

Characterizations of Nonsmooth Robustly Quasiconvex

Functions

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Abstract Two criteria for the robust quasiconvexity of lower semicontinuous functions are established in terms of Fréchet subdifferentials in Asplund spaces. The first criterion extends to such spaces a result established by Barron, Goebel and Jensen (The quasiconvex envelope through first-order partial differential equations which characterize quasiconvexity of nonsmooth functions. *Discrete Contin. Dyn. Syst. Ser. B* (17): 1693–1706 (2012)). The second criterion is totally new even if it is applied to lower semicontinuous functions on finite-dimensional spaces.

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1 Introduction

The question of characterizing convexity and generalized convexity properties in terms of subdifferentials receives tremendous attention in optimization theory and variational analysis. For decades, there have been received many significant contributions devoted to this question such as [1–6] for convex functions, [7–12] for quasiconvex functions and [10] for robustly quasiconvex functions.

This paper follows this stream of research. Our aim is to establish the first-order characterizations for the robust quasiconvexity of lower semicontinuous functions in Asplund spaces. First, some existing results regarding the properties of subdifferential operators of convex, quasiconvex functions are recalled in Section 2, where the definitions and some basic results are given as well. Besides, necessary and sufficient first-order conditions for a lower semicontinuous function to be quasiconvex are reconsidered. Those characterizations, moreover, could be used to characterize the Asplund property of the given space. Second, two criteria for the robust quasiconvexity of lower semicontinuous functions in Asplund spaces are obtained, by using Fréchet subdifferentials, in Section 3. Each criterion corresponds to each type of analogous conditions for quasiconvexity. The first one is based on the zero and first order condition

for quasiconvexity (see Theorem 2.2(b) in Section 2). It extends [10, Proposition 5.3] from finite dimensional spaces to Asplund spaces. Moreover, its proof also overcomes a glitch in the proof of the sufficient condition of [10, Proposition 5.3]. The second criterion is totally new. It is settled from the equivalence of the quasiconvexity of lower semicontinuous functions and the quasimonotonicity of their subdifferential operators (see Theorem 2.2(c) in Section 2).

2 Preliminaries

Let X be a Banach space and X^* its dual space. X is called an Asplund space, or has the Asplund property, if every separable subspace Y of X has separable topological dual space Y^* . The duality pairing on $X \times X^*$ is denoted by $\langle \cdot, \cdot \rangle$. In what follows, $B_r(x)$ is the open ball of radius $r > 0$ centred at $x \in X$ and $\mathbb{B} \subset X$, $\mathbb{B}^* \subset X^*$ are the closed balls of radius 1 centred at 0_X , 0_{X^*} respectively. The extended real-valued function $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ considered mostly is proper lower semicontinuous (l.s.c), i.e. φ is not identically $+\infty$, and the lower level sets $\varphi_\alpha^\leq := \{x \in X : \varphi(x) \leq \alpha\}$ are closed for all $\alpha \in \mathbb{R}$. As usual, the domain of φ is

$$\text{dom } \varphi := \{x \in X : \varphi(x) < \infty\}.$$

For a set-valued mapping $A : X \rightrightarrows X^*$, the domain of A is

$$\text{dom } A := \{x \in X : A(x) \neq \emptyset\}.$$

The graphs of φ and A are respectively defined as

$$\text{graph } \varphi := \{(x, \alpha) \in X \times \mathbb{R} : \varphi(x) = \alpha\},$$

$$\text{graph } A := \{(x, x^*) \in X \times X^* : x^* \in A(x)\}.$$

A subset U of X is convex, iff it contains all closed segments connecting two points in U . The function φ is said to be convex, iff the domain of φ is convex and for any $\alpha \in [0, 1]$, $x, y \in \text{dom } \varphi$ we always have the inequality

$$\varphi(\alpha x + (1 - \alpha)y) \leq \alpha\varphi(x) + (1 - \alpha)\varphi(y).$$

As usual, the Fréchet subdifferential of a proper lower semicontinuous function φ is the set-valued mapping $\widehat{\partial}\varphi : X \rightrightarrows X^*$ defined by

$$\widehat{\partial}\varphi(x) := \left\{ x^* \in X^* : \liminf_{y \rightarrow x} \frac{\varphi(y) - \varphi(x) - \langle x^*, y - x \rangle}{\|y - x\|} \geq 0 \right\}, \text{ for all } x \in \text{dom } \varphi.$$

