Continuous-time Opinion Dynamics on Multiple Interdependent Topics ⋆

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Abstract

In this paper, and inspired by the recent discrete-time model in [1,2], we study two continuous-time opinion dynamics models (Model 1 and Model 2) where the individuals discuss opinions on multiple logically interdependent topics. The logical interdependence between the different topics is captured by a "logic" matrix, which is distinct from the Laplacian matrix capturing interactions between individuals. For each of Model 1 and Model 2, we obtain a necessary and sufficient condition for the network to reach to a consensus on each separate topic. The condition on Model 1 involves a combination of the eigenvalues of the logic matrix and Laplacian matrix, whereas the condition on Model 2 requires only separate conditions on the logic matrix and Laplacian matrix. Further investigations of Model 1 yields two sufficient conditions for consensus, and allow us to conclude that one way to guarantee a consensus is to reduce the rate of interaction between individuals exchanging opinions. By placing further restrictions on the logic matrix, we also establish a set of Laplacian matrices which guarantee consensus for Model 1. The two models are also expanded to include stubborn individuals, who remain attached to their initial opinions. Sufficient conditions are obtained for guaranteeing convergence of the opinion dynamics system, with the final opinions generally being at a persistent disagreement. Simulations are provided to illustrate the results.

Key words: influence networks; social networks; multi-dimensional opinion dynamics; agent-based model

1 Introduction

Recently, the study of "opinion dynamics" has been of particular interest to the control systems community, in part due to the similarities and parallels with multi-agent systems. The key problems involve study of models in which individuals interact and discuss opinions on a topic or set of topics, with each individual's opinion evolution described by an update rule.

In order to understand our contribution in context, we first review several of the widely studied and most relevant models, and refer readers to the survey [3] for more comprehensive discussions. The French-DeGroot discrete-time model (known also as the DeGroot model) was proposed in [4,5], and in it, each individual sets his/her opinion at the next time instant to be a weight average of his/her neighbours' opinions. A continuous-time counterpart was proposed in [6]. The Altafini model [7] developed the concept that an individual may trust or distrust neighbouring individuals (captured by a positive or negative edge weight, respectively); the DeGroot and Abelson models assume individuals either trust or ignore others. The Friedkin-Johnsen model [8] extended

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the DeGroot model to include "stubborn" individuals who remain somewhat attached to their initial opinion. The continuous-time counterpart of the Friedkin-Johnsen model was first proposed in [9] (in fact appearing earlier than the Friedkin-Johnsen model). Later, the model in [9] was studied as an algorithm for containment control for autonomous vehicle formations [10].

A natural extension to the above works is to consider the *simultaneous* discussion of *multiple* topics. If the topics are independent of each other then the models discussed above may be easily extended with introduction of a Kronecker product. However, it is more likely that an individual's opinion on Topic A is influenced by his/her opinion on Topic B and vice versa. In such situations, an individual applies an introspective (internal) cognitive process to ensure that his/her opinions on all topics are logically consistent. The logical interdependences and the introspective process form the individual's *belief system*. The recent works [1,2] combined opinion evolution (as captured by the Friedkin–Johnsen opinion dynamics model) due to interaction among individuals with belief system dynamics. Since each individual's opinion set is now affected through his or her internal belief system and also the neighbours' opinions, the question of when consensus occurs becomes nontrivial, as the two processes are not guaranteed a priori to be consistent with one another. In [1,2], the logical interdependence is described by a matrix, and thus the model can be considered a form of *matrix-weight consensus*. Matrix-weight consensus problems have recently become of interest in multi-agent systems coordination, applicable to consensus on Euclidean spaces, and bearing measurement based localisation and formation control [11].

1.1 Contributions of This Paper

Inspired by [1,2], this paper proposes and studies two continuous-time opinion dynamics models (Model 1 and Model 2) for networks of individuals simultaneously discussing logically interdependent topics. In both models, each individual is affected by three processes: (i) an introspective process using the logic matrix, common to all individuals, to secure logical consistency in the individual's opinions on the set of topics, (ii) a stubborn attachment to the individual's initial opinions, and (iii) interpersonal influence arising from sharing of opinions with neighbouring agents. The key difference between the two proposed models is in the third process, and in particular whether or not an individual assimilates the logical interdependences into his/her opinions before exchanging opinions with his/her neighbours. Further comparison between the two models is postponed till their formal introduction in the next section.

We begin our statement of results by obtaining separate necessary and sufficient conditions for Model 1 and Model 2 to ensure a consensus of opinions is reached when there are no stubborn individuals, i.e. individuals do not remain attached to their initial opinions. The condition on Model 1 involves *a combination of the eigenvalues* of (i) the Laplacian matrix describing the network topology, and (ii) the matrix describing the logical interdependences. In contrast, the condition on Model 2 requires the eigenvalues of the two matrices to *separately satisfy certain conditions*. Two sufficient conditions for consensus with no stubborn individuals, requiring only limited knowledge of the parameters of network and the individuals, are then derived for Model 1; we show that given a matrix describing the logical interdependence, one can always achieve a consensus of opinions by decreasing the strength of interactions. On the other hand, large interaction strengths sometimes results in instability. These observations on Model 1 contrast the results obtained on Model 2, and also to the discrete-time model, where in the absence of stubborn individuals, convergence of the network matrix and logic matrix separately is enough to ensure a consensus of opinions. Networks with stubborn individuals are also treated, with sufficient conditions obtained for ensuring the system is convergent for both Model 1 and Model 2. Using the obtained results, we compare the two models, and examine the conclusions in the social context.

The rest of the paper is structured as follows. Mathematical background and two continuous-time opinion dynamics models are presented in Section 2. Sections 3 and 4 study convergence conditions of the two models, respectively. Simulations are provided in Section 5, with conclusions drawn in Section 6.

2 Background and Formal Problem Statement

We begin by introducing some notation. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote, respectively, the $n \times 1$ column vectors of all ones and all zeros. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $0 \leq \boldsymbol{x}$ and $0 < \boldsymbol{x}$ indicate component-wise inequalities, i.e., for all $i \in \{1, 2, \ldots, n\}$, $0 \leq x_i$ and $0 < x_i$, respectively. The canonical basis of $\mathbb{R}^n$ is given by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. We denote $\sqrt{-1} = \jmath$ as the imaginary unit, and for a complex number $z = a + b\jmath$ we denote $\mathfrak{Re}(z) = a$ and $\mathfrak{Im}(z) = b$. The modulus is $|z| = \sqrt{a^2 + b^2}$. For a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times m}$, we denote its ∞-norm as $\|\boldsymbol{A}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{m} |a_{ij}|$. The Kronecker product is $\otimes$. Note that the terms "node", and "individual" are used interchangeably.

2.1 Graph Theory

The interaction between n individuals in a social network is modelled using a weighted directed graph, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Each individual is a node in the finite, nonempty set of nodes $\mathcal{V} = \{v_i : i \in \mathcal{I} = \{1, \ldots, n\}\}$. The set of ordered edges is $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote an ordered edge as $e_{ij} = (v_i, v_j) \in \mathcal{E}$. An edge e_{ij} is said to be outgoing with respect to v_i and incoming with respect to v_j, and connotes that individual j learns of, and takes into account, the opinion value of individual i when updating its own opinion. The (incoming) neighbour set of v_i is defined as $\mathcal{N}_i = \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$. The weighted adjacency matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ of $\mathcal{G}$ has nonnegative elements a_{ij} satisfying $a_{ij} > 0 \Leftrightarrow e_{ji} \in \mathcal{E}$, and it is assumed that $a_{ii} = 0, \forall i$. The Laplacian matrix, $\mathcal{L} = [l_{ij}]_{n\times n}$, of the associated digraph $\mathcal{G}$ is defined as $l_{ii} = \sum_{k=1,k\neq i}^{n} a_{ik}$ and $l_{ij} = -a_{ij}$ for $j \neq i$. A directed path is a sequence of edges $(v_{p_1}, v_{p_2}), (v_{p_2}, v_{p_3}), ...,$ where $v_{p_i} \in \mathcal{V}, e_{p_i p_{i+1}} \in \mathcal{E}$. Node i is reachable from node j if there exists a directed path from v_j to v_i. A node v_i is a root if there is a path from v_i to every $v_j \in \mathcal{V}, j \neq i$. A directed spanning tree is a graph formed by directed edges that connects all the nodes, and where every vertex apart from the unique root node has exactly one parent. A graph is said to contain a directed spanning tree if a subset of the edges forms a directed spanning tree [1]. A graph is strongly connected if and only if there exists a directed path from every node v_i to every other node v_j. The following is a standard result to be used throughout this paper.

