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Variational Analysis Down Under Open Problem Session

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Abstract We state the problems discussed in the open problem session at Variational Analysis Down Under conference held in honour of Prof. Asen Dontchev on 19–21 February 2018 at Federation University Australia.

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1 Introduction

The conference Variational Analysis Down Under was held on February 19–21 at Federation University in Ballarat, Australia. In this note, we state and summarize the problems which were shared during the open problems session. In particular, in Section 2, we consider a question about the existence of local calm sections; in Section 3 we ask if 6-polytopes are 3-linked; in Section 4 we present a polytope whose decomposability or indecomposability is difficult to determine; in Section 5 we ask whether for pairs of compact sets in a general Hilbert space we may find a point whose projections onto them are singleton faces strongly exposed by the relevant normals; in Section 6 we consider the continuous time version of the Douglas-Rachford method; in Section 7 we ask about representations for a minimal distance problem; in Section 8, we recall the Demyanov-Ryabova conjecture, which has since been answered in the negative, and so we present a new, revised question; finally, we revisit Dürer's conjecture in Section 9. We conclude in Section 10.

2 Existence of Local Calm Selections

This problem was proposed by Asen Dontchev. All background material, including notation, history, etc. can be found in Dontchev's and Rockafellar's book [1]. We are grateful to Asen for providing this description.

Theorem 2.1 *Bartle-Graves (1952). Let X and Y be Banach spaces and let $f : X \to Y$ be a function which is strictly differentiable at $\bar{x}$ and such that the derivative $Df(\bar{x})$ is surjective. Then there exist a neighborhood V of $f(\bar{x})$ and a constant $\gamma > 0$ such that f^{-1} has a continuous selection s on V which is calm with constant γ; that is,*

$$\|s(y) - \bar{x}\| \leq \gamma \|y - f(\bar{x})\| \text{ for every } y \in V.$$

When X and Y are finite dimensional, even Hilbert, the proof is easy. For Banach spaces, the proof is highly nontrivial. A generalization of the Bartle-Graves theorem to set-valued mappings was obtained in [2].

The open problem is as follows.

Conjecture 2.1 Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is Lipschitz continuous around $\bar{x}$ and suppose that all matrices A in Clarke's generalized Jacobian of f at $\bar{x}$ are surjective. Then f^{-1} has a continuous local selection around $\bar{y}$ for $\bar{x}$ which is calm at $\bar{y} = f(\bar{x})$.

If $n = m$ the conjecture reduces to Clarke's inverse function theorem. For $m \leq n$, according to a theorem by Pourciau [3], under the same condition the function f is metrically regular. This last result was generalized recently to Banach spaces in [4].

3 Are 6-Polytopes 3-Linked?

This problem was presented by Hoa Bui.

A graph G is k-linked iff for any selection of k pairs of all distinct vertices $Y := \{(s_1, t_1), \ldots, (s_k, t_k)\}$, $(k \geq 1)$ there exist k disjoint paths, connecting the k pairs of points in Y. If the graph of a polytope is *k-linked* we say that the polytope is also *k-linked*.

Recall that a d-polytope is a d-dimensional polytope, i.e. the linear span of the polytope is a d-dimensional space. The initial question is whether or not every d-polytope is *$\lfloor d/2 \rfloor$-linked*. And the negative answer was given by Gallivan (see [5]) with a construction of a d-polytope which is not *$\lfloor 2(d+4)/5 \rfloor$-linked*. It had been already proven that 4-polytopes and 5-polytopes are *2-linked* (see [6], [7]), meanwhile not all 8-polytopes are *4-linked*. The remaining question is that if all the 6-polytopes are *3-linked*.

4 Is FFS3 Polytope Decomposable?

This problem was suggested by David Yost, and communicated during the open problem section by Scott Lindstrom and Vera Roshchina.

A polytope is called decomposable iff it can be represented as Minkowski sum of dis-similar convex bodies [8]. Two polytopes are similar if one can be obtained from the other by a dilation and a translation.

David Yost in collaboration with Debra Briggs have classified all but one 3-polytopes with up to 16 edges in terms of decomposability (manuscript in preparation). The only remaining case is the (combinatorial) polytope FFS3 with its graph shown in Fig. 1. It is conjectured that this polytope has no

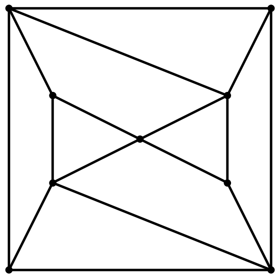

Fig. 1 Graph of the polytope FFS3

decomposable geometric realisation.

All polytopes with up to 15 edges are classified in terms of their decomposability [9], and the resolution of the decomposability question for FFS3 polytope will settle the 16-edge case. Continuing to classify polyhedra with higher numbers of edges as decomposable or indecomposable on a case-by-case basis constitutes a tedious challenge. We suggest the more interesting challenge of developing an algorithm to check decomposability. We note that in an overwhelming number of cases indecomposability can be checked using combinatorial conditions from [8].

