j)g(t,z+\pi(t)Q,j) \\
&- f^{j}(t,z,j) - \gamma(j)g^2(t,z,j)\Big] + \lambda_2\mathbb{E}\Big[f^{j}(t,z-R,j) \\
&+ \gamma(j)g(t,z,j)g(t,z-R,j) - f^{j}(t,z,j) - \gamma(j)g^2(t,z,j)\Big] \\
&+ \lambda_0\mathbb{E}\Big[f^{j}(t,z+\pi(t)Q-R,j) + \gamma(j)g(t,z,j)g(t,z+\pi(t)Q-R,j) \\
&- f^{j}(t,z,j) - \gamma(j)g^2(t,z,j)\Big]\Bigg\} = 0, \quad j = 1,\dots,d,
\end{aligned} \tag{3.3.8}$$

$$
\begin{aligned}
&f_t^q(t,z,j) + f_z^q(t,z,j)\Big[r(t)z + \pi^*(t)(b(t,j) - r(t)) - \mu(t,j)\Big] \\
&+ \frac{1}{2} f_{zz}^q(t,z,j)\Big[\sigma^2(t,j)(\pi^*(t))^2 + \widetilde{\sigma}^2(t,j) - 2\rho\sigma(t,j)\widetilde{\sigma}(t,j)\pi^*(t)\Big] \\
&+ \sum_{k=1}^{d} q_{jk} f^q(t,z,k) + \lambda_1 \mathbb{E}\Big[f^q(t,x,z+\pi^*(t)Q,j) - f^q(t,z,j)\Big] \\
&+ \lambda_2 \mathbb{E}\Big[f^q(t,z-R,j) - f^q(t,z,j)\Big] + \lambda_0 \mathbb{E}\Big[f^q(t,z+\pi(t)Q-R,j) \\
&- f^q(t,z,j)\Big] = 0, \quad j = 1,\ldots,d,
\end{aligned}
\tag{3.3.9}
$$

$$
\begin{aligned}
&g_t(t,z,j) + g_z(t,z,j)\Big[r(t)z + \pi^*(t)(b(t,j) - r(t)) - \mu(t,j)\Big] \\
&+ \frac{1}{2} g_{zz}(t,z,j)\Big[\sigma^2(t,j)(\pi^*(t))^2 + \widetilde{\sigma}^2(t,j) - 2\rho\sigma(t,j)\widetilde{\sigma}(t,j)\pi^*(t)\Big] \\
&+ \sum_{k=1}^{d} q_{jk} g(t,z,k) + \lambda_1 \mathbb{E}\Big[g(t,z+\pi^*(t)Q,j) - g(t,z,j)\Big] \\
&+ \lambda_2 \mathbb{E}\Big[g(t,z-R,j) - g(t,z,j)\Big] + \lambda_0 \mathbb{E}\Big[g(t,x,z+\pi(t)Q-R,j) \\
&- g(t,z,j)\Big] = 0, \quad j = 1,\ldots,d,
\end{aligned}
\tag{3.3.10}
$$

$$
f(T,z,j,q) = z - \frac{\gamma(q)}{2} z^2, \quad j = 1,\ldots,d, \tag{3.3.11}
$$

$$
g(T,z,j) = z, \quad j = 1,\ldots,d. \tag{3.3.12}
$$

Compared to the extended HJB equation derived in [87], the regime-switching jump-diffusion version of the extended HJB equations exhibit the following features. First, due to the introduction of the Markov chain, after substituting the controlled infinitesimal generator $\mathcal{A}^\pi$ into (3.3.7), each equation in the extended HJB system (3.3.2) is converted to a system of j equations, where $j = 1,\ldots,d$. The equations are correlated via the summation term $\sum q_{jk}[\cdot]$, which reflects the effect of switching regimes. Secondly, the expectation part following the summation term reflects the effect of the jump possibility in the asset price and liability processes, and also contributes to the complexity of the equations. These two features reflect the comprehensiveness of the model yet raise the difficulty in solving the related HJB equations. Moreover, the risk aversion coefficient is considered regime-dependent rather than state-dependent. The derivation process of the extended HJB equation would be rather similar. The difference mainly lies in the derivation of the partial derivative term $f_z^j(t,z,j)$. In this paper we focus more on the effect of regime-switching.

3.4 Time-Consistent Solution to the MVALM Problem

In this section, we aim to derive the explicit time-consistent solution to the MVALM problem by solving the extended HJB system of equations (3.3.8)-(3.3.12).

First we present the following *Ansatz*, for any fixed $(t,z,j) \in [0,T] \times \mathbb{R} \times \mathcal{M}$,

$$\begin{aligned} &g(t,z,j) = m(t,j)z + n(t,j), \\ &f(t,z,j,q) = m(t,j)z + n(t,j) - \frac{\gamma(q)}{2}\Big[M(t,j)z^2 + 2C(t,j)z + N(t,j)\Big]. \end{aligned} \tag{3.4.1}$$

We assume that $M(t,j) > 0$ for all (t,j) to ensure that the MVALM problem is feasible.

Then we can easily have the following,

$$\begin{aligned} &g_t(t,z,j) = \dot{m}(t,j)z + \dot{n}(t,j), \\ &g_z(t,z,j) = m(t,j), \\ &g_{zz}(t,z,j) = 0, \\ &f_t^q(t,z,j) = \dot{m}(t,j)z + \dot{n}(t,j) - \frac{\gamma(q)}{2}\Big[\dot{M}(t,j)z^2 + 2\dot{C}(t,j)z + \dot{N}(t,j)\Big], \\ &f_z^q(t,z,j) = m(t,j) - \gamma(q)M(t,j)z - \gamma(q)C(t,j), \\ &f_{zz}^q(t,z,j) = -\gamma(q)M(t,j). \end{aligned} \tag{3.4.2}$$

For simplicity, we denote $l(t,j)$, $l(t)$, and $l(j)$ as l, and denote $l(t,k), l(k)$ as l_k, where $l(\cdot) = m(\cdot), n(\cdot), M(\cdot), C(\cdot), N(\cdot), b(\cdot), \mu(\cdot), \sigma(\cdot), \widetilde{\sigma}(\cdot), \gamma(\cdot), \pi(\cdot), r(\cdot)$.

Then substituting (3.4.2) into equation (3.3.8), we have,

$$
\begin{aligned}
&\sup_{\pi}\Bigg\{\dot{m}z+\dot{n}-\frac{\gamma}{2}(\dot{M}z^2+2\dot{C}z+\dot{N})+\gamma(mz+n)(\dot{m}z+\dot{n})+\\
&\Big[(m-\gamma Mz-\gamma C)+\gamma(mz+n)m\Big]\Big[rz+(b-r)\pi-\mu\Big]+\frac{1}{2}(-\gamma M)\\
&\times\Big[\sigma^2\pi^2+\widetilde{\sigma}^2-2\rho\sigma\widetilde{\sigma}\pi\Big]+\sum_{k=1}^{d}q_{jk}\Big[m_kz+n_k-\frac{\gamma}{2}(M_kz^2+2C_kx+N_k)\\
&+\gamma(m_kz+n_k)(mz+n)\Big]+\lambda_1\Big[m\mu_1\pi-\frac{\gamma}{2}(M\sigma_1^2\pi^2+2M\mu_1z\pi\\
&+2C\mu_1\pi)+\gamma(m^2\mu_1z\pi+mn\mu_1\pi)\Big]+\lambda_2\Big[-m\mu_2-\frac{\gamma}{2}(M\sigma_2^2-2M\mu_2z\\
&-2C\mu_2)+\gamma(-m^2\mu_2z-mn\mu_2)\Big]+\lambda_0\Big[m(\mu_1\pi-\mu_2)\\
&-\frac{\gamma}{2}(M(\sigma_1^2\pi^2+\sigma_2^2)+2Mz(\mu_1\pi-\mu_2)+2C(\mu_1\pi-\mu_2))\\
&+\gamma(m^2z(\mu_1\pi-\mu_2)+mn(\mu_1\pi-\mu_2))\Big]\Bigg\}=0.
\end{aligned}
\tag{3.4.3}
$$

Since the equilibrium control π^* realizes the supremum of the LHS of (3.4.3) which is a concave quadratic function of π, by summing up the terms related to π in the LHS of (3.4.3), we have

$$
\begin{aligned}
&\Big[(m-\gamma z-\gamma C)+\gamma(mz+n)m\Big](b-r)\pi-\frac{\gamma}{2}M(\sigma^2\pi^2-2\rho\sigma\widetilde{\sigma}\pi)\\
&+(\lambda_1+\lambda_0)\Big[m\mu_1\pi-\frac{\gamma}{2}(M\sigma_1^2\pi^2+2M\mu_1z\pi+2C\mu_1\pi)\\
&+\gamma(m^2\mu_1z\pi+mn\mu_1\pi)\Big]\\
&=-\frac{1}{2}\gamma M\Big[\sigma^2+(\lambda_1+\lambda_0)\sigma_1^2\Big]\Bigg\{\pi-\frac{1}{\gamma M[\sigma^2+(\lambda_1+\lambda_0)\sigma_1^2]}\Big[(m-\gamma Mz\\
&-\gamma C+\gamma(mz+n)m)(b-r)+\gamma M\sigma\widetilde{\sigma}\rho+(\lambda_1+\lambda_0)(m\mu_1-\gamma\mu_1(Mz\\
&+C-m^2z-mn))\Big]\Bigg\}^2+\frac{1}{2\gamma M\Big[\sigma^2+(\lambda_1+\lambda_0)\sigma_1^2\Big]}\Bigg\{\Big[m-\gamma Mz-\gamma C\\
&+\gamma(mz+n)m\Big](b-r)+\gamma M\sigma\widetilde{\sigma}\rho+(\lambda_1+\lambda_0)\Big[m\mu_1\\
&-\gamma\mu_1(Mz+C-m^2z-mn)\Big]\Bigg\}^2.
\end{aligned}
\tag{3.4.4}
$$

Then from some calculation we can easily have the following expression of the equilibrium control π^*

$$\pi^* = 2\Delta_1(\Delta_2 z + \Delta_3), \tag{3.4.5}$$

where

$$\Delta_1 = \frac{1}{2\gamma M \xi_2}, \tag{3.4.6}$$

$$\Delta_2 = \xi_1(m^2 - M)\gamma, \tag{3.4.7}$$

$$\Delta_3 = \Big[(1+\gamma n)m - \gamma C\Big]\xi_1 + \gamma\sigma\widetilde{\sigma}\rho M, \tag{3.4.8}$$

$$\begin{aligned} \xi_1 &= b - r + (\lambda_1 + \lambda_0)\mu_1, \\ \xi_2 &= \sigma^2 + (\lambda_1 + \lambda_0)\sigma_1^2. \end{aligned} \tag{3.4.9}$$

Based on the expressions above, we can rewrite $rz + \pi^*(b-r) - \mu$ and $\sigma^2(\pi^*)^2 + \widetilde{\sigma}^2 - 2\sigma\widetilde{\sigma}\rho\pi^*$ as

$$\begin{aligned} & rz + \pi^*(b-r) - \mu = rz + 2\Delta_1(\Delta_2 z + \Delta_3)(b-r) - \mu \\ & = \Delta_4 z + \Delta_5, \end{aligned} \tag{3.4.10}$$

$$\begin{aligned} & \sigma^2(\pi^*)^2 + \widetilde{\sigma}^2 - 2\sigma\widetilde{\sigma}\rho\pi^* \\ & = \sigma^2 4\Delta_1^2(\Delta_2 z + \Delta_3)^2 + \widetilde{\sigma}^2 - 4\sigma\widetilde{\sigma}\rho\Delta_1(\Delta_2 z + \Delta_3) \\ & = \Delta_6 z^2 + \Delta_7 z + \Delta_8, \end{aligned} \tag{3.4.11}$$

where

$$\begin{aligned} \Delta_4 &= r + 2\Delta_1\Delta_2(b-r), \\ \Delta_5 &= 2\Delta_1\Delta_3(b-r) - \mu, \\ \Delta_6 &= 4\sigma^2\Delta_1^2\Delta_2^2, \\ \Delta_7 &= 8\sigma^2\Delta_1^2\Delta_2\Delta_3 - 4\sigma\widetilde{\sigma}\rho\Delta_1\Delta_2, \\ \Delta_8 &= 4\sigma^2\Delta_1^2\Delta_3^2 + \widetilde{\sigma}^2 - 4\sigma\widetilde{\sigma}\rho\Delta_1\Delta_3. \end{aligned} \tag{3.4.12}$$

Substituting (3.4.5), (3.4.10) and (3.4.11) into equation (3.3.9) and (3.3.10) yields,

$$
\begin{aligned}
\mathcal{A}^{\pi^*} f^q(t,z,j) = \Big[&- \frac{\gamma_q}{2}\dot{M} - \gamma_q M\Delta_4 - \frac{1}{2}\gamma_q M\Delta_6 - \sum_{k=1}^{d} q_{jk}\frac{\gamma_q}{2}M_k \\
&+ (\lambda_1+\lambda_0)(-\frac{\gamma_q}{2})(4M\mu_1\Delta_1\Delta_2 + 4M\sigma_1^2\Delta_1^2\Delta_2^2)\Big] z^2 \\
&+ \Big\{ \dot{m} - \gamma_q\dot{C} - \gamma_q M\Delta_5 + (m-\gamma_q C)\Delta_4 \\
&+ \frac{1}{2}(-\gamma_q M)\Delta_7 + \sum_{k=1}^{d} q_{jk}(m_k - \gamma_q C_k) + (\lambda_1+\lambda_0) \\
&\times \Big[2m\mu_1\Delta_1\Delta_2 - \frac{\gamma_q}{2}(4M\mu_1\Delta_1\Delta_3 + 8M\sigma_1^2\Delta_1^2\Delta_2\Delta_3 \\
&+ 4C\mu_1\Delta_1\Delta_2)\Big] + (\lambda_2+\lambda_0)\gamma_q M\mu_2 \Big\} z \\
&+ \Big\{ \dot{n} - \frac{\gamma_q}{2}\dot{N} + (m-\gamma_q C)\Delta_5 - \frac{\gamma_q}{2}M\Delta_8 \\
&+ \sum_{k=1}^{d} q_{jk}(n_k - \frac{\gamma_q}{2}\dot{N}_k) + (\lambda_1+\lambda_0)\Big[2m\mu_1\Delta_1\Delta_3 \\
&- \frac{\gamma_q}{2}(4M\sigma_1^2\Delta_1^2\Delta_3^2 + 4C\mu_1\Delta_1\Delta_3)\Big] + (\lambda_2+\lambda_0)\Big[m\mu_2 \\
&- \frac{\gamma_q}{2}(M\sigma_2^2 - 2C\mu_2)\Big]\Big\} = 0,
\end{aligned}
\tag{3.4.13}
$$

$$
\begin{aligned}
\mathcal{A}^{\pi^*} g(t,z,j) = &\Big[\dot{m} + m\Delta_4 + \sum_{k=1}^{d} q_{jk}m_k + 2(\lambda_1+\lambda_0)m\mu_1\Delta_1\Delta_2\Big] z \\
&+ \Big[\dot{n} + m\Delta_5 + \sum_{k=1}^{d} q_{jk}n_k + 2(\lambda_1+\lambda_0)m\mu_1\Delta_1\Delta_3 \\
&- (\lambda_2+\lambda_0)m\mu_2\Big] = 0.
\end{aligned}
\tag{3.4.14}
$$

By setting $\mathcal{A}^{\pi^*} f^q(t,z,j)$ and $\mathcal{A}^{\pi^*} g(t,z,j)$ equal to zero, and considering the arbitrariness of variables z and q, we can deduce the following systems of ODEs,

$$
\begin{aligned}
&\dot{M} + 2M\Delta_4 + M\Delta_6 + \sum_{k=1}^{d} q_{jk}M_k \\
&+ (\lambda_1+\lambda_0)(4M\mu_1\Delta_1\Delta_2 + 4M\sigma_1^2\Delta_1^2\Delta_2^2) = 0,
\end{aligned}
\tag{3.4.15}
$$

$$\begin{aligned}
&\dot{m} - \gamma_q \dot{C} - \gamma_q M\Delta_5 + (m - \gamma_q C)\Delta_4 - \frac{1}{2}\gamma_q M\Delta_7 \\
&+ \sum_{k=1}^{d} q_{jk}(m_k - \gamma_q C_k) + (\lambda_1 + \lambda_0)\Big[2m\mu_1\Delta_1\Delta_2 \\
&- \frac{\gamma_q}{2}(4M\mu_1\Delta_1\Delta_3 + 8M\sigma_1^2\Delta_1^2\Delta_2\Delta_3 + 4C\mu_1\delta_1\delta_2)\Big] \\
&+ (\lambda_2 + \lambda_0)\gamma_q M\mu_2 = 0,
\end{aligned} \tag{3.4.16}$$

$$\begin{aligned}
&\dot{n} - \frac{\gamma_q}{2}\dot{N} + (m - \gamma_q C)\Delta_5 - \frac{1}{2}\gamma_q M\Delta_8 + \sum_{k=1}^{d} q_{jk}(n_k - \frac{\gamma_q}{2}N_k) \\
&+ (\lambda_1 + \lambda_0)\Big[2m\mu_1\Delta_1\Delta_3 - \frac{\gamma_q}{2}(4M\sigma_1^2\Delta_1^2\Delta_3^2 + 4C\mu_1\Delta_1\Delta_3)\Big] \\
&- (\lambda_2 + \lambda_0)\Big[m\mu_2 + \frac{\gamma_q}{2}(M\sigma_2^2 - 2C\mu_2)\Big] = 0,
\end{aligned} \tag{3.4.17}$$

$$\dot{m} + m\Delta_4 + 2(\lambda_1 + \lambda_0)\mu_1\Delta_1\Delta_2 m + \sum_{k=1}^{d} q_{jk}m_k = 0, \tag{3.4.18}$$

$$\begin{aligned}
&\dot{n} + m\Delta_5 + 2(\lambda_1 + \lambda_0)\mu_1\Delta_1\Delta_3 m \\
&- (\lambda_2 + \lambda_0)m\mu_2 + \sum_{k=1}^{d} q_{jk}n_k = 0.
\end{aligned} \tag{3.4.19}$$

After some observation, we can easily see that (3.4.16) and (3.4.17) can be simplified by using (3.4.18) and (3.4.19). Further by using the specific expressions of Δ_1 to Δ_8 and adding the terminal conditions, we can have the following systems of ODEs,

$$\begin{cases}
\dot{M} + \Big[2r + \dfrac{\xi_1^2}{\xi_2}\dfrac{m^4 - M^2}{M^2}\Big]M + \displaystyle\sum_{k=1}^{d} q_{jk}M_k = 0, \\
M(T, j) = 1, \quad j = 1, \dots, d,
\end{cases} \tag{3.4.20}$$

$$\begin{cases}
\dot{C} + (r - \dfrac{\xi_1^2}{\xi_2})C + \Big[\dfrac{m^3(1+\gamma n)\xi_1^2}{\gamma M^2\xi_2} + \dfrac{\xi_1\sigma\widetilde{\sigma}\rho}{\xi_2} - \mu \\
- (\lambda_2 + \lambda_0)\mu_2\Big]M + \displaystyle\sum_{j=1}^{d} q_{jk}C_k = 0, \\
C(T, j) = 0, \quad j = 1, \dots, d,
\end{cases} \tag{3.4.21}$$

$$\begin{cases} \dot{N} + \Big[\dfrac{2\sigma\widetilde{\sigma}\rho\xi_1}{\xi_2} - \dfrac{\xi_1^2 C}{M\xi_2} - 2(\mu + (\lambda_2 + \lambda_0)\mu_2)\Big]C \\ + \Big[\dfrac{\xi_1^2(1+\gamma n)^2 m^2}{\gamma^2 M^2 \xi_2} + \widetilde{\sigma}^2 + (\lambda_2 + \lambda_0)\sigma_2^2 - \dfrac{\sigma^2\widetilde{\sigma}^2\rho^2}{\xi_2}\Big]M \\ + \displaystyle\sum_{k=1}^{d} q_{jk} N_k = 0, \\ N(T,j) = 0, \quad j = 1,\dots,d, \end{cases} \tag{3.4.22}$$

$$\begin{cases} \dot{m} + \Big[r + \dfrac{(m^2 - M)\xi_1^2}{M\xi_2}\Big]m + \displaystyle\sum_{k=1}^{d} q_{jk} m_k = 0, \\ m(T,j) = 1, \quad j = 1,\dots,d, \end{cases} \tag{3.4.23}$$

$$\begin{cases} \dot{n} + \dfrac{m^2\xi_1^2}{M\xi_2} n + \Bigg\{\dfrac{1}{\gamma M \xi_2}\Big[m\xi_1^2 + \gamma\sigma\widetilde{\sigma}\rho M\xi_1\Big] \\ - \Big[\mu + (\lambda_0 + \lambda_2)\mu_2\Big]\Bigg\} m - \dfrac{m\xi_1^2}{M\xi_2} C + \displaystyle\sum_{k=1}^{d} q_{jk} n_k = 0, \\ n(T,j) = 0, \quad j = 1,\dots,d, \end{cases} \tag{3.4.24}$$

where ξ_1 and ξ_2 are defined as (3.4.9).

Remark 3.4.1. *The existence of the solutions to the systems of ODEs (3.4.20)-(3.4.24) can be guaranteed by the uniform boundedness on all the related parameters in Assumption 2.1.*

To resolve the systems of ODEs, we follow similar steps to those in [21].

Firstly, from equations (3.4.20) and (3.4.23), we can directly establish the associated solutions as $M(t,j) = \exp(2\int_t^T r(s)ds)$ and $m(t,j) = \exp(\int_t^T r(s)ds)$ for all $j \in \mathcal{M}$ since $\sum_{k=1}^{d} q_{jk} = 0$.

Secondly, from equations (3.4.21) and (3.4.24), with the obtained expressions of M and m, we can deduce that $C(t,j) = \exp(\int_t^T r(s)ds) n(t,i)$ as follows, then based on the relationship we can derive a new ODE system for $n(t,i)$,

$$\begin{cases} \dot{n} + \Bigg\{\dfrac{1}{\gamma M \xi_2}\Big[m\xi_1^2 + \gamma\sigma\widetilde{\sigma}\rho M\xi_1\Big] \\ - \Big[\mu + (\lambda_0 + \lambda_2)\mu_2\Big]\Bigg\} m + \displaystyle\sum_{k=1}^{d} q_{jk} n_k = 0, \\ n(T,j) = 0, \quad j = 1,\dots,d. \end{cases} \tag{3.4.25}$$

The above system is well studied with the obtained expressions of $M(t,j)$ and $m(t,j)$, thus $C(t,j)$ and $n(t,j)$ can be solved explicitly.

Finally, since $C(t,j)$ and $n(t,j)$ can be solved explicitly, with the solved expressions of $M(t,j)$ and $m(t,j)$, we can solve $N(t,j)$ from equations (3.4.22).

Before we conclude the section with our main theorem, we provide the following systems of first order linear ODEs satisfied by $N(t,j)$ and $n(t,j)$, substituting into the obtained expressions of $M(t,j)$ and $m(t,j)$,

$$\begin{cases} \dot{N}(t,j) + 2\Big\{ \Big[\dfrac{\sigma(t,j)\widetilde{\sigma}(t,j)\rho\xi_1(t,j)}{\xi_2(t,j)} - (\mu(t,j) + (\lambda_2+\lambda_0)\mu_2)\Big] \\ \times \exp\big(\displaystyle\int_t^T r(s)ds\big) + \dfrac{\xi_1^2(t,j)}{\gamma(j)\xi_2(t,j)}\Big\} n(t,j) + \dfrac{\xi_1^2(t,j)}{\gamma^2(j)\xi_2(t,j)} \\ + \Big[\widetilde{\sigma}^2(t,j) + (\lambda_2+\lambda_0)\sigma_2^2 - \dfrac{\sigma^2(t,j)\widetilde{\sigma}^2(t,j)\rho^2}{\xi_2(t,j)}\Big] \\ \times \exp\big(\displaystyle\int_t^T 2r(s)ds\big) + \sum_{k=1}^d q_{jk}N(t,k) = 0, \\ N(T,j) = 0, \quad j = 1,\dots,d, \end{cases} \tag{3.4.26}$$

$$\begin{cases} \dot{n}(t,j) + \Big\{ \dfrac{\sigma(t,j)\widetilde{\sigma}(t,j)\rho\xi_1(t,j)}{\xi_2(t,j)} - \Big[\mu(t,j) + (\lambda_2+\lambda_0)\mu_2\Big]\Big\} \\ \times \exp\big(\displaystyle\int_t^T r(s)ds\big) + \dfrac{\xi_1^2(t,j)}{\gamma(j)\xi_2(t,j)} + \sum_{k=1}^d q_{jk}n(t,k) = 0, \\ n(T,j) = 0, \quad j = 1,\dots,d, \end{cases} \tag{3.4.27}$$

where

$$\begin{aligned} \xi_1(t,j) &= b(t,j) - r(t) + (\lambda_1+\lambda_0)\mu_1, \\ \xi_2(t,j) &= \sigma^2(t,j) + (\lambda_1+\lambda_0)\sigma_1^2. \end{aligned} \tag{3.4.28}$$

Theorem 3.4.1. *The equilibrium control for the MVALM problem is given by*

$$\begin{aligned} \pi^*(t,z,j) = \frac{1}{\sigma^2(t,j) + (\lambda_1+\lambda_0)\sigma_1^2} \Big\{ &\frac{b(t,j) - r(t) + (\lambda_1+\lambda_0)\mu_1}{\gamma(j)} \\ &\times \exp\big(-2\int_t^T r(s)ds\big) + \sigma(t,j)\widetilde{\sigma}(t,j)\rho\Big\}, \end{aligned} \tag{3.4.29}$$

and the corresponding equilibrium value function is given by

$$V(t,z,j) = \exp\big(\int_t^T r(s)ds\big)z + n(t,j) - \frac{\gamma(j)}{2}\Big[N(t,j) - n^2(t,j)\Big]. \tag{3.4.30}$$

where $N(t,j)$ and $n(t,j)$ are solutions to the systems of linear ODEs in (3.4.26) *and (3.4.27) respectively.*

Moreover, the expectation and variance of the terminal wealth surplus conditional on $\{Z(t) = z, \alpha(t) = j\}$ are given as follows,

$$\begin{aligned} \mathbb{E}_{t,z,j}[Z^{\pi^*}(T)] &= \exp\left(\int_t^T r(s)ds\right)z + n(t,j), \\ Var_{t,z,j}[Z^{\pi^*}(T)] &= N(t,j) - n^2(t,j). \end{aligned} \tag{3.4.31}$$

Proof. Substituting the obtained expressions of $M(t,j)$ and $m(t,j)$, the relation equation of $C(t,j)$ and $n(t,j)$, and all the specific expressions from (3.4.6)-(3.4.9) into (3.4.5), we can easily deduce the final expression of the equilibrium control as (3.4.29). As for the equilibrium value function, we can also deduce it without difficulty from (3.3.6), (3.4.1) and (3.4.20)-(3.4.24). □

3.5 Numerical Illustrations and Sensitivity Analysis

In this section, by giving several numerical examples, we illustrate the effect of the changes of the model parameters such as the transition rate q_{jk}, the intensities of the Poisson processes λ_1, λ_2, λ_0 and the risk aversion coefficient $\gamma(j)$ on the equilibrium control (EC) and equilibrium value function (EVF).

