RESOURCES FOR REASONING

Before we teach reasoning we must confront some deeply-entrenched counter-productive human behaviours. An important one of these is 'confirmation bias', a failing which prevents us from applying logical analysis. Confirmation bias is our tendency to seek evidence in support of our beliefs, neglecting counter-examples which might cause us to re-think. Only when our students realise that psychological obstacles can be overcome will our efforts with reasoning prove fruitful. This essay presents some approaches which may assist students to address biases and apply insightful thinking.

Psychologist Peter Wason (1966) used four cards to demonstrate the existence of confirmation bias. His experiment involved asking the subjects to turn over the card or cards necessary to establish a proposition. The experiment can be replicated using two similar decks of cards, one with red backs and the other blue backs. For the playing cards below, which card or cards must be turned over to confirm that every even card has a red back?

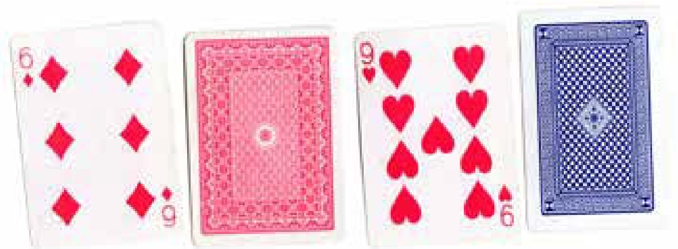


Figure 1. Peter Wason used four cards to demonstrate confirmation bias.

It certainly is surprising to see how much contention this experiment provokes and the extent of incorrect thinking which is revealed. You can check your reasoning at http://spikedmath.com/math-games/puzzle-games/the-wason-selection-task-2.html. Full explanations are given for all sixteen possible choices. Spiked Math also contains other excellent reasoning games.

A similar effect can be seen in a video by YouTube identity Muller (n.d.) where members of the public offer sequences in an attempt to guess the investigator's rule. Most attempts are sequences which accord with the subjects' preconceptions, and it takes a long time before attempts at disconfirmation are made.

How can our faulty reasoning be corrected? Wason (1966) discovered that there is far less confusion when a social context is provided. So, let's get away from numbers and non-representational shapes and try a different format. For example, what card or cards must be turned over in this collection (figure 2) in order to prove that any card with a dog on one side has a bone on the other side?

If you try these less-abstract cards with a similar group of students you are likely to find better reasoning and a higher incidence of correct results (in this case dog, car and cat).

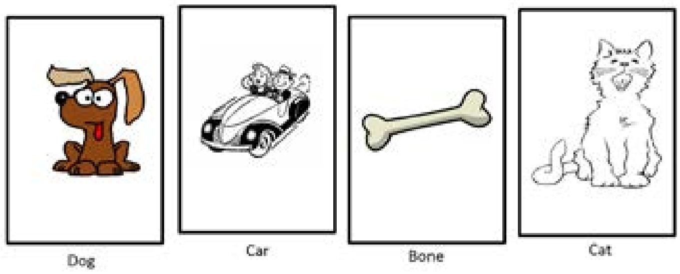


Figure 2. There is less confusion when concrete images are used. Source: Google Images, copyright free.

Once we establish the idea that reasoning can be learned, what should we do to promote such learning? One surprising recommendation advanced by Wells (2016) is to provide faulty games. These are unfair games, although they may appear innocent until players become familiar with them. One such game is 'Cornering the Lady'. A player places the queen near the middle of the top row of an otherwise empty chessboard. Players then alternate moves, the other player moving the queen first, making normal queen moves but never right nor up (that is, only West, South or South-West). The player who moves the queen to the bottom left corner wins. Students will eventually gleefully reason that there is an unbeatable strategy by moving to certain 'safe' cells. An analysis is provided at www.cut-the-knot.org/pythagoras/withoff.shtml (partial image below).

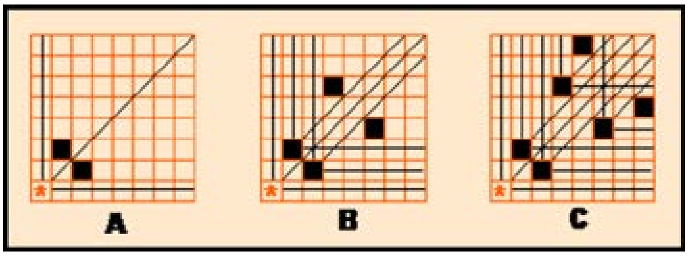
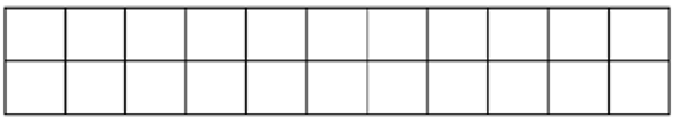


Figure 3. If the queen is in a row, column or diagonal containing the star, the person can win at once. Mark these with three straight lines as shown in A. The two shaded cells are 'safe' because if you occupy one, your opponent must move to a cell that enables you to win on the next move.