5 Projections onto Compact Convex Sets

This problem was proposed by Andrew Eberhard.

Let C_1 and C_2 be compact convex sets in a Hilbert space $\mathcal{H}$. The conjecture states that there always exists a point $x \in \mathcal{H}$ such that for each of its projections p_i onto C_i, $i \in \{1,2\}$ the relevant normals $x - p_1$ and $x - p_2$ define the hyperplanes that strongly expose the faces $\{p_1\}$ and $\{p_2\}$ of C_1 and C_2 respectively.

Recall (see [10, Definition 8.27]) that a point $x \in C$ is strongly exposed by a continuous linear functional f iff $f(x) = \sup_{x' \in C} f(x')$ and $x_k \to x$ for all sequences $\{x_k\} \subset C$ such that $\lim f(x_k) = \sup_{x \in C} f(x)$, and a well-known result (see [11,12]) that in a Banach space every weakly compact convex set is the closed convex hull of its strongly exposed points.

This conjecture indeed arises when solving a special case of Chebyshev's conjecture (see [13]). Given two compact closed convex sets C_1, C_2 in a Hilbert space $\mathcal{H}$, the question is whether there exists a point $x \in \mathcal{H}$ such that $P_{C_1}(x) = P_{C_2}(x)$. The answer to this question can be found in [14, Theorem 15], established for weakly closed sets, borrowing from the proof of Asplund in [15].

6 Convergence of the Continuous Time Douglas-Rachford Algorithm

This problem was proposed by Scott Lindstrom.

For the feasibility problem of finding a point in the nonempty intersection $A \cap B \neq \emptyset$ of proximal sets A and B, the Douglas-Rachford method for a given starting point x_0 generates a sequence

$$x_n \in T x_{n-1} := (\lambda(2P_B - \mathrm{Id})(2P_A - \mathrm{Id}) + (1-\lambda)\mathrm{Id})\, x_{n-1}$$

where P_A, P_B denote the usual projection operators for A, B respectively and $\lambda \in (0, 1]$ is usually taken to be $1/2$. When A, B are also convex, the sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to a fixed point of the method (see [16] and [17]).

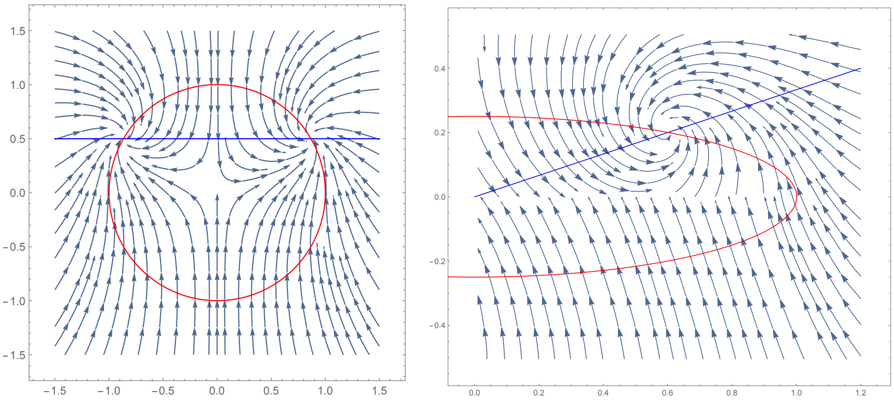

Fig. 2 The flowfield (1) with a circle/line (left) and ellipse/line (right). Images courtesy of Veit Elser.

For the nonconvex case where A is a circle and B a line, Borwein and Sims [18] considered the "continuous time" version of the algorithm. This version corresponds to the solution of the differential equation given by

$$\frac{dx}{dt} = T(x) \quad \text{when } \lambda \to 0^+, \tag{1}$$

for which the corresponding flow field is shown at left in Figure 2. For their purposes, Borwein and Sims considered the continuous time variant as a means to approach the question of convergence in the usual case of $\lambda = 1/2$ in the presence of subtransversality. Benoist [19] has since resolved this question more definitively by constructing a suitable Lyapunov function, a method

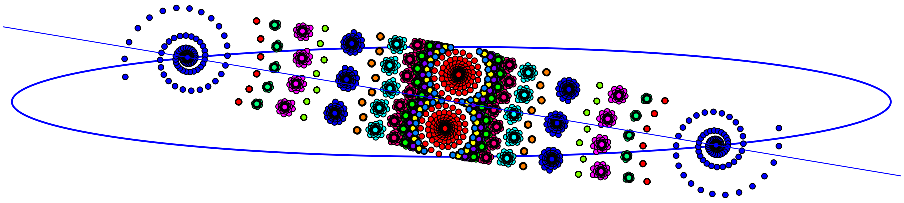

Fig. 3 Behaviour of Douglas-Rachford method with an ellipse and line varies from the case of a circle and line.

since adopted by Minh N. Dao and Matthew K. Tam [20] for showing local convergence with plane curves more generally.