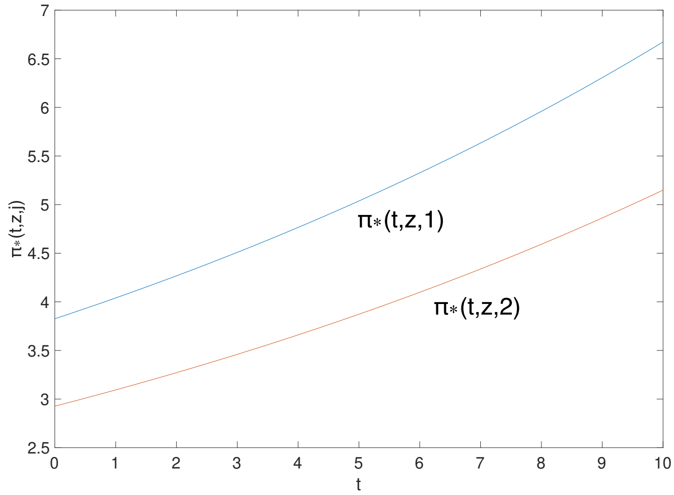
As in a number of literature, we assume that $d = 2$, namely, the market mode is divided into two states, the so-called "bullish" and "bearish", corresponding to regime 1 and regime 2 respectively.

Before we take different values of the concerned parameters, we give the basic parameter set as $T = 10$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_0 = 3$, $\mu_1 = 0.03$, $\sigma_1^2 = 0.008$, $\mu_2 = 0.02$, $\sigma_2^2 = 0.006$, $\rho = 0.4$, $r = 0.03$. The other parameters specified for our illustrative purpose can be found in Table 3.1. Here we have $q_{11} = -q_{12}$ and $q_{22} = -q_{21}$ since $\sum_{k=1}^{d} q_{jk} = 0$. Furthermore, we take $Z_t = z = 40$ here.

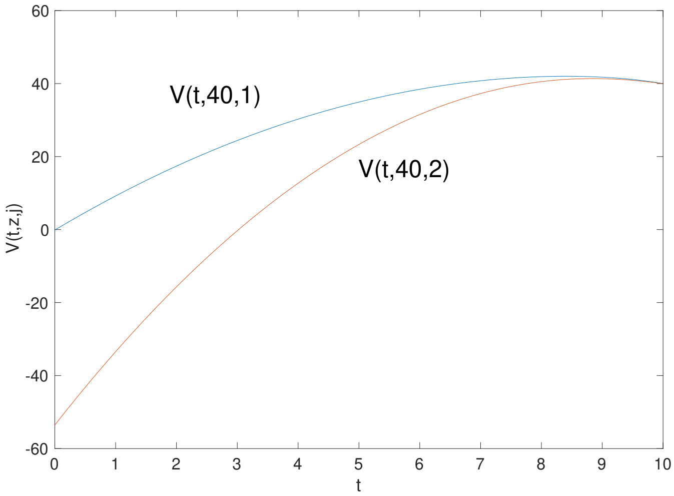
Table 3.1: The parameters for numerical Examples

	$b(\cdot)$	$\sigma(\cdot)$	$\widetilde{\sigma}(\cdot)$	$\mu(\cdot)$	$\gamma(\cdot)$	Transition rate
Regime 1(bearish)	0.3	0.35	0.4	0.08	0.4	$q_{12} = 0.5$
Regime 2(bullish)	0.15	0.17	0.2	0.04	0.8	$q_{21} = 0.5$

Fig. 3.1a depicts the ECs $\pi^*(t,z,j), j = 1,2$ against the initial time t. As we can see, the company would invest more in the risky asset as time goes by, which may be contrary to the situation of the usual pre-committed strategy. It



(a) equilibrium control against initial time



(b) equilibrium value function against initial time

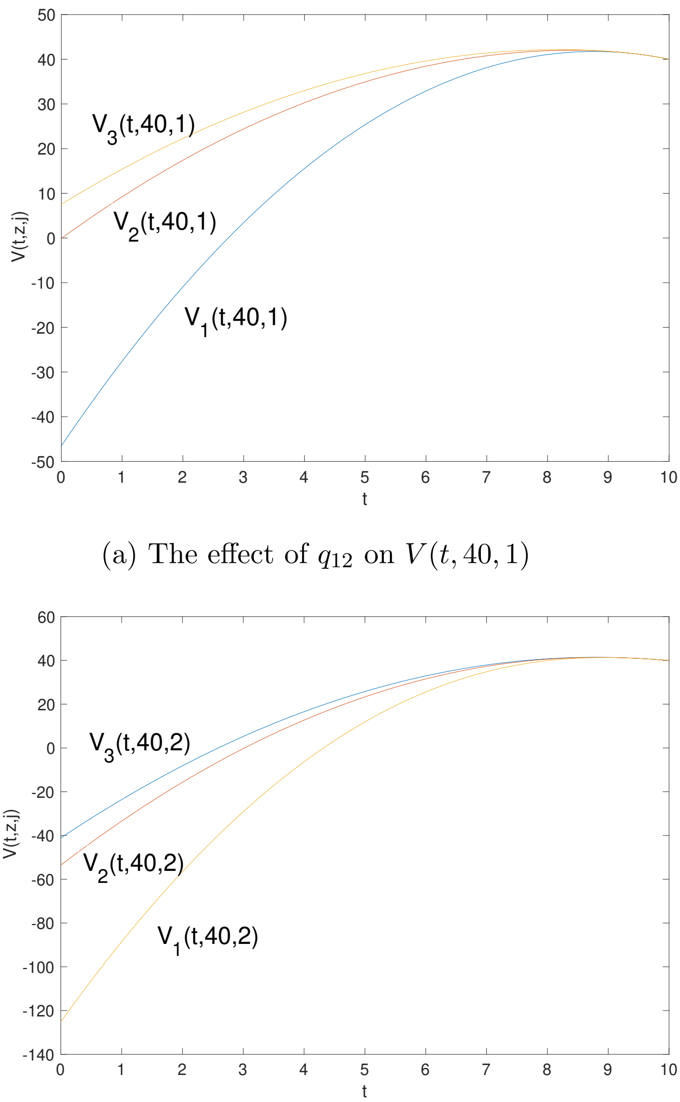
Figure 3.1: Basic parameter set

is because the pre-committed strategy only considers the objective functional at the current time but pay no attention to the utilities at later times. Therefore as time passes, there is less time for the investor to optimize the objective function, then the investor would invest less in the risky stock to be more certain to have a satisfactory wealth surplus at the expiry time. While the equilibrium strategy optimizes the objective functionals over the whole time horizon. As a result, the time-consistent investor will sacrifice the current utility, distributing some of the wealth in the bond as a consideration for the future investment. Besides, it is not difficult to understand that the investor would invest more in the risky asset when entering the bullish market compared to the bearish market since they may be more optimistic in the bullish market.

Fig. 3.1b shows the EVFs against the initial time. Again it is reasonable that the EVF is higher when entering the bullish market compared to the bearish market. Besides, we can see from the figure that the EVFs in the two market states converge as the time expires. This is obvious from (3.3.2) since $V(T, z, j) = F(j, z) + G(j, z) = z$, which means at the expiry time, the equilibrium value function $V(T, z, j)$, namely, the mean-variance utility of the company's terminal wealth surplus $J(t, z, j, \pi^*(\cdot))$, would just equal the current wealth surplus $Z(T) = z$ for $j = 1, 2$.

Next we examine how the variations of some of the model parameters would affect the EC and the EVF through several numerical examples.

Example 3.5.1. *We first study how the transition intensities change would affect the EC and the EVF. From the expression (3.4.29) it is obvious that the EC is not dependent on the transition intensities, which means neither the change of* q_{12} *nor that of* q_{21} *would affect* $\pi^*(t,z,j)$. *Keep other parameters fixed, and set* $q_{12} = 0, 0.5, 0.7$ *respectively, then we can easily make comparisons of the different EVFs through Figure 3.2.*



(a) The effect of q_{12} on $V(t,40,1)$

(b) The effect of q_{12} on $V(t,40,2)$

Figure 3.2: The effect of q_{12} on $V(t,40,j)$

As we can see from Fig. 3.2, the change of q_{12} would affect both $V(t,z,1)$ and $V(t,z,2)$. We denote the EVF with respect to $q_{12} = 0, 0.5, 0.7$ by $V_1(t,z,j)$,$V_2(t,z,j)$ and $V_3(t,z,j)$ respectively. Then we could conclude that the EVF increases (moves upward) in the early time period as the transition intensity from bullish market to bearish market increases whichever market we are entering.

Example 3.5.2. *Next we try to find out how the changes of the intensities of the Poisson processes, λ_1, λ_2, λ_0, would affect the EC and the EVF. First we take a look at the expression of the EC in (3.4.29), we can see that λ_2 has no effect on the control since it is irrelevant in the expression. As for λ_1 and λ_0, we could derive the partial derivatives as follows,*

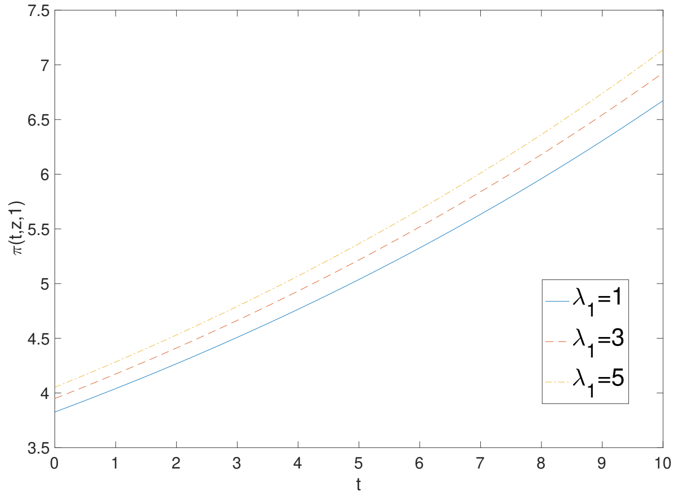
$$\begin{aligned}\frac{\partial \pi^*(t,z,j)}{\partial \lambda_1} &= \frac{\partial \pi^*(t,z,j)}{\partial \lambda_0}\\ &= \frac{\exp\left(-2(\int_t^T r(s)ds)\right)}{\gamma(i)\xi_2(t,j)^2}\Big[\mu_1\xi_2(t,j) - \xi_1(t,j)\sigma_1^2\Big]\\ &- \frac{\sigma_1^2\sigma(t,j)\widetilde{\sigma}(t,j)\rho}{\xi_2^2(t,j)}\\ &= \Delta_9\Big[\mu_1\sigma(t,j)^2 - (b(t,j)-r)\sigma_1^2\\ &- \sigma_1^2\sigma(t,j)\widetilde{\sigma}(t,j)\rho\gamma(j)\exp\left(2\int_t^T r(s)ds\right)\Big]\\ &= \Delta_9\Delta_{10},\end{aligned} \tag{3.5.1}$$

where

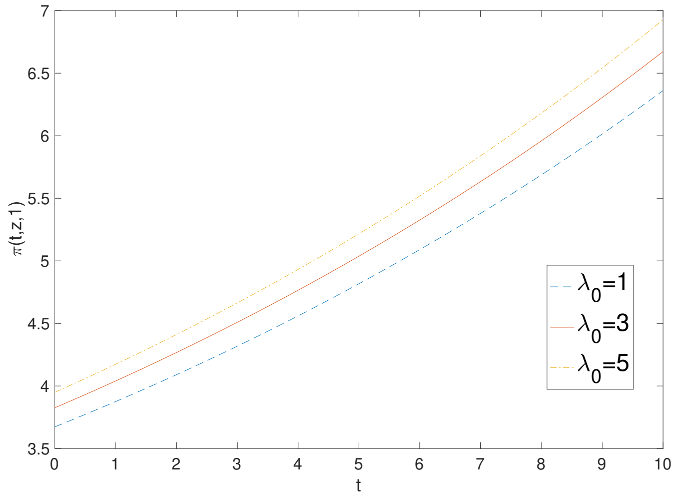
$$\begin{aligned}\Delta_9 &= \frac{\exp\left(-2(\int_t^T r(s)ds)\right)}{\gamma(j)\xi_2(t,j)^2} > 0,\\ \Delta_{10} &= \Big[\mu_1\sigma(t,j)^2 - (b(t,j)-r)\sigma_1^2\\ &- \sigma_1^2\sigma(t,j)\widetilde{\sigma}(t,j)\rho\gamma(j)\exp\left(2\int_t^T r(s)ds\right)\Big].\end{aligned} \tag{3.5.2}$$

The sign of Δ_{10} is not sure, but when $\frac{\mu_1}{\sigma_1^2} > (<)\ \frac{b(t,j)-r}{\sigma(t,j)^2} + \frac{\sigma(t,j)\widetilde{\sigma}(t,i)\rho\gamma(j)\exp(2\int_t^T r(s)ds)}{\sigma(t,j)^2}$, namely, when $\frac{\mu_1}{\sigma_1^2}$ is large (small) enough, we have $\Delta_{10} > 0$, i.e., $\frac{\partial \pi^*(t,z,j)}{\partial \lambda_1} = \frac{\partial \pi^*(t,z,j)}{\partial \lambda_0} > 0$, which means $\pi^*(t,z,j)$ increases as λ_1 (λ_0) increases.

To illustrate the effect of λ_1 and λ_0 through figures, we use the basic parameter set, and take $\lambda_1 = 1, 3, 5$ respectively to see the effect of the change of λ_1 on $\pi^*(t,z,j)$. Then we keep other parameters fixed, and take $\lambda_0 = 1, 3, 5$ respectively to see the effect of λ_0 change. Taking $\pi^*(t,z,1)$ as an example, we can easily get from some simple calculation that $\Delta_{10} > 0$ in this case, exactly as what we can see from Fig. 3.3, namely, $\pi^*(t,z,1)$ increases (moves upward) as λ_1 and λ_0 increases, which means when the intensity of the jumps of the stock increases, the investor (the company) would invest more in the stock.

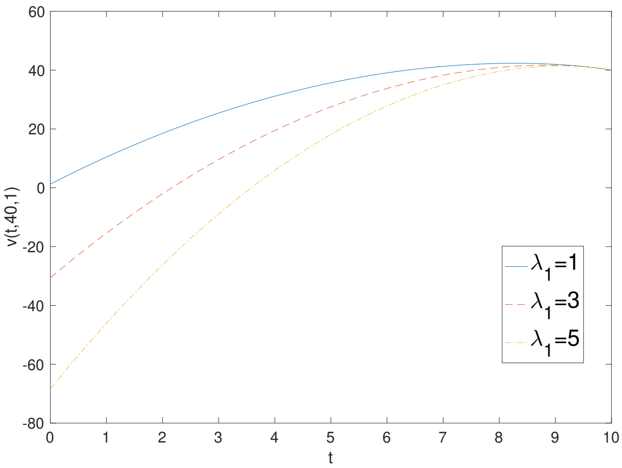


(a) The effect of λ_1

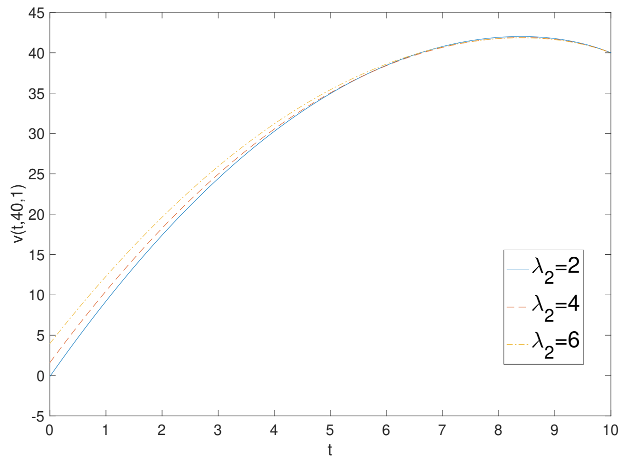


(b) The effect of λ_0

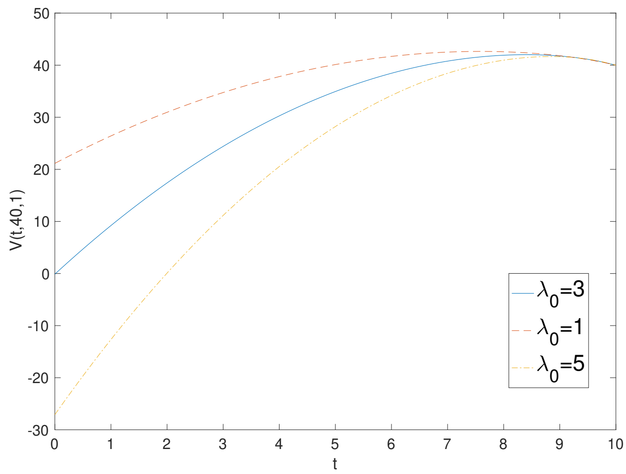
Figure 3.3: The effect of $\lambda_1(\lambda_0)$ on $\pi^*(t, z, 1)$



(a) The effect of λ_1



(b) The effect of λ_2



(c) The effect of λ_0

Figure 3.4: The effect of λ_1(λ_2,λ_0) on $V(t, 40, 1)$

As for the EVF, we can see from Fig. 3.4a that as λ_1 increases, which means when the intensity of the jumps of the stock increases, the equilibrium value function decreases (moves downward) in the early time period. While from Fig. 3.4b we know that the equilibrium value function increases (moves upward) in the early time period as λ_2 increases, which means when the jumps of the liability increases. Furthermore, we can see that the change of λ_2 has less effect than the changes of λ_1, which means the change of the intensity of the jumps of the stock affect more on the equilibrium value function than that of the liability. And this explains the fact that the equilibrium value function decreases (moves downward) as λ_0 increases although λ_0 affects both the intensity of the jumps of the stock and the liability, which can be seen from Fig. 3.4c. From the example we can see that the jump diffusion, especially the jump in the price process of the risky asset, does have a great influence on both the equilibrium control and the equilibrium value function regardless of the sign. This verifies the necessity of combining the jump diffusion into the regime-switching model.

Example 3.5.3. *Finally we illustrate the effect of the change of the risk aversion coefficient $\gamma(j)$ on the EC and the EVF. Similarly, we could derive the partial derivative as follows,*

$$\frac{\partial \pi^*(t,z,j)}{\partial \gamma(j)} = \frac{\xi_1(t,j)}{\xi_2(t,j)} \exp\Big(-(\int_t^T r(s)ds)\Big) \frac{-1}{\gamma^2(j)} < 0. \tag{3.5.3}$$

It is obvious that $\pi^*(t,z,j)$ is decreasing with respect to $\gamma(j)$.

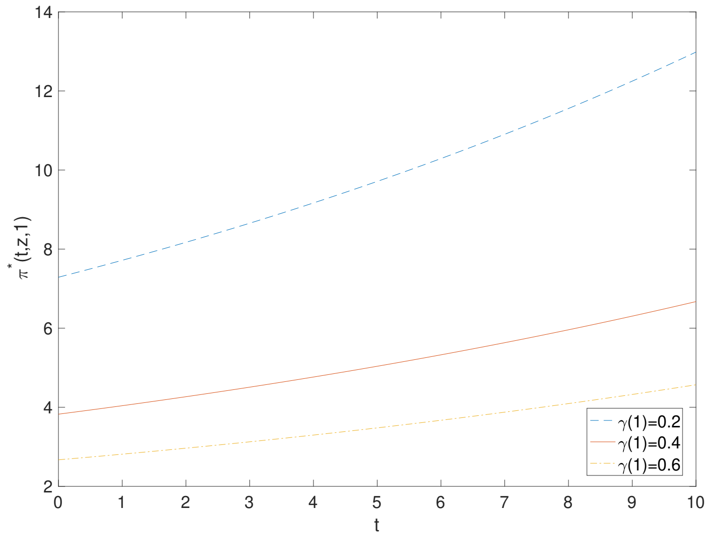


Figure 3.5: The effect of $\gamma(1)$ on $\pi^*(t,z,1)$

To illustrate this through figures, we take $\pi^*(t, z, 1)$ as an example, keep other parameters fixed, and change $\gamma(1)$ from 0.2 to 0.4 to 0.6. Then we can see from Fig. 3.5 that the EC moves downward as the risk aversion coefficient $\gamma(1)$ increases. This makes sense since that the investor (the company) surely would invest less in the risky asset as it gets more risk averse.

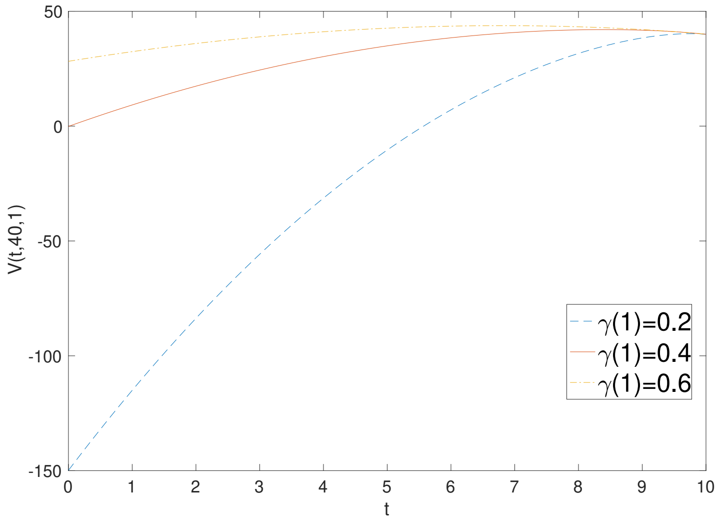


Figure 3.6: The effect of $\gamma(1)$ on $V(t, 40, 1)$

As for the EVF, we also take $V(t, 40, 1)$ as an example. Then we can see from Fig. 3.6 that in the bullish market, the EVF $V(t, 40, 1)$ moves upward in the early time period as the risk aversion coefficient $\gamma(1)$ increases, which means the more risk averse the investor (the company) is, the higher EVF it will have in the early time period.

3.6 Concluding Remarks

In this chapter, we study a continuous-time mean-variance asset-liability management (MVALM) problem in a Markov regime-switching jump-diffusion (RSJD) market. The work can be seen as an extension of [21] by adding correlated jumps to the price processes of both the risky stock and the uncontrollable liability, which is more realistic since the general jumps and Markovian jumps can be used in conjuction to capture both the short-term and long-term market movements. Compared to the pre-committed strategy adopted in previous literatures, we consider a game theoretic framework similar to that in [87] to handle the time-inconsistency problem. Thus the MVALM problem is formulated as a non-cooperate game where the investor at each time point in the investing horizon is regarded as a different player. We aim to find the Nash equilibrium control along with the equilibrium value function. Applying the stochastic dynamic programming techniques we have introduced in Chapter 2, we reduce the problem to solving the regime-switching jump-diffusion version of the extended Hamilton-Jacobi-Bellman (HJB) system based on the verification theorem. The closed-form equilibrium control and equilibrium value function are obtained in terms of five coefficients satisfying the associated ordinary differential equations (ODEs) derived from the HJB equations. Based on the solution, we present numerical and sensitivity analysis by assuming a market with only two states, the "bullish" market and the "bearish" market. It is shown that the equilibrium strategy will invest more in the risky asset as time approaches to the expiry date and the equilibrium value functions under two states take different values when the initial time is far from the expiry date but converge as the time expires. The effect of regime switching and jump diffusion are examined by studying the change of equilibrium control and equilibrium value function caused by the changes of transition rates and jump intensities. Moreover, the regime-dependent risk aversion coefficient affects the equilibrium control inversely, which is reasonable since the more risk averse the investor is, the less he would invest in the risky assets.

CHAPTER 4

Variance Swap Pricing under a Markov-Modulated Jump-Diffusion Model

4.1 General Overview

In this chapter, we aim to price a discretely-sampled variance swap under Heston's stochastic volatility model with Markov-modulated jump diffusion. This will extend the work of Elliott and Lian [114] with further consideration of jump diffusion in the regime-switching setting. Different from the characteristic function method used in [114], we apply the generalized Fourier transform method and the two-stage approach to obtain a semi-closed form pricing formula. To reduce the complexity of computation caused by the Markov chain, we first calculate the fair strike price based on a given fixed path of the Markov chain and then obtain the final expression by allowing various paths. To illustrate the accuracy and efficiency of our discrete solution, we present a semi-Monte-Carlo simulation and derive the pricing formula of a variance swap with continuous sampling times and compare the prices from the three methods under a range of different observation frequencies. Furthermore, as an application of the regime-switching jump-diffusion (RSJD) models, we examine the effect of regime switching and both the Merton-type and Kou-type jumps via numerical analysis.

The rest of this chapter is organized as follows. In Section 4.2, our RSJD version of Heston's stochastic volatility model is established, including a measure change process. In Section 4.3, the pricing formula is derived via the generalized Fourier transform under a two-stage framework. Section 4.4 presents several numerical examples to demonstrate the efficiency and accuracy of our pricing formula. Finally a conclusion is presented in Section 4.5.

4.2 Model Formulation

Before we start formulating our model, we first introduce our basic idea of pricing a variance swap.

A variance swap is defined as a forward contract on the future realized variance of the return from the specified underlying financial asset. Generally, the payoff function of a long position in a variance swap at expiry takes the form $V(T) = (\sigma_R^2 - K_{var}) \times G$, where σ_R^2 denotes the realized variance, K_{var} is the strike price of the variance swap, and G denotes the notional amount of the swap in dollars per volatility point squared. Usually, the values are all considered on an annualized basis.

Furthermore, the value of a variance swap at time t, which equals the expected present value of the payoff under the risk-neutral measurement of $\mathbb{Q}$, can be expressed as follows:

$$V(t) = \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}(\sigma_R^2 - K_{var})G\Big], \tag{4.2.1}$$

where r_t is the related interest rate and $\mathbb{E}_t^{\mathbb{Q}}$ denotes the conditional expectation at time t.

The nature of a forward contract indicates that the value of a variance swap at entry equals to zero. Thus, by setting $V(0) = 0$, we can easily have the following fair strike price,

$$K_{var} = \mathbb{E}_0^{\mathbb{Q}}[\sigma_R^2]. \tag{4.2.2}$$

The pricing of a variance swap is then reduced to calculating the expectation in (4.2.2).

The realized variance σ_R^2 is obtained by discretely sampling over the contract lifetime period $[0, T_e]$, which is also referred to as the total sampling period. The specific calculation of the realized variance σ_R^2 differs from contract to contract. Usually, the details of the calculation would be specified in the contract initially. In this chapter, we use a typical formula which is also used by many other researchers as follows:

$$\sigma_R^2 = \frac{AF}{N}\sum_{k=1}^{N}\left(\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2 \times 100^2, \tag{4.2.3}$$

where S_{t_k} denotes the underlying stock price at the k-th observation time, and N

denotes the total number of the observations. T_e is the life time of the contract, and $AF = \frac{N}{T_e}$ is the annualized factor converting this expression to an annualized variance, which is assumed to be within a wide range from 5 to 252 according to the sampling frequency.