Sullivan (2016) recommends puzzles of the following type.

The diagram below represents $3\frac{2}{3}$. Draw a diagram for $2\frac{1}{2}$.

He notes that there are many different rational solving approaches.



Matching formulas to the quantity being measured is another opportunity for reasoning. Many students confuse the formulas for area of a circle and circumference of a circle. If they realise that the pronumerals are places where a number of centimetres can be substituted – and they are aware of the units of volume, area and length – the problem can be overcome by reasoning.

Students can be tested by asking what type of quantity these formulas measure:

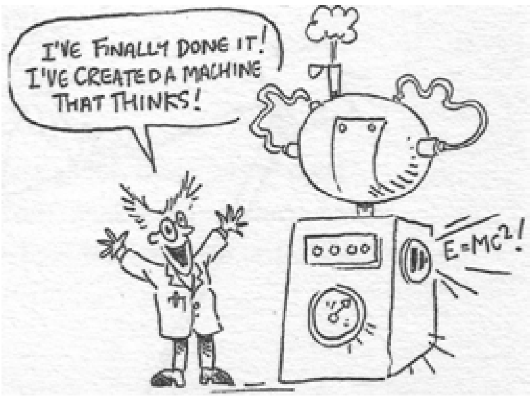
$$\frac{h}{2}(a+b) \qquad \frac{\pi r^2 h}{3} \qquad 2(ab+bc+ac)+\pi r(\sqrt{d^2+r^2}-r)$$

Respectively, they produce area, volume and area. And in case you were wondering, that right-hand formula does have a real use; it models the total surface area of a cuboid pencil sharpener with a conical hole.

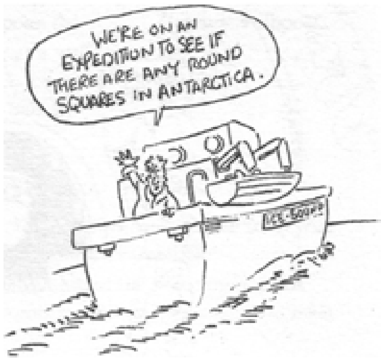
Figure 4. Surface area of a pencil sharpener.

A few avenues exist to assist teachers with the reasoning proficiency strand. The Australian Curriculum recently introduced annotated student work samples and the Australian Association of Mathematics Teachers have some ideas in the Top Drawer Teachers section of their website.

Many books address reasoning, including my own (Brown, 2010). A collection of dozens of problem-solving activities which involve reasoning have been selected from *Mathematics Teacher* (Kasten & Newton, 2011). A search for 'critical thinking activities', 'logic games' or 'lateral thinking puzzles' may be productive. Going wider than mathematics, Law (2011) provides a delightful introduction to reasoning in philosophy, complete with engaging illustrations such as those below.



Source: The Philosophy Files 2, p. ix. (2003) London: Orion.



Source: The Philosophy Files 2, p. 118. (2003) London: Orion.

To facilitate expression of the reasoning behind methods of solution, group work on mathematical problems is invaluable. To promote persistence, posters can feature problems which require sustained attack. And to give students permission to explore alternative reasoning pathways, it is good to ask for more than one method of solution, perhaps utilising questions based on past NAPLAN or Australian Mathematics Competition papers. To allow students to think differently, a variety of working-paper templates should be available.

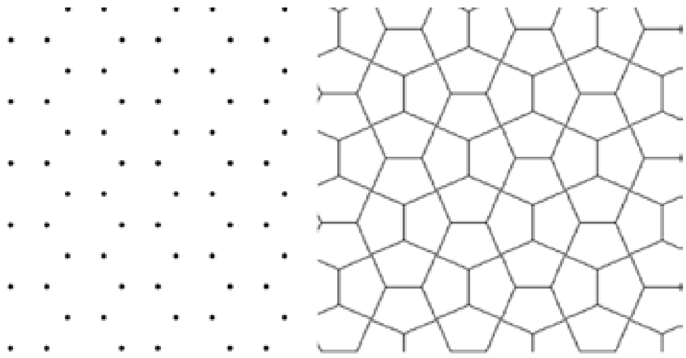


Figure 5. Hexagon dot paper (left) and pentagon grid paper.

