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A comparative review of the Molodensky-Badekas and Burša-Wolf methods for coordinate transformation

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Abstract

J. Badekas reinterpreted M.S. Molodensky's three-dimensional similarity transformation as a vector solution using a centroid. The solution has since been [mis]interpreted by some others with inconsistent reference to the methods of both Molodensky and Badekas, principally relating to the translation vector and the stochastic model. This appears to have led to incorrect claims that the Molodensky-Badekas method is superior to the Helmert similarity and Burša-Wolf methods. This paper reviews the development and description of the original Badekas method, reconfirming its equivalence to the Burša-Wolf method in the forward direction, and provides an alternative solution that suits the same-formula reversal common in commercial surveying software. It is also demonstrated that the Molodensky-Badekas method has no inherent superiority over the Burša-Wolf method, has an ambiguous functional model, and nominally underestimates its parameter statistics when these are compared directly to those from the Burša-Wolf method.

Keywords: similarity, conformal, geodetic, coordinate transformation

1. INTRODUCTION

This paper describes two simplified forms of the general (Helmert) similarity three-dimensional transformation: first, the Burša-Wolf (BW), following the papers of Burša (1962) and Wolf (1963) that allow the full rotation matrix to be simplified for small (say, <5") axial rotations. The original Molodensky 3D similarity transformation (Molodensky et al., 1962) is then described, followed by the second simplified form being its vector reinterpretation by Badekas (1969) and a variant of this which, within a group of other variants, may be grouped under the title of the Molodensky-Badekas (MB) transformation.

Leick and van Gelder (1975) have shown that the MB method suffers from an ambiguity in the translation terms and its stochastic model and their work is followed in this paper. When these model errors are corrected, the MB and BW methods produce equivalent parameter sets. However, the work of Leick and van Gelder (1975) and also Soler (1976) has not been widely appreciated because use of the MB method has continued without acknowledgement of its shortcomings (e.g., Annan et al., 2016; Ansari et al., 2017; Burford, 1985; IOGP, 2018; Kutoglu et al., 2002; Sanchez, 2004; Turgut, 2010; Varga et al., 2015).

The claimed superiority of MB over BW is based on the apparent reduction of the correlations among, and the improved uncertainties of, the derived parameters (cf. IOGP, 2018). This is when the correlations have no effect on the magnitude of the rotation or scale parameters between the two methods (Harvey, 1985) and when the MB translations are ambiguous (Leick & van Gelder, 1975). Though not affecting the rigorous reversal, the lack of rigour leads to a general lack of same-formula reversibility for a MB parameter set (IOGP, 2018), where the error on reversal is insignificant for closely aligned CRS but increases with increasing centroidal vector and/or increasing magnitude of the scale and rotation parameters. Herein, 'same-formula' reversibility is defined as the reverse transformation, that is, back to the target ('from') datum from the source ('to') datum, but using the forward algorithm while simply changing the sign of the parameters (Iliffe & Lott, 2008, p. 93); and see Section 3.2 herein.

It shall be demonstrated in this paper that the apparent reduction in the correlations, and the other promoted benefits of MB follow from improper functional and stochastic modelling, compounded by a limited appreciation of the differences between the BW and MB parameters.

2. BACKGROUND TO THE SIMILARITY TRANSFORMATIONS

2.1. HELMERT AND BURŠA-WOLF TRANSFORMATIONS

The general similarity transformation, developed by Helmert (1876), to transform error ellipses derived from astronomic observations to a local geodetic reference frame, is well known in the geodetic community as a conformal (angle-preserving) datum transformation (e.g., Rapp, 1993). It is general in nature, that is, it can be applied to any two 2D or 3D coordinate frames, irrespective of their difference in size or orientation (e.g, Grossman, 1980, Ch.7.3). Critically for the present paper, the similarity transformation has been shown by theorem to be unique (Modenov & Parkhomenko, 1966, cited in Leick & van Gelder, 1975). That is, for a given set of inputs, the solutions from all properly formed variants of the general similarity transformation will be equivalent.

The simplified and approximate version of Helmert's transformation developed separately by Burša (1962) and Wolf (1963) is herein called the BW method so as to distinguish it from the use of the full rotation matrices in the Helmert method. BW uses approximations for small angles to reduce the complexity of the scaled rotation matrices. These simplifications mean that the method is no longer strictly exact and conformal but will, subject to being applied only to transformations between closely aligned coordinate reference systems where the total network rotation is less than a few seconds of arc (e.g., Malys, 1988), and provide 'acceptable' parameters (e.g. Petit & Luzum, 2010). The BW method is commonly used for geodetic applications in two-, three- and now four-dimensions (e.g., Soler & Snay, 2004) and is almost universally available in surveying software packages.