Thus our pricing of a variance swap problem is reduced to the calculation of the conditional expectation of the realized variance defined by (4.2.3) under the risk-neutral measurement of $\mathbb{Q}$ at time 0. Next we start formulating our model.

4.2.1 The Heston model with Markov-modulated jump diffusion

In this chapter we use a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with $\mathcal{P}$ being the real-world probability measure. The market regime is divided into n different states described by the states of a Markov chain $\alpha(t)$. Following [105], $\alpha(t)$ is a continuous-time finite-state observable Markov chain whose value can be selected from the state space $E = \{e_1, e_2, \cdots, e_n\}$, where $e_i = (0, \cdots, 1, \cdots, 0)' \in \mathcal{R}^n$ is a n-dimensional canonical unit vector. Moreover, the semi-martingale representation theorem for the process $\alpha(t)$ can be obtained as follows:

$$d\alpha(t) = Q(t)\alpha(t)dt + dM(t), \tag{4.2.4}$$

where $M(t), t \in [0, \infty)$ is a $\mathcal{R}^n$-valued martingale increment process with respect to the natural filtration generated by $\alpha(t)$, and

$$Q(t) = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ldots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix} \tag{4.2.5}$$

is the generator matrix of $\alpha(t)$, where q_{ij} denotes the intensity of transition from state i to state j satisfying $\sum_i q_{ij} = 0$.

Furthermore, let w_s and w_y be two Wiener processes. For consideration of the skew effect, we assume that w_s and w_y are correlated with a constant correlation coefficient ρ. The stochastic process $\alpha(t)$ is assumed to be independent of w_s and w_y.

For simplicity, we consider a financial market with only two assets: a risk-less bond $B(t)$ and a risky stock $S(t)$. The price of the bond is driven by the following deterministic process

$$dB(t) = r_{\alpha(t)}B(t)dt, \tag{4.2.6}$$

where $r_{\alpha(t)} =< r, \alpha(t) >$ is the interest rate process which depends on the market state. $< \cdot, \cdot >$ denotes the inner product in $\mathcal{R}^n$, and $r = (r_1, \dots, r_n)'$ is a vector representing different interest rates under different market states. To be specific, r_i is the interest rate corresponding to the state i for each $i = 1, \dots, n$. Note that the subsequent parameters of the risky stock price process are defined in a similar way.

The price of stock is assumed to be driven by the following Markov-modulated jump diffusion process:

$$dS(t) = \Big[\mu_{\alpha(t)}dt + \sigma(y)dw_s + \int_R \beta_{\alpha(t)}(t,z)\tilde{N}_{\alpha(t)}(dz,dt)\Big]S(t), \tag{4.2.7}$$

where $\mu_{\alpha(t)} =< \mu, \alpha(t) >$, $\mu = (\mu_1, \dots, \mu_n)'$ denotes the appreciation rate of the stock process, and $\beta_{\alpha(t)}(t,z) = \beta(t,z,\alpha(t)) =< \beta(t,z), \alpha(t) >$, $\beta(t,z) = (\beta_1(t,z), \dots, \beta_n(t,z))'$ is a generalized form of the jump-size z (we consider an exponential form of $\beta(t,z)$ in the following sections). $\tilde{N}_{\alpha(t)}(dz,dt) =< \tilde{N}(dz,dt), \alpha(t) >$, $\tilde{N}(dz,dt) = (\tilde{N}_1(dz,dt), \dots, \tilde{N}_n(dz,dt))'$ is a compensated Poisson Random measure which is also determined by the Markov chain $\alpha(t)$ and can be rewritten as follows:

$$\tilde{N}_{\alpha(t)}(dz,dt) = N_{\alpha(t)}(dz,dt) - \lambda_{\alpha(t)}v_{\alpha(t)}(dz)\eta(dt), \tag{4.2.8}$$

where $v_{\alpha(t)}(dz) =< v(dz), \alpha(t) >$, $v(dz) = (v_1(dz), \dots, v_n(dz))'$ denotes the jump size distribution and $\lambda_{\alpha(t)} =< \lambda, \alpha(t) >$, $\lambda = (\lambda_1, \cdots, \lambda_n)'$ is the jump intensity which describes the expected number of jumps. $N_{\alpha(t)}(dz,dt) =< N(dz,dt), \alpha(t) >$, $N(dz,dt) = (N_1(dz,dt), \dots, N_n(dz,dt))'$ is the Markov-modulated Poisson Random measure. $\eta(dt)$ is a generalized form of dt, and for simplicity we take $\eta(dt) = dt$ in this chapter. $\sigma(y)$ denotes the volatility rate of the stock process and is assumed to be a function of y which is driven by the following stochastic process

$$dy = a(b_{\alpha(t)} - y)dt + \sigma_v\sqrt{y}dw_y, \tag{4.2.9}$$

where a is corresponding to the speed of mean reversion adjustment, $b_{\alpha(t)} =< b, \alpha(t) >$, $b = (b_1, \cdots, b_n)'$ is the mean and σ_v denotes the so-called volatility of volatility. Furthermore, we take $\sigma(y) = \sqrt{y}$.

4.2.2 Change of measure

As the market is incomplete, there are infinite equivalent martingale pricing measures. By making use of the results in [15], we apply the regime-switching Esscher transform to determine an equivalent risk-neutral measure. Let

$$\begin{aligned} dw_s^* &= dw_s + \theta_{\alpha(t)} dt, \\ dw_y^* &= dw_y + \gamma dt, \end{aligned} \tag{4.2.10}$$

where $\theta_{\alpha(t)} =< \theta, \alpha(t) >$, $\theta = [\frac{\mu_1 - r_1}{\sigma(y)}, \cdots, \frac{\mu_n - r_n}{\sigma(y)}]'$, and $\gamma = -\frac{\Lambda}{\sigma_v}\sqrt{y}$. Λ is the market price of the volatility risk (risk premium).

Substituting (4.2.8) and (4.2.10) into (4.2.7) and (4.2.9), we obtain the dynamics of the price process of the stock $S(t)$ and y under the risk neutral assumption as follows:

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \left[r_{\alpha(t)} - \lambda_{\alpha(t)} m_{\alpha(t)} \right] dt + \sigma(y) dw_s^* \\ &\quad + \int_R \beta_{\alpha(t)}(t, z) N_{\alpha(t)}(dz, dt), \end{aligned} \tag{4.2.11}$$

where $m_{\alpha(t)} = \int_R \beta_{\alpha(t)}(t, z) v_{\alpha(t)}(dz)$ and

$$dy = a^*(b_{\alpha(t)}^* - y) dt + \sigma_v \sqrt{y} dw_y^*, \tag{4.2.12}$$

where $a^* = a - \Lambda$, $b_{\alpha(t)}^* = \frac{ab_{\alpha(t)}}{a - \Lambda}$.

In the subsequent sections we will only use the risk-neutral probability measure.

Before we move on to the next section, we define three natural filtrations generated by the two wiener processes w_s^*, w_y^* and the Markov chain $\alpha(t)$ up to time t as follows:

$$\begin{aligned} \mathcal{F}^s(t) &= \sigma \left\{ w_s^*(u) : u \leq t \right\}, \\ \mathcal{F}^y(t) &= \sigma \left\{ w_y^*(u) : u \leq t \right\}, \\ \mathcal{F}^\alpha(t) &= \sigma \left\{ \alpha(u) : u \leq t \right\}. \end{aligned} \tag{4.2.13}$$

4.3 Variance Swap Pricing

As we have mentioned in Section 4.2, to price a variance swap, we are concerned with the calculation of the conditional expectation as follows:

$$\begin{aligned} K_{var} = \mathbb{E}_0^{\mathbb{Q}}[\sigma_R^2] &= \mathbb{E}_0^{\mathbb{Q}}\left[\frac{AF}{N}\sum_{k=1}^{N}\left(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2 \times 100^2\right] \\ &= \frac{AF}{N}\sum_{k=1}^{N}\mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2\right] \times 100^2. \end{aligned} \tag{4.3.1}$$

Our pricing problem can be further reduced to the calculation of the N conditional expectations of the same form as:

$$\mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2\right] = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2 \mid \mathcal{F}^s(0)\vee\mathcal{F}^y(0)\vee\mathcal{F}^\alpha(0)\right], \tag{4.3.2}$$

for some fixed equal time period $\Delta t = \frac{T_e}{N} = t_k - t_{k-1}, k = 1, \dots, N$, which is referred to as the sampling period defined as the time span between two observation points.

We consider the calculation under two cases: $k = 1$ and $k > 1$. When $k = 1$, we have only one unknown variable S_{t_1} in the expectation to be calculated since $S_{t_0} = S_0$ is the current stock price which is a known constant. We will discuss this case later.

Now we investigate the latter case where $k > 1$. In this case, both S_{t_k} and $S_{t_{k-1}}$ are unknown variables at initial time which makes the calculation of the expectation rather complicated and difficult to work out. Therefore, to reduce the dimension as well as the difficulty in computation, we utilize the work of Little and Pant (see [109,110,112]) and introduce a new variable I_t driven by the underlying process

$$dI_t = \delta(t_{k-1} - t)S_t dt, \tag{4.3.3}$$

where $\delta(\cdot)$ is a step function with the following definition:

$$\delta(t_{k-1} - t) = \delta_{t_{k-1}} = \begin{cases} 0, & t \neq t_{k-1}, \\ 1, & t = t_{k-1}. \end{cases} \tag{4.3.4}$$

and the property

$$\int_R \delta_a F(t)dt = F(a), \text{ for any } a \in R \text{ and any integrable function } F(t). \quad (4.3.5)$$

Thus I_t is a new process only related to the previous observation $S_{t_{k-1}}$, which can be written as follows:

$$I_t = \begin{cases} 0, & t < t_{k-1}, \\ S_{t_{k-1}}, & t \geq t_{k-1}. \end{cases} \quad (4.3.6)$$

With the new defined variable I_t, we then employ the two-stage approach from [112] to calculate the expectation in (4.3.2).

To illustrate this approach, we first consider a contingent claim

$$U_k = U_k(t, S_t, I_t, y_t, \alpha(t)), \quad (4.3.7)$$

defined over the period $[0, t_k]$ with a future payoff function at expiry as

$$U_k(t_k, S_{t_k}, I_{t_k}, y_{t_k}, \alpha(t_k)) = \left(\frac{S_{t_k}}{I_{t_k}} - 1\right)^2. \quad (4.3.8)$$

The value of this claim at time t could be written as

$$U_k(t, S_t, I_t, y_t, \alpha(t)) = \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(t_k-t)}(\frac{S_{t_k}}{I_{t_k}} - 1)^2 \mid \mathcal{F}^s(t) \vee \mathcal{F}^y(t) \vee \mathcal{F}^\alpha(t)\Big]. \quad (4.3.9)$$

Similar to that in [114], we first consider the conditional expectation given the information about the sample path of the Markov Chain $\alpha(t)$ from time 0 to the expiry time T, $\mathcal{F}^\alpha(T)$, where $T = t_k$ in this case. For a given realized path of $\alpha(t)$, the parameters such as $r_{\alpha(t)}$, $\lambda_{\alpha(t)}$, $m_{\alpha(t)}$, and $b_{\alpha(t)}$ are all deterministic functions. Under this assumption, we denote the value of the contingent claim as $W_k(t, S_t, I_t, y_t) = U_k(t, S_t, I_t, y_t, \alpha(t) \mid \mathcal{F}^\alpha(T)) = \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(t_k-t)}(\frac{S_{t_k}}{I_{t_k}} - 1)^2 \mid \mathcal{F}(t) \vee \mathcal{F}^\alpha(T)\Big]$, where $\mathcal{F}(t) = \mathcal{F}^{s(t)} \vee \mathcal{F}^{y(t)}$.

Then we can easily obtain the corresponding partial integral differential equation (PIDE) for W_k in the following theorem by using the Feynman-Kac theorem (Some subscripts have been omitted without ambiguity):

Theorem 4.1. *Let* $W_k(t, S_t, I_t, y_t) = U_k(t, S, I, y, \alpha(t) \mid \mathcal{F}^\alpha(T)) = \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(t_k-t)}(\frac{S_{t_k}}{I_{t_k}} - 1)^2 \mid \mathcal{F}(t) \vee \mathcal{F}^\alpha(T)\Big]$, *and* S *is driven by the dynamics of (2.11). Then* W_k *is gov-*

erned by the following PIDE:

$$
\begin{aligned}
&\frac{\partial W_k}{\partial t} + \Big[r_{\alpha(t)} - \lambda_{\alpha(t)} m_{\alpha(t)}\Big] S \frac{\partial W_k}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W_k}{\partial S^2} \\
&+ a^*(b^*_{\alpha(t)} - y)\frac{\partial W_k}{\partial y} + \frac{1}{2}\sigma_v^2 y \frac{\partial^2 W_k}{\partial y^2} + \rho\sigma_v y S \frac{\partial^2 W_k}{\partial S \partial y} \\
&+ \delta(t_{k-1} - t) S \frac{\partial W_k}{\partial I} + \lambda_{\alpha(t)} \int_R W_k(t, S\beta(t, z, \alpha(t))) v_{\alpha(t)}(dz) \\
&- (r_{\alpha(t)} + \lambda_{\alpha(t)}) W_k = 0,
\end{aligned}
$$

subject to the terminal condition:

$$
W_k(t_k, S, I, y) = \Big(\frac{S}{I} - 1\Big)^2. \tag{4.3.10}
$$

Proof. Let $\phi = W_k(t, S_t, I_t, y_t)$, according to the Itô's formula with jump, we first obtain

$$
\begin{aligned}
d\phi =& \phi_t dt + \phi_S dS^c(t) + \phi_I dI + \frac{1}{2}\phi_{SS}(dS^c(t))^2 \\
&+ \phi_y dy + \frac{1}{2}\phi_{yy}(dy)^2 + \phi_{Sy} dS^c(t) dy \\
&+ \phi(t, S(t), I, y) - \phi(t, S(t-), I, y),
\end{aligned} \tag{4.3.11}
$$

where $dS^c(t) = \big\{[r_{\alpha(t)} - \lambda_{\alpha(t)} m_{\alpha(t)}]dt + \sigma_{\alpha(t)} dw^*_s\big\} S^c(t)$ denotes the continuous part of the stock price process, and the discrete part of the Itô' formula can be written as

$$
\begin{aligned}
&\phi(t, S(t), I, y) - \phi(t, S(t-), I, y) \\
&= \int_R \Big[\phi(t, S\beta(t, z, \alpha(t)) - \phi(t, S)\Big] N_{\alpha(t)}(dz, dt) \\
&= \int_R \Big[\phi(t, S\beta(t, z, \alpha(t))) - \phi(t, S)\Big] \tilde{N}_{\alpha(t)}(dz, dt) \\
&+ \int_R \Big[\phi(t, S\beta(t, z, \alpha(t))) - \phi(t, S)\Big] \lambda_{\alpha(t)} v_{\alpha(t)}(dz)dt \\
&= \int_R \Big[\phi(t, S\beta(t, z, \alpha(t))) - \phi(t, S)\Big] \tilde{N}_{\alpha(t)}(dz, dt) \\
&+ \lambda_{\alpha(t)} \int_R \phi(t, S\beta(t, z, \alpha(t)) v_{\alpha(t)}(dz)dt \\
&- \lambda_{\alpha(t)} \int_R \phi(t, S) v_{\alpha(t)}(dz)dt \\
&= \int_R \Big[\phi(t, S\beta(t, z, \alpha(t))) - \phi(t, S)\Big] \tilde{N}_{\alpha(t)}(dz, dt) \\
&+ \lambda_{\alpha(t)} \int_R \phi(t, S\beta(t, z, \alpha(t)) v_{\alpha(t)}(dz)dt \\
&- \lambda_{\alpha(t)} \phi(t, S)dt.
\end{aligned}
\tag{4.3.12}
$$

Substituting (4.3.12) into (4.3.11) and extracting the coefficient of the dt term, we can prove theorem 4.1. □

Due to the definition of the function $\delta(\cdot)$, the PIDE at any time other than t_{k-1} could be reduced to:

$$
\begin{aligned}
&\frac{\partial W_k}{\partial t} + \Big[r_{\alpha(t)} - \lambda_{\alpha(t)} m_{\alpha(t)}\Big] S \frac{\partial W_k}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W_k}{\partial S^2} \\
&+ a^*(b^*_{\alpha(t)} - y)\frac{\partial W_k}{\partial y} + \frac{1}{2}\sigma_v^2 y \frac{\partial^2 W_k}{\partial y^2} + \rho\sigma_v y S \frac{\partial^2 W_k}{\partial S \partial y} \\
&+ \lambda_{\alpha(t)} \int_R W_k(t, S\beta(t, z, \alpha(t))) v_{\alpha(t)}(dz) \\
&- (r_{\alpha(t)} + \lambda_{\alpha(t)}) W_k = 0.
\end{aligned}
\tag{4.3.13}
$$

Thus, the term related to the variable I_t is not considered in the PIDE anymore except at the time t_{k-1}, which seemingly indicates the success in dimension reduction. However, we still have to consider the time point t_{k-1}, where the variable I experience a jump. Furthermore, the variable I is still present in the terminal condition. To handle this and ensure that the claim's value remains continuous (which is required by the no-arbitrary pricing theory), we utilize Little

& Pant's approach and consider an additional jump condition at time t_{k-1}:

$$lim_{t\uparrow t_{k-1}} W_k(t, S, y, I) = lim_{t\downarrow t_{k-1}} W_k(t, S, y, I). \tag{4.3.14}$$

According to Little and Pant's two-stage approach, we divide the time period into two parts $[0, t_{k-1}]$ and $[t_{k-1}, t_k]$, during each of which the variable I could be treated as constant. Thus we have completed the dimension reduction for the PIDE during each of the time spans. Then we solve the PIDE in (4.3.13) backwards through two stages. We first derive the solution of the PIDE in the first stage in $[t_{k-1}, t_k]$, which provides the terminal condition for the PIDE in the second stage in $[0, t_{k-1}]$ through the jump condition. Then we further solve the PIDE in the second stage and obtain the analytical solution for the PIDE within the whole time period.

After the analytical solution $W_k(\cdot)$ is obtained, we can further solve for $U_k(\cdot)$ by taking into account the path change of the Markov chain $\alpha(t)$ as

$$\begin{aligned} U_k(t, S_t, I_t, y_t, \alpha(t)) &= \mathbb{E}^{\mathbb{Q}}\Big[U_k(t, S_t, I_t, y_t, \alpha(t) \mid \mathcal{F}^{\alpha}(T)) \mid \mathcal{F}^{\alpha}(t)\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[W_k(t, S_t, I_t, y_t) \mid \mathcal{F}^{\alpha}(t)\Big]. \end{aligned} \tag{4.3.15}$$

Then we can eventually calculate the conditional expectation that we are concerned with in (4.3.2) according to the Feynman-Kac theorem as follows:

$$\mathbb{E}_0^{\mathbb{Q}}\left[\left(\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2\right] = e^{rt_k} U_k(0, S_0, I_0, y_0, \alpha(0)). \tag{4.3.16}$$

Thus we can obtain the fair strike price K_{var} in (4.3.1).

Now we have illustrated our basic idea for solving this variance swap pricing problem, we then start by solving the PIDE in (4.3.13) by the two-stage approach and the generalized Fourier transform method.

4.3.1 Variance swap pricing by the two stage process

As we have stated before, we divide the time period $[0, t_k]$ into two parts. Let $T = t_k = k\Delta t$, $\Delta t = \frac{T_e}{N} = t_k - t_{k-1}$. Then the two time spans are denoted by $[0, T - \Delta t]$ and $[T - \Delta t, T]$. We first solve the PIDE in the first stage.

(A) Stage I Algorithm

Let $T - \Delta t \leq t \leq T$, and $x = lnS$, then the equation in (4.3.13) can be easily converted to the following PIDE:

$$\begin{aligned}&\frac{\partial W_k}{\partial t} + \Big[r_{\alpha(t)} - \lambda_{\alpha(t)} m_{\alpha(t)} - \frac{1}{2}y\Big]\frac{\partial W_k}{\partial x} + \frac{1}{2}y\frac{\partial^2 W_k}{\partial x^2}\\ &a^*(b^*_{\alpha(t)} - y)\frac{\partial W_k}{\partial y} + \frac{1}{2}\sigma_v^2 y\frac{\partial^2 W_k}{\partial y^2} + \rho\sigma_v xy\frac{\partial^2 W_k}{\partial x \partial y}\\ &- \big(r_{\alpha(t)} + \lambda_{\alpha(t)}\big) W_k + \lambda_{\alpha(t)}\int_R W_k(t, x+z)v_{\alpha(t)}(dz) = 0,\end{aligned} \tag{4.3.17}$$

subject to the terminal condition

$$W_k(T, x, I, y) = \Big(\frac{e^x}{I} - 1\Big)^2. \tag{4.3.18}$$

Next we apply the generalized Fourier transform method to solve the above equation. Let $V(w) = \mathcal{F}(W_k) = \int_R W_k(x)e^{-jwx}dx$, and let $\Phi_z(w)$ denote the characteristic function of the jump size distribution. Then (4.3.17) can be converted to the following partial differential equation (PDE):

$$\begin{aligned}&\frac{\partial V}{\partial t} + \Big[r_{\alpha(t)} - \lambda_{\alpha(t)} m_{\alpha(t)} - \frac{1}{2}y\Big](jwV) + \frac{1}{2}y(jw)^2 V\\ &+ a^*(b^*_{\alpha(t)} - y)\frac{\partial V}{\partial y} + \frac{1}{2}\sigma_v^2 y\frac{\partial^2 V}{\partial y^2} + \rho\sigma_v y\frac{\partial V}{\partial y}(jw)\\ &- \big(r_{\alpha(t)} + \lambda_{\alpha(t)}\big) V + \lambda_{\alpha(t)}\Phi_z(w)V = 0,\end{aligned} \tag{4.3.19}$$

with the transformed terminal condition

$$V_T = \mathcal{F}(W_k(T)) = \mathcal{F}\Big[\big(\frac{e^x}{I} - 1)^2\big)\Big]. \tag{4.3.20}$$

The conversion from (4.3.17) to (4.3.19) is quite simple according to the table of Fourier transform pairs. The only process we have to specify is the conversion of the term $\lambda_{\alpha(t)}\int_R W_k(t, x+z)v_{\alpha(t)}(dz)$ to $\lambda_{\alpha(t)}\Phi_z(w)V$.

Let $v(dz) = p(z)dz$ in (4.3.17) (the subscript is omitted here for convenience), and $p(z)$ is the density function of the jump size distribution. Then we have the

Fourier transform of the integral term arising from the jump diffusion as follows:

$$\begin{aligned}\mathcal{F}\Big[\int_R W_k(t,x+z)]p(z)dz\Big] &= \int_R\int_R W_k(t,x+z)p(z)e^{-jwx}dzdx \\ &= \int_R p(z)dz\int_R W_k(x+z)e^{-jwx}dx \\ &= \int_R p(z)dz\int_R W_k(y)e^{-jw(y-z)}dy \\ &= \int_R p(z)e^{jwz}dz\int_R W_k(y)e^{-jwy}dy \\ &= \Phi_z(w)V.\end{aligned} \tag{4.3.21}$$

Moreover, we specify the Fourier transform of the terminal condition as follows:

$$\begin{aligned}V_T &= \mathcal{F}\Big[W_k(T)\Big] \\ &= \mathcal{F}(\frac{e^x}{I}-1)^2 \\ &= \mathcal{F}\Big[\frac{e^{2x}}{I^2}-2\frac{e^x}{I}+1\Big] \\ &= 2\pi\Big[\frac{\delta_{-2j}(w)}{I^2}-2\frac{\delta_{-j}(w)}{I}+\delta_0(w)\Big],\end{aligned} \tag{4.3.22}$$

where $\delta_a(\cdot)$, for any complex number a, is the generalized delta function with the same definition and property as in (4.3.4) and (4.3.5).

Now we are concerned with the solution of the PDE (4.3.19) with the above terminal condition, which can be assumed to be of the following form by applying Heston's solution scheme [3]

$$V = e^{L(w,t)+M(w,t)y}V_T. \tag{4.3.23}$$

Substituting the above expression into (4.3.19), we obtain the following ordinary differential equations (ODEs) with the corresponding terminal conditions respectively:

$$\begin{cases} -\dot{M} = -\frac{1}{2}(jw)(1-jw)+(\rho v(jw)-a^*)M+\frac{1}{2}\sigma_v^2M^2, \\ M(w,T)=0, \end{cases} \tag{4.3.24}$$

and

$$\begin{cases} -\dot{L} = \Big[r_{\alpha(t)} - \lambda_{\alpha(t)} m_{\alpha(t)}\Big](jw) - (r_{\alpha(t)} + \lambda_{\alpha(t)}) + \lambda_{\alpha(t)}\Phi_z(w) \\ \qquad + a^* b^*_{\alpha(t)} M, \\ L(w,T) = 0. \end{cases} \tag{4.3.25}$$

The ODE (4.3.24) can be solved explicitly as follows,

$$\begin{cases} M(w,t) = \frac{A+B}{\sigma_v^2}\frac{1-e^{B(T-t)}}{1-Ce^{B(T-t)}}, \\ A = \rho\sigma_v(jw) - a^*, B = \sqrt{A^2 + \sigma_v^2(jw)(jw-1)}, C = \frac{A+B}{A-B}. \end{cases} \tag{4.3.26}$$

As for the ODE (4.3.25), let $\phi_l(w) = [r_{\alpha(t)} - \lambda_{\alpha(t)} m_{\alpha(t)}](jw) - (r_{\alpha(t)} + \lambda_{\alpha(t)}) + \lambda_{\alpha(t)}\Phi_z(w) + a^* b^*_{\alpha(t)} M$, then the equation can be solved numerically as:

$$L(w,t) = \int_t^T \langle \phi_l(w), \alpha(s)\rangle ds. \tag{4.3.27}$$

Finally, according to the inverse Fourier transform, we obtain the solution to the PIDE in (4.3.17) as:

$$\begin{aligned} W_k(t,x,I,y) &= \mathcal{F}^{-1}(V) \\ &= \mathcal{F}^{-1}(e^{L(w,t)+M(w,t)y} V_T) \\ &= \frac{1}{2\pi}\int_R V_T e^{L(w,t)+M(w,t)y} e^{jwx} dw \\ &= \frac{e^{2x}}{I^2} e^{L(-2j,t)+M(-2j,t)y} - 2\frac{e^x}{I} e^{L(-j,t)+M(-j,t)y} \\ &+ e^{L(0,t)+M(0,t)y}. \end{aligned} \tag{4.3.28}$$

From (4.3.26), we could easily have $M(-j,t) = M(0,t) = 0$ for $\forall t \in [T - \Delta t, T]$, therefore we can further obtain:

$$W_k(t,x,I,y) = \frac{e^{2x}}{I^2} e^{L(-2j,t)+M(-2j,t)y} - 2\frac{e^x}{I} e^{L(-j,t)} + e^{L(0,t)}. \tag{4.3.29}$$

Then as we have stated before, we can obtain the terminal condition for stage two based on the solution of $W_k(t,x,I,y)$ in the first stage, through the jump condition in (4.3.14):

$$\begin{aligned} W_k(T-\Delta t, x, I, y) = {} & e^{L(-2j,T-\Delta t)+M(-2j,T-\Delta t)y} - 2e^{L(-j,T-\Delta t)} \\ & + e^{L(0,T-\Delta t)}. \end{aligned} \tag{4.3.30}$$

Note that here we are making use of the fact that $lim_{t \downarrow t_{k-1}} lnS_t = lnI_t$ according to the definition of I_t, and the terminal condition above only contains one stochastic variable, y.