When φ is convex, the Fréchet subdifferential reduces to the convex analysis subdifferential

$$\widehat{\partial}\varphi(x) = \partial\varphi(x) := \{x^* \in X^* : \langle x^*, y - x \rangle \leq \varphi(y) - \varphi(x)\}, \text{ for all } x \in \text{dom } \varphi.$$

An operator A is monotone, iff for all $x, y \in \text{dom } A$, one has $\langle x^* - y^*, x - y \rangle \geq 0$ with $x^* \in A(x), y^* \in A(y)$. It is well-known that, when φ is convex, the operator $\widehat{\partial}\varphi$ is monotone [6]. The inverse implication also holds in Asplund space [3, Theorem 2.4]; but it is not true in general Banach spaces. The reader

is referred to the proof of the reverse implication in [13, Theorem 2.4] for a counter-example.

Let us recall some notions of generalized convex functions.

Definition 2.1 A function $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is

1. **quasiconvex** iff

$$\forall x, y \in X, \lambda \in]0, 1[, \quad f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}. \quad (1)$$

2. **α -robustly quasiconvex** with $\alpha > 0$ iff for every $v^* \in \alpha\mathbb{B}^*$, the function

$$\varphi_{v^*} : x \mapsto \varphi(x) + \langle v^*, x \rangle \text{ is quasiconvex.}$$

Clearly, φ is α -robustly quasiconvex if and only if the function φ_{v^*} is quasiconvex for all $v^* \in X^*$ such that $\|v^*\| < \alpha$.

Tracing back to the original definition of robustly quasiconvex functions, they were first defined in [14] under the name “s-quasiconvex” or “stable quasiconvex”, and then renamed “robustly quasiconvex” in [15]. This class of functions holds a notable role, as many important optimization properties of generalized convex functions are stable when disturbed by a linear functional with a sufficiently small norm (for instance, all lower level sets are convex, each minimum is a global minimum, each stationary point is a global minimizer). For interested readers, we refer to [14] again, and further related works [15, 16].

Definition 2.2 An operator $A : X \rightrightarrows X^*$ is quasimonotone iff for all $x, y \in X$ and $x^* \in A(x)$, $y^* \in A(y)$ we have $\min\{\langle x^*, y - x \rangle, \langle y^*, x - y \rangle\} \leq 0$.

Significant contributions concerning dual criteria for quasiconvex functions are in [7,9]. Those characterizations are applicable for a wide range of subdifferentials, for instance, Rockafellar-Clarke subdifferentials in Banach spaces, and Fréchet subdifferentials in reflexive spaces. These results are still unclear for Fréchet subdifferentials in Asplund spaces. Below, we give a short proof to clarify this. Our proof relies on the proof scheme of [7] and the following approximate mean value theorem.

Theorem 2.1 [4, Theorem 3.49] *Let X be an Asplund space and $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function finite at two given points $a \neq b$. Consider any point $c \in [a, b[$ at which the function*

$$\psi(x) := \varphi(x) - \frac{\varphi(b) - \varphi(a)}{\|a - b\|} \|x - a\|$$

attains its minimum on $[a, b]$; and such a point always exists. Then, there are sequences $x_k \xrightarrow{\varphi} c$ and $x_k^ \in \widehat{\partial}\varphi(x_k)$ satisfying*

$$\liminf_{k \rightarrow \infty} \langle x_k^*, b - x_k \rangle \geq \frac{\varphi(b) - \varphi(a)}{\|a - b\|} \|b - c\|, \quad (2)$$

$$\liminf_{k \rightarrow \infty} \langle x_k^*, b - a \rangle \geq \varphi(b) - \varphi(a). \quad (3)$$

Moreover, when $c \neq a$ one has

$$\lim_{k \rightarrow \infty} \langle x_k^*, b - a \rangle = \varphi(b) - \varphi(a). \quad (4)$$

Theorem 2.1 is the Zagrodny mean value theorem [17] for which it is shown in [18] that the proof in [17] is still valid for all subdifferentials. Applying Theorem 2.1, we could deduce the following three-points lemma which is similar to [8, Lemma 3.1].