A BW transformation between a source Cartesian coordinate system (herein called the U-system) and a target Cartesian coordinate system (herein called the X-system) is (e.g., Rapp, 1993):

$$\mathbf{x} = s\mathbf{R}\mathbf{u} + \mathbf{t_{BW}} \qquad (1)$$

where $\mathbf{x}$ is a coordinate vector in the X-system, s is a scale factor commonly defined as $s = 1 + ds$, where *ds* is the differential scale, $\mathbf{R}$ is the simplified 3x3 rotation matrix applying the coordinate frame rotation convention (cf. Soler, 1976), $\mathbf{u}$ is a coordinate vector in the U-system, and $\mathbf{t_{BW}}$ is a translation vector in the X-system. The simplified rotation matrix is:

$$\mathbf{R} = \begin{bmatrix} 1 & \mathrm{rw} & -\mathrm{rv} \\ -\mathrm{rw} & 1 & \mathrm{ru} \\ \mathrm{rv} & -\mathrm{ru} & 1 \end{bmatrix} \qquad (2)$$

where ru, rv, rw are small rotations of the U, V, W axes respectively.

2.2. MOLODENSKY'S TRANSFORMATION

Section 3 of Chapter 1 of Molodensky et al. (1962) describes a method to transform an existing triangulation network to a 'new coordinate system', initially using a Cartesian coordinate system referenced to the 'initial point of the triangulation' (ibid, p.32) being the origin associated with an astrogeodetic datum. Molodensky then progresses (Eq. 1.3.2 & 1.3.4) to a nominally global Cartesian system defined by the axes of the orthogonal, curvilinear coordinates of this local geodetic datum.

However, Molodensky et al. (1962) describe [perhaps rather vaguely] the translation components as affected by 'progressive translations' being part of the differential components described in the chapter title. Initially, the three translations reflect the sum of two vectors, one in the U-system, the other in the X-system, that are progressively transformed to become a single set of three translations in the X-system (Soler, 1976). This process is not intuitive but appears [to us] central to the problem of the variants that followed Badekas's reinterpretation of Molodensky's method.

Molodensky's original method is essentially an eight-parameter similarity transformation. The parameters vary from the 'conventional' seven-parameters (three translations, three rotations and a scale factor), as Molodensky parameterised the scale factor in terms of the semi-major (a) and semi-minor (b) axes of the reference ellipsoids associated with the source and target datums. An interpretation of the Molodensky transformation, following Soler (1976), converted to Cartesian vectors from the original differential form (Molodensky et al., 1962, pp.13-17) and with a single scale factor is:

$$\mathbf{x} = \mathbf{u} + \mathbf{t_M} + (\mathbf{R} - \mathbf{I})\mathbf{u} + s\mathbf{u} = \mathbf{t_M} + s\mathbf{R}\mathbf{u} \qquad (3)$$

where, as defined before, and $\mathbf{t_M}$ is equivalent to the Burša-Wolf translations. Soler's (1976, p.37) study of the Molodensky method had "*the intention of showing that the model obtained is strictly only a similarity transformation of the Bursa type without anything special introduced besides the scaling variation mentioned above*". Soler demonstrates equivalence between Molodensky's eight-parameters and the seven-parameter BW method within the range of heights expected within a terrestrial geodetic network (ibid., p.40).

Like Burša (1962), Molodensky et al. (1962) give no discussion of a stochastic model.

2.3. BADEKAS'S REINTERPRETATION

Badekas's (1969) paper was a precursor to the development of the World Geodetic System 1972 (Seppelin, 1974). Like Molodensky, Badekas was concerned with gravimetric parameters, and also like Molodensky, he included an early chapter on coordinate systems and their transformation, reviewing the predominant three-dimensional similarity transformations available at that time: Burša-Wolf, Molodensky and Veis (1960).

During the 1960s and 1970s, the concept of the Average Terrestrial System (ATS) as an ideal coordinate system was prominent (e.g., Badekas, 1969 and Krakiwsky & Thomson, 1974). Although the ATS could be precisely defined theoretically, it could not be realised on the ground accurately. However, as a step-change in methodology, the concept of the ATS as a frame for integrating multiple disparate astrogeodetic datums may have prompted Badekas to apply Molodensky's initial method, replacing the geocentric with a nominally terracentric coordinate system using a centroid at the 'initial point', and to reinstate the two original vectors merged by Molodensky: one from the X-system origin to the nominally geocentric origin of the U-system, the second from the U-system origin to the 'initial point'.