Lemma 1 ([12]) *The Laplacian $\mathcal{L}$ associated with a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has a single eigenvalue at 0 if and only if $\mathcal{G}$ has a directed spanning tree. Associated with the single 0 eigenvalue are left and right eigenvectors $\boldsymbol{\gamma} \geq 0$ and $\mathbf{1}_n$, respectively, with normalisation $\boldsymbol{\gamma}^\top \mathbf{1}_n = 1$. All other eigenvalues have strictly positive real part.*

If the graph contains a directed spanning tree, then there exists an $r \leq n$ such that the nodes reordered $v_1, \ldots, v_r$ induce a maximally closed and strongly connected subgraph $\mathcal{G}_L$. By closed, we mean that no edges are incoming to $\mathcal{G}_L$. We denote by $\mathcal{G}_F$ the subgraph induced by the set of nodes $v_{r+1}, \ldots, v_n$. With the nodes reordered, the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ is expressed as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathbf{0}_{r\times(n-r)} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}. \tag{1}$$

where $\mathcal{L}_{11} \in \mathbb{R}^{r\times r}$ is irreducible. If $r = n$ then $\mathcal{G}$ is strongly connected, and $\mathcal{L}_{22}$ vanishes. The matrices $\mathcal{L}$ and $\mathcal{L}_{11}$ are singular M-matrices [13], which implies in light of Lemma 1 that if $r < n$ then $\mathcal{L}_{22}$ is a nonsingular M-matrix, i.e. its eigenvalues have positive real part. Moreover, if $\boldsymbol{D}_1, \boldsymbol{D}_2$ are nonnegative diagonal matrices of appropriate size, and $\boldsymbol{D}_1$ has at least one positive diagonal entry, then all eigenvalues of $\mathcal{L}_{11}+\boldsymbol{D}_1$ and $\mathcal{L}_{22}+\boldsymbol{D}_2$ have positive real part (see [14, Theorem 2.3] and [15, Corollary 4.33], respectively). The left eigenvector $\boldsymbol{\gamma}^\top = [\gamma_1, \ldots, \gamma_n]$ has entries $\gamma_i > 0,\ i \in \{1, \ldots, r\}$.

[1] Some literature use other terms, e.g. rooted out-branching or directed rooted tree.

2.2 Opinion Dynamics Model and Problem Statement

We now present two general opinion dynamics models and the formal problem statement. We then discuss the models' motivation, including exploration of a key matrix describing the logical interdependence of the topics.

Given a population of $n \geq 2$ individuals, indexed by $\mathcal{I} = \{1, \ldots, n\}$, let $\boldsymbol{x}_i(t) = [x_i^1(t), \ldots, x_i^d(t)]^\top \in \mathbb{R}^d$ be the vector of opinion values [2] held by individual $i \in \mathcal{I}$, at time t, on d different topics indexed by $\mathcal{J} = \{1, \ldots, d\}$. Where there is no confusion, we drop the time argument t. We propose two models to describe how the opinions of individual i evolve.

Model 1:

$$\dot{\boldsymbol{x}}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\boldsymbol{C}\left(\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)\right) + (\boldsymbol{C} - \boldsymbol{I}_d)\boldsymbol{x}_i(t) + b_i(\boldsymbol{x}_i(0) - \boldsymbol{x}_i(t)). \tag{2}$$

Model 2:

$$\dot{\boldsymbol{x}}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\left(\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t)\right) + (\boldsymbol{C} - \boldsymbol{I}_d)\boldsymbol{x}_i(t) + b_i(\boldsymbol{x}_i(0) - \boldsymbol{x}_i(t)). \tag{3}$$

In the above, a_{ij} is the $(i, j)^{th}$ entry of the adjacency matrix $\mathcal{A}$ associated with the graph $\mathcal{G}$. The constant matrix $\boldsymbol{C} \in \mathbb{R}^{d\times d}$, which is the same for each individual $i \in \mathcal{I}$, represents the logical interdependence/coupling between different topics. The scalar $b_i \geq 0$ is a measure of individual i's stubbornness, or attachment to his/her initial opinion value $\boldsymbol{x}_i(0)$. When $\boldsymbol{C} = \boldsymbol{I}_d$, Eq. (2) and (3) are equivalent, and the motivations and dynamical properties for this special case are comprehensively detailed in [3,6,9,10]. For the general case of $\boldsymbol{C} \neq \boldsymbol{I}_d$ as investigated in this paper, the role and properties of $\boldsymbol{C}$, and the differences between (2) and (3), are explained in Section 2.3 below, after we complete a formal introduction of the model. Using a single model, [1,2] treat the case of $\boldsymbol{C} \neq \boldsymbol{I}_d$ in a discrete-time framework. We explain later the parallels with our two continuous-time models.

Model 1: The dynamical system describing a network of individuals using (2) can be expressed as $\dot{\boldsymbol{x}} = (\boldsymbol{I}_n \otimes (\boldsymbol{C} - \boldsymbol{I}_d))\boldsymbol{x} - (\mathcal{L} \otimes \boldsymbol{C})\boldsymbol{x} + (\boldsymbol{B} \otimes \boldsymbol{I}_d)(\boldsymbol{x}(0) - \boldsymbol{x})$, where $\boldsymbol{x} = [\boldsymbol{x}_1^\top, \ldots, \boldsymbol{x}_n^\top]^\top \in \mathbb{R}^{nd}$ is the stacked vector of all opinion vectors $\boldsymbol{x}_i$ and $\mathcal{L}$ is the Laplacian matrix associated with

[2] One illustrative example is where the k^{th} entry x_i^k represents individual i's belief/certainty in a statement defining topic k which in principle is provable to be true or false. Alternatively, x_i^k may represent an attitude towards adoption of an idea defining topic k, with negative values (respectively positive values) representing refusal (respectively willingness) to adopt. More details are provided in Section 3.3.

the graph $\mathcal{G}$. The diagonal matrix $\boldsymbol{B} = \text{diag}[b_i]$ encodes individuals' stubbornness. One can rearrange to obtain

$$\dot{\boldsymbol{x}} = -\big(\boldsymbol{I}_{nd} + (\mathcal{L} - \boldsymbol{I}_n) \otimes \boldsymbol{C} + \boldsymbol{B} \otimes \boldsymbol{I}_d\big)\boldsymbol{x} + (\boldsymbol{B} \otimes \boldsymbol{I}_d)\boldsymbol{x}(0). \quad (4)$$

Model 2: Similar to the above, one can show that the network dynamics of (3) are

$$\dot{\boldsymbol{x}}(t) = -\big((\mathcal{L} + \boldsymbol{B}) \otimes \boldsymbol{I}_d + \boldsymbol{I}_n \otimes (\boldsymbol{I}_d - \boldsymbol{C})\big)\boldsymbol{x}(t) + (\boldsymbol{B} \otimes \boldsymbol{I}_d)\boldsymbol{x}(0). \quad (5)$$

The problem considered in this paper is as follows. Let a social network be represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Supposing that all individuals either use the opinion updating rule (2) or (3), we seek to determine 1) the connectivity conditions (including constraints on the edge weights) on the graph $\mathcal{G}$, 2) the conditions on the matrix $\boldsymbol{C}$, and 3) conditions on $\boldsymbol{B}$, which guarantee that as $t \to \infty$, opinions reach a steady value, i.e. $\dot{\boldsymbol{x}}_i = 0, \forall\, i \in \mathcal{I}$. A special case of convergence is consensus of opinions. We say that a consensus on opinions has been reached if $\lim_{t\to\infty} \|\boldsymbol{x}_i - \boldsymbol{x}_j\| = 0,\ \forall i, j \in \mathcal{I}$, and it will be shown that consensus can occur when there are no stubborn individuals in the network, or if there exist stubborn individuals and $\boldsymbol{x}_i(0) = \boldsymbol{x}_j(0), \forall\, i, j$.

2.3 Interdependent Topics and the $\boldsymbol{C}$ *Matrix*

The concept of an opinion dynamics model for capturing simultaneous discussion on multiple logically interdependent topics was first proposed in discrete-time [1,2]. In [1,2], the authors capture this with a matrix of *multi-issues dependence structure* (MiDS). Similarly, we define in this paper a *logic matrix* $\boldsymbol{C}$ which encodes the logical coupling between issues, which has some different properties to MiDS matrix in [1,2]. We now provide an example to motivate $\boldsymbol{C}$ and demonstrate its purpose in a person's cognitive process for handling logically interdependent topics.

Consider two topics being simultaneously discussed; 1) mentally challenging tasks are just as exhausting as physically challenging tasks and 2) that chess should be considered a sport in the Olympics. Let individual i's opinion vector be $\boldsymbol{x}_i = [x_i^1, x_i^2]^\top$. For topic 1, if x_i^1 is positive (respectively negative) then individual i believes mentally challenging tasks are just as exhausting (respectively not as exhausting) as physically challenging tasks. For topic 2, if x_i^2 is positive (respectively negative) then individual i believes chess should be considered (respectively not considered) an Olympic sport. One possible logic matrix is given by

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0.7 & 0.3 \end{bmatrix} \quad (6)$$

which indicates that individual i believes whether an event should be in the Olympics depends heavily on whether it is exhausting. While the above $\boldsymbol{C}$ is row-stochastic, $\boldsymbol{C}$ in general need not be row-stochastic (though other constraints set out below will apply).