With the terminal condition, we can now move on to solving the PIDE (4.3.13) in the second stage.

(B) Algorithm of stage II

Let $0 \leq t \leq T - \Delta t$. In this stage, based on the terminal condition in (4.3.30), we calculate

$$\begin{aligned} W_k(t, x, I, y) &= \mathbb{E}^{\mathbb{Q}}\left\{ e^{-r(T-\Delta t-t)} W_k(T - \Delta t, x, I, y) \mid \mathcal{F}(t) \right\} \\ &= \mathbb{E}^{\mathbb{Q}}\left\{ e^{-r(T-\Delta t-t)} \left[e^{L(-2j,T-\Delta t)+M(-2j,T-\Delta t)y} \right.\right. \\ &\quad \left.\left. - 2e^{L(-j,T-\Delta t)} + e^{L(0,T-\Delta t)} \right] \mid \mathcal{F}(t) \right\} \\ &= e^{-r(T-\Delta t-t)} \left[e^{L(-2j,T-\Delta t)} G(t, y) - 2e^{L(-j,T-\Delta t)} \right. \\ &\quad \left. + e^{L(0,T-\Delta t)} \right], \end{aligned} \tag{4.3.31}$$

where $G(t, y) = \mathbb{E}^{Q}\left(e^{M(-2j,t)y} \mid \mathcal{F}(t) \right)$.

Since here we only have one unknown stochastic variable y which is contained in the term $G(t, y)$, we have to solve for $G(t, y)$ to finally obtain the closed-form expression of W_k.

According to the Feynman-Kac theorem, $G(t, y)$ should satisfy the following PDE with the corresponding terminal condition:

$$\begin{cases} G_t + \frac{1}{2}\sigma_v^2 y G_{yy} + (a^*(b^*_{\alpha(t)} - y))G_y = 0, \\ G(T - \Delta t, y) = e^{M(-2j,T-\Delta t)y}. \end{cases} \tag{4.3.32}$$

We assume the following affine form for G,

$$G(t, y) = e^{R(t)+H(t)y}. \tag{4.3.33}$$

Substituting (4.3.33) into (4.3.32), we obtain the following ODEs

$$-\frac{\partial R}{\partial t} = a^* b_{\alpha(t)} H, \tag{4.3.34}$$

$$-\frac{\partial H}{\partial t} = -a^* H + \frac{1}{2}\sigma_v^2 H_i^2, \tag{4.3.35}$$

with the terminal conditions $R(T-\Delta t) = 0$ and $H(T-\Delta t) = M(-2j, T-\Delta t)$. After some simple derivation, we obtain:

$$H(t) = \frac{2a^*}{\sigma_v^2}\frac{e^{-a^*(T-\Delta t)}}{e^{-a^*(T-\Delta t)} - c_0}, \tag{4.3.36}$$

$$R(t) = \int_t^{T-\Delta t} \langle a^* b_{\alpha(t)} H, \alpha(t)\rangle dt, \tag{4.3.37}$$

where $c_0 = 1 - \frac{2a^*}{\sigma_v^2 M(-2j, T-\Delta t)}$.

Substituting (4.3.32) into (4.3.31), we can finally obtain the solution to the PIDE in **Theorem 4.1** through the two-stage approach as follows:

$$\begin{aligned} W_k(t,x,I,y) = e^{-r(T-\Delta t-t)}\Big[&e^{L(-2j,T-\Delta t)+R(t)+H(t)y} \\ &- 2e^{L(-j,T-\Delta t)} + e^{L(0,T-\Delta t)}\Big]. \end{aligned} \tag{4.3.38}$$

4.3.2 Variance swap pricing under regime switching Markov chain

Now we have obtained W_k as the value of U_k based on a given realized path of the Markov chain $\alpha(t)$, and we calculate U_k by taking into account the change of the sample path of the Markov chain.

Combining (4.3.15) and (4.3.38), we obtain:

$$\begin{aligned} U_k(t, S_t, I_t, y_t, \alpha(t)) &= \mathbb{E}^{\mathbb{Q}}\Big[W_k(t, S_t, I_t, y_t) \mid \mathcal{F}^\alpha(t)\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big\{e^{-r(T-\Delta t-t)}\Big[e^{L(-2j,T-\Delta t)+R(t)+H(t)y} \\ &- 2e^{L(-j,T-\Delta t)} + e^{L(0,T-\Delta t)}\Big] \mid \mathcal{F}^\alpha(t)\Big\} \\ &= e^{-r(T-\Delta t-t)}\Big[\Phi_{k1}(-2j,t)\Phi_{k2}(t)e^{H(t)y} \\ &- 2\Phi_{k1}(-j,t) + \Phi_{k1}(0,t)\Big], \end{aligned} \tag{4.3.39}$$

where

$$\begin{aligned}\Phi_{k1}(w,t) &= \mathbb{E}^{\mathbb{Q}}\Big(e^{\int_{T-\Delta t}^{T}\langle\phi_l(w),\alpha(s)\rangle ds} \mid \mathcal{F}^{\alpha}(t)\Big)\\ &= \mathbb{E}^{\mathbb{Q}}\Big(e^{\int_{t_{k-1}}^{t_k}\langle\phi_l(w),\alpha(s)\rangle ds} \mid \mathcal{F}^{\alpha}(t)\Big),\end{aligned} \tag{4.3.40}$$

$$\begin{aligned}\Phi_{k2}(t) &= \mathbb{E}^{\mathbb{Q}}\Big(e^{\int_{t}^{T-\Delta t}\langle a^* b^*_{\alpha(s)}H,\alpha(s)\rangle ds} \mid \mathcal{F}^{\alpha}(t)\Big)\\ &= \mathbb{E}^{\mathbb{Q}}\Big(e^{\int_{t}^{t_{k-1}}\langle a^* b^*_{\alpha(s)}H,\alpha(s)\rangle ds} \mid \mathcal{F}^{\alpha}(t)\Big).\end{aligned} \tag{4.3.41}$$

$\Phi_{k1}(w,t)$ and $\Phi_{k2}(t)$ can be calculated by utilizing the following formula in the Proposition 3.2 of [114]:

$$\mathbb{E}^{\mathbb{Q}}\Big(e^{\int_t^T\langle v,\alpha(t)\rangle ds} \mid \mathcal{F}^{\alpha}(t)\Big) = \langle exp(\int_t^T Q' + diag[v]ds)\alpha(t), E\rangle, \tag{4.3.42}$$

where $E = (1,1,\cdots,1)^T \in \mathcal{R}^n$ and Q' denotes the transpose of the transition matrix Q.

According to (4.3.16), we have

$$\begin{aligned}\mathbb{E}_0^{\mathbb{Q}}\Big[\Big(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\Big)^2\Big] &= e^{rt_k}U_k(0,S_0,I_0,y_0,\alpha(0))\\ &= e^{r\Delta t}\Big[\Phi_{k1}(-2j,0)\Phi_{k2}(0)e^{H(0)y_0} - 2\Phi_1(-j,0)\\ &+ \Phi_1(0,0)\Big].\end{aligned} \tag{4.3.43}$$

Since we have only considered the case where $k>1$ as we have stated before, we have to further work on the case where $k=1$ to obtain the summation in (4.3.1) fully.

For $k=1$, we have $t_k=t_1=T$ and $t_{k-1}=t_0=0=T-\Delta t$, which indicates that $[0,T]=[T-\Delta t,T]$. Therefore we could derive the term $\mathbb{E}_0^{\mathbb{Q}}[\Big(\frac{S_{t_1}-S_{t_0}}{S_{t_0}}\Big)^2]$ by making use of the result of the first stage in (4.3.30) as follows:

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}\Big[\Big(\frac{S_{t_1}-S_{t_0}}{S_{t_0}}\Big)^2\Big] &= e^{rt_1}U_1(0,S_0,I_0,y_0,\alpha(0)) \\
&= e^{rt_1}\mathbb{E}^{\mathbb{Q}}\Big[U_1(0,S_0,I_0,y_0,\alpha(0)\mid\mathcal{F}^{\alpha}(T))\mid\mathcal{F}^{\alpha}(0)\Big] \\
&= e^{rt_1}\mathbb{E}^{\mathbb{Q}}\Big[W_1(0,S_0,I_0,y_0)\mid\mathcal{F}^{\alpha}(0)\Big] \\
&= e^{rt_1}\mathbb{E}^{\mathbb{Q}}\Big[e^{L(-2j,0)+M(-2j,0)y_0}-2e^{L(-j,0)} \\
&+ e^{L(0,0)}\mid\mathcal{F}^{\alpha}(0)\Big] \\
&= e^{r\Delta t}\Big[\Phi_{11}(-2j,0)e^{M(-2j,0)y_0}-2\Phi_{11}(-j,0)+\Phi_{11}(0,0)\Big].
\end{aligned} \tag{4.3.44}$$

Let $N = AF * T_e$ and $\Delta t = \frac{T_e}{N} = \frac{1}{AF}$. Combining (4.3.1), (4.3.43) and (4.3.44), we can eventually obtain our fair strike price as follows,

$$\begin{aligned}
K_{var} &= \frac{AF}{N}\sum_{k=1}^{N}\mathbb{E}_0^{\mathbb{Q}}\Big[\Big(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\Big)^2\Big]\times 100^2 \\
&= \frac{e^{r\Delta t}}{T_e}\Big[f_1(y_0)+\sum_{k=2}^{N}f_k(y_0)\Big]\times 100^2,
\end{aligned} \tag{4.3.45}$$

where

$$f_1(y_0) = \Phi_{11}(-2j,0)e^{M(-2j,0)y_0}-2\Phi_{11}(-j,0)+\Phi_{11}(0,0), \tag{4.3.46}$$

$$f_k(y_0) = \Phi_{k1}(-2j,0)\Phi_{k2}(0)e^{H(0)y_0}-2\Phi_{k1}(-j,0)+\Phi_{k1}(0,0), \tag{4.3.47}$$

for $k = 2,\ldots,N$.

We conclude our calculation algorithm of the discretely sampled variance swap price as follows.

For the initialization part, we simply write a class to include all the formulas obtained in our derivation. The benefit of using class is that parameters in different status can inherit our class method easily without re-definition.

Algorithm 1:

```
Initialization;
for AF in [1, 252] do
    Δt = T_e / AF;
    for k in [1, AF] do  (the loop here is for calculating f_k)
        T = kΔt;
        if k==1 then
            Calculate f_1(y_0) by (4.3.46);
        else
            Calculate f_k(y_0) by (4.3.47);
        end
        f+ = f_k;
        K_var = 10000 * e^{rΔt} f/T_e
    end
end
```

4.4 Numerical Examples

In this section, we present several numerical examples for illustrating our semi-closed pricing formula for a variance swap in a Heston model with Markov-modulated jump diffusion. The effect of incorporating regime switching and jump diffusion would be investigated. We will also derive the counterpart pricing formula for a continuously sampled model and compare the two different prices under varying observation frequency, which will be helpful for readers to understand the improvement in accuracy of our discrete sampling solution.

For simplicity, we assume that there are two market regimes: regime 1 and regime 2, which can be interpreted as the 'bullish' and the 'bearish' market, respectively. In this case, we have $n = 2$ and the state space of the Markov chain is reduced to $E = \{e_1, e_2\}$, we assume the generator matrix Q as:

$$Q = \begin{bmatrix} -0.1 & 0.1 \\ 0.4 & -0.4 \end{bmatrix}. \tag{4.4.1}$$

A basic set of parameters adopted in this section is displayed in Table 4.1. We would change some of the parameters while keeping others fixed to investigate the effect of the change of the particular parameter. As we can see from Table 4.1, the interest rate r and the mean reversion value b in 'bullish' regime I ($r =$

0.06,$b = 0.09$) are higher than those in 'bearish' regime II ($r = 0.03$,$b = 0.04$), and the jump in the 'bullish' market is more oscillatory with higher jump intensity, which is economically reasonable. Note that the same parameters except the jump intensity are also adopted in [114] in a Heston's stochastic volatility model with only regime switching. In addition, we consider both the Merton-type jump and the Kou-type jump, whose characteristic functions and density functions are displayed in Table 4.2. The parameters corresponding to the jump process are also adopted in [117]. The lifetime of the variance swap contract is assumed to be $T_e = 1$.

Table 4.1: Model Parameters

Notations	*Parameters*	*Regime I*	*Regime II*
r	Interest rate	0.06	0.03
μ	Appreciation Rate	0.08	0.04
a	Mean reversion rate	3.46	3.46
b	Mean reversion value	0.009	0.004
σ_v	Volatility of volatility	0.14	0.14
ρ	Correlation Coefficient	-0.82	-0.82
λ	Jump intensity	0.2	0.1
	Merton Jump		
$\tilde{\mu}$	Mean of jump size	0.05	0.04
σ	Jump size volatility	0.086	0.078
	Kou Jump		
η_1	Inverse mean one	25	20
η_2	Inverse mean two	50	45
p	Exponential occurrences	0.2	0.16

Table 4.2: Jump Model Parameters

Model	$\gamma(dz)$	$\Phi_z(w)$
Merton	$\frac{e^{-(z-\tilde{\mu})^2}}{\sqrt{2\pi}\sigma}dz$	$e^{j\tilde{\mu}w-\frac{w^2}{2}\sigma^2}$
Kou	$p\eta_1 e^{-\eta_1 z}\mathcal{I}_{z>0} + (1-p)\eta_2 e^{\eta_2 z}\mathcal{I}_{z<0}dz$	$\frac{p\eta_1}{\eta_1 - jw} + \frac{(1-p)\eta_2}{\eta_2 + jw}$

4.4.1 Model validation

To show the improvement of accuracy of our solution, we first compare our pricing formula with the continuously sampled counterpart and the semi-Monte-Carlo simulation.

(A) Derivation of the pricing formula for a continuously-sampled variance swap

Our derivation of the continuous strike price is based on the results of [108], [15, 105, 114], and [13].

According to the work of [108], two terms contribute to the continuous fair strike price, including the stochastic volatility and the jump diffusion term.

Let

$$K_{var} = K_{var}^{vol} + K_{var}^{jump}, \tag{4.4.2}$$

where K_{var}^{vol} is the part of the fair strike price calculated by taking expectation of the cumulative volatility term:

$$K_{var}^{vol} = \mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{T_e}\int_0^{T_e} y_t dt \mid \mathcal{F}^{\alpha}(0) \vee \mathcal{F}^{y}(0)\Big], \tag{4.4.3}$$

and K_{var}^{jump} is the component of K_{var} derived from the expectation of jump diffusion process:

$$K_{var}^{jump} = \mathbb{E}\Big[\sum_{j=1}^{M}\frac{1}{T_j}(\sum_{i=1}^{N(T_j)}(ln(z_i))^2) \mid \mathcal{F}^{\alpha}(0) \vee \mathcal{F}^{s}(0)\Big]. \tag{4.4.4}$$

Let $T_e = \sum_{i=1}^{M} T_i$, and $T_i = \int_0^T \mathbb{I}_{\alpha(t)=i} ds$ denotes the occupation time of the Markov chain and let M be the total occupation of Markov chain. For each T_i, the parameters of the model stick to a specific status.

For the K_{var}^{vol}, we can utilize the results derived in [105] and [114],

$$K_{var}^{vol} = \frac{1-e^{-aT}}{aT}V_0 + \frac{a}{T}\int_0^T \langle exp(Q't)diag[\frac{b(1-e^{-at})}{a}]\alpha(t), E\rangle. \tag{4.4.5}$$

Here we mainly focus on the proof of K_{var}^{jump} from jump diffusion terms, utilizing the work of [27], where the author investigated the start forward option with regime-switching jump diffusion by Fourier transform. $N(T_i)$ is orthogonal to $N(T_j)$ when $i \neq j$, $i, j = 1 \cdots M$. Based on the work of [108], we obtain the characteristic function of (4.4.4) for a given sample path of the Markov chain from time 0 to time T_e

$$\begin{aligned}\Phi =&\mathbb{E}^{\mathbb{Q}}\Big[e^{-jw\sum_{j=1}^{M}\frac{1}{T_j}(\sum_{i=1}^{N(T_j)}(ln(Y_i))^2)} \mid \mathcal{F}^{\alpha}(T)\Big]\\ =&\prod_{i=1}^{M}\mathbb{E}^{\mathbb{Q}}\Big[e^{-jw\frac{1}{T_j}(\sum_{i=1}^{N(T_j)}(ln(Y_i))^2)} \mid \mathcal{F}^{\alpha}(T)\Big]\\ =&\prod_{i=1}^{M}e^{\lambda_j T_j(\phi_i(w)-1)},\end{aligned} \tag{4.4.6}$$

where

$$\phi_i(w) = \sqrt{\frac{T_j}{-2jw\sigma_j^2+T_j}}e^{\frac{\mu_j jw}{-2jw\sigma_j^2+T_j}}. \tag{4.4.7}$$

Then, we take the path change of the Markov chain into consideration and let $\varphi = \mathbb{E}^{\mathbb{Q}}[\Phi \mid \mathcal{F}^{\alpha}(0)]$, we obtain

$$\begin{aligned}\varphi &= \mathbb{E}^{\mathbb{Q}}\Big[\prod_{i=1}^{M}e^{\lambda_i T_i(\phi_i(w)-1)}\Big]\\ &= \mathbb{E}^{\mathbb{Q}}\Big[e^{\sum_{i=1}^{M}\lambda_i T_i(\phi_i(w)-1)}\Big]\\ &= \mathbb{E}^{\mathbb{Q}}\Big[e^{j\sum_{i=1}^{M}\lambda_i T_i\vartheta(w)}\Big]\\ &= e^{jT\vartheta_M(w)}\mathbb{E}^{\mathbb{Q}}\Big[e^{j\sum_{i=1}^{M-1}T_i\tilde{\vartheta}_i(w)}\Big]\\ &= \langle e^{Q'+jdiag(\theta(\mathbf{w}))}\alpha(0), E\rangle,\end{aligned} \tag{4.4.8}$$

where $\tilde{\vartheta}_i(w) = \vartheta_i(w)-\vartheta_M(w)$, $\vartheta_i(w) = \lambda_i(\phi_i(w)-1)$, and $\theta = (\vartheta_1(w), \vartheta_2(w), \cdots, \vartheta_M(w))'$. Here we assume $T_M = T - \sum_{i=1}^{M-1}T_i$. Utilising the result from [13], we have the following,

$$\mathbb{E}^{\mathbb{Q}}\Big[e^{j\sum_{i=1}^{M-1}\lambda_i T_i\tilde{\vartheta}_i(w)}\Big] = \langle e^{Q'+jdiag(\theta(\mathbf{w}))}\alpha(0), E\rangle, \tag{4.4.9}$$

where $E = (1, 1, \cdots, 1)' \in R^M$, and $\tilde{\theta} = (\tilde{\vartheta}_1(w), \tilde{\vartheta}_2(w), \cdots, \tilde{\vartheta}_{M-1}(w), 0)'$.

Thus,

$$K_{var}^{jump} = \frac{\partial\Phi}{\partial w}|_{w=0} = \langle e^{Q'}diag(\kappa)\alpha(0), E\rangle, \tag{4.4.10}$$

where $\kappa =< \lambda_j(\tilde{\mu}_j^2+\sigma_j^2)) >_{i=1}^{N}$. Consequently, we obtain the fair strike price for a continuously sampled variance swap as

$$
\begin{aligned}
K_{var} =& \frac{1-e^{-aT_e}}{aT_e} V_0 + \frac{a}{T_e} \int_0^{T_e} \langle e^{Q't} diag[\frac{b(1-e^{-at})}{a}] \alpha(t), E \rangle \\
&+ \langle (e^{Q'} diag(\lambda_j(\tilde{\mu}_j^2 + \sigma_j^2))) \alpha(t), E \rangle.
\end{aligned} \tag{4.4.11}
$$

(B) Semi-Monte-Carlo simulation

Different from the traditional Monte-Carlo simulation, the semi-Monte-Carlo simulation only requires simulation of the sample path of the Markov chain. The simulation process is discussed in detail in [114] and [118]. We will implement the semi-Monte-Carlo simulation through the following procedures:

(i) Simulate 10000 sample paths for the Markov chain $\alpha(t)$ through the generator matrix Q following the method of [118].

(ii) For the i-th sample path obtained, we calculate f_k^i according to the formulas in (4.3.46) and (4.3.47).

(iii) Calculate the fair strike price K_var^i for the i-th sample path using the results from the previous step and the formula (4.3.45).

(iv) Obtain the final fair strike price K_{var} by taking expectation of $K_{var}^i, i = 1, \cdots, 100000$.

Figure 4.1 compares the fair strike prices obtained from our discretely sampled pricing formula (4.3.45), the continuously sampled counterpart (4.4.11) and the semi-Monte-carlo simulation. As we can see from the figure, the results from our discrete pricing formula match the results from the semi-Monte-Carlo simulation, which provides verification for our solution. In fact, our pricing formula is more efficiently ideal with an elegant closed from. On the other hand, one can also see from the figure that the fair strike price of the discrete model can be extremely high and deviate drastically from the continuous counterpart when the observation frequency is low. Therefore it would be very inappropriate to use the continuous price as an approximation of the discrete one in this situation. However, as the sampling period narrows, the discrete K_{var} declines rapidly and asymptotically approaches the continuous K_{var}.

4.4.2 Regime switching effect

Next, we examine the effect of incorporating regime switching in our variance swap pricing model. We assume that the model without regime-switching

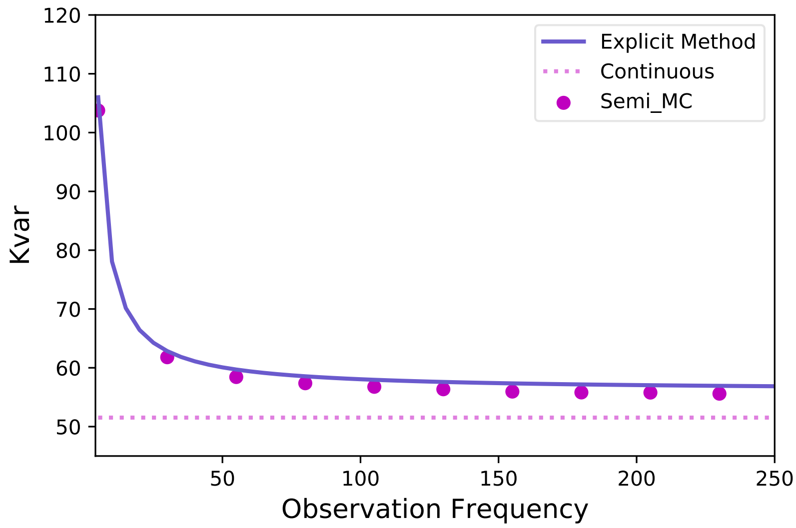


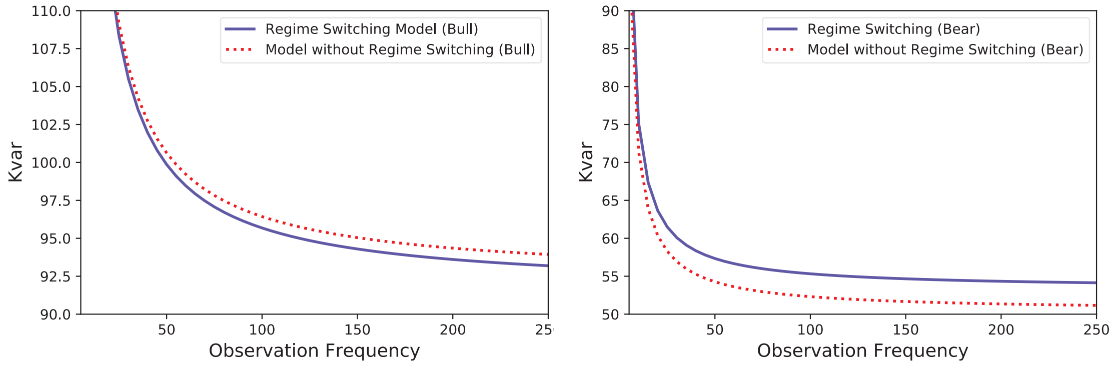
Figure 4.1: Calculated K_{var} from the discrete model, the continuous model and the semi-Monte-Carlo simulation

coincides with the 'bullish' case of the model with regime-switching, but with zero probability of switching to the other regime. Moreover, to get rid of the impact of the jump diffusion part, we add the same jumps to both of the models with and without regime switching. We have documented different prices of both models under a range of observation frequencies in Table 4.3 where we could find that the K_{var} with regime-switching is smaller than that without regime switching. This is reasonable due to the possibility of switching to the 'bearish' regime where the K_{var} calculated from our pricing formula is smaller because of the smaller values of the related parameters. As a matter of fact, things would be the opposite if we assume that the model without regime switching coincides with the 'bearish' case. To be more specific, comparisons between the pricing model with and without regime switching when entering markets with different initial regimes are displayed in Figure 4.2 and Figure 4.3 under different jump types.