Faulty questions and questions with more than one solution (Figure 6) can be particularly thought-provoking. I hope you saw the connection with the semi-circle in the impossible diagram of part C!

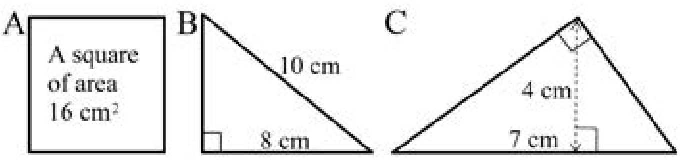


Figure 6. Find the perimeters of these shapes (not drawn to scale).

The Australian Curriculum (Australian Curriculum and Reporting Agency, 2016) focuses on reasoning in geometry. This lends itself to artistic expression such as making symmetrical cushion covers, and also to board games. A particularly useful game is the L Game (de Bono, 1967) as it incorporates all the congruence transformations. The game board is given on the next page. Photocopying this on coloured paper enables distinct colours of Ls to be cut out. Coins can be used as counters: 20 expertly-made disks can be obtained for $1 (at most banks).

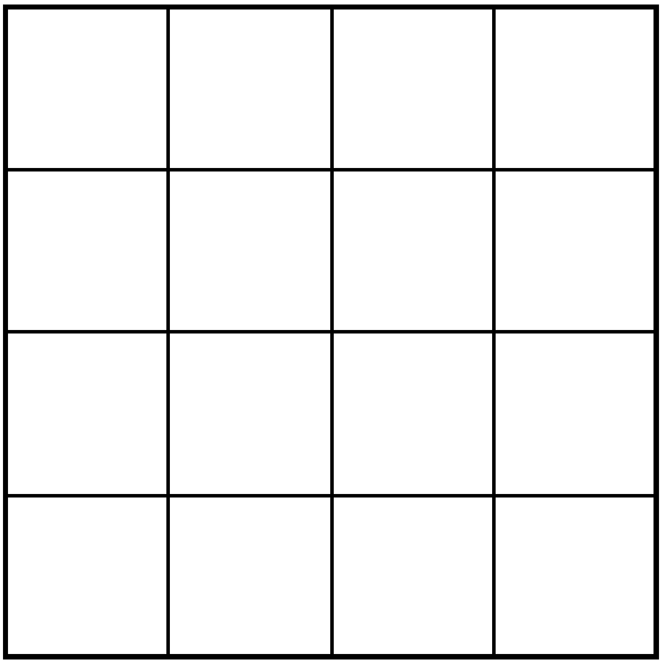


Figure 7. The L game playing grid. Teachers may make (enlarged) photocopies of these.

L GAME RULES

Take turns, starting as shown below.

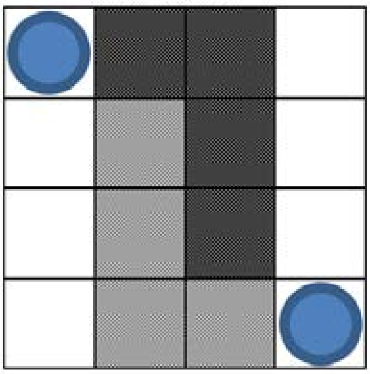


Figure 8. The L game starting position.

Move your L piece, then, if you wish, move either counter to any vacant square.

L pieces may be rotated or flipped but must align with the playing grid.

All pieces must be placed within the playing grid, no overlapping.

The first player who cannot move their L piece to a new position has lost the game.

Perhaps less fashionable these days, but still valuable in the development of linguistic reasoning, are analogy questions of the type below. If you know of particularly good in-print or online sources, please share them.

Square is to cube, as triangle is to	Odd is to even as prime is to
A. Regular tetrahedron	A. Natural
B. Triangular prism	B. Composite
C. Hexagon	C. Fraction

These are sometimes presented in the format
ruler : line :: compass : ________

The confusing example below is from EduGoog. www.edugoog.com/details/6-18-4-number-analogy.html. The question is given as 6 : 18 :: 4 : ? and the options given are 2, 6, 8 and 16. The answer 8 is justified by the relationship being $x : x^2/2$. The website authors apparently believe that the solution which involves the highest order polynomial is the best! It is possible to find a rule (even a linear rule) which would justify any of these options. For example, if the rule was $x : x + 12$, then 16 would be correct.