Lemma 2.1 *Let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous function on an Asplund space X . Let $u, v, w \in X$ such that $v \in [u, w]$, $\varphi(v) > \varphi(u)$ and $\lambda > 0$. Then, there are $\bar{x} \in \text{dom } \varphi$ and $\bar{x}^* \in \widehat{\partial}\varphi(\bar{x})$ such that*

$$\bar{x} \in B_\lambda([u, v]) \quad \text{and} \quad \langle \bar{x}^*, w - \bar{x} \rangle > 0,$$

where

$$B_\lambda([u, v]) := \{x \in X : \exists y \in [u, v] \text{ such that } \|x - y\| < \lambda\}.$$

We are in position to establish characterizations of quasiconvexity in terms of Fréchet subdifferentials in Asplund spaces.

Theorem 2.2 *Let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function on an Asplund space X . The following statements are equivalent*

- (a) φ is quasiconvex;
- (b) If there are $x, y \in X$ such that $\varphi(y) \leq \varphi(x)$, then $\langle x^*, y - x \rangle \leq 0$ for all $x^* \in \widehat{\partial}\varphi(x)$.
- (c) $\widehat{\partial}\varphi$ is quasimonotone.

Proof. (a) \Rightarrow (b) Assume that $x, y \in X$, $\varphi(x) \geq \varphi(y)$, and $x^* \in \widehat{\partial}\varphi(x)$. Consider the lower level set $S_x := \{u \in X : \varphi(u) \leq \varphi(x)\}$. Since φ is quasiconvex,

then S_x is a convex set. Thus, we have the function $f := \delta_{S_x} + \varphi(x)$ is convex, where δ_{S_x} is equal to 0 for $u \in S_x$ and to ∞ otherwise. On the other hand, $f(x) = \varphi(x)$ and $f(u) \geq \varphi(u)$ for all $u \in X$, thus $\widehat{\partial}\varphi(x) \subset \widehat{\partial}f(x)$. By the definition of convex subdifferential, since $x^* \in \widehat{\partial}\varphi(x) \subset \widehat{\partial}f(x)$, we have $\langle x^*, y - x \rangle \leq 0$.

(b) \Rightarrow (c) Assume that there are $x, y \in X$ and $x^* \in \widehat{\partial}\varphi(x)$, $y^* \in \widehat{\partial}\varphi(y)$ such that $\langle x^*, x - y \rangle < 0$ and $\langle y^*, x - y \rangle > 0$. Then, by (b), $\varphi(x) < \varphi(y)$ and $\varphi(y) < \varphi(x)$, which is a contradiction.

(c) \Rightarrow (a) By using Lemma 2.1, the proof of this assertion is similar to one in [7, Theorem 4.1]. \square

Remark 2.1 Observe that the implications (a) \Rightarrow (b) and (b) \Rightarrow (c) hold in Banach spaces while (c) \Rightarrow (a) only holds in Asplund spaces. The proof of [7, Theorem 4.1] for the equivalence (a) \Leftrightarrow (c) is actually also valid for Fréchet subdifferential in Asplund spaces. In fact, the equivalence of these three statements can characterize the Asplund property in the sense that if X is not an Asplund space, then there is a function φ whose Fréchet subdifferential satisfies (b) and (c) but is not quasiconvex. Such a function φ can be found in [13, Theorem 2.4].

3 Characterizations of Robustly Quasiconvex Functions

A zero and first order characterization of robust quasiconvexity was given in [10, Proposition 5.3] for finite dimensional spaces. We remark that there is an

oversight in the proof given there; although the function f is only assumed to be lower semicontinuous, the existence of z in the second paragraph actually requires continuity. Here we show that this conclusion is still correct not only when f is assumed just to be lower semicontinuous, but also when X is only assumed to be an Asplund space. To derive this generalization, we need the following lemmas, revealing that quasiconvex functions have certain nice properties which are similar to one of convex functions.