In Figure 4.2, we add the Merton-type jump to both of the models with the parameters specified in Table 4.1. While in Figure 4.3, we consider the Kou-type jump with parameters specified in Table 4.1. We can conclude from Figure 4.2a and Figure 4.3a that with the consideration of regime-switching possibility, the K_{var} under 'bullish' market will be dragged down by the 'bearish' market. Inversely, from Figure 4.2b and Figure 4.3b we can see that the K_{var} under the 'bearish' economy will be pulled up by the effect of the 'bullish' market. Moreover, the difference between the two models seems wider in Figure 4.2b and Figure 4.3b than that in Figure 4.2a and Figure 4.3a. This can be explained by

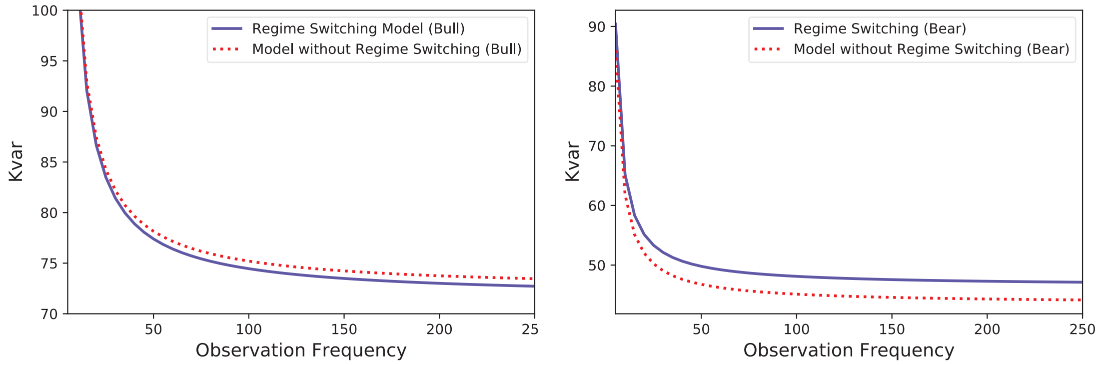
Table 4.3: Prices of Variance Swap with Regime Switching and Without Regime Switching

Observation Frequency	N	K_{var} (Regime-switching)	K_{var} (Non-Regime-switching)
Quarterly	4	210.20	211.83
Monthly	12	127.32	128.22
Fortnightly	26	107.68	108.48
Weekly	52	99.54	100.30
Daily	252	93.17	93.92
Continuously	∞	90.62	89.25



(a) State 1 (b) State 2

Figure 4.2: Comparison of K_{var} with and without regime switching: Merton-type jump



(a) Regime 1 (b) Regime 2

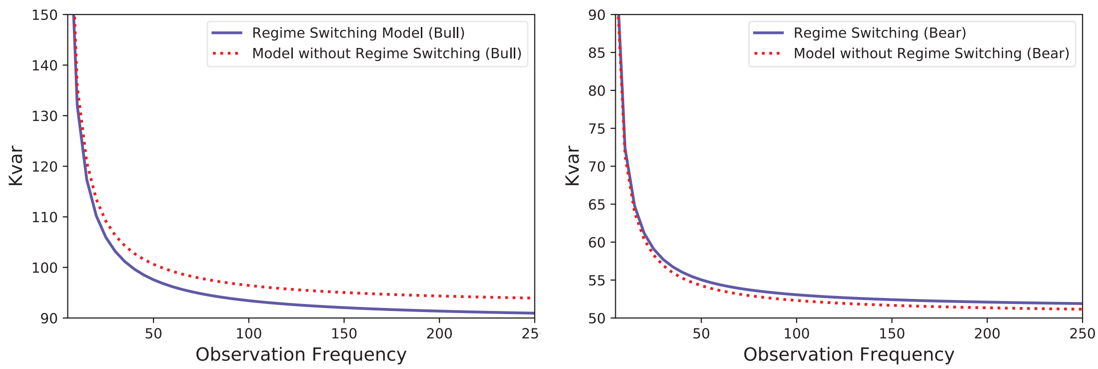
Figure 4.3: Comparison of K_{var} with and without regime switching: Kou-type jump

the different transition rates defined in the generator matrix Q. The transition rate from regime 1 (bullish) to regime 2 (bearish) is four times of the rate from regime 2 (bearish) to regime 1 (bullish). Consequently, the difference between the K_{var} of the model with and without regime-switching in Figure 4.2b and 4.3b is approximately four times bigger than that in Figure 4.2a and 4.3a.

To verify the influence of the transition rates on the fair strike price, we apply another set of generator matrix:

$$Q_2 = \begin{bmatrix} -0.4 & 0.4 \\ 0.1 & -0.1 \end{bmatrix}, \tag{4.4.12}$$

where the transition rate from regime 1 to regime 2 is now bigger than that from regime 2 to regime 1. The numerical result is displayed in Figure 4.4 where we can see that the difference of the price between the model with and without regime switching is wider when entering the 'bullish' market (regime 1) than the 'bearish' market (regime 2), which implies that the transition rates do have a great influence on the K_{var} in our model.



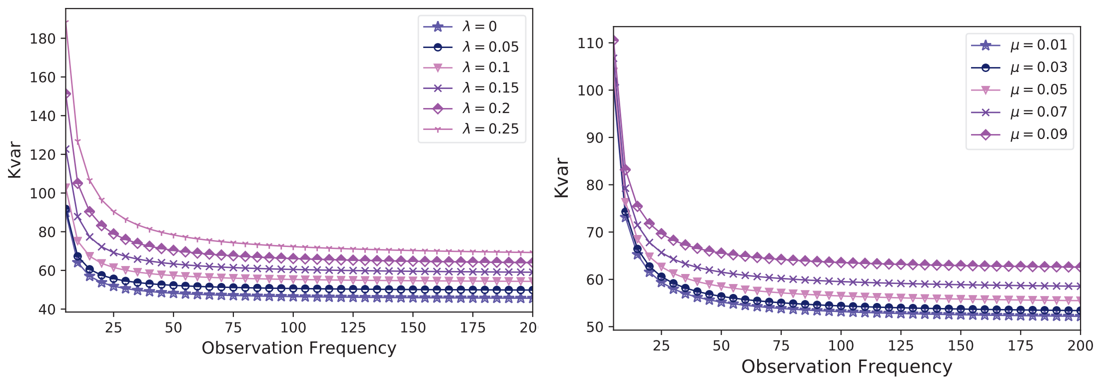
(a) Regime 1 (b) Regime 2

Figure 4.4: Comparison of K_{var} with and without regime switching with generator Q_2

4.4.3 Jump-diffusion effect

Finally, we investigate the impact of the jump diffusion. Here we also consider Merton-type jump and Kou-type jump in Figure 4.5 and Figure 4.6 respectively. Note that we only focus on the case of a 'bullish' market since the regime switching is not the major concern now.

In Figure 4.5, we assume that the jump size is driven by a normal distribution. The influence of the jump intensity is investigated in Figure 4.5a by changing the value of λ while keeping other parameters fixed. Note that by setting $\lambda = 0$, we consider the pricing model without a jump. As the figure depicts, the K_{var} is positively correlated to the jump intensity and the jump diffusion does have a great influence on the price. Figure 4.5b displays the different prices with a varying mean of the jump size, from which we can also conclude that the K_{var} increases with the mean. These results are in line with [108] where a similar conclusion about the effect of jumps is drawn.

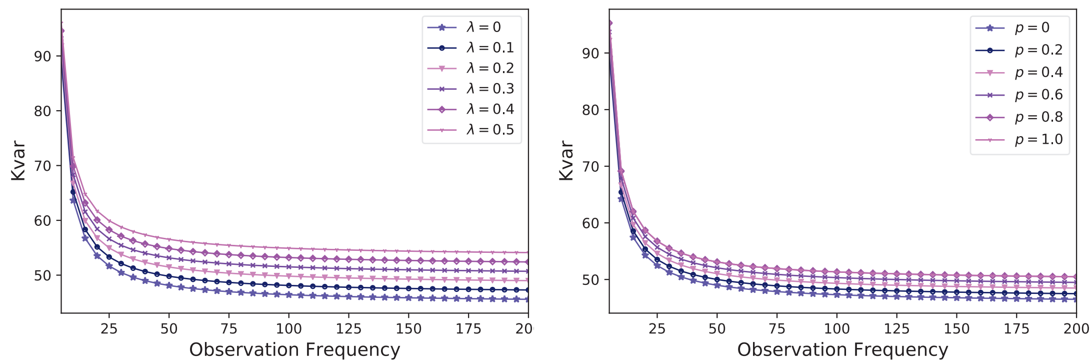


(a) Comparison of K_{var} with varying jump intensity

(b) Comparison of K_{var} with different mean

Figure 4.5: Comparison of K_{var} with the Merton type jump

Figure 4.6 depicts the jump effects under Kou's Model with the jump size being driven by a double exponential distribution. Similar result is obtained from Figure 4.6a about the positive correlation between the jump intensity and the K_{var}. Also, Figure 4.6b shows the positive effect of the weight we assign on each exponential distribution.



(a) Comparison of K_{var} with varying jump intensity

(b) Comparison of K_{var} with different probability

Figure 4.6: Comparison of K_{var} with the Kou type jump

Three main conclusions drawn from the numerical analysis are summarized as follows:

- The discretely sampled variance swap price converges to that of the continuously sampled counterpart as the observation frequency approaches infinity.
- Interactions of different market states resulting from the regime switching probability are evident. For instance, the fair strike price under the 'bullish' market will be smaller compared to the non-regime-switching market with same model parameters due to the possible transition to the 'bearish' market.
- Jumps have a significant effect on the variance swap price. More specifically, the fair strike price will increase as the jump intensity increases.

4.5 Concluding Remarks

In this chapter, we investigate the pricing of a discretely-sampled variance swap under the framework of Heston's stochastic volatility model with Markov regime switching jump diffusion (RSJD). The model parameters, including those related to the jump diffusion, are modulated by a Markov chain which is used to represent different market states. The pricing problem is reduced to the calculation of a series of conditional expectations based on the fact that a variance swap is essentially a forward contract that requires zero initial cost. By utilizing the two-stage approach and the generalized Fourier transform method, we obtain a semi-closed pricing formula for the fair strike price K_{var}. To show the improvement of the accuracy of our solution, we also derive the price of a continuously sampled variance swap and compare the results from the two pricing formulas with a semi-Monte-Carlo simulation. We find that the discrete price asymptotically approaches the continuous price as the observation frequency increases, though the differences between the two prices could be drastic when the frequency is low. We conduct numerical analysis where a market with two regimes is assumed. We conclude that the possibility of regime switching has a significant effect on the price, which is primarily related to the value of the transition rates. Moreover, we examine the effect of incorporating the jump diffusion into the pricing model by considering both the Merton-type and the Kou-type jump. By changing the value of related jump parameters, we find that either jump has a significant effect on the price, which is positively correlated to the jump intensity.

CHAPTER 5

Pricing of Volatility Derivatives in a Heston-CIR Model with Markov-Modulated Jump-Diffusion

5.1 General Overview

In this chapter, we combine the regime-switching jump-diffusion (RSJD) model with a hybrid model of Heston's stochastic volatility and the Cox-Ingersoll-Ross (CIR) stochastic interest rate to investigate the pricing of both volatility swaps and variance swaps with discrete sampling times. To our knowledge, this is the first attempt to establish such a RSJD Heston-CIR model for the pricing of volatility derivatives. The consideration of a stochastic interest rate instead of a deterministic one is more realistic and appropriate for volatility derivatives with a long term maturity. The CIR stochastic interest rate follows an ergodic process with a non-central chi-square distribution. A change of *numéraire* from the money market account to the zero-coupon bond is conducted due to the presence of stochastic interest rate. Then under the risk-neutral T-forward probability measure, we obtain the fair delivery prices based on different pre-specified calculating formulae for the realized variance and volatility. This process involves calculating the characteristic function for a random variable $\alpha(\cdot)$ which is defined on the log return of the underlying stock. Similarly, we first fix a path for the Markov chain and then obtain the results conditional on the changing paths. Numerical and sensitivity analysis is presented to examine the effect of each factor considered in our model on the fair strike price.

The rest of this chapter is organized as follows. The Heston-CIR model with Markov-modulated jump diffusion is formulated in Section 5.2, including a change of *numéraire*. In Section 5.3, we derive the characteristic function, and utilize the results to obtain various pricing formulae for variance swaps and volatility swaps. Section 5.4 presents the numerical and sensitivity analysis where the influence of each factor and each model parameter is examined and analysed, including a semi-Monte-Carlo simulation. Finally Section 5.5 concludes this chapter.

5.2 Model Setup

In this section, we develop the Heston-CIR model in a Markov regime-switching jump-diffusion (RSJD) market and present a *numéraire* change to convert the dynamics of the underlying asset under the real-world probability measure to those under the risk-neutral T-forward measure.

5.2.1 The Heston-CIR model with Markov-modulated jump diffusion

(A) Markov chain

We first consider a continuous-time finite-state observable Markov chain $X(t)$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\mathcal{P}$ denotes the real-world probability measure. The finite state space $S = \{s_1, \ldots, s_n\}$ of the Markov chain represents n different regimes of the financial markets. In the simplest case, for example, $n = 2$, the states s_1 and s_2 could be interpreted as representatives for 'bullish' and 'bearish' market respectively. Without loss of generality, we assume the state space as $S = E = \{e_1, \ldots, e_n\}$ where $e_i = (0, \ldots, 1, \ldots, 0)' \in \mathcal{R}^n$ is a canonical unit vector. Following [31], the semi-martingale representation of the Markov chain $X(t)$ can be obtained as follows,

$$dX(t) = Q(t)X(t)dt + dM(t), \tag{5.2.1}$$

where $M(t), t \in [0, \infty]$ is a $\mathcal{R}^n$-valued martingale with respect to the natural filtration generated by $X(t)$ under $\mathcal{P}$. $Q(t)$ denotes the generator matrix of $X(t)$ and is defined as:

$$Q(t) = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ldots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}, \tag{5.2.2}$$

where $q_{ij}, i, j = 1, \ldots, n$, denotes the transition rate from state e_i to state e_j satisfying $\sum_{j=1}^{n} q_{ij} = 0$. Moreover, $q_{ij} > 0$ $(j \neq i)$ and $q_{ii} = -\sum_{j \neq i} q_{ij}$.

(B) Financial market

Let $S(t)$ be the price process of the underlying risky asset in the financial market. Under the Heston-CIR model with Markov-modulated jump diffusion, $S(t)$ is driven by the following system of stochastic differential equations (SDEs):

$$\begin{cases} dS(t) = \Big[\mu_{X(t)}dt + \sigma(y(t))dW_s(t) + \int_{\mathbb{R}} J^s(Z^s_{X(t)})\tilde{N}^s(dt, dZ^s_{X(t)})\Big]S(t), \\ dy(t) = k(\theta_{X(t)} - y(t))dt + v\sqrt{y(t)}dW_y(t) + \int_{\mathbb{R}} J^y(Z^y_{X(t)})\tilde{N}^y(dt, dZ^y_{X(t)}), \\ dr(t) = b(a_{X(t)} - r(t))dt + \eta\sqrt{r(t)}dW_r(t), \end{cases} \tag{5.2.3}$$

where $y(t)$ is the stochastic instantaneous volatility process and $r(t)$ is the stochastic instantaneous interest rate process.

In the first equation, $\mu_{X(t)} =< \mu, X(t) >$ is the appreciation rate of the underlying asset $S(t)$ which is dependent on the market's current state. Here $< \cdot >$ denotes the inner product in $\mathcal{R}^n$, and $\mu = (\mu_1, \dots, \mu_n)'$ is a vector representing different appreciation rates corresponding to each possible market state, i.e., μ_i is the appreciation rate under state e_i, and $\sigma(y(t))$ is a function of $y(t)$ which we take as $\sigma(y(t)) = \sqrt{y(t)}$ in this chapter. $W_s(t)$ is a standard Brownian motion under $\mathcal{P}$. $J^s(Z^s_{X(t)})$ is a generalized form of the jump size $Z^s_{X(t)} =< Z^s, X(t) >$, $Z^s = (Z^s_1, \dots, Z^s_n)'$ and is assumed to be in an exponential form in the subsequent sections. $N^s(dt, dZ^s_{X(t)})$ is the Markov-modulated Poisson random measure and we define $\tilde{N}^s(dt, dZ^s_{X(t)}) = N^s(dt, dZ^s_{X(t)}) - \lambda^s_{X(t)}\gamma^s(dZ^s_{X(t)})dt$ as a compensated Poisson random measure. Here $\gamma^s(dZ^s_{X(t)})$ denotes the jump size distribution satisfying $\gamma^s(dZ^s_{X(t)}) = f^s(Z^s_{X(t)})dZ^s_{X(t)}$, where $f^s(Z^s_{X(t)})$ is the density function of $Z^s_{X(t)}$. $\lambda^s_{X(t)} =< \lambda^s, X(t) >, \lambda^s = (\lambda^s_1, \dots, \lambda^s_n)'$ is the jump intensity which describes the expected jump numbers. Here for a matrix or vector M, M' denotes the transpose of M.

In the second equation, k is corresponding to the mean-reverting speed of $y(t)$, and $\theta_{X(t)} =< \theta, X(t) >$, $\theta = (\theta_1, \dots, \theta_n)'$ is the long-term mean modulated by the Markov chain similarly as $\mu_{X(t)}$. v is the so-called volatility of volatility. $W_y(t)$ is another Brownian motion under $\mathcal{P}$. $N^y(dt, dZ^y_{X(t)})$ is another Markov-modulated Poisson random measure. $J^y(Z^y_{X(t)})$ is a generalized function of the jump size $Z^y_{X(t)} =< Z^y, X(t) >, Z^y = (Z^y_1, \dots, Z^y_n)'$. In this paper, we take $J^y(Z^y_{X(t)}) = Z^y_{X(t)}$ and assume that $Z^y_{X(t)}$ complies with a standard normal distribution so that $E[Z^y_{X(t)}] = 0$. Define $\tilde{N}^y(dt, dZ^y_{X(t)}) = N^y(dt, dZ^y_{X(t)}) - \lambda^y_{X(t)}\gamma^y(dZ^y_{X(t)})dt$ as another compensated Poisson process related to the volatility process. Similarly, $\gamma^s(dZ^y_{X(t)}) = f^y(Z^s_{X(t)})dZ^y_{X(t)}$ is the jump size distribution, where $f^y(Z^y_{X(t)})$ is the density function of $Z^y_{X(t)}$. $\lambda^y_{X(t)} =< \lambda^y, X(t) >, \lambda^y = (\lambda^y_1, \ldots, \lambda^y_n)'$ denotes the jump intensity.

As for the third equation, b is the speed of mean-reverting adjustment of $r(t)$. $a_{X(t)} =< a, X(t) >, a = (a_1, \ldots, a_n)'$ denotes the regime-switching long-term mean. η is the volatility of the interest rate process.

To ensure the positivity of both the square root processes, we assume that $2k\theta_{X(t)} \geq v^2$ and $2ba_{X(t)} \geq \eta^2$ (see [3]). We also assume that the Markov chain $X(t)$ is independent of $W_s(t), W_y(t)$ as well as $W_r(t)$, and the correlations between the Brownian motions are given as: $dW_s(t)dW_y(t) = \rho dt$, $dW_s(t)dW_r(t) = 0$, and $dW_y(t)dW_r(t) = 0$.

5.2.2 Change of *numéraire*

(A) Risk-neutral measure $\mathbb{Q}$ for the money market account numéraire

Under a Heston-CIR model with regime switching, there is no way to hedge against the risks rising from the randomness of the stochastic volatility, stochastic interest rate or Markov chain. In this sense, the market is incomplete, and therefore there exist infinitely many risk-neutral probability measures. Following [105], we can determine one risk-neutral probability measure $\mathbb{Q}$ for the money account numéraire using the Esscher transform. Let

$$\begin{cases} dW_s^*(t) = dW_s(t) + \dfrac{\mu_{X(t)} - r(t)}{\sqrt{y(t)}}dt, \\ dW_y^*(t) = dW_y(t) + \dfrac{\Lambda_y}{v}\sqrt{y(t)}dt, \\ dW_r^*(t) = dW_r(t) + \dfrac{\Lambda_r}{\eta}\sqrt{r(t)}dt, \end{cases} \tag{5.2.4}$$

where Λ_y and Λ_r denote the market price of the volatility risk and interest rate risk respectively.

Then the dynamics of the $S(t)$, $y(t)$ and $r(t)$ under the risk-neutral probability measure $\mathbb{Q}$ are as follows,

$$\begin{cases} dS(t) = \Big[r(t)dt + \sqrt{y(t)}dW_s^*(t) + \displaystyle\int_{\mathbb{R}} J^s(Z^s_{X(t)})\tilde{N}^s(dt, dZ^s_{X(t)})\Big]S(t), \\ dy(t) = k^*(\theta^*_{X(t)} - y(t))dt + v\sqrt{y(t)}dW_y^*(t) + \displaystyle\int_{\mathbb{R}} Z^y_{X(t)}\tilde{N}^y(dt, dZ^y_{X(t)}), \\ dr(t) = b^*(a^*_{X(t)} - r(t))dt + \eta\sqrt{r(t)}dW_r^*(t), \end{cases} \tag{5.2.5}$$

where $k^* = k + \Lambda_y$, $b^* = b + \Lambda_r$, $\theta^*_{X(t)} =< \theta^*, X(t) >, \theta^* = (\theta^*_1, \ldots, \theta^*_n)'$, $a^*_{X(t)} =< a^*, X(t) >, a^* = (a^*_1, \ldots, a^*_n)'$ and $\theta^*_i = \frac{k}{k+\Lambda_y}\theta_i$, $a^*_i = \frac{b}{b+\Lambda_r}a_i$, for $i = 1, \ldots, n$.

Furthermore, let $\tilde{W}_s(t)$, $\tilde{W}_y(t)$ and $\tilde{W}_r(t)$ be mutually independent Brownian motions under $\mathbb{Q}$ satisfying

$$\begin{pmatrix} dW^*_s(t) \\ dW^*_y(t) \\ dW^*_r(t) \end{pmatrix} = C \times \begin{pmatrix} d\tilde{W}_s(t) \\ d\tilde{W}_y(t) \\ d\tilde{W}_r(t) \end{pmatrix}, \tag{5.2.6}$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ such that } CC' = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2.7}$$

Then applying the Cholesky decomposition, we can re-write (5.2.5) as:

$$\begin{aligned} \begin{pmatrix} dS(t) \\ dy(t) \\ dr(t) \end{pmatrix} &= \begin{pmatrix} r(t)S(t) \\ k^*(\theta^*_{X(t)} - y(t)) \\ b^*(a^*_{X(t)} - r(t)) \end{pmatrix} dt + \begin{pmatrix} \int_{\mathbb{R}} J^s(Z^s_{X(t)})\tilde{N}^s(dt, dZ^s_{X(t)}) \\ \int_{\mathbb{R}} Z^y_{X(t)}\tilde{N}^y(dt, dZ^y_{X(t)}) \\ 0 \end{pmatrix} \\ &\quad + \Sigma \times C \times \begin{pmatrix} d\tilde{W}_s(t) \\ d\tilde{W}_y(t) \\ d\tilde{W}_r(t) \end{pmatrix} \\ &= \mu^{\mathbb{Q}} dt + \Sigma \times C \times \begin{pmatrix} d\tilde{W}_s(t) \\ d\tilde{W}_y(t) \\ d\tilde{W}_r(t) \end{pmatrix} + \begin{pmatrix} \int_{\mathbb{R}} J^s(Z^s_{X(t)})N^s(dt, dZ^s_{X(t)}) \\ \int_{\mathbb{R}} Z^y_{X(t)}N^y(dt, dZ^y_{X(t)}) \\ 0 \end{pmatrix}, \end{aligned} \tag{5.2.8}$$

where

$$\mu^Q = \begin{pmatrix} (r(t) - \lambda^s_{X(t)}m)S(t) \\ k^*(\theta^*_{X(t)} - y(t)) \\ b^*(a^*_{X(t)} - r(t)) \end{pmatrix}, m = \int_{\mathbb{R}} J^s(Z^s_{X(t)})\gamma^s(dZ^s_{X(t)}), \tag{5.2.9}$$

$$\Sigma = \begin{pmatrix} \sqrt{y(t)}S(t) & 0 & 0 \\ 0 & v\sqrt{y(t)} & 0 \\ 0 & 0 & \eta\sqrt{r(t)} \end{pmatrix}. \tag{5.2.10}$$

Before moving on to the next subsection, we define four natural filtrations generated by the Brownian motions $\tilde{W}_s(t), \tilde{W}_y(t), \tilde{W}_r(t)$ and the Markov chain

$X(t)$ as follows,

$$\begin{aligned}
&\mathcal{F}_s(t) = \sigma\{\tilde{W}_s(u) : u \leq t\}, \\
&\mathcal{F}_y(t) = \sigma\{\tilde{W}_y(u) : u \leq t\}, \\
&\mathcal{F}r(t) = \sigma\{\tilde{W}_r(u) : u \leq t\}, \\
&\mathcal{F}_X(t) = \sigma\{X(u) : u \leq t\}.
\end{aligned} \tag{5.2.11}$$

(B) Risk-neutral T-forward measure $\mathbb{Q}^T$ for the zero-coupon bond numéraire

In this subsection, we present a numéraire change from the money market account to the zero-coupon bond and derive the dynamics under the risk-neutral T-forward measure $\mathbb{Q}^T$.

Now consider a zero-coupon bond defined on $[0, T]$ with a payoff of 1 dollar whose value at time t under the Heston-CIR model regime-switching jump-diffusion model is denoted by $B(t, r(t), X(t))$. We define the discounting process as

$$D(t) = e^{-\int_0^t r(u)du}. \tag{5.2.12}$$

Then we have the value of the bond as:

$$B(t, r(t), X(t)) = \frac{1}{D(t)} \mathbb{E}^{\mathbb{Q}}[D(T)|\mathcal{F}_s(t) \vee \mathcal{F}_y(t) \vee \mathcal{F}_r(t) \vee \mathcal{F}_X(t)]. \tag{5.2.13}$$

According to the risk-neutral pricing theory, the discounted price process $D(t)B(t, T, r(t), X(t))$ should be a martingale under the risk-neutral probability measure $\mathbb{Q}$.

Applying the Feynman-Kac theorem to $D(t)B(t, r(t), X(t))$, we have the following partial differential equation (PDE) with the corresponding terminal condition:

$$\begin{cases}
\dfrac{\partial B}{\partial t} + b^*(a^*_{X(t)} - r)\dfrac{\partial B}{\partial r} + \dfrac{1}{2}\eta^2 r \dfrac{\partial^2 B}{\partial r^2} - rB + <\mathbf{B}, QX(t)>= 0, \\
\\
B(T, r(T), X(T)) = 1,
\end{cases} \tag{5.2.14}$$

where $\mathbf{B} = (B_1, \ldots, B_n)$,$B_i = B(t, r(t), e_i), i = 1, \ldots, n$ is a vector with each element representing the bond price process under the specific market state e_i.

We solve (5.2.14) by utilizing the following *Ansatz*:

$$B(t, r(t), X(t)) = e^{A(t,X(t))+P(t)r(t)}, \tag{5.2.15}$$

where $A(t, X(t)) =< \mathbf{A}, X(t) >$, $\mathbf{A} = (A_1(t), \ldots, A_n(t))$, $A_i(t) = A(t, e_i)$ for $i = 1, \ldots, n$.

Furthermore, for each specific market state e_i, we have $X(t) = e_i$, and

$$B_i = B(t, r(t), e_i) = e^{A(t,e_i)+P(t)r(t)} = e^{A_i(t)+P(t)r(t)}. \tag{5.2.16}$$

Then under state e_i, (5.2.14) becomes

$$\begin{cases} \dfrac{\partial B_i}{\partial t} + b^*(a_i^* - r)\dfrac{\partial B_i}{\partial r} + \dfrac{1}{2}\eta^2 r \dfrac{\partial^2 B_i}{\partial r^2} - rB_i + < \mathbf{B}, Q' e_i >= 0, \quad i = 1, \ldots, n, \\ \\ B(T, r(T), e_i) = 1. \end{cases} \tag{5.2.17}$$

Substituting (5.2.16) into (5.2.17) results in the following ODEs:

$$\begin{cases} \dfrac{dP(t)}{dt} - b^* P(t) + \dfrac{1}{2}\eta^2 P(t)^2 - 1 = 0, \\ P(T) = 0, \end{cases} \tag{5.2.18}$$

and

$$\begin{cases} \dfrac{dA_i(t)}{dt} + a_i^* b^* P(t) + e^{-A_i(t)} < \mathbf{M}, Q e_i >= 0, \quad , i = 1, \ldots, n, \\ A_i(T) = 0, \end{cases} \tag{5.2.19}$$

where $\mathbf{M} = (M_1, \ldots, M_n)', M_i = e^{A_i}$, for $i = 1, \ldots, n$.

