A method to validate gravimetric geoid computation software based on Stokes's integral formula

W. E. Featherstone

School of Surveying and Land Information, Curtin University of Technology, GPO Box U1987, Perth WA 6001, Australia.

tel: +61 9 351 2734

fax: +61 9 351 2703

email: Featherstone_WE@cc.curtin.edu.au

J. G. Olliver

Department of Earth Sciences, Oxford University, Parks Road, Oxford OX1 3PR, England.

tel: +44 1865 274581

fax: +44 1865 272072

email: Joe.Olliver@earth.ox.ac.uk

Abstract — A method is presented with which to verify that the computer software used to compute a gravimetric geoid is capable of producing the correct results, assuming accurate input data. The Stokes, gravimetric terrain correction and indirect effect formulae are integrated analytically after applying a transformation to surface spherical coordinates centred on each computation point. These analytical results can be compared with those from geoid computation software using constant gravity data in order to verify its integrity. Results of tests conducted with geoid computation software are presented which illustrate the need for integration weighting factors, especially for those compartments close to the computation point.

Keywords — geoid determination, Stokes's integral, terrain corrections.

Introduction

Many different computer software packages have been developed over the years with which to evaluate the geoid using Stokes's integral or a modification thereof. Theoretically, all geoid software should produce identical results, given the same input gravity data, irrespective of the computational approach taken. However, this may not always be the case. If the geoid computation software is incorrect, perhaps because of the numerical methods used, so will be the final geoid solution. Therefore, it is important to prove that the algorithms utilised can indeed produce the correct geoid heights. Unfortunately however, not all authors demonstrate that the validity of their software has been tested prior to its use. Instead, GPS in conjunction with spirit levelling is used almost excusively to validate the gravimetric geoid solution, which validates the software by implication only.

The validity of the numerical integration of Stokes's formula has been discussed by de Min (1994 and 1995), whereas the spectral solution of Stokes's integral has been investigated by Tziavos (1996), where gravity anomalies implied by a global geopotential model are used to determine the geoid and thus allow a comparison of the corresponding geoid heights. Alternatively, the geoid software can be validated by comparing geoid results from the software using constant gravity data to those of pure mathematical integration (the analytical evaluation of Stokes's integral), which is the objective of this discussion. Such validation enables the software to be eliminated as a source of error, thus allowing identification of other error sources, such as gravity data preparation for example.

This short treatise presents a method to test the validity of numerical gravimetric geoid computation software. Stokes's formula is integrated analytically by assuming that the input gravity data are constant. This allows an exact evaluation of the geoid height for any integration area. The same constant gravity anomalies are then input to the geoid software for the same area. Therefore, a simple comparison with the analytical solution can be used to validate the software. As such, it is strictly a mathematical test of a numerical method, and not a test of geoid computation techniques nor the geoid results themselves.

Integration of Stokes's formula

The geoid height (N) can be computed from terrestrial gravity data using the classical Stokes formula

(see Heiskanen and Moritz, 1967, eq. 2-163b)

$$N = \frac{r}{4\pi\gamma} \int_\sigma S(\psi)\, \Delta g \, d\sigma \quad , \tag{1}$$

where r is the mean earth radius, γ is normal gravity on the reference ellipsoid, Δg are the gravity anomalies, $d\sigma$ is an element of surface area on the sphere, $S(\psi)$ is Stokes's integration kernel (*ibid*, eq. 2-164)

$$S(\psi) = \csc\left(\frac{\psi}{2}\right) - 6\sin\left(\frac{\psi}{2}\right) + 1 - 5\cos\psi - \\ -3\cos\psi\,\ln\left\{\sin\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)\right\} \quad , \tag{2}$$

and the surface spherical radius (ψ) between two points on the sphere is given by

$$\cos\psi = \sin\phi\sin\phi' + \cos\phi\cos\phi'\cos(\lambda' - \lambda) \quad .$$

Firstly, the integration element $d\sigma$ in equation (1) is transformed to integration elements expressed in terms of surface spherical coordinates $(d\psi, d\alpha)$, whose origin is at the geoid computation point, and where α is the azimuth, such that

$$\int_\sigma d\sigma = \int_\alpha \int_\psi \sin\psi \, d\psi \, d\alpha \tag{3}$$

for $0 \leq \psi \leq \pi$ and $0 \leq \alpha \leq 2\pi$. This is equivalent to using spherical polar coordinates, centered on each computation point instead of the north pole.