To gain further insight into constraints on $\boldsymbol{C}$ and each individual's internal process for securing logical consistency, and by way of example, suppose that $\mathcal{N}_i = \{\emptyset\}$ and $b_i = 0$. Then (2) and (3) become

$$\dot{\boldsymbol{x}}_i = \boldsymbol{C}\boldsymbol{x}_i - \boldsymbol{x}_i \quad (7)$$

Here, the matrix $\boldsymbol{C}$ is the logic matrix detailed in Section 2.3, and $(\boldsymbol{C} - \boldsymbol{I}_d)\boldsymbol{x}_i$ *is the difference* between individual i's current opinion $\boldsymbol{x}_i$ and its opinions *after assimilating* the logical interdependencies of the discussed topics, $\boldsymbol{C}\boldsymbol{x}_i$. Existing literature indicates that individuals will use an *introspective (internal) cognitive process* to remove cognitive inconsistencies in their set of beliefs [16–18], and this process is modelled in individual i by the dynamics of (7).

Returning to the example of chess and Olympic sports, suppose that individual i has initial opinions $\boldsymbol{x}_i(0) = [1, -1]^\top$. Then (7) with $\boldsymbol{C}$ given in (6) yields $\lim_{t\to\infty} \boldsymbol{x}_i = [1, 1]^\top$. In other words, individual i has an initial opinion against chess being an Olympic sport, but his/her logical reasoning that mentally challenging tasks are just as exhausting creates an *inconsistency*. Individual i uses a cognitive process, viz. (7), to adjust his/her opinions until a consistent set of opinions is held.

The fact that (7) represents a cognitive process implies that some constraints must be placed on $\boldsymbol{C}$. We assume that (7) will eventually lead to a *consistent* belief system. We therefore do not expect $\boldsymbol{x}_i(t)$ to oscillate indefinitely, or for $\lim_{t\to\infty} \|\boldsymbol{x}_i(t)\| = \infty$. If (7) is asymptotically stable then $\lim_{t\to\infty} \boldsymbol{x}_i = \boldsymbol{0}_d$, which is a non-generic cognitive process, and we therefore assume does not occur. Thus, one expects in general that $\lim_{t\to\infty} \boldsymbol{x}(t)$ exists under (7) and is nonzero. In order for (7) to have these properties, we impose the following assumption.

Assumption 1 *The matrix* $\boldsymbol{C}$*, with eigenvalues* $\lambda_k(\boldsymbol{C})$*, has a semi-simple*[3] *eigenvalue at 1 with multiplicity* $p \geq 1$*, ordered as* $\lambda_1(\boldsymbol{C}) = \ldots = \lambda_p(\boldsymbol{C}) = 1$*, with associated right and left eigenvectors* $\boldsymbol{\zeta}_r$ *and* $\boldsymbol{\xi}_r^\top$*, respectively, satisfying* $\boldsymbol{\xi}_r^\top \boldsymbol{\zeta}_r = 1$ *for* $r = 1, \ldots, p$*. Other eigenvalues* $\lambda_k(\boldsymbol{C})$ *satisfy* $\mathfrak{Re}(\lambda_k(\boldsymbol{C})) < 1, \forall\, k > p$*, and* $c_{ii} \geq 0\, \forall\, i \in \mathcal{J}$*.*

The eigenvalue assumptions are necessary and sufficient for (7) to have the desired convergence properties. The

[3] By semi-simple, we mean that the geometric and algebraic multiplicities are the same. Equivalently, the Jordan blocks of the eigenvalue 1 are all 1 by 1.

requirement that $c_{ii} \geq 0$ for all $i \in \mathcal{J}$ simply (and reasonably) indicates that topic i is nonnegatively coupled to itself. No restrictions are placed on the off-diagonal entries of $\boldsymbol{C}$, i.e. how two different topics are coupled. A special case of Assumption 1 is $\boldsymbol{C} = \boldsymbol{I}_d$. We note here that $\boldsymbol{\zeta}_i, i = 1, \ldots, p$ is a nullvector of $\boldsymbol{C} - \boldsymbol{I}_d$, and the introspective dynamics (7) will yield that $\boldsymbol{x}_i(\infty)$ is in the span of $\{\boldsymbol{\zeta}_r\}, r = 1, \ldots, p$. It will become apparent that $\boldsymbol{\xi}_r, \boldsymbol{\zeta}_r, r = 1, \ldots, p$ also play a role in determining the final set of opinions for the network of individuals. We now examine the differences in the first terms of (2) and (3), which capture the interpersonal interactions.

Model 1: One can consider $\boldsymbol{C}\boldsymbol{x}_i$ as individual i's opinions *after* assimilating the logical interdependencies using $\boldsymbol{C}$, implying that $\boldsymbol{y}_j = \boldsymbol{C}\boldsymbol{x}_j$ is the output of individual j which is communicated to individual i if $j \in \mathcal{N}_i$. The first term on the right of (2) thus captures the notion that individual i is influenced by the weighted difference in assimilated opinions between himself/herself, and his/her neighbours, $\sum_{j\in\mathcal{N}_i} a_{ij}(\boldsymbol{C}\boldsymbol{x}_j(t) - \boldsymbol{C}\boldsymbol{x}_i(t))$. Thus, (2) captures the simultaneous effect of three different processes, viz. (i) interpersonal influence due to differences in *assimilated opinions* between individual i and neighbour individuals j, (ii) an introspective cognitive process for securing logical consistency between topics, and (iii) a stubborn attachment to i's initial prejudices/opinions.

Model 2: In contrast, the first term of (3) reflects that individual i displays opinions $\boldsymbol{x}_i(t)$ without assimilation, then learns of opinions $\boldsymbol{x}_j(t)$ also without assimilation. Individual i's rate of opinion change is influenced by the weighted difference in unassimilated (displayed) opinions between himself/herself and his/her neighbours, $\sum_{j\in\mathcal{N}_i} a_{ij}(\boldsymbol{x}_j(t) - \boldsymbol{x}_i(t))$. Thus, in (3), individual i only uses the introspective process (second term) to internally assimilate the logical interdependencies into his/her opinions.

The models in this paper are both inspired by the discrete-time model in [1,2]. In [2, Supplementary Material, Remark 1], the authors describe two variations for the discrete-time opinion dynamics when multiple logically interdependent topics are simultaneously discussed. In discrete-time, the two variations yield identical difference equations when $\boldsymbol{C}$ is homogeneous among the individuals, and no variation candidate is stated as being clearly more accepted. Thus, we developed both variations in continuous-time to obtain Models 1 and 2 as alternative models for continuous-time opinion evolution for logically interdependent topics. We study both in order to better understand the dynamics of the two processes, including any differences. It turns out that whether individuals exchange assimilated opinions (Model 1) or unassimilated opinions (Model 2) can lead to different convergence and stability properties.

Remark 1 *We wish to clarify that $\boldsymbol{C}\boldsymbol{x}_i(t)$ represents the effect of individual i using $\boldsymbol{C}$ to assimilate the logical interdependences into his/her opinion vector $\boldsymbol{x}_i(t)$ to obtain an opinion vector $\boldsymbol{C}\boldsymbol{x}_i(t)$. On the other hand* (7) *is the proposed model of the introspective process by which individual i ensures that he or she eventually has a set of opinions which are consistent with the logical interdependence structure, i.e. a consistent belief system. Such a model guarantees that $\boldsymbol{x}_i(\infty)$ is a fixed point of the linear map $\boldsymbol{C}$, i.e. $\boldsymbol{C}\boldsymbol{x}_i(\infty) = \boldsymbol{x}_i(\infty)$.*

Last, we state an assumption on the graph $\mathcal{G}$ representing the interpersonal interaction topology of the network.

Assumption 2 *The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has a directed spanning tree, and the nodes are ordered such that the associated Laplacian matrix $\mathcal{L}$ takes the form in* (1).

If $\mathcal{G}$ does not have a directed spanning tree, then there exist at least two closed and strongly connected subgraphs, with each subgraph containing at least one individual. *Regardless of whether Model 1 or Model 2* is used to capture the opinion dynamics, the opinions of the individual(s) in each closed subgraph evolve independently of the opinions of other individuals, and for almost all $\boldsymbol{x}(0)$, will not not reach a consensus with the opinions of any other individual in the network. By "almost all $\boldsymbol{x}(0)$", it is meant that there may be a proper subset $\mathcal{F}$ of $\mathbb{R}^{nd}$ with Lebesgue measure zero, for which consensus can still be reached if $\boldsymbol{x}(0) \in \mathcal{F}$. This implies that for almost all $\boldsymbol{x}(0)$, $\mathcal{G}$ having a directed spanning tree (Assumption 2) is a *necessary condition* for consensus to be achieved.

We present the convergence analysis in the following two sections, and defer discussion of the results and comparison between the two models to the end of Section 4.

3 Networks of Individuals with Model 1

3.1 Consensus With No Stubborn Individuals

We first present the main convergence result when there are no stubborn individuals, i.e. $b_i = 0, \forall\, i \in \mathcal{I}$.