(5.2.19) can be solved numerically and (5.2.18) belongs to the type 1 ODE that we will describe in the next subsection whose solution can be easily obtained using the results in (5.3.36) as follows:

$$P(t) = \frac{-2\sin h(\frac{(T-t)\sqrt{(b^*)^2+2\eta^2}}{2})}{\sin h(\frac{(T-t)\sqrt{(b^*)^2+2\eta^2}}{2})b^* + \cos h(\frac{(T-t)\sqrt{(b^*)^2+2\eta^2}}{2})\sqrt{(b^*)^2+2\eta^2}}. \tag{5.2.20}$$

Next we implement the numéraire change for the dynamics under the T forward measure $\mathbb{Q}^T$.

First we denote the two numéraires as $N_1(t) = \frac{1}{D(t)} = e^{\int_0^t r(u)du}$ and $N_2(t) = B(t, r(t), X(t))$. Then we can obtain

$$d \ln N_1(t) = r(t)dt, \tag{5.2.21}$$

and

$$\begin{aligned} d \ln N_2(t) &= d(A(t, X(t) + P(t)r(t)) \\ &=< \frac{d\mathbf{A}}{dt}, X(t) > dt + < \mathbf{A}, Q(t)X(t) > dt + < \mathbf{A}, dM(t) > \\ &+ P(t)\Big[b^*(a^*_{X(t)} - r)dt + \eta\sqrt{r(t)}dW_r(t)\Big] \\ &+ (b^*P(t) - \frac{1}{2}\eta^2 P^2(t) + 1)r(t)dt, \end{aligned} \tag{5.2.22}$$

where $d\mathbf{A}/dt = (dA_1(t)/dt, \ldots, dA_n(t)/dt)'$.

Note that $M(t)$ and $W_r(t)$ are independent, so the volatilities for the numéraires $N_1(t)$ and $N_2(t)$ are $\Sigma^{\mathbb{Q}} = (0, 0, 0)'$ and $\Sigma^{\mathbb{Q}^T} = (0, 0, P(t)\eta\sqrt{r(t)})'$ respectively.

To derive the dynamics under the T-forward measure $\mathbb{Q}^T$, we only have to derive the different drift part for (5.2.24). Applying the fundamental formula (see [119]), and using the results above, we can obtain the drift part under $\mathbb{Q}^T$ as follows,

$$\begin{aligned} \mu^{\mathbb{Q}^T} &= \mu^{\mathbb{Q}} - \Big(\Sigma \times C \times C^T \times (\Sigma^{\mathbb{Q}} - \Sigma^{\mathbb{Q}^T})\Big) \\ &= \begin{pmatrix} (r(t) - \lambda^s_{X(t)} m)S(t) \\ k^*(\theta^*_{X(t)} - y) \\ b^*(a^*_{X(t)} - r(t)) + P(t)\eta^2 r(t) \end{pmatrix}. \end{aligned} \tag{5.2.23}$$

Finally, we can derive the dynamics of (5.2.24) under $\mathbb{Q}^T$ as:

$$\begin{pmatrix} dS(t) \\ dy(t) \\ dr(t) \end{pmatrix} = \mu^{\mathbb{Q}^T} dt + \Sigma \times C \times \begin{pmatrix} d\tilde{W}_s(t) \\ d\tilde{W}_y(t) \\ d\tilde{W}_r(t) \end{pmatrix} + \begin{pmatrix} \int_{\mathbb{R}} J^s(Z^s_{X(t)}) N^s(dt, dZ^s_{X(t)}) \\ \int_{\mathbb{R}} Z^y_{X(t)} N^y(dt, dZ^y_{X(t)}) \\ 0 \end{pmatrix} \tag{5.2.24}$$

or

$$\begin{cases} dS(t) = \Big[(r(t) - \lambda^s_{X(t)} m)dt + \sqrt{y(t)}dW^*_s(t) + \displaystyle\int_{\mathbb{R}} J^s(Z^s_{X(t)})N^s(dt, dZ^s_{X(t)})\Big] S(t), \\ dy(t) = k^*(\theta^*_{X(t)} - y(t))dt + v\sqrt{y(t)}dW^*_y(t) + \displaystyle\int_{\mathbb{R}} Z^y_{X(t)} N^y(dt, dZ^y_{X(t)})), \\ dr(t) = \Big[b^*(a^*_{X(t)} - r(t)) + P(t)\eta^2 r(t)\Big]dt + \eta\sqrt{r(t)}dW^*_r(t). \end{cases} \tag{5.2.25}$$

One more important thing that we need to specify is that under $\mathbb{Q}^T$, we have the new semi-martingale representation of the Markov chain $X(t)$ as

$$dX(t) = Q^T X(t)dt + dM^T(t), \tag{5.2.26}$$

where M^T is a martingale under $\mathbb{Q}^T$ and $Q^T = (q^T_{ij}(t))_{n\times n}$, $i, j = 1, \ldots, n$ is the new generator matrix under $\mathbb{Q}^T$ defined by (according to [120])

$$\begin{cases} q^T_{ij}(t) = q_{ij} e^{A_j(t) - A_i(t)}, \quad j \neq i, \\ q^T_{ii}(t) = -\displaystyle\sum_{j\neq i} q^T_{ij}(t), \quad j = i. \end{cases} \tag{5.2.27}$$

Now we have completed the basic model setup, and we will then describe our basic idea and process of pricing the volatility swaps and variance swaps in this paper.

5.2.3 Volatility swaps and variance swaps

Volatility is often used as a measurement of the uncertainty of the intrinsic value of the underlying asset or derivative, indicating the potential risk of investing in the specific financial instrument. A volatility swap is a forward contract trading on the future realized volatility of the return from the specified underlying asset whose payoff is determined by the difference of the pre-specified strike price and the realized volatility over the contract period. A variance swap is defined similarly on the future realized variance.

To be specific, the payoff function at expiry for a volatility swap can be expressed as: $V_{vol}(T_l) = (\sigma_{vol} - Kvol) \times G$. Here σ_{vol} denotes the annualized realized volatility over the time period $[0, T_l]$, T_l denotes the expiry time for the contract. $Kvol$ is the fair delivery price for the contract, and G stands for the notional amount of the swap in dollars per volatility point. Similarly, the payoff function for a variance swap takes the form: $V_{var}(T_l) = (\sigma_{var}^2 - Kvar) \times L$, where σ_{var}^2 denotes the annualized realized variance over the contract life period $[0, T_l]$ and L denotes the notional amount of the swap in dollars per volatility point squared.

Based on the future payoff, the value of a volatility swap at time t can be calculated as:

$$V_{vol}(t) = \mathbb{E}_t^{\mathbb{Q}^T}\left[e^{-\int_t^{T_l} r(s)ds} V_{vol}(T_l)\right] = \mathbb{E}_t^{\mathbb{Q}^T}\left[e^{-\int_t^{T_l} r(s)ds}(\sigma_{vol} - Kvol) \times G\right], \quad (5.2.28)$$

and the current value of the variance swap can be calculated as:

$$V_{var}(t) = \mathbb{E}_t^{\mathbb{Q}^T}\left[e^{-\int_t^{T_l} r(s)ds} V_{var}(T_l)\right] = \mathbb{E}_t^{\mathbb{Q}^T}\left[e^{-\int_t^{T_l} r(s)ds}(\sigma_{var}^2 - Kvar) \times L\right], \quad (5.2.29)$$

where $\mathbb{E}_t^{\mathbb{Q}^T}[\cdot]$ denotes the conditional expectation up to time t under the risk-neutral T-forward measure $\mathbb{Q}^T$, and $r(t)$ denotes the interest rate process. Since the risk-neutral pricing of a forward contract requires the initial value to equal to zero, the fair delivery price can be calculated as: $Kvol = \mathbb{E}_0^{Q^T}[\sigma_{vol}]$. Similarly, for a variance swap the price should be: $Kvar = \mathbb{E}_0^{Q^T}[\sigma_{var}^2]$. Thus the valuation problem is reduced to calculating the conditional expectation of the future volatility or variance.

Normally, the realized volatility or variance is obtained by discretely sampling over the contract life period, or the so-called total sampling period, $[0, T_l]$. When entering a new contract, the factors that affect the calculation of the realized volatility or variance would be specified in advance, including the underlying asset, the sampling period or the observation frequency for the price, the annualized factor, the contract expiry time, and the specific formula of calculating the realized volatility or variance.

We consider the following typical formulae in this chapter, which have been adopted in plenty of literatures,

$$\text{actual-return realized volatility: } \sigma_{vol} = \sqrt{\frac{AF}{N}\sum_{k=1}^{N}\left(\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2} \times 100, \quad (5.2.30)$$

$$\text{log-return realized volatility: } \sigma_{vol} = \sqrt{\frac{\pi}{2NT_l}} \sum_{k=1}^{N} \left|\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}\right| \times 100, \quad (5.2.31)$$

$$\text{actual-return realized variance: } \sigma_{var}^2 = \frac{AF}{N}\sum_{k=1}^{N}\left(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2 \times 100^2, \quad (5.2.32)$$

$$\text{log-return realized variance: } \sigma_{var}^2 = \frac{AF}{N}\sum_{k=1}^{N} ln^2\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right) \times 100^2, \quad (5.2.33)$$

where $S_{t_k}, k = 0, \ldots, N$, denotes the k-th observation of the price of the underlying asset at time t_k. $t_k \in [0, T_l]$, and $[t_{k-1}, t_k]$ is called a sampling period between two observation sampling points. N denotes the total number of the observations. AF is an annualized factor converting the value to an annualized basis, which is determined by the observation frequency. We assume equally spaced discrete observations in this paper, i.e., $N\Delta t = T_l$, $\Delta t = t_k - t_{k-1}$ for $k = 0, \ldots, N$. Therefore we have $AF = \frac{1}{\Delta t} = \frac{N}{T_l}$, which is within the range from 5 to 252.

5.3 Derivation of Pricing Formulae for Volatility Swaps and Variance Swaps

In this section we aim at calculating the conditional expectation $E_0^{\mathbb{Q}^T}[\sigma_{vol}]$ or $E_0^{\mathbb{Q}^T}[\sigma_{var}^2]$ where the realized volatility or variance is defined as in (5.2.30)-(5.2.33) by employing the characteristic function method, and finally derive the pricing formulae for the volatility swaps and variances swaps.

5.3.1 Characteristic function

Let $\alpha(T-\Delta t) = \ln S(T) - \ln S(T-\Delta t)$. We define the characteristic function of $\alpha(T-\Delta t)$ as

$$\begin{aligned} f(\phi; t, T-\Delta t, \Delta t, y(t), r(t)) &= \mathbb{E}^{\mathbb{Q}^T}\left[e^{\phi\alpha(T-\Delta t)} | \mathcal{F}_s(t) \vee \mathcal{F}_y(t) \vee \mathcal{F}_r(t) \vee \mathcal{F}_X(t)\right] \\ &= \mathbb{E}^{\mathbb{Q}^T}\Big[e^{\phi(\ln S(T) - \ln S(T-\Delta t))} | \mathcal{F}_s(t) \vee \mathcal{F}_y(t) \\ &\quad \vee \mathcal{F}_r(t) \vee \mathcal{F}_X(t)\Big]. \end{aligned} \quad (5.3.1)$$

Based on the similar idea to that in [105], we first consider the conditional characteristic function based on the given information about the sample path of the Markov Chain $X(t)$ from time 0 to time T, denoted by $\mathcal{F}_X(T)$. For a fixed realized path of $X(t)$, the state dependent parameters such as $\theta^*_{X(t)}$ and $a^*_{X(t)}$ are all deterministic functions. After obtaining the characteristic function, we derive the unconditional characteristic function, taking into account different realized paths of the Markov chain. Namely, we calculate the characteristic function (5.3.1) as:

$$\begin{aligned} f(\phi; t, T-\Delta t, \Delta t, y(t), r(t)) &= \mathbb{E}^{\mathbb{Q}^T}\Big\{ f(\phi; t, T-\Delta t, \Delta t, y(t), r(t)|\mathcal{F}_X(T))|\mathcal{F}_X(t)\Big\} \\ &= \mathbb{E}^{\mathbb{Q}^T}\Bigg\{ \mathbb{E}^{\mathbb{Q}^T}\Big[e^{\phi\alpha(T-\Delta t)}|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t) \\ &\quad \vee \mathcal{F}_X(T)\Big]|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\vee\mathcal{F}_X(t)\Bigg\}. \end{aligned} \tag{5.3.2}$$

(A) Conditional characteristic function given $\mathcal{F}_X(T)$

Proposition 5.1. *Given the information $\mathcal{F}_X(T)$, if the dynamics of the asset $S(t)$ evolve as (5.2.24), we can obtain the* ***conditional characteristic function*** *corresponding to $\alpha(T-\Delta t) = \ln S(T) - \ln S(T-\Delta t)$ as follows,*

$$\begin{aligned} & f(\phi; t, T-\Delta t, \Delta t, y(t), r(t)|\mathcal{F}_X(T)) \\ &= e^{F(\phi, T-\Delta t)} g(C(\phi, T-\Delta t); t, T-\Delta t, y(t)) h(D(\phi, T-\Delta t); t, T-\Delta t, r(t)), \end{aligned} \tag{5.3.3}$$

where $C(\phi,t)$, $g(\phi; t, T-\Delta t, y(t))$ and $h(\phi; t, T-\Delta t, r(t))$ are given by

$$\begin{aligned} C(\phi, t) &= \frac{d-b}{v^2}\frac{e^{(T-t)d}-1}{le^{(T-t)d}-1}, \\ d &= \sqrt{(\rho v\phi - k^*)^2 - \phi(\phi-1)v^2}, \quad b = \rho v \phi - k^*, \quad l = \frac{b-d}{b+d}, \end{aligned} \tag{5.3.4}$$

$$\begin{aligned} g(\phi; t, T-\Delta t, y(t)) &= e^{M(\phi,t)+N(\phi,t)y(t)}, \\ M(\phi, t) &= \int_t^{T-\Delta t} < -[k^*\theta^*_{X(u)}N(\phi,u) + J^3_{X(u)}], X(u) > du, \\ J^3_{X(t)} &= \lambda^y_{X(t)}\int_{\mathbb{R}}(e^{N(\phi,t)Z^y_{X(t)}}-1)f^y(Z^y_{X(t)})dZ^y_{X(t)}, \\ N(\phi,t) &= \frac{2k^*}{v^2}\frac{e^{(T-t)k^*}}{1-ne^{(T-t)k^*}}, \quad n = 1 - \frac{2k^*}{v^2\phi}, \end{aligned} \tag{5.3.5}$$

$$h(\phi; t, T-\Delta t, r(t)) = e^{R(\phi,t)+Q(\phi,t)r(t)}, \tag{5.3.6}$$

and $F(\phi,t)$, $D(\phi,t)$, $R(\phi,t)$ and $Q(\phi,t)$ are determined by the following ODEs along with the corresponding terminal conditions,

$$\begin{cases} \dot{F} = -\Big[-\lambda^s_{X(t)}m\phi + b^* a^*_{X(t)}D + k^*\theta^*_{X(t)}C + J^1_{X(t)} + J^2_{X(t)}\Big], \quad F(\phi,T)=0, \\ \dot{D} = -\Big[\phi + (P(t)\eta^2 - b^*)D + \frac{1}{2}\eta^2 D^2\Big], \quad D(\phi,T)=0, \\ \dot{R} = -b^* a^*_{X(t)}Q, \quad R(\phi,T-\Delta t)=0, \\ \dot{Q} = -\Big[(P(t)\eta^2 - b^*)Q + \frac{1}{2}\eta^2 Q^2\Big], \quad Q(\phi,T-\Delta t)=\phi, \end{cases} \tag{5.3.7}$$

where

$$\begin{aligned} J^1_{X(t)} &= \lambda^s_{X(t)}\int_{\mathbb{R}}(e^{\phi Z^s_{X(t)}}-1)f^s(Z^s_{X(t)})dZ^s_{X(t)}, \\ J^2_{X(t)} &= \lambda^y_{X(t)}\int_{\mathbb{R}}(e^{C(\phi,t)Z^y_{X(t)}}-1)f^y(Z^y_{X(t)})dZ^y_{X(t)}. \end{aligned} \tag{5.3.8}$$

Proof. Given the filtration $\mathcal{F}_X(T)$, we can temporarily ignore the effect of the Markov chain and rewrite the conditional characteristic function as (according to the tower rule of expectation)

$$\begin{aligned} & f(\phi;t,T-\Delta t,\Delta t,y(t),r(t)|\mathcal{F}_X(T))) \\ &= \mathbb{E}^{\mathbb{Q}^T}\Big[e^{\phi\alpha(T-\Delta t)}|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\vee\mathcal{F}_X(T)\Big] \\ &= \mathbb{E}^{\mathbb{Q}^T}\Bigg\{\mathbb{E}^{\mathbb{Q}^T}\Big[e^{\phi\alpha(T-\Delta t)}|\mathcal{F}_s(T-\Delta t)\vee\mathcal{F}_y(T-\Delta t) \\ & \vee\mathcal{F}_r(T-\Delta t)\vee\mathcal{F}_X(T)\Big]|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\vee\mathcal{F}_X(T)\Bigg\} \\ &= \mathbb{E}^{\mathbb{Q}^T}\Bigg\{\mathbb{E}^{\mathbb{Q}^T}\Big[e^{\phi\alpha(T-\Delta t)}|\mathcal{F}_s(T-\Delta t)\vee\mathcal{F}_y(T-\Delta t)\vee\mathcal{F}_r(T-\Delta t)\Big] \\ & |\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\Bigg\}. \end{aligned} \tag{5.3.9}$$

Then we can obtain the conditional characteristic function by solving the inner expectation and outer expectation successively via two partial differential equations (PDEs).

We first focus on the inner expectation $\mathbb{E}^{\mathbb{Q}^T}\Big[e^{\phi\alpha(T-\Delta t)}|\mathcal{F}_s(T-\Delta t)\vee\mathcal{F}_y(T-\Delta t)\vee\mathcal{F}_r(T-\Delta t)\Big]$.

Define a function

$$U(\phi; t, W(t), y(t), r(t)) = \mathbb{E}^{\mathbb{Q}^T}\Big[e^{\phi\alpha(T-\Delta t)}|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\Big], \quad t\in[T-\Delta t, T], \tag{5.3.10}$$

where $W(t) = \ln S(t) - \ln S(T-\Delta t)$.

According to the Feynman-Kac theorem, $U(\cdot)$ satisfies the following PDE (subscripts related to $X(t)$ are omitted here for convenience),

$$\begin{aligned} &\mathcal{A}U + \lambda^s\int_{\mathbb{R}}\Big[U(\phi; t, W+Z^s, y, r) - U(\phi; t, W, y, r)\Big]f^s(Z^s)dZ^s \\ &\quad + \lambda^y\int_{\mathbb{R}}\Big[U(\phi; t, W, y+Z^y, r) - U(\phi; t, W, y, r)\Big]f^y(Z^y)dZ^y = 0, \\ &U(\phi; t=T, W, y, r) = e^{\phi\alpha(T-\Delta t)}, \end{aligned} \tag{5.3.11}$$

where

$$\begin{aligned} \mathcal{A} = &\frac{\partial}{\partial t} + (r - \lambda^s m - \frac{1}{2}y)\frac{\partial}{\partial W} + \Big[b^*(a^*-r) + P\eta^2 r\Big]\frac{\partial}{\partial r} + k^*(\theta^*-y)\frac{\partial}{\partial y} \\ &+ \frac{1}{2}y\frac{\partial^2}{\partial W^2} + \frac{1}{2}v^2 y\frac{\partial^2}{\partial y^2} + \frac{1}{2}\eta^2 r\frac{\partial^2}{\partial r^2} + \rho y v\frac{\partial^2}{\partial W\partial y}. \end{aligned} \tag{5.3.12}$$

The solution to (5.3.11) is assumed to be in the following form (see [3] and [121]):

$$U(\phi; t, W, y, r) = e^{F(\phi,t)+C(\phi,t)y+D(\phi,t)r+W\phi}. \tag{5.3.13}$$

Substituting (5.3.13) into (5.3.11), the PDE can be reduced to the following ODEs,

$$\begin{cases} \dot{F} - \lambda^s m\phi + b^*a^*D + k^*\theta^*C + J^1 + J^2 = 0, \\ F(\phi, T) = 0, \end{cases} \tag{5.3.14}$$

where

$$\begin{aligned} J^1 &= \lambda^s\int_{\mathbb{R}}(e^{\phi Z^s}-1)f^s(Z^s)dZ^s, \\ J^2 &= \lambda^y\int_{\mathbb{R}}(e^{CZ^y}-1)f^y(Z^y)dZ^y, \end{aligned} \tag{5.3.15}$$

and

$$\begin{cases} \dot{C} + \frac{1}{2}\phi(\phi - 1) + (\rho v\phi - k^*)C + \frac{1}{2}v^2C^2 = 0, \\ C(\phi, T) = 0, \end{cases} \tag{5.3.16}$$

$$\begin{cases} \dot{D} + \phi + (P(t)\eta^2 - b^*)D + \frac{1}{2}\eta^2D^2 = 0, \\ D(\phi, T) = 0, \end{cases} \tag{5.3.17}$$

(5.3.14) and (5.3.17) could only be solved numerically while (5.3.16) belongs to the type 1 ODE and the solution can be easily obtained based on (5.3.36) as follows

$$\begin{aligned} C(\phi, t) &= \frac{d-b}{v^2}\frac{e^{(T-t)d} - 1}{le^{(T-t)d} - 1}, \\ d &= \sqrt{(\rho v\phi - k^*)^2 - \phi(\phi - 1)v^2}, \quad b = \rho v\phi - k^*, \quad l = \frac{b-d}{b+d}. \end{aligned} \tag{5.3.18}$$

Thus the inner expectation is given by

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}^T}\left[e^{\phi\alpha(T-\Delta t)}|\mathcal{F}_s(T-\Delta t) \vee \mathcal{F}_y(T-\Delta t) \vee \mathcal{F}_r(T-\Delta t)\right] \\ &= U(\phi; t = T - \Delta t, W, y, r) \\ &= e^{F(\phi,T-\Delta t)+C(\phi,T-\Delta t)y(T-\Delta t)+D(\phi,T-\Delta t)r(T-\Delta t)}. \end{aligned} \tag{5.3.19}$$

Then the outer expectation is calculated as

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}^T}\left[e^{F(\phi,T-\Delta t)+C(\phi,T-\Delta t)y(T-\Delta t)+D(\phi,T-\Delta t)r(T-\Delta t)}|\mathcal{F}_s(t) \vee \mathcal{F}_y(t) \vee \mathcal{F}_r(t)\right] \\ &= e^{F(\phi,T-\Delta t)}\mathbb{E}^{\mathbb{Q}^T}\left[e^{C(\phi,T-\Delta t)y(T-\Delta t)}|\mathcal{F}_s(t) \vee \mathcal{F}_y(t) \vee \mathcal{F}_r(t)\right] \\ &\times \mathbb{E}^{\mathbb{Q}^T}\left[e^{D(\phi,T-\Delta t)r(T-\Delta t)}|\mathcal{F}_s(t) \vee \mathcal{F}_y(t) \vee \mathcal{F}_r(t)\right], \quad t \in [0, T - \Delta t]. \end{aligned} \tag{5.3.20}$$

To solve this expectation we first define the characteristic functions of $y(t)$ and $r(t)$ as

$$\begin{aligned} g(\phi; t, T - \Delta t, y(t)) &= \mathbb{E}^{\mathbb{Q}^T}\left[e^{\phi y(T-\Delta t)}|\mathcal{F}_s(t) \vee \mathcal{F}_y(t) \vee \mathcal{F}_r(t)\right], \\ h(\phi; t, T - \Delta t, r(t)) &= \mathbb{E}^{\mathbb{Q}^T}\left[e^{\phi r(T-\Delta t)}|\mathcal{F}_s(t) \vee \mathcal{F}_y(t) \vee \mathcal{F}_r(t)\right]. \end{aligned}$$

Similarly, we can obtain the following PDEs corresponding to $g(\cdot)$ and $h(\cdot)$

by utilizing the Feynman-Kac theorem,

$$\begin{cases} g_t + k^*(\theta^* - y)g_y + \frac{1}{2}v^2 y g_{yy} + \lambda^y \int_{\mathbb{R}} \Big[g(\phi; t, y + Z^y) - g(\phi; t, y) \Big] f_y(Z^y) dZ^y = 0, \\ g(\phi; t = T - \Delta t, T - \Delta t, y(t)) = e^{\phi y(T-\Delta t)}, \end{cases} \tag{5.3.21}$$

and

$$\begin{cases} h_t + \Big[b^*(a^* - r) + P(t)\eta^2 r \Big] h_r + \frac{1}{2}\eta^2 r h_{rr} = 0, \\ h(\phi; t = T - \Delta t, T - \Delta t, r(t)) = e^{\phi r(T-\Delta t)}. \end{cases} \tag{5.3.22}$$

We also solve (5.3.21) and (5.3.22) by utilizing the following *Ansatz*

$$g(\phi; t, T - \Delta t, y(t)) = e^{M(\phi,t) + N(\phi,t)y(t)},$$
$$h(\phi; t, T - \Delta t, r(t)) = e^{R(\phi,t) + Q(\phi,t)r(t)}.$$

Substituting the above equations into (5.3.21) and (5.3.22) respectively, we obtain the following ODEs,

$$\begin{cases} \dot{N} + \frac{1}{2}v^2 N^2 - k^* N = 0, \\ N(\phi, T - \Delta t) = \phi, \end{cases} \tag{5.3.23}$$

$$\begin{cases} \dot{M} + k^*\theta^* N + J^3 = 0, \\ M(\phi, T - \Delta t) = 0, \end{cases}, \text{ where } J^3 = \lambda^y \int_{\mathbb{R}} (e^{NZ^y} - 1) f^y(Z^y) dZ^y, \tag{5.3.24}$$

and

$$\begin{cases} \dot{R} + b^* a^* Q = 0, \\ R(\phi, T - \Delta t) = 0, \end{cases} \tag{5.3.25}$$

$$\begin{cases} \dot{Q} + (P(t)\eta^2 - b^*)Q + \frac{1}{2}\eta^2 Q^2 = 0, \\ Q(\phi, T - \Delta t) = \phi. \end{cases} \tag{5.3.26}$$

(5.3.23) can be solved according to (5.3.36) as follows,

$$N(\phi, t) = \frac{2k^*}{v^2} \frac{e^{k^*(T-t)}}{1 - e^{k^*(T-t)} n}, \quad n = 1 - \frac{2k^*}{v^2 \phi}. \tag{5.3.27}$$

While (5.3.24), (5.3.25) and (5.3.26) can only be solved numerically.