When the coordinate transformation (3) is applied to equation (1), this gives

$$N = \frac{r}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} S(\psi)\sin\psi \, \Delta g(\psi, \alpha) \, d\psi \, d\alpha \quad . \tag{4}$$

The coordinate transformation also removes the singularity in Stokes's kernel at the computation point ($\psi = 0$). However, the integral in equation (4) is almost impossible to solve analytically when using observed gravity anomalies, because the Earth's gravity field is a complicated (and unknown) function of both ψ and α.

This restriction can be overcome by assuming that the gravity anomalies are constant, and therefore independent of the variables ψ and α. As such, the constant Δg can be moved outside the integrals in equation (4) to yield

$$N = \frac{r\,\Delta g}{4\pi\gamma} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} S(\psi)\sin\psi \, d\psi \, d\alpha \quad . \tag{5}$$

The analytical integration is now simplified, as Stokes's kernel is an isotropic function of ψ only. The first integration with respect to α is easily performed ($\int_0^{2\pi} d\alpha = 2\pi$) to leave

$$N = c \int_{\psi} S(\psi) \sin\psi \, d\psi \ , \tag{6}$$

where the constant term $c = (r\Delta g)/(2\gamma)$ has been introduced for brevity.

The integrand in equation (6) is similar to the $F(\psi)$ function of Heiskanen and Moritz (1967, p.96), which has no singularities and is fully integrable anywhere in the region $0 \leq \psi \leq \pi$. As such, analytical integration can be carried out from the computation point ($\psi = 0$) to any surface spherical radius ψ_o, provided that $\psi_o \leq \pi$.

In order to do this, the closed expression for Stokes's kernel (2) is substituted into equation (6), then expanded using standard trigonometric identities for double angles, to give the following definite integral for the analytically defined geoid height

$$\begin{aligned} N = c \int_0^{\psi_o} \Bigg[& 2\cos\left(\frac{\psi}{2}\right) - 12\sin^2\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right) + \\ & + \sin\psi - \frac{5}{2}\sin 2\psi - \\ & - \frac{3}{2}\sin 2\psi \ \ln\left\{\sin\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)\right\}\Bigg] d\psi \ . \end{aligned} \tag{7}$$

Equation (7) is divided into five separate definite integrals and integration performed between 0 and ψ_o. The results are

$$N_1 = c \int_0^{\psi_o} 2\cos\left(\frac{\psi}{2}\right) \ d\psi = \left[4c\sin(\frac{\psi}{2})\right]_0^{\psi_o} \ , \tag{8}$$

$$\begin{aligned} N_2 &= c \int_0^{\psi_o} -12\sin^2\left(\frac{\psi}{2}\right)\cos\left(\frac{\psi}{2}\right) \ d\psi = \\ &= \left[-8c\sin^3(\frac{\psi}{2})\right]_0^{\psi_o} \ , \end{aligned} \tag{9}$$

$$N_3 = c \int_0^{\psi_o} \sin\psi d\psi = \Big[-c\cos\psi\Big]_0^{\psi_o} \ , \tag{10}$$

$$N_4 = c \int_0^{\psi_o} -\frac{5}{2}\sin 2\psi d\psi = \left[\frac{5c}{4}\cos 2\psi\right]_0^{\psi_o} \ , \tag{11}$$

$$\begin{aligned} N_5 &= c \int_0^{\psi_o} -\frac{3}{2}\sin 2\psi \ \ln\left\{\sin\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)\right\} d\psi = \\ &= c\left[-\frac{3}{2}\sin^2\psi \ \ln\left\{\sin\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)\right\} - \right. \end{aligned}$$

$$-3\sin^4\left(\frac{\psi}{2}\right) +2\sin^3\left(\frac{\psi}{2}\right) + 3\sin^2\left(\frac{\psi}{2}\right)\Bigg]_0^{\psi_o} \quad . \tag{12}$$