Theorem 1 *Let $\boldsymbol{C}$, which satisfies Assumption 1, and $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be given, with $\lambda_i(\mathcal{L})$ and $\lambda_k(\boldsymbol{C})$ being the eigenvalues of the Laplacian matrix $\mathcal{L}$ and logic matrix $\boldsymbol{C}$, respectively. The eigenvalues are ordered such that $\lambda_1(\mathcal{L}) = 0$, and $\lambda_1(\boldsymbol{C}), \ldots, \lambda_p(\boldsymbol{C})$ are the $p \geq 1$ semi-simple eigenvalues at 1.*

Then, with each individuals' opinions evolving according to (2)*, and $b_i = 0\,\forall\, i \in \mathcal{I}$, the social network reaches a consensus on all topics exponentially fast if and only if*

$$\mathfrak{Re}\left((1 - \lambda_i(\mathcal{L}))\lambda_k(\boldsymbol{C})\right) < 1, \forall\, i \in \mathcal{I} \setminus \{1\}, \forall k \in \mathcal{J} \quad (8)$$

Moreover, with $\boldsymbol{\gamma}^\top$ as defined in Assumption 2, and $\boldsymbol{\xi}_r^\top$ and $\boldsymbol{\zeta}_r$ as defined in 1, the solution satisfies

$$\lim_{t\to\infty} \boldsymbol{x}_i(t) = \Big(\sum_{r=1}^{p} \boldsymbol{\zeta}_r \boldsymbol{\xi}_r^\top\Big) \sum_{j=1}^{n} \gamma_j \boldsymbol{x}_j(0), \forall i \in \mathcal{I}, \qquad (9)$$

Proof: Observe that (8) holds only if $\lambda_i(\mathcal{L}) \neq 0$, $i = 2, ..., n$, which in turn holds if and only if $\mathcal{G}$ has a directed spanning tree (see discussion below Assumption 2).

We first establish the sufficiency of (8). Since $b_i = 0, \forall i \in \mathcal{I}$, the opinions $\boldsymbol{x}(t)$ evolve according to (4) with $\boldsymbol{B} = \boldsymbol{0}_{n\times n}$. Denote $\boldsymbol{M} = -\boldsymbol{I}_{nd} + (\boldsymbol{I}_n - \mathcal{L}) \otimes \boldsymbol{C}$. Clearly the j^{th} eigenvalue of $\boldsymbol{M}$ is equal to $-1 + \lambda_j(\boldsymbol{A})$ where $\lambda_i(\boldsymbol{A})$ is the j^{th} eigenvalue of $\boldsymbol{A} = (\boldsymbol{I}_n - \mathcal{L}) \otimes \boldsymbol{C}$. The associated eigenvector is $\boldsymbol{v}_j$, where $\boldsymbol{v}_j$ is the eigenvector of $\boldsymbol{A}$ associated with $\lambda_j(\boldsymbol{A})$. From [19, Proposition 7.1.10], we conclude that $\lambda_j(\boldsymbol{A}) = \mu_i\varphi_k$ where μ_i and φ_k are eigenvalues of $\boldsymbol{I}_n - \mathcal{L}$ and $\boldsymbol{C}$ respectively, $i \in \mathcal{I}$, $k \in \mathcal{J}$. Then, one can verify that $\boldsymbol{v}_j = \boldsymbol{u}_i \otimes \boldsymbol{w}_k$ is an eigenvector of $\boldsymbol{A}$ associated with $\lambda_j(\boldsymbol{A})$, where $\boldsymbol{u}_i$ and $\boldsymbol{w}_k$ are eigenvectors of $\boldsymbol{I}_n - \mathcal{L}$ and $\boldsymbol{C}$ associated with μ_i and φ_k, respectively. According to Assumption 1, $\boldsymbol{C}$ has a semi-simple eigenvalue at 1 with multiplicity $p \geq 1$; because we need to subsequently distinguish these eigenvalues, we denote them as $\varphi_1, \ldots, \varphi_p$. If $\mathcal{G}$ has a directed spanning tree, then $\boldsymbol{I}_n - \mathcal{L}$ has a single eigenvalue at 1, which we denote as μ_1. Then clearly, $\lambda_j = \mu_1\varphi_r = 1, r = 1, \ldots, p$ is an eigenvalue of $\boldsymbol{A}$ with right eigenvector $\boldsymbol{v}_j = \boldsymbol{1}_n \otimes \boldsymbol{\zeta}_r$. For $\lambda_j = \mu_1\varphi_k, k = p+1, \ldots, d$, clearly $\lambda_i = \varphi_k$ has real part strictly less than 1, because Assumption 1 states that $\mathfrak{Re}(\varphi_k) < 1$. For $\lambda_j = \mu_i\varphi_k$ where $i \in \{2, \ldots, n\}, k \in \mathcal{J}$, if (8) is satisfied then λ_j has real part strictly less than 1. It follows that all eigenvalues of $\boldsymbol{M}$ have strictly negative real part, except for p eigenvalues at the origin, with associated right eigenvectors $\boldsymbol{v}_j = \boldsymbol{1}_n \otimes \boldsymbol{\zeta}_r, r = 1, \ldots, p$. Let $\boldsymbol{J}_{\mathcal{L}} = \boldsymbol{P}_1^{-1}\mathcal{L}\boldsymbol{P}_1$ and $\boldsymbol{J}_{\boldsymbol{C}} = \boldsymbol{P}_2^{-1}\boldsymbol{C}\boldsymbol{P}_2$ be the Jordan canonical form of $\mathcal{L}$ and $\boldsymbol{C}$, respectively, ordered such that the first Jordan block of $\boldsymbol{J}_{\mathcal{L}}$ is associated with the single zero eigenvalue of $\mathcal{L}$ and the first p Jordan blocks of $\boldsymbol{J}_{\boldsymbol{C}}$ are associated with the p semi-simple unity eigenvalues of $\boldsymbol{C}$. With $\boldsymbol{P} = \boldsymbol{P}_1 \otimes \boldsymbol{P}_2$, verify that

$$\boldsymbol{P}\boldsymbol{M}\boldsymbol{P}^{-1} = \boldsymbol{J} = \begin{bmatrix} \boldsymbol{0}_{p\times p} & \boldsymbol{0}_{p\times(nd-p)} \\ \boldsymbol{0}_{(nd-p)\times p} & \boldsymbol{\Delta} \end{bmatrix}, \qquad (10)$$

with the p eigenvalues of $\boldsymbol{M}$ at the origin being semi-simple and the $nd-p$ nonzero diagonal entries of $\boldsymbol{\Delta}$ being the stable eigenvalues of $\boldsymbol{M}$. From linear systems theory, one then has that $\boldsymbol{x}(t) = e^{\boldsymbol{M}t}\boldsymbol{x}(0) = \boldsymbol{P}e^{\boldsymbol{J}t}\boldsymbol{P}^{-1}\boldsymbol{x}(0)$, which yields $\lim_{t\to\infty} \boldsymbol{x}(t) = \sum_{r=1}^{p} \boldsymbol{p}_r\boldsymbol{q}_r^\top \boldsymbol{x}(0)$ where $\boldsymbol{p}_r$ and $\boldsymbol{q}_r^\top$ are right and left eigenvectors of $\boldsymbol{M}$ associated with the semi-simple zero eigenvalue, satisfying $\boldsymbol{p}_r^\top \boldsymbol{q}_r = 1, \forall r = 1, \ldots, p$. The above analysis yielded $\boldsymbol{p}_r = \boldsymbol{1}_n \otimes \boldsymbol{\zeta}_r$. One can easily verify that $\boldsymbol{q}_r^\top = (\boldsymbol{\gamma} \otimes \boldsymbol{\xi}_r)^\top$ and thus $\lim_{t\to\infty} \boldsymbol{x}(t) = \sum_{r=1}^{p} (\boldsymbol{\gamma} \otimes \boldsymbol{\xi}_r)^\top \boldsymbol{x}(0)(\boldsymbol{1}_n \otimes \boldsymbol{\zeta}_r)$. In other words, $\lim_{t\to\infty} \boldsymbol{x}_i(t) = \sum_{r=1}^{p} (\boldsymbol{\gamma} \otimes \boldsymbol{\xi}_r)^\top \boldsymbol{x}(0)\boldsymbol{\zeta}_r$, which can be rearranged to obtain (9). The sufficiency of (8) has been established.