Lemma 3.1 *If $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a quasiconvex and lower semicontinuous function, and $u, v \in X$ are such that $\varphi(v) \geq \varphi(u)$ then*

$$\lim_{t \downarrow 0} \varphi(v + t(u - v)) = \varphi(v). \quad (5)$$

Proof. Suppose that $u, v \in X$ and that $\varphi(v) \geq \varphi(u)$. Since φ is quasiconvex, for all $t \in]0, 1[$, we have $\varphi(v + t(u - v)) \leq \max\{\varphi(v), \varphi(u)\} = \varphi(v)$. It follows that $\limsup_{t \downarrow 0} \varphi(v + t(u - v)) \leq \varphi(v)$. Combining the latter with the lower semicontinuity of φ we get (5). \square

Lemma 3.2 *Let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a quasiconvex function and let $u, v, w \in X$ be such that $v \in]u, w[$, $\varphi(u) \leq \varphi(w)$. Suppose there exist $v^* \in X^*$ and $z \in]u, v[$ such that $\varphi_{v^*}(z) > \max\{\varphi_{v^*}(u), \varphi_{v^*}(w)\}$. Then,*

$$\varphi(u) < \varphi(z) \leq \varphi(v) \leq \varphi(w). \quad (6)$$

Proof. Since $z \in]u, v[\subset]u, w[$, $\varphi(u) \leq \varphi(w)$ and φ is quasiconvex, then we have

$$\varphi(z) \leq \max\{\varphi(u), \varphi(w)\} = \varphi(w).$$

Hence, the latter and the inequality $\varphi_{v^*}(z) > \varphi_{v^*}(w)$ imply $\langle v^*, z \rangle > \langle v^*, w \rangle$.

Again, $z \in]u, w[$ implies $\langle v^*, z \rangle < \langle v^*, u \rangle$. Thus, the inequality $\varphi_{v^*}(u) < \varphi_{v^*}(z)$ yields $\varphi(u) < \varphi(z)$. Since $z \in]u, v[$ and $v \in]z, w[$, we deduce

$$\varphi(z) \leq \varphi(v) \leq \varphi(w)$$

from the latter inequality and the quasiconvexity of φ . Hence, (6) holds. \square

Lemma 3.3 *Let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a quasiconvex, proper, and lower semicontinuous function, and $v^* \in X^*$. If φ_{v^*} is not quasiconvex then there exist $u, v, w \in X$ such that $v \in]u, w[$ and*

$$\varphi(w) \geq \varphi(v) > \varphi(u), \quad (7)$$

$$\varphi_{v^*}(v) > \max\{\varphi_{v^*}(u), \varphi_{v^*}(w)\}, \quad (8)$$

$$\forall \gamma > 0, \exists v_\gamma \in B_\gamma(v) \cap]v, w[: \varphi_{v^*}(v) > \varphi_{v^*}(v_\gamma). \quad (9)$$

Proof. Since φ_{v^*} is not quasiconvex, there exist $u, w \in X$ such that $u \neq w$, $\varphi(u) \leq \varphi(w)$ and $v_0 \in]u, w[$ such that $\varphi_{v^*}(v_0) > \max\{\varphi_{v^*}(u), \varphi_{v^*}(w)\}$.

Applying Lemma 3.1, we get $\lim_{t \downarrow 0} \varphi(w + t(u - w)) = \varphi(w)$, and so

$$\lim_{t \downarrow 0} \varphi_{v^*}(w + t(u - w)) = \varphi_{v^*}(w).$$

Since $\varphi_{v^*}(w) < \varphi_{v^*}(v_0)$, there exists $t_0 \in]0, 1[$ such that

$$\varphi_{v^*}(w + t(u - w)) < \varphi_{v^*}(v_0), \quad \forall t \in]0, t_0[. \quad (10)$$

Consider the set

$$\mathcal{L} := \{z \in]u, w[: \varphi_{v^*}(z) \geq \varphi_{v^*}(v_0)\}.$$

Clearly, $\mathcal{L} \neq \emptyset$ and for each $z \in \mathcal{L}$ we have $\|z - w\| \geq t_0\|u - w\|$ by (10). It follows that

$$r := \inf\{\|z - w\| : z \in \mathcal{L}\} \in [t_0\|u - w\|, \|u - w\|[\subset]0, \|u - w\|[,$$

$$v := w + r \frac{u - w}{\|u - w\|} \in]u, w[.$$

We will show that $v \in \mathcal{L}$, and thus (8) holds. Suppose that, on the contrary, $v \notin \mathcal{L}$. Then, $v_0 \in]u, v[$ and we get $\varphi(u) < \varphi(v_0) \leq \varphi(v) \leq \varphi(w)$ by Lemma 3.2. Applying Lemma 3.1, we get $\lim_{t \downarrow 0} \varphi(v + t(u - v)) = \varphi(v)$, and so $\lim_{t \downarrow 0} \varphi_{v^*}(v + t(u - v)) = \varphi_{v^*}(v)$. By the definition of r , there exists a sequence