The integrals (8) to (12) are then recombined ($N = \Sigma_{i=1}^{5} N_i$) to give

$$N = c\left[-\frac{3}{2}\sin^2\psi \;\ln\left\{\sin\left(\frac{\psi}{2}\right) + \sin^2\left(\frac{\psi}{2}\right)\right\} + \right.$$
$$+4\sin\left(\frac{\psi}{2}\right) + 3\sin^2\left(\frac{\psi}{2}\right) - 6\sin^3\left(\frac{\psi}{2}\right) -$$
$$\left.-3\sin^4\left(\frac{\psi}{2}\right) - \cos\psi + \frac{5}{4}\cos 2\psi\right]_0^{\psi_o} \quad . \tag{13}$$

Differentiation of equation (13) produces equation (7), thus proving the correctness of this integration. Equation (13) also agrees with the result in Lambert and Darling (1936 p.103), where it was derived in a different context. On inserting the integration limits, the synthetic geoid height from analytical integration for $0 \leq \psi \leq \psi_o$ with constant gravity data becomes

$$N = \frac{r\Delta g}{8\gamma}\left(-6\sin^2\psi_o \;\ln\left\{\sin\left(\frac{\psi_o}{2}\right) + \sin^2\left(\frac{\psi_o}{2}\right)\right\} + \right.$$
$$+16\sin\left(\frac{\psi_o}{2}\right) + 12\sin^2\left(\frac{\psi_o}{2}\right) - 24\sin^3\left(\frac{\psi_o}{2}\right) -$$
$$\left.-12\sin^4\left(\frac{\psi_o}{2}\right) - 4\cos\psi_o + 5\cos 2\psi_o - 1\right) \quad . \tag{14}$$

Equation (14) can be evaluated exactly for any surface spherical radius ($0 \leq \psi_o \leq \pi$), and for any constant values of gravity anomalies (see Figure 1). Therefore these analytically derived geoid values can be compared with the output of any geoid computation software using the corresponding constant gravity anomalies, in order to assess the software's reliability.

Figure 1: *The analytically integrated Stokes formula with increasing surface spherical radius for* Δg*=100mgal*

Integration of the gravimetric terrain correction and indirect effect formulae

A similar approach to that taken for the Stokes integration is taken for the formulae used to evaluate the gravimetric terrain correction and primary indirect effect on the geoid.

The second-order topographic effect on gravity anomalies (Δg_t) is deduced from Moritz (1968) as

$$\Delta g_t = \frac{G\rho r^2}{2}\int_\sigma \frac{(H' - H)^2}{l^3}d\sigma -$$

$$-\frac{3G\rho r^2}{8}\int_\sigma \frac{(H'-H)^4}{l^5}d\sigma \ , \tag{15}$$

where G is the Newtonian gravitational constant, ρ is the topographic density, which is assumed constant at 2670kgm^{-3}, H is the height of the computation point, H' is the height of the roving point, and l is the direct separation between the computation and roving points.