Badekas defines the translation vector in the X-system between two geocentric Cartesian coordinate systems as "*dx_0, dy_0, dz_0 are the coordinates of the origin of the geodetic system [x_1] after it has been rotated and become parallel to the average terrestrial system*" (Badekas, 1969, p.14). Note the use of the term 'after' indicating the differential operation, that is, and unlike BW, the origin of the U-system has two states; before at, say, t=1, then after the U-system is scaled and rotated around the centroid at t=2.

Badekas then defines a nominal centroid as "*x_0, y_0, z_0 are the geodetic coordinates of the point P (initial point) which is kept fixed during rotations*" (Badekas, 1969, p.14), that is, the coordinates are referenced to the U-system and leads to:

$$\mathbf{x} = \mathbf{t_{MB}} + \mathbf{u_0} + \mathbf{R}(\mathbf{u} - \mathbf{u_0}) + s(\mathbf{u} - \mathbf{u_0}) \qquad (4)$$

where $\mathbf{t_{MB}}$ = [dx_0, dy_0, dz_0] and in this instance $\mathbf{u_o}$ = [u_0, v_0, w_0].

As a last step, Badekas simplifies his method by differencing the second and third terms in Eq. (4) to arrive at his ultimate transformation equation (Badekas, 1969, Eq. 3.2-2c):

$$\mathbf{x} = \mathbf{t_{MB}} + \mathbf{u} + (\mathbf{R} - \mathbf{I})(\mathbf{u} - \mathbf{u_0}) + s(\mathbf{u} - \mathbf{u_0}) \qquad (5)$$

Our search of the literature found only a single citation to the use of this final equation (Leick, 1995, p.479). Krakiwsky and Thompson (1974) started from this equation but, having identified the ambiguity, resolved it by requiring the centroid to be defined in a CRS that is parallel to the ATS. In general, published works that apply Badekas's method use his un-numbered intermediate equation (Eq. 4 here) or variants of this equation (e.g., Burford, 1985; Kutoglu et al., 2002; Varga et al., 2017; Yun et al., 2006), and appear to ignore his final step. To us, the reason for this remains unanswered.

Like Molodensky et al. (1962) and Burša (1962), Badekas (1969) gives no discussion on a stochastic model for his method. Also, neither Molodensky nor Badekas discuss reversibility, their primary interest being in transforming from multiple existing regional to a single global coordinate reference system (CRS).

2.4. THE IOGP/EPSG VARIANT

Following Badekas's report, variants of MB have been published. Amongst others (e.g., NGA, 2014), the variant with possibly the most visibility in geodesy and surveying is that published and promoted by the International Association of Oil and Gas Producers (IOGP). The IOGP inherited the geodetic registry developed by the now-defunct European Petroleum Surveyors Group (EPSG), though the registry's acronym has been maintained. The EPSG registry is widely used as a source for coordinate operations and their parameters and is freely available at *https://www.epsg-registry.org*.

The EPSG's form of MB (EPSG::1034) applies the intermediate form of Badekas's interpretation (Eq. 4 here). The EPSG's variant A is defined in Guidance Note 373-7-2 (IOGP, 2018) as a 10-parameter transformation:

$$\mathbf{x} = s\mathbf{R}(\mathbf{u} - \mathbf{u_0}) + \mathbf{u_0} + \mathbf{t_{MB}} \qquad (6)$$

where $\mathbf{u_o}$ are "*coordinates of a pivot point in the source coordinate reference system*', and $\mathbf{t_{MB}}$ '*is a translation vector, to be added to the point's position vector in the source CRS in order to transform from source CRS to target CRS; also, the coordinates of the origin of source CRS in the target CRS".* (IOGP, 2018, p.132).

The EPSG variant does not require the use of the 'initial point', or of a computed centroid, but provides for an arbitrary 'pivot point' in the U-system. This has no real effect in the forward direction but is inconsistent with the requirement, to be discussed later, that the centroid must be known in both CRSs.

3. OBSERVATIONS ON THE MOLODENSKY-BADEKAS AND AN ALTERNATIVE METHOD

MB has been critiqued by Leick and van Gelder (1975) and Soler (1976). Leick and van Gelder (Leick & van Gelder, 1975, p. 8) point out that "*$|U_o|$ is the position vector of the initial point of the geodetic datum in the in the U system, but the assumption is made that this vector is identical to the same vector in a geodetic system which is already parallel to the Average Terrestrial System*". This assumption contradicts the 'closure under addition' axiom of linear algebra (e.g., Grossman, 1980, p. 161) that requires $\mathbf{u_o}$ to be referenced either to the X-system or be associated with an appropriate correction.