It remains for the necessity of (8) to be established. Suppose that (8) is not satisfied. Then there is some $\lambda_j = \mu_i\varphi_k, i \in \{2, \ldots, n\}, k \in \mathcal{J}$ such that the eigenvalue of $\boldsymbol{M}$, $-1+\lambda_j$, is in the closed right half-plane. The system is either unstable, or $-1+\lambda_j$ is on the imaginary axis (possibly at the origin). In the latter case either *a*) there are now at least $p+1$ eigenvalues of $\boldsymbol{M}$ at the origin, or *b*) $\boldsymbol{M}$ has a pair of purely imaginary eigenvalues. Regarding *a*), the system is either unstable (there is a Jordan block of size at least 2×2 in $\boldsymbol{J}$ associated with the eigenvalue 0), or for some $i \neq 1$ and $k \in \mathcal{J}$, there holds $\lambda_j = \mu_i\varphi_k = 0$. Then, $\boldsymbol{x}$ converges exponentially fast to a subspace spanned by $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p, \boldsymbol{v}_j\}$ where $\boldsymbol{v}_j$ is an eigenvector of $\boldsymbol{M}$ associated with eigenvalue λ_j. Because $i \neq 1$, $\boldsymbol{v}_j = \boldsymbol{u}_i \otimes \boldsymbol{w}_k$ cannot take the form $\boldsymbol{1}_n \otimes \boldsymbol{w}_k$, for some $\boldsymbol{w}_k \in \mathbb{R}^d$, which implies that consensus is not reached for generic initial conditions. Regarding *b*), denote one of the imaginary eigenvalues as $\lambda_j = \mu_i\varphi_k, i \neq 1$. Then, the system oscillates but not in consensus because, similar to the above arguments, $\boldsymbol{v}_j$ associated with the imaginary λ_j cannot take the form $\boldsymbol{1}_n \otimes \boldsymbol{w}_k$. The proof is complete. □

It may be difficult to verify the conditions in Theorem 1 because precise values of eigenvalues of both $\mathcal{L}, \boldsymbol{C}$ are needed. We now present two results on sufficient conditions which guarantee consensus *using limited information about the network and the logic structure.*

Corollary 1 *For given $\boldsymbol{C}$ and $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, suppose that Assumptions 1 and 2 are satisfied. Then, there exists a graph $\overline{\mathcal{G}} = \{\mathcal{V}, \mathcal{E}, \overline{\mathcal{A}}\}$ with the same node and edge set as $\mathcal{G}$ but with different edge weights, such that consensus of opinions is achieved using* (2) *and* $b_i = 0 \,\forall\, i \in \mathcal{I}$.

Proof: Let $\mathcal{L}$ be the Laplacian associated with $\mathcal{G}$. Observe that $\mathfrak{Re}\left((1 - \lambda_i(\mathcal{L}))\lambda_k(\boldsymbol{C})\right) = d_k - y_id_k \pm z_ie_k$, where, without loss of generality, $\lambda_i(\mathcal{L}) = y_i \pm z_i\jmath$ and $\lambda_k(\boldsymbol{C}) = d_k \pm e_k\jmath$ are complex conjugate eigenvalues of $\mathcal{L}$ and $\boldsymbol{C}$ respectively, and $z_i, e_k \geq 0$. For $i \in \mathcal{I} \setminus \{1\}$ and $k \in \mathcal{J}$, it follows that $\mathfrak{Re}\left((1 - \lambda_i(\mathcal{L}))\lambda_k(\boldsymbol{C})\right) < 1 \Leftrightarrow d_k - y_id_k + z_ie_k < 1$. Define $\overline{\mathcal{A}} = \alpha\mathcal{A}$, where $\alpha > 0$ is a constant scaling every edge weight. Let $\overline{\mathcal{L}}$ be the Laplacian associated with $\overline{\mathcal{G}}$. Since $\mathfrak{Re}\left((1 - \lambda_i(\overline{\mathcal{L}}))\lambda_k(\boldsymbol{C})\right) = \mathfrak{Re}\left((1 - \alpha\lambda_i(\mathcal{L}))\lambda_k(\boldsymbol{C})\right)$, it follows that consensus of opinions is achieved on $\overline{\mathcal{G}}$ if and only if $d_k - \alpha(y_id_k - z_ie_k) < 1 \,\forall\, k \in \mathcal{J}$. According to Lemma 1, $z_1 = y_1 = 0$, and $y_i > 0$ for all $i \geq 2$. From Assumption 1, we have $d_k = 1$ and $e_k = 0$ if $k = 1, \ldots, p$, and $d_k < 1$ otherwise. Thus, there always exists a sufficiently small α satisfying $d_k - \alpha(y_id_k - z_ie_k) < 1$. □

Next, we present an explicit sufficiency condition which requires limited knowledge of the edge weights and $\boldsymbol{C}$.

Corollary 2 *For given $\boldsymbol{C}$ and $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, suppose that Assumptions 1 and 2 hold. Then,* (8) *holds if*

$$\bar{l} < \min\left\{\frac{|1 - |\lambda_k|\cos(\theta_k)|(1+\cos(\theta_k))}{|\lambda_k|\sin^2(\theta_k)}, 0.5\right\} \forall k \in \mathcal{J} \tag{11}$$

where $|\lambda_k| = |\lambda_k(\boldsymbol{C})|$ *and* $\tan(\theta_k) = e_k/d_k$ *with* $\lambda_k(\boldsymbol{C}) = d_k \pm e_k\mathfrak{J}$. *Here,* $\bar{l} = \max_{i\in\mathcal{I}} l_{ii}$ *where* l_{ii} *is the* i^{th} *diagonal entry of* $\mathcal{L}$.

Proof: The proof relatively simple, and is in the arXiv version [20]. In the proof, we show that the right hand side of (11) is well defined and continuous in θ_k even though, separately, $\lim_{\theta_k\to\pi} 1 + \cos(\theta_k) = 0$ and $\lim_{\theta_k\to\pi}\sin^2(\theta_k) = 0$. This ensures that (11) is always evaluable, even if the entries of $\boldsymbol{C}$ vary smoothly in a way such that $\theta_k \to \pi$. □

Remark 2 *Corollary 1 establishes an existence result: there is always a set of edge weights which guarantees consensus. Corollary 1 proves this by the scaling of every* a_{ij} *by a constant* $\alpha > 0$, *and requires knowledge of* $\boldsymbol{C}$. *In contrast, Corollary 2 shows that we need to scale* a_{ij} *for individual* i *(and not necessarily by the same constant) only if* $l_{ii} = \sum_{j=1}^n a_{ij}$ *exceeds the right hand side of* (11), *and any such adjustment requires only limited knowledge of the eigenvalues of* $\boldsymbol{C}$. *While both results need knowledge of* $\mathcal{G}$, *including the spectral radius of* $\mathcal{L}$, *it is only limited knowledge. In the case of Corollary 2, limited information concerning* $\boldsymbol{C}$ *is also required. Additional discussion of the inequality* (11), *with simulations, is provided in [20].*

3.2 Convergence in Networks with Stubborn Individuals

We now study networks with stubborn individuals, i.e. $\exists i \in \mathcal{I} : b_i > 0$. We first give a standard result for the convergence of an exponentially stable linear system with a constant input, with the proof given in [20].

Lemma 2 *Consider the linear system* $\dot{\boldsymbol{x}}(t) = -\boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{u}$, *where* $-\boldsymbol{F}$ *is Hurwitz, and* $\boldsymbol{u}$ *is a constant vector. Then,* $\lim_{t\to\infty}\boldsymbol{x}(t) = \boldsymbol{F}^{-1}\boldsymbol{u}$ *exponentially fast.*

Consider the system (4). By establishing conditions for which the matrix

$$\bar{\boldsymbol{M}} = -\left[\boldsymbol{I}_{nd} + \left((\mathcal{L} - \boldsymbol{I}_n)\otimes\boldsymbol{C}\right) + \boldsymbol{B}\otimes\boldsymbol{I}_d\right] \tag{12}$$

is Hurwitz, by treating $(\boldsymbol{B}\otimes\boldsymbol{I}_d)\boldsymbol{x}(0)$ as a constant input, and replacing $\boldsymbol{F}$ with $\bar{\boldsymbol{M}}$, Lemma 2 allows us to establish conditions for which

$$\lim_{t\to\infty}\boldsymbol{x}(t) = [\boldsymbol{I}_{nd} + (\mathcal{L} - \boldsymbol{I}_n)\otimes\boldsymbol{C} + \boldsymbol{B}\otimes\boldsymbol{I}_d]^{-1} \times (\boldsymbol{B}\otimes\boldsymbol{I}_d)\boldsymbol{x}(0). \tag{13}$$

Note that if $\boldsymbol{x}_i(0) = \boldsymbol{x}_j(0), \forall i, j \in \mathcal{I}$, i.e. all individuals are initially at consensus, then clearly $\dot{\boldsymbol{x}} = (\boldsymbol{I}_n \otimes (\boldsymbol{C} - \boldsymbol{I}_d))\boldsymbol{x}$ and $\lim_{t\to\infty}\boldsymbol{x}_i(t) = \sum_{k=1}^p \boldsymbol{\xi}_k^\top\boldsymbol{x}_i(0)\boldsymbol{\zeta}_k$ for all $i \in \mathcal{I}$, where $\boldsymbol{\xi}_k^\top$ and $\boldsymbol{\zeta}_k$ were given in Assumption 1. When the initial conditions are not equal, the opinions converge to (13), which in general corresponds to a persistent disagreement of opinions. We present results for individuals who (i) are slightly stubborn, (ii) have approximately the same stubbornness, and (iii) are extremely stubborn. The proofs, which are not difficult, are provided in [20].