Using the cosine rule, this separation is

$$l = \sqrt{r^2 + r'^2 - 2rr'\cos\psi} \ . \tag{16}$$

At the surface of the Earth, r' can be assumed equal to r with an accuracy of 0.126% for the maximum $H \simeq 8$km. Using this planar approximation and trigonometric identities, equation (16) reduces to

$$l = 2r\sin\left(\frac{\psi}{2}\right) \ . \tag{17}$$

Equation (17) is inserted into equation (15), together with the coordinate transformation (3), then standard trigonometric identities for double angles are applied to produce

$$\Delta g_t = \frac{G\rho}{8r}\int_0^{2\pi}\int_0^{\pi} \frac{(H'(\alpha,\psi)-H)^2\cos\left(\frac{\psi}{2}\right)}{\sin^2\left(\frac{\psi}{2}\right)}d\psi d\alpha -$$

$$-\frac{3G\rho}{128r^3}\int_0^{2\pi}\int_0^{\pi} \frac{(H'(\alpha,\psi)-H)^4\cos\left(\frac{\psi}{2}\right)}{\sin^4\left(\frac{\psi}{2}\right)}d\psi d\alpha \ . \tag{18}$$

As with the gravity field, the Earth's topography is a complicated (and unknown) function of both ψ and α. Once again, this is overcome by assuming that the term $(H'(\psi,\alpha)-H)$ is a non-zero constant for all ψ and α. As H is always a constant at each computation point, this is equivalent to assuming that H' is constant and not equal to H. As such, the constant $(H'-H)$ can be moved outside the integrals in equation (18) and integration performed with respect to α. This yields

$$\Delta g_t = \frac{\pi G\rho(H'-H)^2}{4r}\int_{\psi=0}^{\psi_o} \csc\left(\frac{\psi}{2}\right)\cot\left(\frac{\psi}{2}\right)d\psi -$$

$$-\frac{3\pi G\rho(H'-H)^4}{64r^3}\int_{\psi=0}^{\psi_o} \csc^3\left(\frac{\psi}{2}\right)\cot\left(\frac{\psi}{2}\right)d\psi \ . \tag{19}$$

However, a weak singularity remains in the integrand of equation (19) at $\psi = 0$. This is avoided in practice as $H' = H$ at each computation point and alternative approaches are used to compute this innermost zone effect; see, for example, Schwarz *et al.* (1990), Klose and Ilk (1994) or van Gysen (1995).

Therefore, the integral of equation (19) is determined between the limits $\psi_i \leq \psi \leq \psi_o$, where ψ_i is the equivalent radius of the innermost zone, as

$$\Delta g_t = \frac{\pi G\rho(H'-H)^2}{2r}\left\{\csc\left(\frac{\psi_i}{2}\right)-\csc\left(\frac{\psi_o}{2}\right)\right\}-$$
$$-\frac{\pi G\rho(H'-H)^4}{32r^3}\left\{\csc^3\left(\frac{\psi_i}{2}\right)-\csc^3\left(\frac{\psi_o}{2}\right)\right\} \quad . \tag{20}$$

The value of Δg_t for some representative values of $(H-H')$, ψ_0 and ρ are shown in Figure 2.

Figure 2: *The analytically integrated terrain correction formula with increasing surface spherical radius for* $(H-H')$*=100m,* ψ_i*=0.01°* *and* ρ*=2670kgm*$^{-3}$

Exactly the same approach is taken for the analytical determination of Wichiencharoen's (1982) second-order indirect effect on the geoid (N_i). The indirect effect formula is

$$N_i = -\frac{\pi G\rho H^2}{\gamma} - \frac{G\rho r^2}{6\gamma}\int_\sigma \frac{H'^3-H^3}{l^3}\,d\sigma +$$
$$+\frac{3G\rho r^2}{40\gamma}\int_\sigma \frac{H'^5-H^5}{l^5}\,d\sigma \tag{21}$$

where all terms are as defined earlier. This produces the following result

$$N_i = -\frac{\pi G\rho H^2}{\gamma}-$$
$$-\frac{\pi G\rho(H'^3-H^3)}{6\gamma r}\left\{\csc\left(\frac{\psi_i}{2}\right)-\csc\left(\frac{\psi_o}{2}\right)\right\}+$$
$$+\frac{\pi G\rho(H'^5-H^5)}{160\gamma r^3}\left\{\csc^3\left(\frac{\psi_i}{2}\right)-\csc^3\left(\frac{\psi_o}{2}\right)\right\} \quad . \tag{22}$$

Figure 3 shows the variation of N_i with increasing ψ_o, which are computed from equation (22).