The Badekas assumption may be understood by, following Harvey (1985), where Eq. (4) can be re-arranged and written as:

$$\mathbf{x} = \mathbf{p} + \mathbf{q} \qquad (7)$$

where

$$\mathbf{p} = \mathbf{t_{MB}} + s\mathbf{Ru} \qquad (8)$$

and

$$\mathbf{q} = \mathbf{u_0} - s\mathbf{Ru_0} \qquad (9)$$

Noting that $\mathbf{q}$ is a constant term common to all points, and equating Eq. (1) and (4), then

$$\mathbf{u_0} - s\mathbf{Ru_0} = \mathbf{t_{BW}} - \mathbf{t_{MB}} \qquad (10)$$

showing that $\mathbf{q}$ affects the translations only following from the scaling and rotation of the U-system origin described by Badekas (1969). By inspection, Eq. (8) is the BW with the MB centroid now at the origin of the U-system. The estimates for s and $\mathbf{R}$ are derived from Eq. (8) only, demonstrating the equivalence between BW and MB scale and rotations (Leick & van Gelder, 1975). In Eq. (9), $s\mathbf{Ru_0}$ is the centroidal vector scaled and rotated into the X-system, but noting that it is not equal to $\mathbf{x_0}$. Eq. (9) is the problem; the translation difference is in the X-system, while the centroidal difference is in both U- and X-systems. Accepting the right-hand side of Eq. (10), the correct difference is:

$$\mathbf{t_{BW}} - \mathbf{t_{MB}} = \mathrm{x_0} - s\mathrm{Ru_0} \qquad (11)$$

Eqs. (9) and (10) follow from Molodensky's 'progressive translations' (Molodensky et al., 1962, pg.14) and, in this case, requires the difference to be absorbed by $\mathbf{t_{MB}}$ (Pearse & Crook, 1997). Ironically, the MB has been promoted for its reduced correlations (ibid, pg. 13) but it relies on a correlation between its rotation and translation parameters to enable the translation term to absorb the model error. Leick and van Gelder (1975, p.22) comment that "*the translation parameters of Model 2* [MB] *have hardly any geometric significance*" and "*the reduction of the correlation between rotational and translation parameters is artificial and misleading*". Reflecting the nominal equivalence between BW and MB in the forward direction, the two sets of translation parameters are related by rearranging Eq. (10), (e.g., Leick & van Gelder, 1975):

$$\mathbf{t_{BW}} = \mathbf{t_{MB}} + \mathbf{u_0} - s\mathbf{Ru_0} \qquad (12)$$

and will be demonstrated later in Section 5.

3.1. USE OF CENTROIDS

We postulate that the use of a centroid for coordinate transformations was popular for two reasons:

i. A centroid reduced the magnitude of coordinates applied in a least squares solution and so reduced the effect of round-off caused either by (a) human geodetic computers (War Department, 1944) using logarithm tables in a manual calculation, or (b) by the limited floating-point precision of the 16- and 32-bit processors available in computers during the 1970s through to the late 20th century when 64-bit processors improved, but did not entirely resolve, the round-off issue (Toms & Kellough, 2006).

ii. A centroid produces statistics for the derived parameters that appear 'better' (i.e., smaller in magnitude) than a method without a centroid due to an arbitrary pre-transformation of the source coordinates that is not properly reflected in the functional and stochastic models.

The US National Geospatial-Intelligence Agency website (http://earth-info.nga.mil) describes a variant of Badekas's equation that includes the comment: *"When the initial point of the local datum (U',V',W') is not provided, assume values of (0,0,0)"*. To us, this seemingly unusual advice reflects the relatively common problem where the MB centroidal vector is not always reported with the computed seven parameters (cf. Burford, 1985; Pearse & Crook, 1997).

A centroidal method may use any coordinate as the pivot point: the coordinate may be completely arbitrary (e.g., IOGP, 2018); the mean of the given control points (e.g., Harvey, 1985); or Molodensky's 'initial point' (Molodensky et al., 1962, p.13). A centroid may also be a parameter in the functional model (Leick & van Gelder, 1975) though this is not straightforward. For example, a simple functional model might be:

$$\mathbf{x} = s\mathbf{R}\mathbf{u}_{\mathbf{0}} + s\mathbf{R}(\mathbf{u} - \mathbf{u}_{\mathbf{0}}) + \mathbf{t}_{x} \qquad (13)$$

where, on the right hand side, the first term is the scaled and rotated centroidal vector, the second is the scaled and rotated local vector, and the third, $\mathbf{t}_{\mathbf{x}}$, represents the translation in the X-system of the scaled and rotated U-system. The first and second terms are linearly dependent and, when simplified, Eq. (13) reduces to the BW method reflecting Soler's (1976) comment quoted earlier in this paper.