Theorem 2 *For given* $\boldsymbol{C}$ *and* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, *suppose that Assumptions 1 and 2 are satisfied. Suppose further that* (8) *is satisfied. Then, the opinion dynamics system* (4) *with stubborn individuals converges to* (13) *if*

(1) Parameter $b_i \geq 0$ *is sufficiently small, for all* $i \in \mathcal{I}$, *and* $\exists j \in \{1, \ldots, r\} : b_j > 0$.
(2) For some $\alpha > 0$, $b_i = \alpha + \epsilon_i$ *for some sufficiently small* $\epsilon_i \in \mathbb{R}, \forall i \in \mathcal{I}$.

Lemma 3 *For given* $\boldsymbol{C}$ *and* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, *suppose that Assumptions 1 and 2 are satisfied. Then, the opinion dynamics system* (4) *with stubborn individuals converges to* (13) *if* $b_i > 0$ *is sufficiently large, for all* $i \in \mathcal{I}$.

3.3 Convergence for a Class of $\boldsymbol{C}$ *Matrices*

In many opinion dynamics problems, it is desirable to scale the opinions to be in some predefined interval $[-a, a]$, for some scalar $a > 0$, and one *desirable property of an opinion dynamics model* is that opinions starting in $[-a, a]$ remain inside $[-a, a]$ for all time [1]. Supposing that the k^{th} topic represents the discussion of attitudes towards a statement, e.g. "recreational marijuana should be legal", one might scale the opinions so that $x_i^k = a$ represents maximal support for the statement, $x_i^k = 0$ represents a neutral stance, while $x_i^k = -a$ represents maximal rejection of the statement. Now, we explore one set of sufficient requirements on $\mathcal{G}$ and the $\boldsymbol{C}$ which ensures that (2) has this desirable property.

Assumption 3 *The* i^{th} *diagonal entry of the Laplacian matrix* $\mathcal{L}$, *associated with* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, *satisfies* $l_{ii} \leq 1, \forall i$. *The* k^{th} *diagonal of the logic matrix* $\boldsymbol{C}$ *satisfies* $c_{kk} > 0$ *for all* k, *and* $\|\boldsymbol{C}\|_\infty = 1$.

Note that the constraint on the Laplacian entries l_{ii} can always be satisfied by scaling the adjacency matrix weights. We show an invariant set property for (4), and then consider networks without stubborn individuals, and then with stubborn individuals, i.e. (4) with $\boldsymbol{B} = \boldsymbol{0}_{n\times n}$ and then with $\boldsymbol{B} \neq \boldsymbol{0}_{n\times n}$, respectively.

Lemma 4 *Suppose that Assumptions 1, 2, and 3 hold for given* $\boldsymbol{C}$ *and* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. *Suppose further that each individual's opinion evolves according to* (2). *If* $\boldsymbol{x}(0) \in \mathcal{R} \triangleq \{\boldsymbol{x} : x_i^k \in [-a, a], \forall i \in \mathcal{I}, \forall k \in \mathcal{J}\}$ *for some arbitrary but fixed* $a \in \mathbb{R}_+$, *then* $\boldsymbol{x}(t) \in \mathcal{R}$ *for all* $t \geq 0$.

Proof: The proof is in the extended arXiv version [20].

Theorem 3 (No Stubborn Individuals) *Suppose that Assumptions 1 and 3 hold for $\boldsymbol{C}$ and $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. Then for almost all $\boldsymbol{x}(0)$, with each individual's opinions evolving according to* (2) *and $b_i = 0 \,\forall\, i \in \mathcal{I}$, the network reaches a consensus exponentially fast if and only if $\mathcal{G}$ has a directed spanning tree, with consensus values in* (9).

Proof: The necessity of the directed spanning tree is detailed below Assumption 2. Before proving sufficiency, we first derive some properties of the eigenvalues of $\boldsymbol{M} = -\boldsymbol{I}_{nd} + (\boldsymbol{I}_n - \mathcal{L}) \otimes \boldsymbol{C}$. Consider a given $l \in \{1, \ldots, nd\}$. The l^{th} diagonal entry of $\boldsymbol{M}$ is $m_{ll} = -1 + (1 - \sum_{j\in\mathcal{N}_i} a_{ij})c_{kk}$ for some $i \in \mathcal{I}$ and $k \in \mathcal{J}$. The off-diagonal entries of the l^{th} row, m_{lj}, are given by $(1 - l_{ii})c_{kp}$ for all $q \in \mathcal{I}, p \in \mathcal{J}, p \neq k$, and $a_{iq}c_{kp}$ for all $q \in \mathcal{I}, p \in \mathcal{J}$. From Assumption 3, we have $0 < c_{kk} \leq 1$ and $\sum_{j\in\mathcal{N}_i} a_{ij} = l_{ii} \leq 1 \,\forall\, i \in \mathcal{I} \Rightarrow 0 \leq 1 - \sum_{j\in\mathcal{N}_i} a_{ij} \leq 1$. It follows that $m_{ll} \leq 0$ for all $l \in \{1, \ldots, nd\}$. Define the sum of the absolute values of the off-diagonal entries of the l^{th} row of $\boldsymbol{M}$ as $R_l(\boldsymbol{M}) = \sum_{j=1,j\neq l}^{nd} |m_{lj}|$. One can verify (with calculations provided in [20]) that

$$m_{ll} + R_l(\boldsymbol{M}) = -1 + \hat{c}_k \leq 0, \tag{14}$$

where $\hat{c}_k = \sum_{p=1}^{d} |c_{kp}|$ is the sum of the absolute values of the k^{th} row of $\boldsymbol{C}$. This implies that $m_{ll} \leq -R_l(\boldsymbol{M})$, and that this holds for all $l \in \{1, \ldots, nd\}$. Thus, the Geršgorin Circle Theorem indicates that the Geršgorin discs of $\boldsymbol{M}$ are all in the left half-plane [21, Theorem 6.1.1]. Specifically, the discs are either in the open left half-plane ($m_{ll} < -R_l(\boldsymbol{M})$) or touch the imaginary axis at the origin but do not enclose it ($m_{ll} = -R_l(\boldsymbol{M})$, with this including the possibility that $m_{ll} = 0$). This implies that the eigenvalues of $\boldsymbol{M}$ either have strictly negative real part, or are equal to zero. Define $\boldsymbol{A} = (\boldsymbol{I}_n - \mathcal{L}) \otimes \boldsymbol{C}$, with eigenvalue $\lambda_i = (1 - \mu_k)\varphi_l$, where μ_k and φ_l are eigenvalues of $(\boldsymbol{I}_n - \mathcal{L})$ and $\boldsymbol{C}$, respectively.

Because the eigenvalues of $\boldsymbol{M}$ either have strictly negative real part, or are equal to zero, we need to show only that $\lambda_i(\boldsymbol{A}) = \mu_k\varphi_l \neq 1$ for all $k \in \mathcal{I} \setminus \{1\}$ and $l \in \mathcal{J}$ to satisfy (8). Because $\mathcal{L}$ has a simple zero eigenvalue with multiplicity $p \geq 1$ and all other eigenvalues have positive real part, it follows that $\mathfrak{Re}(\mu_k) < 1$ for $k > p$. This implies that $\lambda_i(\boldsymbol{A}) = \mu_k\varphi_j \neq 1$, for all $k \neq 1, j = 1, \ldots, p$ according to Assumption 1. Consider now $k \in \mathcal{I} \setminus \{1\}$ and $l \in \{p+1, \ldots, d\}$. Because $l_{ii} \leq 1$, we conclude the Geršgorin Circle Theorem that $|\mu_k| \leq 1$ [21, Theorem 6.1.1]. Because $c_{ll} > 0, \forall\, l$ and $\|\boldsymbol{C}\|_\infty = 1$, the l^{th} Geršgorin disc of $\boldsymbol{C}$ is centred at $(c_{ll}, 0)$ with radius $1 - c_{ll}$. It follows that $|\varphi_l| < 1$ for $l = \{p+1, \ldots, d\}$. Thus $|\lambda_i| = |\mu_k\varphi_l| \leq |\mu_k||\varphi_l| < 1$ for all $k \in \mathcal{I} \setminus \{1\}$ and $l \in \{p+1, \ldots, d\}$; (8) of Theorem 1 is satisfied. ☐

Theorem 4 (Stubborn Individuals) *For given $\boldsymbol{C}$ and $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, suppose that Assumptions 1, 2, and 3 hold. If each individual's opinions evolve according to* (2), *$b_i \geq 0, \forall\, i \in \mathcal{I}$, and $\exists j \in \{1, \ldots, r\} : b_j > 0$, then the system* (4) *converges to* (13) *exponentially fast.*

Proof: Notice that $\bar{\boldsymbol{M}}$ in (12) can be expressed as $\bar{\boldsymbol{M}} = \boldsymbol{M} - \boldsymbol{B} \otimes \boldsymbol{I}_d$, where $\boldsymbol{M} = -\boldsymbol{I}_{nd} + (\boldsymbol{I}_n - \mathcal{L}) \otimes \boldsymbol{C}$ was defined in the proof of Theorem 1. We showed in that proof that the Geršgorin discs of $\boldsymbol{M}$ were in the closed left half-plane, and the discs could at most touch the origin, but not enclose it. This implied that $\boldsymbol{M}$ has eigenvalues that either have strictly negative real part, or are at the origin. If $b_i \geq 0, \forall\, i \in \mathcal{I}$ then $-\boldsymbol{B} \otimes \boldsymbol{I}_d$ is a diagonal matrix with nonpositive diagonal entries. It follows that from the Geršgorin Circle Theorem that, for $i \in \mathcal{I}$, the $(i-1)d+1, \ldots, (i-1)d+d$ Geršgorin discs of $\bar{\boldsymbol{M}}$ are the $(i-1)d+1, \ldots, (i-1)d+d$ Geršgorin discs of $\boldsymbol{M}$, with the same radius, but shifted along the real axis to the left by $b_i \geq 0$. Thus, by proving that $\bar{\boldsymbol{M}}$ is invertible (as we shall now do) it follows that all eigenvalues of $\bar{\boldsymbol{M}}$ have negative real part.