Finally the outer expectation (5.3.20) could be given by

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^T}\left[e^{F(\phi,T-\Delta t)+C(\phi,T-\Delta t)y(T-\Delta t)+D(\phi,T-\Delta t)r(T-\Delta t)}|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\right],\\
&=e^{F(\phi,T-\Delta t)}\mathbb{E}^{\mathbb{Q}^T}\left[e^{C(\phi,T-\Delta t)y(T-\Delta t)}|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\right]\\
&\times\mathbb{E}^{\mathbb{Q}^T}\left[e^{D(\phi,T-\Delta t)r(T-\Delta t)}|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\right],\\
&=e^{F(\phi,T-\Delta t)}g(C(\phi,T-\Delta t);t,T-\Delta t,y(t))h(D(\phi;T-\Delta);t,T-\Delta t,r(t)),\\
&\quad t\in[0,T-\Delta t].
\end{aligned} \tag{5.3.28}$$

Thus we have completed the proof of the proposition. □

(B) Characteristic function given $\mathcal{F}_X(t)$

We now move on to derive the semi-closed form unconditional characteristic function by letting the Markov chain $X(t)$ change.

Combining (5.3.9) and (5.3.3), we can obtain

$$\begin{aligned}
&f(\phi;t,T-\Delta t,\Delta t,y(t),r(t))\\
&=\mathbb{E}^{\mathbb{Q}^T}\Big\{f(\phi;t,T-\Delta t,\Delta t,y(t),r(t)|\mathcal{F}_X(T))|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\vee\mathcal{F}_X(t)\Big\}\\
&=\mathbb{E}^{\mathbb{Q}^T}\Big\{\exp\left(F(\phi,T-\Delta t)\right)g(C(\phi,T-\Delta t);t,T-\Delta t,y(t))\\
&h(D(\phi,T-\Delta t);t,T-\Delta t,r(t))|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\vee\mathcal{F}_X(t)\Big\}\\
&=\mathbb{E}^{\mathbb{Q}^T}\Big\{\exp[F(\phi,T-\Delta t)+M(C(\phi,T-\Delta t),t)+N(C(\phi,T-\Delta t),t)y(t)\\
&+R(D(\phi,T-\Delta t),t)+Q(D(\phi,T-\Delta t),t)r(t)]|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\vee\mathcal{F}_X(t)\Big\}\\
&=\mathbb{E}^{\mathbb{Q}^T}\Big\{\exp[\int_{T-\Delta t}^{T}<-\lambda^s_{X(t)}m\phi+b^*a^*_{X(t)}D(\phi,t)+k^*\theta^*_{X(t)}C(\phi,t)\\
&+J^1_{X(t)}+J^2_{X(t)},X(t)>dt+\int_t^{T-\Delta t}(<k^*\theta^*_{X(s)}N(C(\phi,T-\Delta t),s)+J^3_{X(s)},X(s)>\\
&+<b^*a^*_{X(s)}Q(D(\phi,T-\Delta t),s),X(s)>)ds+N(C(\phi,T-\Delta t),t)y(t)\\
&+Q(D(\phi,T-\Delta t),t)r(t)]|\mathcal{F}_s(t)\vee\mathcal{F}_y(t)\vee\mathcal{F}_r(t)\vee\mathcal{F}_X(t)\Big\}\\
&=\exp\left[N(C(\phi,T-\Delta t),t)y(t)\right]\exp\left[Q(D(\phi,T-\Delta t),t)r(t)\right]\\
&\mathbb{E}^{\mathbb{Q}^T}\Big\{\exp[\int_t^T<\xi(s),X(s)>ds]|\mathcal{F}_X(t)\Big\},
\end{aligned} \tag{5.3.29}$$

where

$$\xi(t) = \Big\{ \Big[-\lambda^s_{X(t)} m\phi + b^* a^*_{X(t)} D(\phi,t) + k^* \theta^*_{X(t)} C(\phi,t) + J^1_{X(t)} + J^2_{X(t)} \Big] \psi(t)$$
$$+ \Big[k^* \theta^*_{X(t)} N(C(\phi, T-\Delta t), t) + J^3_{X(t)} + b^* a^*_{X(t)} Q(D(\phi, T-\Delta t), t) \Big] (1-\psi(t)) \Big\},$$

$$\psi(t) = \begin{cases} 1, & T-\Delta t \le t \le T, \\ 0, & 0 \le t < T-\Delta t. \end{cases}$$

Based on the Proposition 3.2 in [114], the expectation in (5.3.29) is given by

$$\mathbb{E}^{\mathbb{Q}^T} \Big\{ \exp\,[\int_t^T < \xi(s), X(s) > ds] | \mathcal{F}_X(t) \Big\} = < \Phi(t,T;\xi(t)) X(t), \mathbf{1} >, \quad (5.3.30)$$

where

$$\Phi(t,T;\xi(t)) = \exp\Big(\int_t^T ((Q^T(s))' + diag(\xi(s)))ds \Big),$$
$$\mathbf{1} = (1,\ldots,1)' \in \mathcal{R}^n.$$

Thus we can eventually obtain the unconditional characteristic function as presented in the following proposition

Proposition 5.2. *If the dynamics of the underlying asset evolves as (5.2.24), the characteristic function of the stochastic variable* $\alpha(T-\Delta t) = \ln S(T) - \ln S(T-\Delta t)$ *is given by*

$$f(\phi; t, T-\Delta t, \Delta t, y(t), r(t))$$
$$= \exp\Big[N(C(\phi, T-\Delta t), t) y(t) \Big] \exp\Big[Q(D(\phi, T-\Delta t), t) r(t) \Big] < \Phi(t,T;\xi(t)) X(t), \mathbf{1} >, \quad (5.3.31)$$

where

$$\Phi(t,T;\xi(t)) = \exp\Big(\int_t^T ((Q^T(s))' + diag(\xi(s)))ds\Big), \quad \mathbf{1} = (1,\ldots,1)' \in \mathcal{R}^n,$$

$$\begin{aligned}
\xi(t) &= \Big\{\Big[-\lambda^s_{X(t)} m\phi + b^* a^*_{X(t)} D(\phi,t) + k^*\theta^*_{X(t)} C(\phi,t) + J^1_{X(t)} + J^2_{X(t)}\Big]\psi(t) \\
&\quad + \Big[k^*\theta^*_{X(t)} N(C(\phi,T-\Delta t),t) + J^3_{X(t)} \\
&\quad + b^* a^*_{X(t)} Q(D(\phi,T-\Delta t),t)\Big](1-\psi(t))\Big\}, \\
J^1_{X(t)} &= \lambda^s_{X(t)} \int_{\mathbb{R}} (e^{\phi Z^s_{X(t)}} - 1) f^s(Z^s_{X(t)}) dZ^s_{X(t)}, \\
J^2_{X(t)} &= \lambda^y_{X(t)} \int_{\mathbb{R}} (e^{C(\phi,t) Z^y_{X(t)}} - 1) f^y(Z^y_{X(t)}) dZ^y_{X(t)}, \\
J^3_{X(t)} &= \lambda^y_{X(t)} \int_{\mathbb{R}} (e^{N(C(\phi,T-\Delta t),t) Z^y_{X(t)}} - 1) f_y(Z^y_{X(t)}) dZ^y_{X(t)}, \\
\psi(t) &= \begin{cases} 1, & T-\Delta t \le t \le T, \\ 0, & 0 \le t < T-\Delta t, \end{cases}
\end{aligned}$$

and $C(\phi,t)$ *along with* $N(\phi,t)$ *are given by*

$$\begin{aligned}
C(\phi,t) &= \frac{d-b}{v^2}\frac{e^{(T-t)d}-1}{le^{(T-t)d}-1}, \\
d &= \sqrt{(\rho v\phi - k^*)^2 - \phi(\phi-1)v^2}, \quad b = \rho v\phi - k^*, \quad l = \frac{b-d}{b+d}, \qquad (5.3.32) \\
N(\phi,t) &= \frac{2k^*}{v^2}\frac{e^{k^*(T-t)}}{1-e^{k^*(T-t)}n}, \quad n = 1 - \frac{2k^*}{v^2\phi},
\end{aligned}$$

$D(\phi,t)$ *and* $Q(\phi,t)$ *are determined by the following ODEs with the corresponding terminal conditions,*

$$\begin{cases} \dot{D} + \phi + (P(t)\eta^2 - b^*)D + \frac{1}{2}\eta^2 D^2 = 0, \quad D(\phi,T) = 0, \\ \dot{Q} + (P(t)\eta^2 - b^*)Q + \frac{1}{2}\eta^2 Q^2 = 0, \quad Q(\phi,T-\Delta t) = \phi. \end{cases} \qquad (5.3.33)$$

5.3.2 Solution to the type 1 ODEs

In this subsection, we illustrate the algorithm of solving the type 1 ODE that appears in this chapter defined as equations taking the following form:

$$\dot{x} + Ax^2 + Bx + C = 0, \quad x(T) = x_T. \qquad (5.3.34)$$

We rewrite the equation and derive the solution as follows,

$$
\begin{aligned}
&\dot{x} + A\Big(x^2 + \frac{B}{A}x\Big) + C = 0 \\
&\Rightarrow \dot{x} + A\left[\Big(x + \frac{B}{2A}\Big)^2 - \frac{B^2}{4A^2}\right] + C = 0, \\
&\Rightarrow \dot{x} + A\Big(x + \frac{B}{2A}\Big)^2 - \frac{(B^2 - 4AC)A}{4A^2} = 0, \\
&\Rightarrow -\frac{\dot{x}}{(x + \frac{B}{2A})^2 - \frac{B^2-4AC}{4A^2}} = A, \\
&\Rightarrow -\dot{x}\Big(\frac{1}{x + \frac{B}{2A} - \frac{\sqrt{B^2-4AC}}{2A}} - \frac{1}{x + \frac{B}{2A} + \frac{\sqrt{B^2-4AC}}{2A}}\Big) = \sqrt{B^2 - 4AC}, \\
&\Rightarrow \frac{x + \frac{B}{2A} - \frac{\sqrt{B^2-4AC}}{2A}}{x + \frac{B}{2A} + \frac{\sqrt{B^2-4AC}}{2A}} = \tilde{C}e^{-t\sqrt{B^2-4AC}} (C \text{ denotes some constant here}), \\
&\Rightarrow \frac{2Ax + B - \sqrt{B^2 - 4AC}}{2Ax + B + \sqrt{B^2 - 4AC}} = \tilde{C}e^{-t\sqrt{B^2-4AC}}.
\end{aligned}
\tag{5.3.35}
$$

Then using the terminal condition, we have

$$
\begin{aligned}
&\frac{2Ax_T + B - \sqrt{B^2 - 4AC}}{2Ax_T + B + \sqrt{B^2 - 4AC}} = \tilde{C}e^{-T\sqrt{B^2-4AC}}, \\
&\quad \Rightarrow \tilde{C} = e^{T\sqrt{B^2-4AC}}\frac{2Ax_T + B - \sqrt{B^2 - 4AC}}{2Ax_T + B + \sqrt{B^2 - 4AC}}.
\end{aligned}
$$

Substituting the above back to (5.3.35), we have

$$
\frac{2Ax + B - \sqrt{B^2 - 4AC}}{2Ax + B + \sqrt{B^2 - 4AC}} = e^{\sqrt{B^2-4AC}(T-t)}\frac{2Ax_T + B - \sqrt{B^2 - 4AC}}{2Ax_T + B + \sqrt{B^2 - 4AC}}.
$$

Let

$$
\begin{aligned}
&D = \sqrt{B^2 - 4AC}, \\
&\Delta = e^{D(T-t)}\frac{2Ax_T + B - \sqrt{B^2 - 4AC}}{2Ax_T + B + \sqrt{B^2 - 4AC}}.
\end{aligned}
$$

We can finally obtain the solution to (5.3.34) as follows,

$$
x(t) = \frac{\Delta(B + D) - (B - D)}{2A(1 - \Delta)}. \tag{5.3.36}
$$

Note that equations (5.2.18), (5.3.16) and (5.3.23) could be solved easily using the results in (5.3.36).

5.3.3 Pricing formulae

In this subsection, we derive the pricing formulae of the variance swaps and volatility swaps by making use of the characteristic function we have obtained.

For a variance swap, denote the fair strike prices based on the realized variance formulae (5.2.32) and (5.2.33) by $Kvar^{act}$ and $Kvar^{log}$ respectively, then we can obtain the pricing formulae as follows

$$\begin{aligned}
Kvar^{act} &= \mathbb{E}_0^{\mathbb{Q}^T}[\sigma_{var}^2] \\
&= \mathbb{E}_0^{\mathbb{Q}^T}\Big[\frac{AF}{N}\sum_{k=1}^{N}\Big(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\Big)^2 \times 100^2\Big] \\
&= \mathbb{E}^{\mathbb{Q}^T}\Big[\frac{AF}{N}\sum_{k=1}^{N}\Big(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\Big)^2 \times 100^2|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \\
&= \frac{AF}{N}\sum_{k=1}^{N}\mathbb{E}^{\mathbb{Q}^T}\Big[\Big(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\Big)^2|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \times 100^2.
\end{aligned} \tag{5.3.37}$$

The expectation in (5.3.37) can be easily obtained by using the characteristic function

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^T}\Big[\Big(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\Big)^2|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \\
&= \mathbb{E}^{\mathbb{Q}^T}\Big[(e^{\alpha(t_{k-1})}-1)^2|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \\
&= \mathbb{E}^{\mathbb{Q}^T}\Big[e^{2\alpha(t_{k-1})}-2e^{\alpha(t_{k-1})}+1|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \\
&= f(2;0,t_{k-1},\Delta t,y(0),r(0)) - 2f(1;0,t_{k-1},\Delta t,y(0),r(0)) + 1,
\end{aligned} \tag{5.3.38}$$

where $\alpha(t_{k-1}) = \ln S_{t_k} - \ln S_{t_{k-1}}$, $\Delta t = t_k - t_{k-1}$ and $f(\phi;t,T-\Delta t,\Delta t,y(t),r(t))$ is given in Proposition 5.2.

Therefore we have

$$\begin{aligned}
Kvar^{act} &= \frac{AF}{N}\sum_{k=1}^{N}\mathbb{E}^{\mathbb{Q}^T}\Big[\Big(\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}\Big)^2|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \times 100^2 \\
&= \frac{AF}{N}\sum_{k=1}^{N}\Big[f(2;0,t_{k-1},\Delta t,y(0),r(0)) - 2f(1;0,t_{k-1},\Delta t,y(0),r(0)) + 1\Big] \\
&\quad \times 100^2.
\end{aligned} \tag{5.3.39}$$

As for $Kvar^{log}$, we have

$$\begin{aligned} Kvar^{log} &= \mathbb{E}_0^{\mathbb{Q}^T}[\sigma_{var}^2] \\ &= \mathbb{E}_0^{\mathbb{Q}^T}\Big[\frac{AF}{N}\sum_{k=1}^{N} ln^2\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right)\times 100^2\Big] \\ &= \mathbb{E}^{\mathbb{Q}^T}\Big[\frac{AF}{N}\sum_{k=1}^{N} ln^2\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right)\times 100^2|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \\ &= \frac{AF}{N}\sum_{k=1}^{N}\mathbb{E}^{\mathbb{Q}^T}\Big[ln^2\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right)|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big]\times 100^2. \end{aligned} \tag{5.3.40}$$

Similarly, we calculate the expectation as follows,

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}^T}\Big[ln^2\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right)|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \\ &= \mathbb{E}^{\mathbb{Q}^T}\Big[(\alpha(t_{k-1}))^2|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big] \\ &= f^{(2)}(0;0,t_{k-1},\Delta t,y(0),r(0)), \end{aligned} \tag{5.3.41}$$

where $f^{(2)}(0;0,t_{k-1},\Delta t,y(0),r(0)) = \partial^2 f(\phi;0,t_{k-1},\Delta t,y(0),r(0))/\partial\phi^2|_{\phi=0}$ and this is the second-order derivative of the characteristic function given in Proposition 5.2 with respect to $\phi=0$.

Therefore we obtain the fair strike price as,

$$\begin{aligned} Kvar^{log} &= \frac{AF}{N}\sum_{k=1}^{N}\mathbb{E}^{\mathbb{Q}^T}\Big[ln^2\left(\frac{S_{t_k}}{S_{t_{k-1}}}\right)|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big]\times 100^2 \\ &= \frac{AF}{N}\sum_{k=1}^{N}\Big[f^{(2)}(0;0,t_{k-1},\Delta t,y(0),r(0))\Big]\times 100^2. \end{aligned} \tag{5.3.42}$$

Next we discuss our pricing formulae for a volatility swap based on (5.2.30) and (5.2.31). Similarly, we denote the fair strike prices as $Kvol^{act}$ and $Kvol^{log}$ respectively. Since $Kvol^{act}$ can be easily obtained from (5.3.39), we only focus on the derivation of $Kvol^{log}$.

Based on (5.2.31), $Kvol^{log}$ can be calculated as follows,

$$
\begin{aligned}
Kvol^{log} &= \mathbb{E}_0^{\mathbb{Q}^T}[\sigma_{vol}] \\
&= \mathbb{E}_0^{\mathbb{Q}^T}\Big[\sqrt{\frac{\pi}{2NT_l}}\sum_{k=1}^{N}|\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}|\times 100\Big] \\
&= \sqrt{\frac{\pi}{2NT_l}}\sum_{k=1}^{N}\mathbb{E}^{\mathbb{Q}^T}\Big[|\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}|\ |\ \mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big]\times 100.
\end{aligned}
\tag{5.3.43}
$$

Again, we only need to solve for the expectation in (5.3.43).

Before we proceed to calculate the expectation, we define the density function of $\alpha(t_{k-1})$ as $p[\alpha(t_{k-1})]$. Then the cumulative distribution function (CDF) related to $p[\alpha(t_{k-1})]$ is given by

$$
F^p(0)=\int_{-\infty}^{0}p[\alpha(t_{k-1})]d\alpha(t_{k-1})=\frac{1}{2}-\frac{1}{\pi}\int_0^{\infty}\mathcal{R}\Big[\frac{f(i\phi;0,t_{k-1},\Delta t,y(0),r(0))}{i\phi}\Big]d\phi,
\tag{5.3.44}
$$

where $\mathcal{R}$ denotes the real part of the complex number.

Thus we have

$$
\int_0^{\infty}p[\alpha(t_{k-1})]d\alpha(t_{k-1})=1-F^p(0)=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathcal{R}\Big[\frac{f(i\phi;0,t_{k-1},\Delta t,y(0),r(0))}{i\phi}\Big]d\phi.
\tag{5.3.45}
$$

In addition, we define a new function $q[\alpha(t_{k-1})]$ as

$$
q[\alpha(t_{k-1})]=\frac{e^{\alpha(t_{k-1})}p[\alpha(t_{k-1}]}{f(1;0,t_{k-1},\Delta t,y(0),r(0))}.
\tag{5.3.46}
$$

It can be easily verified that $q[\alpha(t_{k-1})]$ is also a density function satisfying the two basic properties: a) $q[\alpha(t_{k-1})]>0$ and b) $\int_{-\infty}^{+\infty}q[\alpha(t_{k-1})]d\alpha(t_{k-1})=1$. The corresponding characteristic function, denoted by $f^q(\phi;0,t_{k-1},\Delta t,y(0),r(0))$, can be obtained by applying the Fourier transform with a sign reversal as follows (see [114, 122]),

$$
\begin{aligned}
f^q(\phi; 0, t_{k-1}, \Delta t, y(0), r(0)) &= \mathcal{F}\Big[\frac{e^{\alpha(t_{k-1})} p[\alpha(t_{k-1})]}{f(1; 0, t_{k-1}, \Delta t, y(0), r(0))}\Big] \\
&= \frac{1}{f(1; 0, t_{k-1}, \Delta t, y(0), r(0))} \\
&\times \int_{-\infty}^{\infty} e^{i\phi\alpha(t_{k-1})} e^{\alpha(t_{k-1})} p(\alpha(t_{k-1})) d\alpha(t_{k-1}) \\
&= \frac{f(i\phi + 1; 0, t_{k-1}, \Delta t, y(0), r(0))}{f(1; 0, t_{k-1}, \Delta t, y(0), r(0))}.
\end{aligned} \tag{5.3.47}
$$

Using the characteristic function $f^q(\cdot)$, we can obtain the following CDF corresponding to $q[\alpha(t_{k-1})]$,

$$
F^q(0) = \int_{-\infty}^{0} q[\alpha(t_{k-1})] d\alpha(t_{k-1}) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} \mathcal{R}\Big[\frac{f(i\phi + 1; 0, t_{k-1}, \Delta t, y(0), r(0))}{i\phi f(1; 0, t_{k-1}, \Delta t, y(0), r(0))}\Big] d\phi. \tag{5.3.48}
$$

Similarly, we have

$$
\int_0^{\infty} q[\alpha(t_{k-1})] d\alpha(t_{k-1}) = 1 - F^q(0) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \mathcal{R}\Big[\frac{f(i\phi + 1; 0, t_{k-1}, \Delta t, y(0), r(0))}{i\phi f(1; 0, t_{k-1}, \Delta t, y(0), r(0))}\Big] d\phi. \tag{5.3.49}
$$

Now we can calculate the expectation in (5.3.43) using the results in (5.3.45) and (5.3.49) as follows,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^T}\left[|\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}|\middle|\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\right]\\
&=\mathbb{E}^{\mathbb{Q}^T}\Big[\,|e^{\alpha(t_{k-1})}-1|\Big|\,\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big]\\
&=\int_{-\infty}^{\infty}|e^{\alpha(t_{k-1})}-1|p[\alpha(t_{k-1})]d\alpha(t_{k-1})\\
&=\int_{0}^{\infty}(e^{\alpha(t_{k-1})}-1)p[\alpha(t_{k-1})]d\alpha(t_{k-1})+\int_{-\infty}^{0}(1-e^{\alpha(t_{k-1})})p[\alpha(t_{k-1})]d\alpha(t_{k-1})\\
&=\int_{0}^{\infty}e^{\alpha(t_{k-1})}p[\alpha(t_{k-1})]d\alpha(t_{k-1})-\int_{0}^{\infty}p[\alpha(t_{k-1})]d\alpha(t_{k-1})\\
&+\int_{-\infty}^{0}p[\alpha(t_{k-1})]d\alpha(t_{k-1})-\int_{-\infty}^{0}e^{\alpha(t_{k-1})}p[\alpha(t_{k-1})]d\alpha(t_{k-1})\\
&=1-2\int_{0}^{\infty}p[\alpha(t_{k-1})]d\alpha(t_{k-1})+\int_{0}^{\infty}e^{\alpha(t_{k-1})}p[\alpha(t_{k-1})]d\alpha(t_{k-1})\\
&-\int_{-\infty}^{0}e^{\alpha(t_{k-1})}p[\alpha(t_{k-1})]d\alpha(t_{k-1})\\
&=1-2\int_{0}^{\infty}p[\alpha(t_{k-1})]d\alpha(t_{k-1})+f(1;0,t_{k-1},\Delta t,y(0),r(0))\Big[\int_{0}^{\infty}q[\alpha(t_{k-1})]d\alpha(t_{k-1})\\
&-\int_{-\infty}^{0}q[\alpha(t_{k-1})]d\alpha(t_{k-1})\Big]\\
&=1-2\int_{0}^{\infty}p[\alpha(t_{k-1})]d\alpha(t_{k-1})+f(1;0,t_{k-1},\Delta t,y(0),r(0))\Big[2\int_{0}^{\infty}q[\alpha(t_{k-1})]-1\Big]\\
&=\frac{2}{\pi}\int_{0}^{\infty}\mathcal{R}\Big[\frac{f(i\phi+1;0,t_{k-1},\Delta t,y(0),r(0))-f(i\phi;0,t_{k-1},\Delta t,y(0),r(0))}{i\phi}\Big]d\phi.
\end{aligned}\tag{5.3.50}$$

Finally we can obtain the fair delivery price $Kvol^{log}$ as follows,

$$\begin{aligned}
Kvol^{log}&=\sqrt{\frac{\pi}{2NT_l}}\sum_{k=1}^{N}\mathbb{E}^{\mathbb{Q}^T}\Big[|\frac{S_{t_k}-S_{t_{k-1}}}{S_{t_{k-1}}}|\mid\mathcal{F}_s(0)\vee\mathcal{F}_y(0)\vee\mathcal{F}_r(0)\vee\mathcal{F}_X(0)\Big]\times 100\\
&=\sqrt{\frac{\pi}{2NT_l}}\sum_{k=1}^{N}\frac{2}{\pi}\int_{0}^{\infty}\mathcal{R}\Big[\frac{f(i\phi+1)-f(i\phi)}{i\phi}\Big]d\phi\times 100\\
&=\sqrt{\frac{2}{\pi NT_l}}\sum_{k=1}^{N}\int_{0}^{\infty}\mathcal{R}\Big[\frac{f(i\phi+1)-f(i\phi)}{i\phi}\Big]d\phi\times 100,
\end{aligned}\tag{5.3.51}$$

where

$$\begin{aligned} f(i\phi+1) &= f(i\phi+1;0,t_{k-1},\Delta t,y(0),r(0)), \\ f(i\phi) &= f(i\phi;0,t_{k-1},\Delta t,y(0),r(0)). \end{aligned} \tag{5.3.52}$$

Thus we have obtained the closed-form pricing formulae for a variance swap and a volatility swap with the variance or volatility defined as (5.2.30) to (5.2.33) using the characteristic function method.

5.4 Numerical and Sensitivity Analysis

In this section, we conduct numerical analysis to verify the accuracy and efficiency of our explicit solution and examine the effect of incorporating different factors in our hybrid model based on the pricing formula (5.3.39) for a discretely-sampled variance swap. We assume a typical two-regime market that switches between the so-called "bullish" (good economy) and "bearish" (bad economy) regime, denoted by regime 1 and regime 2 respectively. Thus we have the state space as $S = \{e_1, e_2\}$. The transition matrix Q is given as

$$Q = \begin{bmatrix} -0.1 & 0.1 \\ 0.4 & -0.4 \end{bmatrix}. \tag{5.4.1}$$

Other model parameters for the two regimes are given in Table 5.1. Moreover, the contract expiry time is given as $T_l = 1$. Note that we consider both Merton-type jump and Kou-type jump for the stock price process $S(t)$ and the corresponding density functions and characteristic functions are given in Table 5.2. The parameters we use are adapted from [53] and [114] for our RSJD model. The values of the regime-dependent variables in the bullish state are considered higher than those in the bearish state, resulting in a higher strike price in the bullish market, which is economically reasonable. Based on the set of parameters, we will present a sensitivity analysis investigating the effect of a 1% change of each model parameter on the fair delivery price $Kvar$.