However, recognising that the sum of the first and third terms in Eq. (13) is the centroidal vector in the X-system, $\mathbf{x}_0$, Eq. (13) may be rewritten as:

$$\mathbf{x} = s\mathbf{R}(\mathbf{u} - \mathbf{u_o}) + \mathbf{x_o} \qquad (14)$$

Equation (14) resolves the centroidal ambiguity of the MB variants by requiring the centroidal vector be 'known' in both the U- and X-systems. This solution, which is a mix of parameters and observations, implies the use of general or combined least squares (e.g., Mikhail & Ackermann, 1976), also known as the Gauss-Helmert Method (GHM), in contrast to the Gauss-Markov Method (GMM) of parametric least squares estimation (Chang, 2015). The consequence of Eq. (14) is that this MB variant requires 13 parameters for forward and same-formula reversal: the standard seven parameters and a given centroidal vector defined in both the U- and X-systems.

3.2. REVERSIBILITY

To be useful practically, a single transformation method and parameter set between two CRSs must either be reversible, or be reported with separate sets of forward and reverse parameters and equations (cf. Ruffhead, 2018). Here, reversible is considered to have two forms: (i) a rigorous mathematical form where the algorithm of the forward method is inverted and the defined transformation parameters are applied to the reverse calculation; and (ii) a simplified form, defined earlier as 'same-formula', where the forward algorithm and parameters are used and only the sign of each parameter is changed. Note that the second form is not exact, but instead a computational convenience, not general in scope, and which requires rotation and scale parameters of small magnitude to support the linear assumptions implied in the method. This issue affects BW and MB equally.

Mathematical inversion may be applied to any rigorously derived 3D similarity transformation no matter the magnitude of the parameters (e.g, Grossman, 1980, Ch.7.3). In contrast, same-formula reversal requires the two CRSs be closely aligned so as to reduce the effect of the correlated rotations and linear assumptions (Malys, 1988). Reversal of the EPSG's MB variant is defined using the same-formula method (IOGP, 2018, p.135) and this is used generally in commercial-off-the-shelf (COTS) software for the reverse transformation; very few using the rigorous inverse (Reit, 2009).

It is acknowledged that a defect in its functional model does not preclude MB from being rigorously reversible. However, the dual state of the centroidal vector does limit the same-formula reversibility of MB and its variants where, from Eq. (10), the error increases proportionally to the size of the scale and rotation parameters and for an increasingly large translation vector. Krakiwsky and Thompson (1974) identified this ambiguity and resolved it by requiring the 'initial point' be defined in a CRS that is parallel to the ATS.

Like Eq. (4), Eq. (14) is rigorously, but not same-formula, reversible. However, a same-formula reversal may be computed with a rearrangement of the coordinate vectors, $\mathbf{u}$, $\mathbf{u_0}$, $\mathbf{x}$ and $\mathbf{x_0}$, based on the additive relationships in Eq. (13) and the linear assumptions of the rotation matrix, so that:

$$\mathbf{u} = s\mathbf{R}(\mathbf{x} - \mathbf{x_o}) + \mathbf{u_o} \qquad (15)$$

where *s* and **R** here reflect the reversed sign of the forward parameters, and noting that Eq. (15) differs here from Eq. (14) to illustrate the reverse transformation. In practice, a same-formula reversal would apply Eq. (14) with a reversal of sign of the scale difference and rotation parameters and swapping of the centroids. Eq. (14) with Eq. (15) provides an alternative forward and reverse method for a centroid-based similarity transformation using the same-formula method with a simplified rotation matrix. We do not propose, however, that this variant ever be used in a practical coordinate transformation, except perhaps as a curiosity, as it is redundant under the uniqueness theorem of Modenov & Parkhomenko (1966) and requires special attention to its stochastic model.

4. THE STOCHASTIC MODELS

Many published works that describe a datum transformation method either do not discuss the relevant stochastic model or provide only limited comment (e.g., Appelbaum, 1982; Badekas, 1969; Veis, 1960; Ruffhead, 2019). For works that later apply these methods to a practical problem, there is also little discussion of the stochastic model that drives the resulting statistics from which much may be interpreted (e.g., Annan et al., 2016; Kutoglu et al., 2002; Varga et al., 2015). This is in contrast to Blewitt's (1998, Ch.6) work demonstrating equivalence between the functional and stochastic models.