$(z_n) \subset \mathcal{L}$ such that $\|z_n - w\| \rightarrow r$ and $\|z_n - w\| > r$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\varphi_{v^*}(v) &= \lim_{t \downarrow 0} \varphi_{v^*}(v + t(u - v)) \\
&= \lim \varphi_{v^*} \left(v + \frac{\|z_n - w\| - r}{\|u - v\|} (u - v) \right) \\
&= \lim \varphi_{v^*} \left(v - \frac{r}{\|u - v\|} (u - v) + \frac{\|z_n - w\|}{\|u - v\|} (u - v) \right) \\
&= \lim \varphi_{v^*} \left(v - \frac{r}{\|u - w\|} (u - w) + \frac{\|z_n - w\|}{\|u - w\|} (u - w) \right) \\
&= \lim \varphi_{v^*} \left(w + \frac{\|z_n - w\|}{\|u - w\|} (u - w) \right) \\
&= \lim \varphi_{v^*}(z_n) \\
&\geq \varphi_{v^*}(v_0),
\end{aligned}$$

which is a contradiction. Now we show that v satisfies (9). Let γ be any positive real number and

$$v_\gamma := w + \frac{r - r_\gamma}{\|u - w\|} (u - w) \text{ with } r_\gamma := \min\{r/2, \gamma/2\} > 0.$$

Since $0 < r - r_\gamma < r < \|u - w\|$, it implies that $v_\gamma \in]v, w[\setminus \mathcal{L}$. Therefore,

$\varphi_{v^*}(v_\gamma) < \varphi_{v^*}(v_0) \leq \varphi_{v^*}(v)$. Furthermore,

$$\|v_\gamma - v\| = \left\| w + \frac{r - r_\gamma}{\|u - w\|} (u - w) - w - r \frac{u - w}{\|u - w\|} \right\| = r_\gamma < \gamma.$$

Hence, v satisfies (9). \square

Theorem 3.1 *Let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function on a Banach space X , and $\alpha > 0$. Consider the following statements*

(a) φ is α -robustly quasiconvex;

(b) For every $x, y \in X$

$$\varphi(y) \leq \varphi(x) \implies \langle x^*, y - x \rangle \leq -\min \{ \alpha \|y - x\|, \varphi(x) - \varphi(y) \}, \forall x^* \in \widehat{\partial}\varphi(x). \quad (11)$$

Then, (a) \implies (b). Additionally, if X is an Asplund space, then (b) \implies (a).

Proof. Suppose that φ is α -robustly quasiconvex, and $x, y \in X$ satisfy $\varphi(y) \leq \varphi(x)$. Assume that $x^* \in \widehat{\partial}\varphi(x)$. We will prove

$$\langle x^*, y - x \rangle \leq -\min \{ \alpha \|y - x\|, \varphi(x) - \varphi(y) \}.$$

If $x = y$, the above inequality is trivial. Otherwise, we consider two cases:

Case 1. $\alpha \|y - x\| \leq \varphi(x) - \varphi(y)$. We then need to prove that

$$\langle x^*, y - x \rangle \leq -\alpha \|y - x\|. \quad (12)$$

Since any element x of X attains its norm on the unit sphere of the dual space X^* , there exists $v^* \in X^*$, $\|v^*\| = 1$ such that $\langle v^*, y - x \rangle = \|y - x\|$. Consider the function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$f(z) = \varphi(z) + \alpha \langle v^*, z - x \rangle \quad \forall z \in X.$$