To establish a contradiction, suppose that $\bar{\boldsymbol{M}} = -\boldsymbol{I}_{nd} + (\boldsymbol{I}_n - \mathcal{L}) \otimes \boldsymbol{C} - \boldsymbol{B} \otimes \boldsymbol{I}_d$ is singular. Then there exists a nonzero vector $\boldsymbol{x} \in \mathbb{R}^{nd}$ such that $\bar{\boldsymbol{M}}\boldsymbol{x} = \boldsymbol{0}_{nd}$. This implies that $((\boldsymbol{B} + \boldsymbol{I}_n) \otimes \boldsymbol{I}_d)\,\boldsymbol{x} = ((\boldsymbol{I}_n - \mathcal{L}) \otimes \boldsymbol{C})\,\boldsymbol{x}$, or

$$\boldsymbol{x} = \left(((\boldsymbol{B} + \boldsymbol{I}_n)^{-1}(\boldsymbol{I}_n - \mathcal{L})) \otimes \boldsymbol{C}\right)\boldsymbol{x} \tag{15}$$

with $\boldsymbol{B} + \boldsymbol{I}_n$ invertible because $b_i \geq 0 \,\forall\, i \in \mathcal{I}$. Obviously, (15) holds if and only if $\boldsymbol{N} = ((\boldsymbol{B} + \boldsymbol{I}_n)^{-1}(\boldsymbol{I}_n - \mathcal{L})) \otimes \boldsymbol{C}$ has an eigenvalue at 1. We prove by contradiction that $\boldsymbol{N}$ does not have an eigenvalue at 1. Denote the eigenvalues of $\boldsymbol{N}$, $(\boldsymbol{B} + \boldsymbol{I}_n)^{-1}(\boldsymbol{I}_n - \mathcal{L})$, and $\boldsymbol{C}$ as ψ_i, $\bar{\mu}_k$, and φ_l, respectively. [19, Proposition 7.1.10] indicates that $\psi_i = \bar{\mu}_k\varphi_l$, for $k \in \mathcal{I}$ and $l \in \mathcal{J}$. Assumption 1 yields that $\varphi_r = 1, r = 1, \ldots, p$. Next, the Geršgorin Circle Theorem yields that under Assumption 3, the eigenvalues of $\boldsymbol{I}_n - \mathcal{L}$ are in the closed unit circle [21, Theorem 6.1.1]. Because the $(i-1)d+1, \ldots, (i-1)d+d$ rows of $(\boldsymbol{B} + \boldsymbol{I}_n)^{-1}(\boldsymbol{I}_n - \mathcal{L})$ are equal to the rows of $\boldsymbol{I}_n - \mathcal{L}$ scaled by $(b_i + 1)^{-1} \leq 1$, it follows from the Geršgorin Circle Theorem that $|\bar{\mu}_k| \leq 1$ [21, Theorem 6.1.1].

Using the same arguments as in the last paragraph of the proof of Theorem 3, one can establish that under Assumption 3, $\psi_i = \bar{\mu}_k\varphi_l \neq 1$ for $k \in \mathcal{I}$ and $l \in \{p+1, \ldots, d\}$. Thus, $\boldsymbol{N}$ has an eigenvalue at 1 if and only if $\psi_i = \bar{\mu}_k\varphi_r = 1$ for some $k \in \mathcal{I}$ and $r = 1, \ldots, p$. Since $\varphi_r = 1$, this implies that $\exists k : \bar{\mu}_k = 1$, i.e. for some nonzero $\boldsymbol{r} \in \mathbb{R}^n$ there holds $(\boldsymbol{B} + \boldsymbol{I}_n)^{-1}(\boldsymbol{I}_n - \mathcal{L})\boldsymbol{r} = \boldsymbol{r}$ or equivalently $(\boldsymbol{B} + \mathcal{L})\boldsymbol{r} = \boldsymbol{0}_n$. In other words, $\boldsymbol{B} + \mathcal{L}$ must be singular if $\boldsymbol{N}$ has an eigenvalue at 1. With $\mathcal{L}$ expressed in lower block triangular form as in (1), the arguments below (1) establish that $\mathcal{L}_{11} + \text{diag}[b_1, \ldots, b_r]$ and $\mathcal{L}_{22} + \text{diag}[b_{r+1}, \ldots, b_n]$ are separately nonsingular, because $b_i \geq 0, \; \forall\, i \in \mathcal{I}$ and $\exists j \in \{1, \ldots, r\} : b_j > 0$. Thus, $\boldsymbol{B} + \mathcal{L}$ is nonsingular, and it follows that $\boldsymbol{N}$ does not have an eigenvalue at 1, and thus $\bar{\boldsymbol{M}}$ is Hurwitz. Lemma 2 yields that the final opinions are as in (13). □

Remark 3 *Assumption 3 places constraints on both the logic matrix $\boldsymbol{C}$ and entries of the network Laplacian $\mathcal{L}$ (note that $c_{kk} > 0$ simply implies that the k^{th} topic is positively dependent on itself). The constraints are placed to ensure that if $\boldsymbol{x}(0) \in \mathcal{R}$, where $\mathcal{R}$ is defined in Lemma 4, then $\mathcal{R}$ is an invariant set of (4). As it turns out, the same constraints are sufficient for convergence of Model 1, as identified in Theorems 3 and 4. Generally speaking, if Assumption 3 does not hold, then Model 1 may still converge, even though the desired invariance property may no longer hold.*

4 Networks of Individuals with Model 2

Perhaps unsurprisingly, the lack of $\boldsymbol{C}$ in the first summand on the right of (3) simplifies the analysis, allowing for the establishing of a more comprehensive set of results. Two theorems establish conditions for convergence in networks where (3) has $b_i = 0\,\forall\, i \in \mathcal{I}$ and $b_i > 0$ for some $i \in \mathcal{I}$, respectively. Afterwards, we discuss and compare Model 1 and Model 2.

4.1 Analysis of Model 2

Theorem 5 (No Stubborn Individuals) *Suppose that Assumption 1 holds for a given $\boldsymbol{C}$. Then for almost all $\boldsymbol{x}(0)$, with each individuals' opinions evolving according to (3) with $b_i = 0\,\forall\, i \in \mathcal{I}$, the social network reaches a consensus on all topics exponentially fast if and only if $\mathcal{G}$ has a directed spanning tree, with limiting opinions*

$$\lim_{t\to\infty} \boldsymbol{x}_i(t) = \Big(\sum_{r=1}^{p} \boldsymbol{\zeta}_r \boldsymbol{\xi}_r^\top\Big) \sum_{j=1}^{n} \gamma_j \boldsymbol{x}_j(0),\ \forall\, i \in \mathcal{I}. \quad (16)$$

Proof: The necessity of having a directed spanning tree is explained below Assumption 2. For the proof of sufficiency, observe that convergence of the system (3) with $\boldsymbol{B} = \boldsymbol{0}_{n\times n}$ depends on the eigenvalue properties of the matrix $\boldsymbol{A} = \mathcal{L} \otimes \boldsymbol{I}_d + \boldsymbol{I}_n \otimes (\boldsymbol{I}_d - \boldsymbol{C})$. From [19, Proposition 11.1.7], we obtain that the solution of the system is

$$\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{x}(0) = \big(\boldsymbol{N}_1(t) \otimes \boldsymbol{N}_2(t)\big)\boldsymbol{x}(0), \quad (17)$$

with matrix exponentials $\boldsymbol{N}_1(t) = e^{-\mathcal{L}t}$ and $\boldsymbol{N}_2(t) = e^{-(\boldsymbol{I}_d - \boldsymbol{C})t}$. Since $\mathcal{G}$ has a directed spanning tree, we know that $\lim_{t\to\infty} \boldsymbol{N}_1(t) = \boldsymbol{1}_n \boldsymbol{\gamma}^\top$, where $\boldsymbol{\gamma}$ is defined in Lemma 1. Under Assumption 1, one can obtain $\lim_{t\to\infty} \boldsymbol{N}_2(t) = \sum_{r=1}^{p} \boldsymbol{\zeta}_r \boldsymbol{\xi}_r^\top \triangleq \boldsymbol{Y}$. It follows that