The continued promotion and use of MB for its 'apparent' superior stochastic outcomes appears [to us] unusual, when it has been shown that the MB and BW transformations are equivalent in a forward transformation (e.g., Leick & van Gelder, 1975; Soler, 1976). That is, there is no difference in the coordinates or their variance-covariance information, and further, that one set of transformation parameters, and related statistics, may be derived directly from the other. It is not clear [to us] why this result has since not been appreciated more widely.

MB variants generally apply a centroidal vector as a given absolute value without an associated *a priori* uncertainty. Furthermore, differencing an observed coordinate with a centroid induces a 100% correlation of the network with the centroid. This has a major effect on the propagation of uncertainty in the least squares parameter estimation, effectively removing the Earth's radius vector from the error propagation process. The use of a centroid, noting from Section 3 here that the centroid has to be given in both CRSs, causes an arbitrary shift of the rotational centre of the two coordinate sets from the geocentre to the centroid. This is the sole source of the improved correlations for the MB methods. The centroid causes the MB translations to become decorrelated from the rotation parameters, and for the related correlations to go to zero, while having no effect on the magnitude of the computed scale and rotation parameters or their correlations. That is, basic information is lost. The effect of this mathematical 'sleight of hand' will be demonstrated later.

The limited stochastic model of the MB leads to biased error estimates, where the uncertainty and correlations of the parameters are underestimated if compared directly, that is, like for like, to the BW solution. This is while, following from Eqs. (8) and (9), the uncertainty of the scale and rotation parameters are the same in both methods assuming a geocentric reference frame. An apparent [to us] lack of appreciation for the different reference frames of the MB and BW statistics has led some to inappropriate confidence in the veracity of the transformation parameters and related statistics.

A coordinate transformation, where the functional model is a mix of observations and parameters, implies the use of the GHM in place of the GMM, though the latter may be extended to include additional observables or parameters. The stochastic model for a GHM solution may include the variance/covariance (VCV) information for both coordinate sets. A GMM solution, recognising only the target CRS coordinates as observables, causes the error that exists in both sets of coordinates to be lumped into a single residual (e.g., Annan et al., 2016; Turgut, 2010), which limits any effective analysis. Historically, the use of VCV information, while well understood, has been complicated by a lack of available data though the reasoned use of available empirical evidence, and variance estimation methods have improved outcomes seeking high accuracy (e.g., Badekas, 1969; Malys, 1988).

The functional model asserts a form of stochastic constraint from its definition of space. For example, a similarity transformation correlates all included points under the assumption of orthogonality and uniform scale, and the similarity refers to both the transformation parameters and their VCV and that, for a given set of inputs, there exist a unique set of parameters (Leick & van Gelder, 1975). This implies that only a single form of similarity transformation is required; that any alternative method that may be used to gain a particular insight (e.g., Badekas, 1969; Krakiwsky & Thomson, 1974) can be converted exactly to the general Helmert similarity result.

5. ILLUSTRATIVE TRANSFORMATIONS

To demonstrate empirically where the MB and BW methods are related or differ, a simple simulated data set is used, consisting of six points on a one degree of latitude and two degrees of longitude graticule centred on [S33° 30', E124° 00'] to deliberately induce high parameter correlations. To provide a point of 'truth' for the derived parameters, the six points in the U-system were transformed to the X-system using a pre-defined transformation parameter set where: $\mathbf{t}_{BW}$ = [+80.0 m, -90.0 m, +100.0 m], [ru, rv, rw] =[+0.3", -0.4", +0.5"] following the coordinate frame rotation convention, and *ds* = +0.25 ppm for a total scale factor, *s* = 1.000025, representing a set of parameters within the expected bounds of use (cf. http://www.epsg-registry.org). That is, we have constructed a simulation that is same-formula reversible (cf. Section 3.2). To this nominally 'perfect' data set, random error within ±0.10 m was added to create a 'noisy' data set and provide significant residuals and statistics.

Three example solutions were computed to illustrate the issues discussed here: (i) a test of both methods ability to recover the original parameters using the ‘perfect’ data set; (ii) the same test but with the 'noisy' data set; and (iii) a BW solution using a centroid known in both CRSs. The solutions used the GHM with 29 degrees of freedom and a per-ordinate uncorrelated uncertainty of ±0.025 m (one-sigma) for both U- and X-coordinate sets.