Then, $f(x) = \varphi(x)$, and

$$f(y) = \varphi(y) + \alpha \langle v^*, y - x \rangle = \varphi(y) + \alpha \|y - x\| \leq \varphi(x) = f(x),$$

i.e., $\max\{f(x), f(y)\} = f(x)$. Since φ is α -robustly quasiconvex, f is quasiconvex. Therefore for each $t \in [0, 1]$, we always have

$$\begin{aligned}\varphi(x) = f(x) = \max\{f(x), f(y)\} &\geq f(x + t(y - x)) \\ &= \varphi(x + t(y - x)) + t\alpha\langle v^*, y - x \rangle \\ &= \varphi(x + t(y - x)) + t\alpha\|y - x\|,\end{aligned}$$

which implies that

$$\varphi(x) - t\alpha\|y - x\| \geq \varphi(x + t(y - x)). \quad (13)$$

On the other hand, since $x^* \in \widehat{\partial}\varphi(x)$, for any $\gamma > 0$, there exists a number $r > 0$ such that

$$\varphi(z) \geq \varphi(x) + \langle x^*, z - x \rangle - \gamma\|z - x\| \quad \forall z \in B_r(x). \quad (14)$$

Let $t \in]0, 1[$ such that $x + t(y - x) \in B_r(x)$. It follows from (13) and (14) that

$$\varphi(x) - t\alpha\|y - x\| \geq \varphi(x) + t\langle x^*, y - x \rangle - t\gamma\|y - x\|,$$

and so

$$\langle x^*, y - x \rangle \leq -\alpha\|y - x\| + \gamma\|y - x\|. \quad (15)$$

On taking limit on both sides of the above inequality as $\gamma \downarrow 0$, we get (12).

Case 2. $\alpha\|y - x\| > \varphi(x) - \varphi(y)$.

We have $\bar{\alpha}\|y - x\| = \varphi(x) - \varphi(y)$, where

$$\bar{\alpha} := \frac{\varphi(x) - \varphi(y)}{\|y - x\|} \in]0, \alpha[.$$

Since φ is $\bar{\alpha}$ -robustly quasiconvex, we derive from Case 1 that

$$\begin{aligned} \langle x^*, y - x \rangle &\leq -\bar{\alpha}\|y - x\| = \varphi(y) - \varphi(x) \\ &= -\min \{ \alpha\|y - x\|, \varphi(x) - \varphi(y) \}. \end{aligned}$$

Conversely, assume that X is Asplund, and (b) holds. It follows from Theorem 2.2 that φ is quasiconvex. Suppose that φ is not α -robustly quasiconvex, i.e., there exists $v^* \in X^* \setminus \{0\}$, $\|v^*\| < \alpha$ such that φ_{v^*} is not quasiconvex. By Lemma 3.3, there are $u, w \in X$ and $v \in]u, w[$ satisfying (7), (8), and (9). Since $\varphi_{v^*}(v) > \varphi_{v^*}(u)$, there exists $\delta > 0$ such that $\bar{v}^* := (1 + \delta)v^*$ satisfies $\|\bar{v}^*\| < \alpha$ and $\varphi_{\bar{v}^*}(v) > \varphi_{\bar{v}^*}(u)$. Thus, we have $\varphi(v) > \varphi(u)$, $\varphi_{v^*}(v) > \varphi_{v^*}(u)$, $\varphi_{\bar{v}^*}(v) > \varphi_{\bar{v}^*}(u)$ and the lower semicontinuity of φ, φ_{v^*} , and $\varphi_{\bar{v}^*}$. This implies the existence of $\gamma > 0$ satisfying

$$\varphi(z) > \varphi(u), \quad \varphi_{v^*}(z) > \varphi_{v^*}(u), \quad \varphi_{\bar{v}^*}(z) > \varphi_{\bar{v}^*}(u) \quad \forall z \in B_\gamma(v). \quad (16)$$

By the assertion (9), there is $v_\gamma \in B_\gamma(v) \cap]v, w[$ such that $\varphi_{v^*}(v) > \varphi_{v^*}(v_\gamma)$.