5.4.1 Semi-Monte-Carlo simulation

First we examine the accuracy of our analytical solution by comparing different fair delivery prices calculated from our semi-closed-form pricing formula (5.3.39) with the results obtained from the semi-Monte-Carlo simulation. This could be meaningful for market practitioners who may prefer numerical results

Table 5.1: Model Parameters

Notations	*Parameters*	*Regime I*	*Regime II*
a^*	Long-term interest rate mean	0.08	0.06
b^*	Interest rate mean reversion speed	1.2	1.2
k^*	Volatility mean reversion speed	2	2
θ^*	Long-term volatility mean	0.075	0.04
v	Volatility of volatility	0.1	0.1
η	Volatility of interest rate	0.01	0.01
ρ	Correlation Coefficient	-0.4	-0.4
λ_s	Jump intensity of stock price $S(t)$	0.3	0.2
λ_y	Jump intensity of volatility $y(t)$	0.5	0.4
	Merton Jump		
$\tilde{\mu}$	Mean of jump size	0.03	0.025
σ	Jump size volatility	0.086	0.078
	Kou Jump		
η_1	Inverse mean one	25	20
η_2	Inverse mean two	50	45
p	Exponential occurrences	0.2	0.15

Table 5.2: Two typical jump models

Jump model	*Density function*	*Characteristic function*
Merton	$\frac{e^{-(z-\tilde{\mu})^2}}{\sqrt{2\pi}\sigma}$	$e^{\phi\tilde{\mu}+\frac{\phi^2}{2}\sigma^2}$
Kou	$p\eta_1 e^{-\eta_1 z}\mathcal{I}_{z>0} + (1-p)\eta_2 e^{\eta_2 z}\mathcal{I}_{z<0}$	$\frac{p\eta_1}{\eta_1-\phi} + \frac{(1-p)\eta_2}{\eta_2+\phi}$

over analytical solutions.

The semi-Monte-Carlo simulation scheme has been used as an improvement for the traditional Monte-Carlo simulation for the regime-switching models in terms of efficiency. The basic idea is to simulate a large number of sample paths for the Markov chain, calculate the fair strike price $Kvar$ given each of the sample path and obtain the final $Kvar$ as the mean of the different prices. For more details, the readers are referred to [114, 118, 123]. Specifically, we implement our simulation procedure as follows,

(1) Simulate 20000 sample paths for $X(t)$ following the method of [118].

(2) For the i-th sample path, $i = 1, \dots, 20000$, obtain the associated conditional forward characteristic function $f_i(\cdot|\mathcal{F}_X(T))$ according to (5.3.3).

(3) Using the obtained characteristic function $f_i(\cdot|\mathcal{F}_X(T))$, calculate the fair

strike price $Kvar_i$ associated with the i-th sample path following similar steps in section 5.3.3.

(4) Calculate the final fair strike price $Kvar$ as the mean of all the obtained prices $Kvar_i$, for $i = 1, \ldots, 20000$.

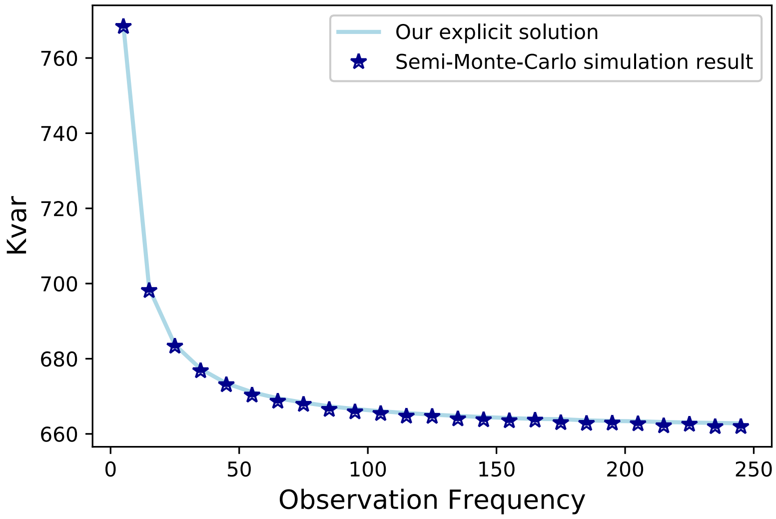
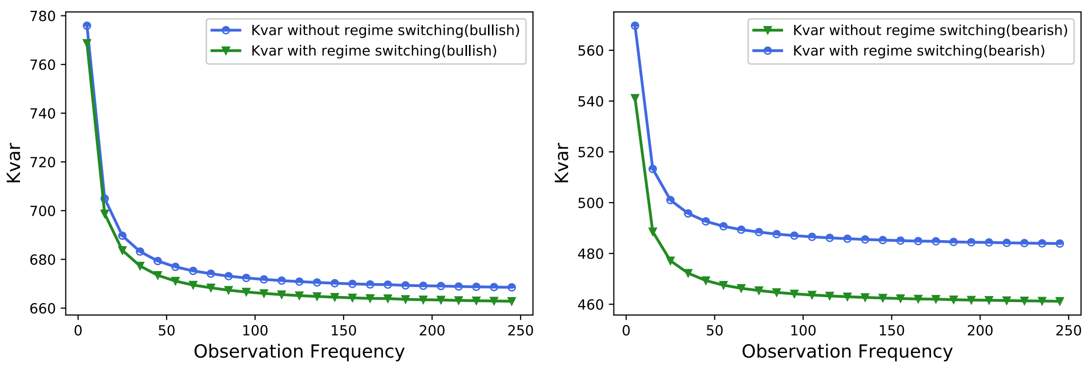


Figure 5.1: Comparison of $Kvar$ obtained from semi-Monte-Carlo simulation and our explicit solution

Figure 5.1 compares the prices obtained from our solution and semi-Monte-Carlo simulation under different observations frequencies ranging from 5 to 255. As the figure depicts, our solution matches the simulation result very well, which provides verification of the accuracy of our pricing formula. Moreover, as an analytical solution, our pricing formula provides higher computation efficiency over the semi-Monte-Carlo simulation method.

5.4.2 Regime switching effect

We then examine the regime switching effect on the fair delivery price under two initial market regimes. As a comparison with our pricing formula, the $Kvar$ without regime switching effect is derived by equating the two sets of parameters. For example, for the bullish market that we are entering, we equates the parameters for regime 2 to those for regime 1. Thus the two regimes are reduced to one. Note that we do not eliminate the effect of stochastic interest rate or jump diffusions. Moreover, we only consider Merton-type jump in this section. The results under two cases are shown in Figure 5.2 where different fair delivery prices against observation frequency are displayed.



(a) Regime 1 (b) Regime 2

Figure 5.2: Comparison of $Kvar$ with and without regime switching

As we can see from Figure 5.2a, the possibility of switching to the bearish market drags down the prices for an initial bullish market under a range of observation frequencies. Contrarily, for a bearish market at initial time, the prices are pulled up due to regime switching as shown in Figure 5.2b. This is economically reasonable since the prices without regime switching in a bullish market is higher than those in a bearish market. As the model parameters vary with the changing regimes, the prices moves upwards or downwards. Table 5.3 keeps track of the two prices sampled from quarterly to continuously.

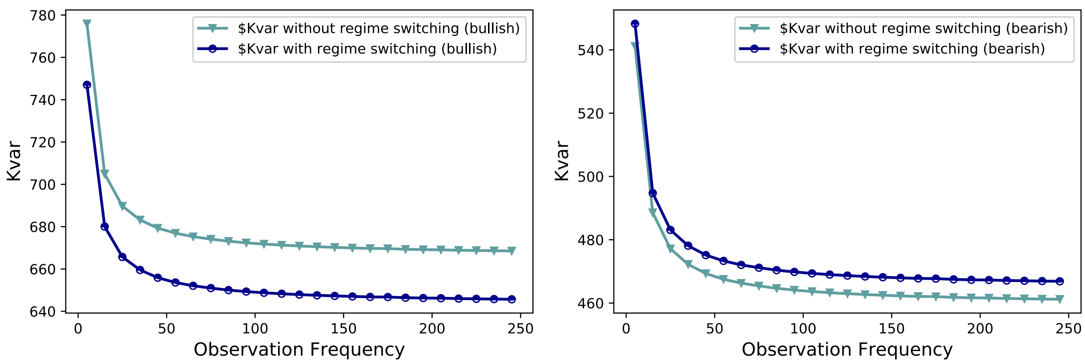
Additionally, we can find that the regime switching effect is more prominent for the bearish market than that for the bullish market. This is due to the higher transition rate we assume from regime 2 to regime 1. If we reverse the value of q_{12} and q_{21} and define a new generator matrix as

$$\tilde{Q} = \begin{bmatrix} -0.4 & 0.4 \\ 0.1 & -0.1 \end{bmatrix}, \tag{5.4.2}$$

Table 5.3: $Kvar$ with regime switching and without regime switching (bullish)

Observation Frequency	N	$Kvar$ *(regime-switching)*	$Kvar$ *(Non-regime-switching)*
Quarterly	4	793.20	800.72
Monthly	12	707.91	714.27
Fortnightly	26	682.81	688.80
Weekly	52	671.71	677.55
Daily	252	662.75	668.46
Continuously	∞	662.39	668.10

we will have an opposite result as displayed in Figure 5.3.



(a) Regime 1 (b) Regime 2

Figure 5.3: Comparison of $Kvar$ with and without regime switching with generator $\tilde{Q}$

5.4.3 Jump diffusion effect

Next we investigate the effect of the Markov-modulated jump diffusion by observing the strike price with various jump intensities under a range of observation frequencies while keeping other parameters fixed. Moreover, we consider Merton-type jump for the volatility process where the jump size follows a normal distribution. As for the stock price process, we additionally consider a Kou-type jump where the jump size follows a double exponential distribution. Also, we focus on the bullish regime in this case since regime switching is not our major concern now.

Figure 5.4 displays $Kvar$ under different observation frequencies when the jump intensity λ_s varies from 0 to 0.4. When $\lambda_s = 0$, the jump effect is eliminated from the stock price process. As the figure shows, the incorporation of jump in $S(t)$ leads to higher prices under each observation frequency and the price further goes up as the jump intensity λ_s increases. An explanation for this is that the jump diffusion contributes to the variation of the underlying asset's price, resulting in a higher realized variance and a higher fair delivery price.

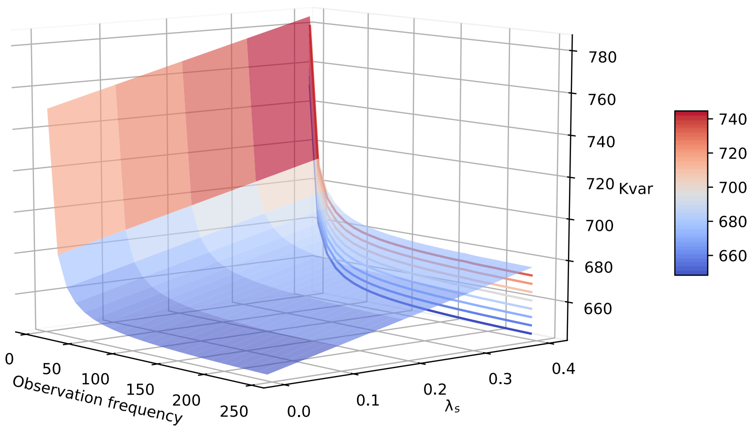


Figure 5.4: Comparison of $Kvar$ with various jump intensities λ_s (Merton-type)

Similarly, we can see from Figure 5.5 that the increasing jump intensity λ_y for the volatility process also increases $Kvar$. However, the effect is much smaller than λ_s. This may be resulting from the different forms we assumed for $J^s(Z^s) = e^{Z^s} - 1$ and $J^y(Z^y) = Z^y$. Figure 5.6 depicts the case with a Kou-type jump where a positive effect of the jump diffusion can also be concluded. The results are consistent with those in [108] where the jump effect on the variance swap prices are also investigated in detail.

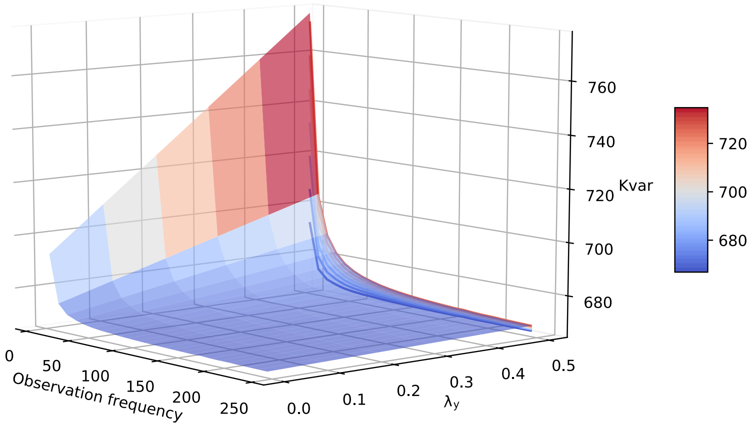


Figure 5.5: Comparison of $Kvar$ with various jump intensities λ_y

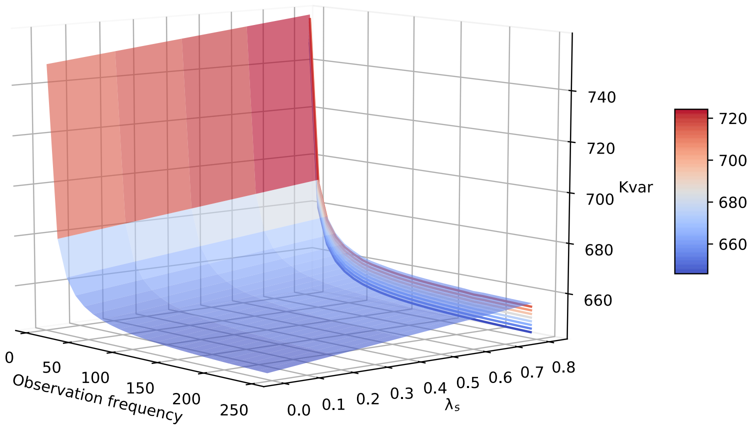


Figure 5.6: Comparison of K_{var} with various jump intensities λ_s (Kou-type)

5.4.4 Stochastic interest rate effect

Finally we study the effect of stochastic interest rate. We focus on the movements of $Kvar$ in the bullish market caused by the change of long-term mean a^* within the range $[0, 0.10]$. As Figure 5.7 shows, the prices can vary positively a lot along with a^* when the observation frequency is low. While as the sampling period narrows, the difference of $Kvar$ with various a^* almost disappears. This leads to the conclusion that the effect of stochastic interest rate is insignificant for a short contract lifetime. Since the need for the incorporation of stochastic interest rate comes from a longer contract period, we document different values of $Kvar$ for a range of a^* when the expiry time T_l extends to 10 instead of 1 in Table 5.4. Note that we here consider a daily-sampled variance swap. In this case, the percentage change of $Kvar$ is much larger than the counterpart for $T_l = 1$. For example, when the long-term interest rate mean changes from 0.06 to 0.08, the change of $Kvar$ for $T_l = 10$ is 0.27%, compared to the 0.012% for $T_l = 1$. Therefore for a long-term contract lifetime, the positive effect of stochastic interest rate cannot be neglected.

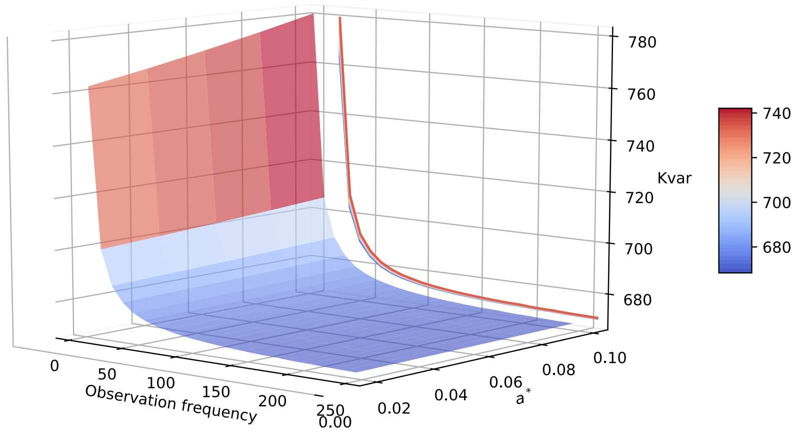


Figure 5.7: Comparison of K_{var} with different a^*

Table 5.4: Effect of stochastic interest rate for different contract lifetimes

a^*	$Kvar(T_l = 10)$	$Kvar(T_l = 1)$
0.00	762.28	666.29
0.02	763.60	666.35
0.04	765.19	666.43
0.06	767.05	666.51
0.08	769.20	666.59
0.10	771.62	666.69

To conclude this section and summarize the effect of the major model parameters on our pricing formula, we conduct a sensitivity analysis based on our parameter set in the bullish regime. Specifically, we calculate the percentage change in $Kvar$ caused by 1% change of each parameter from its base value we assume and record the results in Table 5.5. It is obvious that the price is most sensible to the three parameters related to the stochastic volatility, i.e., θ^*, k^* and $y(0)$, which is economically reasonable and in line with the previous literatures. The drastic positive effect of θ^* is depicted in Figure 5.8.

Table 5.5: Sensitivity analysis

Parameters	*Value*	*Percentage change in Kvar*
a^*	0.08	0.00053713%
b^*	1.2	0.00014265%
$r(0)$	0.05	0.00043367%
k^*	2	0.10863865%
θ^*	0.075	0.63736395%
$y(0)$	0.05	0.32356315%
λ_s	0.3	0.03618091%
λ_y	0.5	0.00279182%
$q12$	0.1	−0.00842272%
$q21$	0.4	0.00076640%

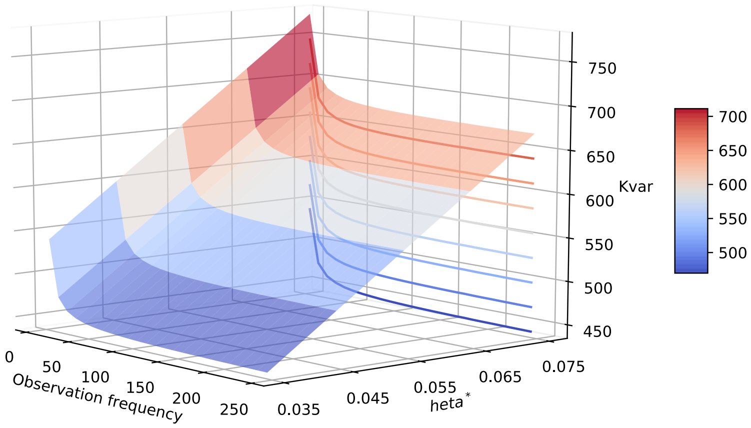


Figure 5.8: Comparison of K_{var} with different θ^*

The transition rate q_{12} affects the bullish $Kvar$ negatively. While in the bearish market, the result can be opposite. $Kvar$ seems little sensible to the interest rate parameters, but things could be different as the contract lifetime gets longer. Therefore all the factors in our Heston-CIR model with regime-switching jump-diffusion have significant effect on the variance swap prices and should be taken into account in the valuation process.

5.5 Concluding Remarks

This chapter investigates the fair strike prices for both variance swaps and volatility swaps under the Heston-CIR model with Markov-modulated jump-diffusion. Under the risk-neutral pricing framework, the problem is reduced to calculating a series of conditional expectations of the realized variance or realized volatility under a risk-neutral T-forward probability measure. The calculation of realized variance or volatility is pre-specified in the contract, and we consider four typical formulae in this chapter. Based on the characteristic function that we derive for a random variable $\alpha(\cdot)$ defined as the log-return of the underlying stock during a sampling period $[T-\Delta t, T]$, we obtain the corresponding fair strike price for a variance swap and a volatility swap for each calculating formula. Then we conduct numerical analysis under a typical two-regime market for a range of observation frequencies where the effect of the realistic factors considered in our model are examined, including regime-switching, jump-diffusion, stochastic interest rate and volatility. The switching possibility to another regime either drags down the price for the good economy or pulls up that for the bad economy. The intensity depends on the specific transition rate. The jump diffusion contributes to the variation of the stock price, resulting in higher delivery price. The stochastic interest rate seems not significant when the contract lifetime is short, but the influence can increase significantly and cannot be neglected when the expiry time gets longer. Additionally, we carry out a semi-Monte-Carlo simulation whose results perfectly match our analytical solution, which validates the accuracy and efficiency of our pricing formula. Finally, we conclude our results by a sensitivity analysis where the percentage change in the fair strike price caused by 1% change of each parameter is recorded, according to which the price is most sensible to the stochastic volatility.

CHAPTER 6

Summary and Future Research

6.1 Summary

In this thesis, we establish and apply various regime-switching jump-diffusion (RSJD) models to study two important financial problems, the mean-variance asset-liability management (MVALM) and the pricing of variance (volatility) swaps. To our knowledge, little work has been done to investigate the applications of RSJD models to these two financial problems. By utilizing the stochastic dynamic programming techniques and the risk-neutral pricing methods, we obtain closed-form solutions to each problem and investigate the validity and efficiency of our solutions with numerical examples. The major findings and results are summarized as follows.

(i) A basic RSJD model is established to investigate the MVALM problem under a game theoretic framework. By applying the stochastic programming techniques, we obtain the Nash equilibrium strategy along with the equilibrium value function in terms of five systems of ordinary differential equations (ODEs) arising from the extended Hamilton-Jacobi-Bellman (HJB) equations and the verification theorem. Compared to the general pre-committed strategy, the equilibrium strategy puts more weight on the risky assets as time goes since it considers the future investment while making the current decision. The equilibrium value functions under different regimes converge to the current wealth surplus value as the time expires. Numerical and sensitivity analysis on the effect of regime switching and jump diffusion is also presented. The equilibrium value function increases as the transition rate goes up. The effect of jump intensities on the equilibrium strategy is not sure due to the sign of the partial derivatives. And the effect of the jumps in the stock is larger than that in the liability process. Moreover, the risk aversion coefficient has a negative effect on the investment strategy, which is economically reasonable and verified by the corresponding partial derivative.

(ii) A Heston's stochastic volatility model with regime-switching jump diffusion is established for the pricing of discretely-sampled variance swaps. Due to the forward contract nature of a variance swap and the risk-neutral pricing, the pricing problem is reduced to solving a series of conditional expectations. The fair strike price is obtained in semi-closed-form by solving the partial differential equation satisfied by the value function via the two-stage approach and the generalized Fourier transform method. The accuracy and efficiency is validated by comparison of our solution with a semi-Monte-Carlo simulation result. A continuous counterpart is also derived and compared with our solution, based on which we prove that the continuous approximation can lead to large errors when the observation frequency is low, even though the discretely-sampled variance swap price converges to the continuous counterpart as the observation frequency approaches infinity. Numerical analysis is conducted under a two-regime market to examine the effect of regime switching and jump diffusion by changing the values of corresponding model parameters. The possibility of regime switching affects the fair delivery prices by pulling up the price in an initial bearish market and dragging down that in an initial bullish market. The jump processes contribute to the variation in the underlying asset's return, resulting in a positive effect on the fair strike prices.

(iii) A hybrid model combining Heston's stochastic volatility and CIR stochastic interest rate with regime-switching jump diffusion is established to price both variance swaps and volatility swaps with discrete sampling times. A change of *numéraire* is presented to determine the dynamics under the T-forward risk-neutral probability measure. Under the risk-neutral pricing framework, the semi-closed-form fair delivery prices based on different realized variance (volatility) calculating formulae are obtained in terms of the characteristic functions of a newly defined random variable corresponding to the log-return of the underlying stock during a sampling period. The process of deriving the associated characteristic functions starts with assuming a given fixed path of the Markov chain and ends with allowing various paths to determine the conditional expectation. Similarly, the solution is validated with a semi-Monte-Carlo simulation. Numerical examples are presented under a typical two-regime market to examine the effect of each factor considered in our hybrid model. The incorporation of regime switching may cause positive or negative effect on the strike price depending on whether the alternative economy is better or worse than the current state. Both jumps in the stock price and volatility process lead to increase of the realized variance, resulting in a higher delivery price. Compared to other factors, the stochastic interest rate exhibits little influence when the contract lifetime is short. However, the effect can increase rapidly and cannot be neglected as the contract extends to a longer period. A sensitivity analysis is conducted regarding the percentage change of the fair strike price due to 1% change of each model parameter, where the long-term mean of the volatility process is shown to have the most significant effect on our pricing formula.

6.2 Future Research Directions

The main objective of this research is to investigate the mean-variance asset-liability management (MVALM) problem and the pricing of variance (volatility) swaps under various regime-switching jump-diffusion (RSJD) models. The RSJD models are more realistic and effective due to the ability to capture the short-term and long-term market movements caused by some single unexpected event or the structural changes of the macroeconomic environment respectively. Though we establish models that incorporate more complex factors such as stochastic interest rate, and apply new techniques such as the game-theoretic time-consistent approach, the following improvements can still be made in the future.

(i) Besides the MVALM and pricing of variance swaps, more applications of the RSJD models to other financial problems such as option pricing, optimal investment and reinsurance strategy, the prediction of Value-at-Risk, and optimal selling rules can be investigated.

(ii) The market regimes in our RSJD models are assumed to be fully observable by the investors and researchers. However, this may not be the case in the real world. Thus a hidden Markov chain can be considered instead of an observable Markov chain in further RSJD models.

(iii) For the MVALM problem under RSJD models, we assume a regime-dependent risk aversion coefficient $\gamma(i)$. To be more realistic, we could consider a risk aversion $\gamma(x, i)$ that also depends on the current wealth state x. Furthermore, we could introduce skewness preference to the objective functional besides the mean and variance.

(iv) Calibration of the model parameters could be considered since the value of the parameter plays an essential role in the model.

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Appendix 1. Statement of Candidate's Contributions to Joint-Authored published

To Whom It May Concern,

I, Yu Yang, made major contributions to the design of the research work, development of theories, analysis of results, and drafting of the paper entitled *'Time-Consistent Mean-Variance Asset-Liability Management in a Regime-Switching Jump-Diffusion Market.'*

<u>Yu Yang</u>

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

<u>Yonghong Wu</u>

<u>Benchawan Wiwatanapataphee</u>

Appendix 2. Statement of Candidate's Contributions to Joint-Authored published

To Whom It May Concern,

I, Yu Yang, made major contributions to the design of the research work, development of theories, analysis of results, and drafting of the paper entitled *'Variance Swap Pricing under a Markov-modulated Jump-Diffusion Model.'*

Yu Yang

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Shican Liu

Yonghong Wu

Benchawan Wiwatanapataphee

Appendix 3. Statement of Candidate's Contributions to Joint-Authored published

To Whom It May Concern,

I, Yu Yang, made major contributions to the design of the research work, development of theories, analysis of results, and drafting of the paper entitled *'Pricing of Volatility Derivatives in a Heston-CIR Model with Markov-modulated Jump Diffusion.'*

Yu Yang

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Shican Liu

Yonghong Wu

Benchawan Wiwatanapataphee

Appendix 4. Statement of Candidate's Contributions to Joint-Authored published

To Whom It May Concern,

I, Yu Yang, made major contributions to the design of the research work, development of theories, analysis of results, and drafting of the paper entitled ' Variance Swap Pricing under Hybrid Jump Model.'

Yu Yang

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Shican Liu

Benchawan Wiwatanapataphee

Yonghong Wu