5.1. INITIAL SOLUTIONS

Initially, BW and MB solutions were computed using the same 'perfect' coordinate set for both.

Table 1: MB and BW solutions for 'perfect' data

7-Par	MB parameters	StdDev	$\mathbf{t}_{MB}$->$\mathbf{t}_{BW}$	BW parameters	StdDev
tx	83.166 m	±0.014 m	80.000 m	80.000 m	±1.216 m
ty	-86.772 m	±0.014 m	-90.000 m	-90.000 m	±1.220 m
tz	98.479 m	±0.014 m	100.000 m	100.000 m	±1.486 m
ru	0.300"	±0.048"		0.300"	±0.048"
rv	-0.400"	±0.041"		-0.400"	±0.041"
rw	0.500"	±0.037"		0.500"	±0.037"
ds	0.250 ppm	±0.154 ppm		0.250 ppm	±0.154 ppm

Table 2: MB correlation matrix (bold text: correlation>|0.5|)

Correlation	tx	ty	tz	ru	rv	rw	ds
tx	**1.00**	0.00	0.00	0.00	0.00	0.00	0.00
ty	0.00	**1.00**	0.00	0.00	0.00	0.00	0.00
tz	0.00	0.00	**1.00**	0.00	0.00	0.00	0.00
ru	0.00	0.00	0.00	**1.00**	0.40	-0.08	0.00
rv	0.00	0.00	0.00	0.40	**1.00**	0.13	0.00
rw	0.00	0.00	0.00	-0.08	0.13	**1.00**	0.00
ds	0.00	0.00	0.00	0.00	0.00	0.00	**1.00**

Table 3: BW correlation matrix (bold text: correlation>|0.5|)

Correlation	tx	ty	tz	ru	rv	rw	ds
tx	**1.00**	-0.01	-0.25	-0.18	**-0.66**	**-0.73**	0.38
ty	-0.01	**1.00**	0.38	**0.71**	0.21	-0.49	**-0.56**
tz	-0.25	0.38	1.00	**0.86**	**0.68**	0.00	0.36
ru	-0.18	**0.71**	**0.86**	1.00	0.40	-0.08	0.00
rv	**-0.66**	0.21	**0.68**	0.40	**1.00**	0.13	0.00
rw	**-0.73**	-0.49	0.00	-0.08	0.13	**1.00**	0.00
ds	0.38	**-0.56**	0.36	0.00	0.00	0.00	**1.00**

From Table 1, both solutions recovered the pre-defined transformation parameters to better than the three decimal places quoted, even given the poor geometry resulting from the small coordinate domain. Given this near-perfect fit, the *a posteriori* variance factor (APFV) is 0.000. The parameter uncertainties differ only in the translations; the BW is the greater by two orders of magnitude. This noted, on the application of Eq. (12), the MB translations were converted exactly to the BW values emphasising the discrepancy between MB and BW stochastic modelling and error propagation.

Tables 2 and 3 show the correlation matrices, following the order of the parameters. The MB correlations are less than those of BW with the absolute value of all correlations less than 0.50. Three points are of note however: (i) the correlations between the scale and rotation parameters are the same between the two methods; (ii) the MB translations are decorrelated from the other parameters; and (iii) the overall difference in the correlations has no consequence on the computed parameters.

5.2. SOLUTIONS WITH RANDOM ERROR INCLUDED

A second BW and MB solution was computed using the 'noisy' coordinate set with all other parameters remaining the same as above.

Table 4: MB and BW solution for ±0.1 m random error

7-par	MB parameters	StdDev	t_{MB}->t_{BW}	BW parameters	StdDev
tx	83.125	±0.014 m	76.592 m	76.592	±1.216 m
ty	-86.771	±0.014 m	-93.972 m	-93.972	±1.220 m
tz	98.446	±0.014 m	99.112 m	99.112	±1.486 m
ru	0.249"	±0.048"		0.249"	±0.048"
rv	-0.411"	±0.041"		-0.411"	±0.041"
rw	0.681"	±0.037"		0.681"	±0.037"
ds	0.361 ppm	±0.154 ppm		0.361 ppm	±0.154 ppm

Table 4 shows that neither MB nor BW solution recovers the transformation parameters exactly, but they are within a total error vector of approximately 8 metres and both have an APVF of 0.991. The MB translations again convert exactly to the BW translations as per Eq. (12), while the scale and rotation uncertainties and both correlation matrices remain unchanged. However, while the MB and BW translation uncertainties also remain unchanged from the initial solution (Section 5.1), the equivalence of the MB and BW translations from Eq. (12), and that the BW uncertainties are generally within the 3-sigma level, demonstrate the MB's decoupling from the network's stochastic model.