Then, v_γ can be written as

$$v_\gamma := v + \lambda(w - v) \text{ with } \lambda \in \left] 0, \min \left\{ 1, \frac{\gamma}{\|w - v\|} \right\} \right].$$

Since $\varphi_{v^*}(v) > \varphi_{v^*}(w)$ and $\varphi(v) \leq \varphi(w)$, we have $\langle v^*, w - v \rangle < 0$ and so

$$\begin{aligned} \varphi_{\bar{v}^*}(v_\gamma) - \varphi_{\bar{v}^*}(v) &= \varphi_{v^*}(v_\gamma) - \varphi_{v^*}(v) + \delta \langle v^*, v_\gamma - v \rangle \\ &= \varphi_{v^*}(v_\gamma) - \varphi_{v^*}(v) + \delta \lambda \langle v^*, w - v \rangle < 0. \end{aligned}$$

Applying Lemma 2.1 for $\varphi_{\bar{v}^*}$, $v \in [v_\gamma, u]$ with $\varphi_{\bar{v}^*}(v) > \varphi_{\bar{v}^*}(v_\gamma)$, we can find a vector $x \in \text{dom } \varphi_{\bar{v}^*}$ and a dual element $x^* \in \widehat{\partial} \varphi_{\bar{v}^*}(x)$ such that

$$x \in [v_\gamma, v] + (r - \|v_\gamma - v\|)\mathbb{B} \quad \text{and} \quad \langle x^*, u - x \rangle > 0. \quad (17)$$

Then, $x \in B_\gamma(v)$ and so $\varphi(x) > \varphi(u)$ by (16). By the assumption (b) and the second inequality of (17),

$$-\langle \bar{v}^*, u - x \rangle < \langle x^* - \bar{v}^*, u - x \rangle \leq -\min\{\alpha \|u - x\|, \varphi(x) - \varphi(u)\}.$$

Since $\langle \bar{v}^*, u - x \rangle \leq \|\bar{v}^*\| \|u - x\| < \alpha \|u - x\|$, the above inequality implies that $\langle \bar{v}^*, u - x \rangle > \varphi(x) - \varphi(u)$, i.e., $\varphi_{\bar{v}^*}(x) < \varphi_{\bar{v}^*}(u)$ and this contradicts (16). \square

We next construct a completely new characterization for the robust quasiconvexity. It is based on the equivalence of the quasiconvexity of a lower semicontinuous function and the quasimonotonicity of its subdifferential operator.

Theorem 3.2 *Let $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous on an Asplund space X and $\alpha > 0$. Then, φ is α -robustly quasiconvex if and only if*

for any $(x, x^*), (y, y^*) \in \text{graph } \widehat{\partial}\varphi$, we have

$$\min\{\langle x^*, y - x \rangle, \langle y^*, x - y \rangle\} > -\alpha\|y - x\| \implies \langle x^* - y^*, x - y \rangle \geq 0. \quad (18)$$

Proof. Suppose that φ is α -robustly quasiconvex and there exist $(x, x^*), (y, y^*)$ in $\text{graph } \widehat{\partial}\varphi$ such that

$$\min\{\langle x^*, y - x \rangle, \langle y^*, x - y \rangle\} > -\alpha\|y - x\|. \quad (19)$$

Since φ is quasiconvex, $\widehat{\partial}\varphi$ is quasimonotone by Theorem 2.2. It follows that

$$\min\{\langle x^*, y - x \rangle, \langle y^*, x - y \rangle\} \leq 0. \quad (20)$$

Combining (19) and (20), we have

$$0 \leq -\min\left\{\left\langle x^*, \frac{y - x}{\|y - x\|} \right\rangle, \left\langle y^*, \frac{x - y}{\|x - y\|} \right\rangle\right\} < \alpha.$$

Without loss of generality, we may assume

$$\left\langle x^*, \frac{y - x}{\|y - x\|} \right\rangle = \min\left\{\left\langle x^*, \frac{y - x}{\|y - x\|} \right\rangle, \left\langle y^*, \frac{x - y}{\|x - y\|} \right\rangle\right\}.$$

Let $r > 0$ be such that

$$-\left\langle x^*, \frac{y - x}{\|y - x\|} \right\rangle < r \leq \alpha. \quad (21)$$

It is known that any element x of a Banach space X attains its norm on the unit sphere of the dual space X^* . Hence, there exists $v^* \in X^*$ satisfying

$\langle v^*, y - x \rangle = r\|y - x\|$ and $\|v^*\| = r \leq \alpha$. It follows that

$$\langle x^*, y - x \rangle + \langle v^*, y - x \rangle > -r\|y - x\| + r\|y - x\| = 0. \quad (22)$$