The generalization to a subtransversal ellipse and line and also to a p-sphere and a line was considered by Borwein et al.[21], who showed that local convergence remains while global behaviour becomes far more complicated. See, for example, Figure 3. Veit Elser has suggested analysing the continuous time version of the method in these more general settings and has generously furnished the images in Figure 2.

7 Minimal Distance Problem

This problem was proposed by Alex Kruger.

Given a finite set of points $a_1, \dots, a_m \in X$, where X is an Euclidean space, find the solution to the problem

$$\min_{x \in X} \max_{i \in \{1,\dots,m\}} \|a_i - x\|. \tag{2}$$

The problem has a unique solution for which x is the centre of the minimal Euclidean sphere that contains all the points. However it is unclear whether there exists a neat way to write this explicitly.

This is a particular case of a more general problem. The space X can be an arbitrary normed linear or even a metric space. In the latter case, the norm of the difference in (2) should be replaced by the distance. Instead of the maximum in (2), it could be an arbitrary norm in $\mathbb{R}^m$.

From an operational research point of view, problem (2) corresponds to the problem of identifying the location x of a single facility which minimizes the maximum distance from the facility to any of the m demand points that has been studied in applicative contexts (see for example [22]). Our question is not about numerical methods of solving, but rather when the solution to (2) may be represented more explicitly. For example, in $\mathbb{R}$, the solution is

$$x = \frac{1}{2}\left(\max_{i\in\{1,\ldots,m\}}\{a_i\} + \min_{i\in\{1,\ldots,m\}}\{a_i\}\right).$$

8 Demyanov-Ryabova Conjecture

This problem was communicated by Vera Roshchina.

The problem was originally stated in [23, Conjecture 1]. Recently two different special cases were confirmed in [24,25]. A counterexample was found during the preparation of this file [26].

Given a finite family Ω of convex polytopes in $\mathbb{R}^n$, for each unit vector $g \in S_{n-1}$ we construct a new polytope as the convex hull of all support faces of all polytopes in the family Ω, i.e. we define the function

$$C(g) := \operatorname{conv}\{\operatorname*{Arg\,max}_{x\in P}\langle x, g\rangle,\ P \in \Omega\}.$$

Collecting all such polytopes, we obtain a new finite family of polytopes,

$$F(\Omega) := \{C(g),\ g \in S_{n-1}\}.$$

Now starting from a given finite collection of polytopes Ω_0 we apply this transformation infinitely obtaining a sequence $\Omega_0,\ \Omega_1,\ \Omega_2,\ \ldots$, where $\Omega_i = F(\Omega_{i-1})$, $i \in \mathbb{N}$.

The original Demyanov-Ryabova conjecture claimed that this sequence eventually reaches a two-cycle, i.e. for a sufficiently large N we have $\Omega_{N+2} = \Omega_N$. Thanks to a recent counterexample [26], we now know that the original conjecture is false. However, a new question is to find a characterisation of the collections of polytopes that yield two-cycles, extending and generalising the results of [24, 25].

9 Dürer's Conjecture

This problem was communicated by Vera Roshchina.

Albrecht Dürer dedicated a nontrivial part of his career to laying out the geometric foundations of drawing and perspective. His five centuries old work [27] is available online via Google books. The mathematical statement known as Dürer's conjecture was motivated by this work and proposed in 1975 by Shephard [28]. A net (or unfolding) of a 3-polytope is the result of cutting it along its edges, so that the resulting connected shape can be flattened (developed) into the plane [29]. It is not difficult to find examples of polytopes for which certain cuts result in overlapping unfoldings (or nets), such as the truncated tetrahedron shown in Fig. 4 (see [30]).

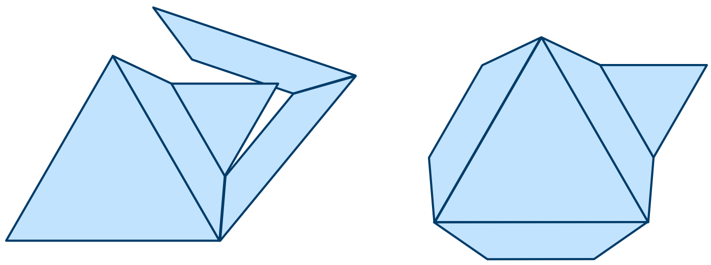

Fig. 4 Two different nets of the same truncated tetrahedron

The Dürer's conjecture is a claim that any polytope has a nonoverlapping net. A significant recent contribution in this direction is the work by Mohammed Ghomi who showed that every polytope is combinatorially equivalent to an unfoldable one [30]. For more details we refer the reader to an overview [29] by the same author.

10 Conclusions

These problems represent a broad sampling of the many and varied avenues of inquiry explored under the broad umbrellas of variational analysis and optimization. We present them here not only to draw them to the attention of other researchers but to provide a compact encapsulation of the diversity of pursuits within the field at the present time.

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