5.3. BURSA-WOLF SOLUTION WITH EQUIVALENT CENTROIDS

To demonstrate the effect of the arbitrary pre-transformation used by MB, a BW solution was computed using the 'noisy' data set (Section 5.2) and equivalent centroids ($\mathbf{u_0}$ and $\mathbf{x_0}$) of the proposed MB variant (Eq. 14). In this case, the adopted centroid is the mean of the U- and X-system coordinates rounded to three decimal places (equivalent to millimetre), which is the likely cause of the 1 mm Z-translation seen in Table 5.

Table 5: BW solution using equivalent centroids

7-par	BW parameters	StdDev
tx	0.000 m	±0.014 m
ty	0.000 m	±0.014 m
tz	-0.001 m	±0.014 m
ru	0.249"	±0.048"
rv	-0.411"	±0.041"
rw	0.681"	±0.037"
ds	0.361 ppm	±0.154 ppm

Table 6: BW (equivalent centroids) correlation matrix

Correlation	tx	ty	tz	ru	rv	rw	ds
tx	**1.00**	0.00	0.00	0.00	0.00	0.00	0.00
ty	0.00	**1.00**	0.00	0.00	0.00	0.00	0.00
tz	0.00	0.00	**1.00**	0.00	0.00	0.00	0.00
ru	0.00	0.00	0.00	**1.00**	0.40	-0.08	0.00
rv	0.00	0.00	0.00	0.40	**1.00**	0.13	0.00
rw	0.00	0.00	0.00	-0.08	0.13	**1.00**	0.00
ds	0.00	0.00	0.00	0.00	0.00	0.00	**1.00**

Tables 5 and 6 demonstrate how, by an arbitrary removal of the geocentre-centroidal vectors from both coordinate sets, the BW emulates the MB solution producing the same scale and rotation parameters and VCV information (refer to Table 4 in Section 5.2). In particular, it can be understood how the MB correlations may appear better than the BW; they do not refer to the geocentre but to the centroid (cf. Harvey, 1986). This example also demonstrates that the MB solution is not reliant on one but two centroids or, more correctly, a single centroid referenced in both CRSs.

These three examples, using a 1° by 2° area on the ellipsoid where high parameter correlations are created deliberately, demonstrate the equivalent performance of the BW and MB methods in the forward direction, and that there is no basis for the claim by Lott (2011) that MB performance is superior for sub-continental areas.

6. CONCLUSION

This paper has reviewed the connection between two simplified 3D similarity transformation methods: the Burša-Wolf (BW) and Molodensky-Badekas (MB). It has been shown by Leick and van Gelder (1974), and demonstrated again here, that the two methods are equivalent in the forward transformation and that the promoted qualities of MB, as a stochastically superior and independent similarity transformation, are simply a consequence of a mathematical 'sleight of hand'.

The problem of same-formula reversibility affecting MB flows from it being referenced to an equivalent centroid in both the U- and X-systems, not just the ambiguous U-system centroid of Badekas's forward solution. The error induced by this ambiguity in a same-formula reverse transformation is directly related to the difference in translation, scale and rotation between the U- and X-systems. To resolve this, we have proposed an MB variant, including a method providing for same-formula reversal, using two centroids. Recognising that use of same-formula reversal is no longer warranted given recent increases in computing power, it is not proposed that this variant ever be used in practical coordinate transformations, except perhaps as a curiosity, as it is redundant under the uniqueness theorem for similarity transformations.

Using a simulated data set over a small area to deliberately induce high parameter correlations, this paper has demonstrated that the known equivalence between BW and MB allows the parameters of one to be computed directly from the other. It has also shown that MB statistics, while valid for their model, are the consequence of the use of a centroid that, unlike the BW, prevents their propagation to the geocentric reference frame of the subject coordinates, leading some to mistakenly perceive improved statistics. Acknowledging that MB does not add new information to the similarity transformation, and its statistics are prone to mis-interpretation, MB does not provide any tangible benefit relative to the use of BW.

For various reasons, practical and theoretical, MB has not achieved the widespread application in COTS survey software of the BW. Accepting that COTS software predominately uses the same-formula method for reversal, we therefore suggest that published values of existing MB solutions be made accessible, and reversible, by converting the parameter sets to the equivalent BW form.

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DATA AVAILABILITY STATEMENT

All data generated or used during the study are available from the corresponding author by request.

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