Figure 3: *The analytically integrated indirect effect formula with increasing surface spherical radius for* H'*=1001m,* H*=1101m,* ψ_i*=0.01°* *and* ρ*=2670kgm*$^{-3}$

Comparisons with geoid software

The geoid computation software tested here was developed between 1988 and 1992 (Featherstone, 1992), and was used to compute the most recent gravimetric geoid of the British Isles (Featherstone and Olliver, 1994). The source code is now also operational at Curtin University of Technology, Perth, Western Australia.

Figure 4 shows the difference between the geoid height derived from analytical integration (equation 14) and the geoid height produced by the software using constant 100mgal gravity anomlies on a $\sim$ 4km grid. The comparisons are presented for the point (32°S, 115°E) out to an integration radius of $\psi_o = 4°$. The innermost zone contribution (δn) was computed using the relationship given by Heiskanen and Moritz (1967, p.122) and Strang van Hees (1990).

Figure 4: *The difference between geoid heights using analytical integration (truth) and synthetic geoid heights using the software (test) with 100mgal gravity anomalies, (a) without and (b) with the integration weighting factor* W_s

The small variations for each comparison (a) and (b) in Figure 4 are due to discretisation-type errors because this software uses gravity data from a 2' latitude by 4' longitude geographical grid and has been compared to equation (14), which is expressed in terms of spherical polar coordinates. This effect, and hence the difference shown in Figure 4, is reduced when using a denser grid of gravity anomalies, which reduces the discretisation error (cf. Featherstone *et al.*, 1996). Moreover, there is significant disagreement evident in Figure 4(a) between the analytical and software-generated geoid heights, where the synthetic geoid height has been underestimated by the software. This difference becomes progressively larger with increasing integration radius, especially for small values of ψ, due to the accumulation of numerical integration errors from each compartment.

The magnitude of the integration error per compartment is largest close to the computation point. This is because Stokes's kernel varies most rapidly in this region. Therefore, the value of Stokes's kernel which was adopted at the centre of each compartment is unrepresentative of the true mean value for these compartments, hence causing an underestimate of the geoid height. This integration error can be reduced in Figure 4(b) by using an integration weighting factor in those compartments close to the computation point. This is pertinent to fast Fourier transform (FFT) geoid computations when a regular grid of gravity anomalies are used over the entire computation area (Schwarz *et al.*, 1990).

Other approaches to the reduction of this discretisation error are: to increase the integration step by using compartments whose size is a function of ψ, such that the compartments become progressively smaller on moving closer to the compuation point; to use the integrated Stokes function (equation 13) for those compartments close to the computation point instead of the central, non-integrated value; or, to use an analytical function for the inner zone together with discrete gravity data (cf. Olliver, 1980). However, this discussion will continue to concentrate specifically upon the use of a regula r grid of gravity data.

Integration weighting factors

As the csc $(\frac{\psi}{2})$ term is dominant in Stokes's kernel close to the computation point, Strang van Hees (1990) derived a weighting factor (W_s) for Stokes's integral using the planar approximation, which is valid close to the computation point. This is

$$W_s = \frac{\overline{\psi}}{y} \ln \left\{ \frac{2\overline{\psi} + y}{2\overline{\psi} - y} \right\} \quad , \tag{23}$$

where $\overline{\psi}$ is the surface spherical radius at the centre of each compartment, and y is the compartment width. As expected, the numerical value of equation (23) is largest for those compartments closest to the computation point. For example, with $y = 0.1°$, $W_s = 1.039721$ at $\overline{\psi} = 0.15°$, which decreases to $W_s = 1.000208$ at $\overline{\psi} = 2°$. Alternatively, equation (23) can be rewritten in a series form by using standard logarithmic identities, which gives

$$W_s = 1 + \frac{1}{3} \left(\frac{y}{2\overline{\psi}} \right)^2 + ... \frac{1}{2n-1} \left(\frac{y}{2\overline{\psi}} \right)^{2n-2} \quad . \tag{24}$$