Finally, a great hope for the formal teaching of reasoning is coding. This is not computer programming, and it is worth checking out the questions posed in the Computational and Algorithmic Thinking Competition at the Australian Mathematical Trust website. In coding, a mistake is not failure but an opportunity for debugging. Mindstorms (Papert, 1980) is an inspiring classic book for educators – no wonder volunteers have kept his FMS Logo education software (image below), available free at https://fmslogo.en.softonic.com/

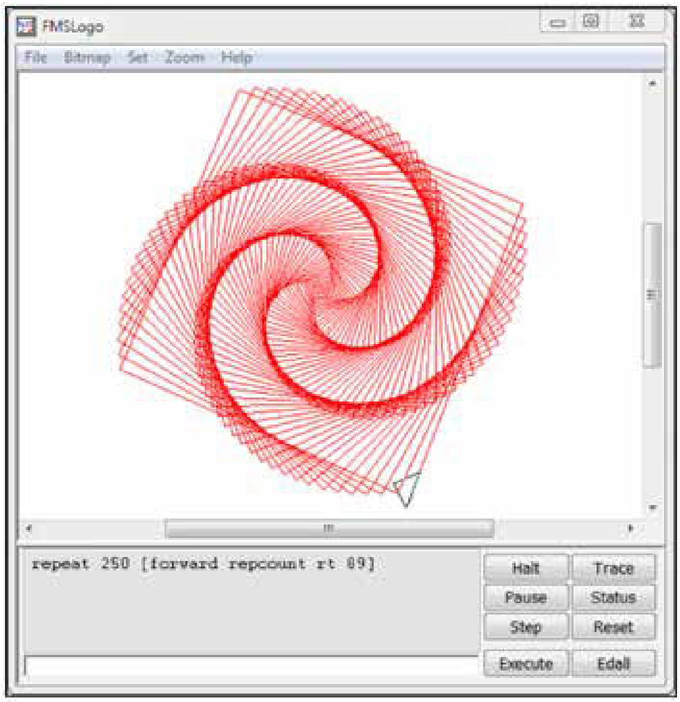


Figure 9. FMS Logo.

Other free coding software includes Scratch and BYOB, as discussed in *Vinculum* (50)2 & (50)3. For help with coding visit code.org or find your local coderdojo (coding club).

Of course, spreadsheets also enable systematic thinking, debugging and many pre-algebra activities. A nested 'if' statement is located in cell A5, driving the game-like spreadsheet below. Built-in help and YouTube videos are available to explain how such functions work.

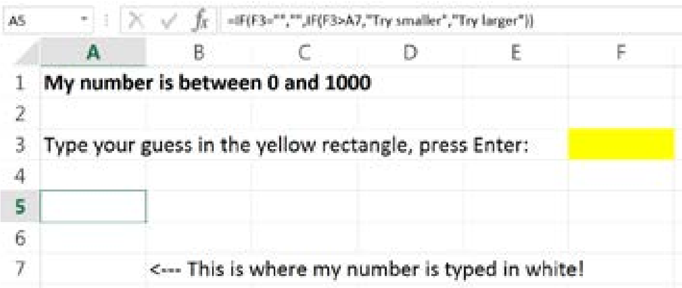


Figure 10. Spreadsheets can be used for activities involving reasoning.

The formula in cell A5 reads =IF(F3="","",IF(F3>A7,"Try smaller","Try larger")). It means that if F3 is blank, leave it blank, otherwise compare the guess with the target and give an appropriate hint.

Reasoning, like problem solving, is a topic students will explore for themselves once they are motivated. An introduction which is too abstract may be counter-productive, but the more tangible approaches mentioned in this essay may work reasonably well.

AUTHOR

Paul Brown is a teaching academic at Curtin University. In addition to preparing people to teach school mathematics, Paul is interested in puzzle-based learning, surreptitious teaching, out-of-field teachers, reasoning and proof. As stated on his book about mathematical proof, Paul's hobbies are 'mathematics, mathematics and mathematics'.

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ONLINE RESOURCES

Australian Association of Mathematics Teachers Top Drawer Teachers section, http://topdrawer.aamt.edu.au/Reasoning/Downloads

The Australian Curriculum annotated student work samples http://resources.australiancurriculum.edu.au/proficiencies/mathematics-portfolios/reasoning/

Australian Mathematical Trust: Computational and Algorithmic Thinking Competition, www.amt.edu.au/informatics/cat/

Play the L game, https://hwwmath.looiwenli.com/l-game

Logo, FMSLogo, https://fmslogo.en.softonic.com/

Scratch, http://scratch.mit.edu

BYOB (an extension of Scratch), http://byob.berkeley.edu/

Spiked Math Games, http://spikedmath.com/math-games/

Muller (Veritasium), www.youtube.com/user/1veritasium

Veritasium contains a series of scientific/logical video clips. The 'guess my number rule' video can be found at www.youtube.com/watch?v=vKA4w2O61Xo&feature=youtu.be

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