Consider $\varphi_{v^*} : X \rightarrow \mathbb{R} \cup \{+\infty\}$ given by $\varphi_{v^*}(u) = \varphi(u) + \langle v^*, u \rangle$ for any $u \in X$. Then, we have $\widehat{\partial}\varphi_{v^*}(u) = \widehat{\partial}\varphi(u) + v^*$ for $u \in \text{dom } \varphi$. Hence, by the quasiconvexity of φ_{v^*} and by Theorem 2.2, we have

$$\min\{\langle x^*, y - x \rangle + \langle v^*, y - x \rangle, \langle y^*, x - y \rangle + \langle v^*, x - y \rangle\} \leq 0.$$

Combining with (22), it implies

$$\langle y^*, x - y \rangle + \langle v^*, x - y \rangle \leq 0, \text{ i.e., } \langle y^*, x - y \rangle \leq \langle v^*, y - x \rangle = r\|x - y\|.$$

Letting $r \rightarrow -\left\langle x^*, \frac{y-x}{\|y-x\|} \right\rangle$, we obtain $\langle y^*, x - y \rangle \leq \langle x^*, x - y \rangle$ and thus (18) holds.

Conversely, assume that (18) holds for all $x, y \in X$, and $x^* \in \widehat{\partial}\varphi(x)$, $y^* \in \widehat{\partial}\varphi(y)$. Taking any v^* in $\alpha\mathbb{B}^*$, we next prove that $\varphi_{v^*} : X \rightarrow \mathbb{R}$, defined by $\varphi_{v^*}(u) = \varphi(u) + \langle v^*, u \rangle$ for any $u \in X$, is quasiconvex by showing the quasimonotonicity of $\widehat{\partial}\varphi_{v^*}$. Indeed, taking any $x, y \in X$, and $x^* \in \widehat{\partial}\varphi_{v^*}(x)$, $y^* \in \widehat{\partial}\varphi_{v^*}(y)$, then $x^* - v^* \in \widehat{\partial}\varphi(x)$, $y^* - v^* \in \widehat{\partial}\varphi(y)$. We then consider two cases.

Case 1. $\min\{\langle x^* - v^*, y - x \rangle, \langle y^* - v^*, x - y \rangle\} \leq -\alpha\|y - x\|.$

Without loss of generality, assume that

$$\langle x^* - v^*, y - x \rangle = \min\{\langle x^* - v^*, y - x \rangle, \langle y^* - v^*, x - y \rangle\}.$$

Since $\|v^*\| \leq \alpha$, we have

$$\begin{aligned} \min\{\langle x^*, y - x \rangle, \langle y^*, x - y \rangle\} &\leq \langle x^*, y - x \rangle = \langle x^* - v^*, y - x \rangle + \langle v^*, y - x \rangle \\ &\leq -\alpha\|y - x\| + \|v^*\|\|y - x\| \leq 0. \end{aligned}$$

Case 2. $\min\{\langle x^* - v^*, y - x \rangle, \langle y^* - v^*, x - y \rangle\} > -\alpha\|y - x\|.$

Since (18) is satisfied, we have

$$\langle (x^* - v^*) - (y^* - v^*), x - y \rangle \geq 0,$$

i.e., $\langle x^* - y^*, x - y \rangle \geq 0$. It implies that

$$2 \min\{\langle x^*, y - x \rangle, \langle y^*, x - y \rangle\} \leq \langle x^*, y - x \rangle + \langle y^*, x - y \rangle \leq 0.$$

Hence, $\widehat{\partial}\varphi_{v^*}$ is quasimonotone and thus φ_{v^*} is quasiconvex for any $v^* \in \alpha\mathbb{B}^*$ by Theorem 2.2. This yields the α -robust quasiconvexity of φ . \square

4 Conclusions

Using Fréchet subdifferentials, we have obtained two first-order characterizations for the robust quasiconvexity of lower semicontinuous functions in Asplund spaces. The first one is a generalization of [10, Proposition 5.3] from

finite dimensional spaces to Asplund spaces and its proof also overcomes a glitch in the proof of the sufficient condition of [10, Proposition 5.3]. The second criterion is totally new and it is settled from the equivalence of the quasiconvexity of lower semicontinuous functions and the quasimonotonicity of their subdifferential operators. Further investigations are needed to apply those characterizations in partial differential equations with connections to differential geometry, mean curvature, tug-of-war games, and stochastic optimal control [10, 15, 19].

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