Exactly the same approach is used for the linear component of the topographic and indirect effect formulae in equations (15) and (21). As these integration kernels are identical, so are their respective weighting factors (W_{ti}). Using Strang van Hees's (*ibid.*) planarisation, this gives

$$W_{ti} = \frac{\overline{\psi}^3 [(\overline{\psi} + y/2)^2 - (\overline{\psi} - y/2)^2]}{2y(\overline{\psi} + y/2)^2 (\overline{\psi} - y/2)^2} \quad . \tag{25}$$

This integration weighting factor for the topographic and indirect effects is relatively large close to the computation point. Again, with $y = 0.1°$, $W_{ti} = 1.265625$ at $\overline{\psi} = 0.15°$, which decreases to $W_{ti} = 1.001251$ at $\overline{\psi} = 2°$. This is due to the dominant inverse distance cubed kernel. Furthermore, this illustrates the importantance of using a high resolution digital terrain model in practical geoid computation as this reduces the discretisation error. Alternatively, and as with the geoid computation, the terrain compartments can be reduced in size or the integrated kernels in equations (20) and (22) can be utilised close to each computation point.

These weighting factors are easily implemented by computing the kernel at the centre of each compartment, then multiplying this by the appropriate integration weighting factor (23) or (25).

Figure 4(b) shows the difference between analytical and synthetic geoid heights when including the integration weighting factor (23) for a ~4km grid. In this instance, the software's solution with constant gravity data for $\psi_o = 2°$ is now only underestimated by ~0.032% when compared to analytical integration (Figure 1). This is negligible in relation to the error of 0.298%, which is expected due to the spherical approximations made in Stokes's formula (Heiskanen and Moritz, 1967, p.94).

The improved agreement with the analytical result in Figure 4(b) was only achieved when the integration weighting factor (23) was utilised. Therefore, the software tested is capable of producing the correct geoid heights, given accurate gravity data. A similar improvement is gained for the topographic and indirect effect formulae when equation (25) is used.

Conclusions and a recommendation

This brief discussion has shown that a comparison between the output of geoid computation software, using constant gravity data, with analytical integration can be utilised to validate the operation of that software. However, this assumes that constant gravity data are used and does not necessarily guarantee that the correct results will be achieved when real gravity data are used. Nevertheless, it is reasonable to assume that this would indeed be the case, provided that accurate gravity and terrain data are used.

When using mean gravity anomalies and a digital terrain model for geoid computations, as is usually the case with numerical integration or the fast Fourier transform (FFT) methods, the integration weighting factor must be included if the integration kernel is evaluated at the centre of each compartment. This is extremely important close to each computation point, and especially so for the topographic and indirect effect formulae.

It is recommended that all geoid computation software is validated in this or some other way prior to its use. Moreover, such a comparison will bestow a level of confidence in the software, thus isolating the reasons for any subsequent disagreements between the gravimetric geoid heights and those implied by GPS in conjunction with levelling. These reasons could comprise the reduction and prediction of gravity anomalies, or the accuracy of the GPS and levelling data used for geoid comparisons, for example.

Acknowledgements

This research is funded as part of Australian Research Council (ARC) grant A49331318. The work was undertaken whilst the second author was visiting Australia as a recipient of a British Council Travel Grant. We would also like to thank the three reviewers for their helpful comments and for drawing our attention to some additional references.

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