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \big((\boldsymbol{1}_n \boldsymbol{\gamma}^\top) \otimes \boldsymbol{Y}\big)\boldsymbol{x}(0) = \boldsymbol{1}_n \otimes \big((\boldsymbol{\gamma}^\top \otimes \boldsymbol{Y})\boldsymbol{x}(0)\big).$$

Thus, the final opinions of each individual are at the consensus value of $(\boldsymbol{\gamma}^\top \otimes \boldsymbol{Y})\boldsymbol{x}(0) = \boldsymbol{Y} \sum_{i=1}^{n} \gamma_i \boldsymbol{x}_i(0)$. □

Theorem 6 (Stubborn Individuals) *Suppose that Assumptions 1 and 2 hold for given $\boldsymbol{C}$ and $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. If each individual's opinions evolve according to (3), $b_i \geq 0\,\forall\, i \in \mathcal{I}$, and $\exists j \in \{1, \ldots, r\} : b_j > 0$, then the system (4) converges exponentially fast to*

$$\lim_{t\to\infty} \boldsymbol{x}(t) = \Big((\mathcal{L} + \boldsymbol{B}) \otimes \boldsymbol{I}_d + \boldsymbol{I}_n \otimes (\boldsymbol{I}_d - \boldsymbol{C})\Big)^{-1} \times (\boldsymbol{B} \otimes \boldsymbol{I}_d)\boldsymbol{x}(0). \quad (18)$$

Proof: With $\mathcal{L}$ expressed in lower block triangular form as in (1), the arguments below (1) establish that all eigenvalues of $\mathcal{L}_{11} + \text{diag}[b_1, \ldots, b_r]$ and $\mathcal{L}_{22} + \text{diag}[b_{r+1}, \ldots, b_n]$ have strictly positive real part, because $b_i \geq 0\,\forall\, i \in \mathcal{I}$ and $\exists j \in \{1, \ldots, r\} : b_j > 0$. Thus, $-(\mathcal{L} + \boldsymbol{B})$ is Hurwitz. Let $\boldsymbol{M} = (\mathcal{L} + \boldsymbol{B}) \otimes \boldsymbol{I}_d + \boldsymbol{I}_n \otimes (\boldsymbol{I}_d - \boldsymbol{C})$. From [19, Proposition 7.2.3], we have that $\lambda_i = \mu_j + \rho_k$, where $\lambda_i, i \in \{1, \ldots nd\}$, $\mu_j, j \in \mathcal{I}$ and $\rho_k, k \in \mathcal{J}$ are eigenvalues of $\boldsymbol{M}$, $\mathcal{L} + \boldsymbol{B}$, and $(\boldsymbol{I}_d - \boldsymbol{C})$, respectively. Since $-(\mathcal{L} + \boldsymbol{B})$ is Hurwitz and $-(\boldsymbol{I}_d - \boldsymbol{C})$ has p zero eigenvalues and all other eigenvalues have negative real part, it follows that $-\boldsymbol{M}$ is Hurwitz. Convergence to (18) is concluded from Lemma 2. □

4.2 Discussions on Model 1 and Model 2

To place the focus on the effects of the logic matrix $\boldsymbol{C}$, let us compare networks with Model 1 and Model 2 having no stubborn individuals, i.e. the systems (4) and (3) with $\boldsymbol{B} = \boldsymbol{0}_{n\times n}$. Notice that if (8) is satisfied, then the consensus value is the same for both systems, as recorded in (9) and (16). Interestingly, this implies that *if a consensus is reached*, then the presence of the matrix $\boldsymbol{C}$ in the interpersonal influence term has no effect on the limiting opinion values. In other words, if (8) is satisfied, then it does not matter whether individuals exchange opinions that have or have not been first assimilated using the logic matrix $\boldsymbol{C}$. However, (8) may not always be satisfied, implying that Model 1 is unstable, even if the conditions for Model 2 to be stable are fulfilled.

When comparing Theorem 1 with Theorem 5, the necessary and sufficient condition for consensus is much simpler for Model 2 compared with Model 1. Specifically, in Model 2, the only requirements on $\boldsymbol{C}$ and $\mathcal{G}$ are **separately** Assumption 1 and Assumption 2 ($\mathcal{G}$ has a directed spanning tree). This is similar to the discrete-time model in [1], where consensus (with no stubborn individuals) is reached by the system $\boldsymbol{x}(k + 1) = (\boldsymbol{W} \otimes \boldsymbol{D})\boldsymbol{x}(k)$ if and only if $\lim_{k\to\infty} \boldsymbol{D}^k$ exists and either (i) $\lim_{k\to\infty} \boldsymbol{D}^k = \boldsymbol{0}_{d\times d}$, or (ii) $\lim_{k\to\infty} \boldsymbol{W}^k = \boldsymbol{1}_n \boldsymbol{v}^\top$ for some nonnegative vector $\boldsymbol{v}$. Here, $\boldsymbol{D}$ and $\boldsymbol{W}$ is the discrete-time counterpart to $\boldsymbol{C}$ and $\mathcal{L}$, respectively. In contrast, for Model 1, the necessary and sufficient condition (8) clearly depends on the combination of the network topology and the logical interdependence, encoded by $\mathcal{L}$ and $\boldsymbol{C}$, respectively. For a given $\boldsymbol{C}$, two different $\mathcal{G}_1$ and $\mathcal{G}_2$ may have different convergence properties.

Without logical interdependence, i.e. $\boldsymbol{C} = \boldsymbol{I}_d$, then the consensus value is $\sum_{j=1}^{n} \gamma_j \boldsymbol{x}_j(0)$. Defining the projection matrix $\boldsymbol{Y} = \sum_{r=1}^{p} \boldsymbol{\zeta}_r \boldsymbol{\xi}_r^\top$, we see that the effect of the logical interdependence is to project $\sum_{j=1}^{n} \gamma_j \boldsymbol{x}_j(0)$ onto the range space of $\boldsymbol{Y}$, which is equivalent to the nullspace of $\boldsymbol{I}_d - \boldsymbol{C}$. Put another way, the final consensus value in (9) or (16) is consistent with the logical interdependence structure, being a fixed point of $\boldsymbol{C}$. Recent work on the discrete-time model [22] indicates that heterogeneity of $\boldsymbol{C}$ may create disagreement in the final opinions, even without stubborn individuals. It would be of great interest to investigate whether the same phenomena holds in either Model 1 or Model 2 (or both).

Concerning Model 1, Corollaries 1 and 2 show that for any given $\boldsymbol{C}$, it is always possible to reach a consensus if there is a sufficiently slow exchange of opinions (weights a_{ij} are small). Sets of topics whose $\boldsymbol{C}$ have large e_k and θ_k close to $\pi/2$ (as defined in Corollary 2) reflect a cognitive process (7) where the opinions oscillate heavily and rapidly before settling to a consistent belief system. In such instances, the bound (11) becomes $\bar{l} < \frac{|1-|\lambda_k|\cos(\theta_k)|(1+\cos(\theta_k))}{|\lambda_k|\sin^2(\theta_k)} = \frac{(1-d_k)(1+\cos(\theta_k))}{e_k \sin(\theta_k)} < 0.5$. Thus, one method of guaranteeing consensus when individuals exchange assimilated opinions $\boldsymbol{C}x_i(t) - \boldsymbol{C}x_j(t)$ is to reduce the rate of interaction, by decreasing a_{ij} to satisfy (11). On the other hand, rapid discussions *may* lead to instability. We stress the word *may*, because sometimes there is no risk. For instance, if $\boldsymbol{C}$ is lower triangular (which reflects a cascade logic structure) and satisfies Assumption 1, then Model 1 will always reach consensus for any magnitude of the a_{ij}. When topics are uncoupled, $\boldsymbol{C} = \boldsymbol{I}_d$, or the logical interdependences have not been assimilated into the exchanged opinions (Model 2), there is never such risk.

In psychology and organisational science, it has been observed that an individual can experience *cognitive overload* when receiving external information at a high density that overwhelms their internal capacity to process the information, leading to a decrease in decision making abilities and response time to stimulus [23,24]. When considering the above discussion, it appears that the instability that can arise in Model 1 when increasing exchange of assimilated opinions resembles behaviour similar to cognitive overload. More precisely, one interpretation of the results on Model 1 is that the internal process should be not "overwhelmed" by exogenous social influence because individuals are assimilating logical interdependences into the exchanged opinions, as captured by the term $\sum_{j \in \mathcal{N}_i} a_{ij}(\boldsymbol{C}x_i(t) - \boldsymbol{C}x_j(t))$.

One may wish to model the interpersonal influence process and introspective process as having different time-scales. Assuming $b_i = 0$ for all i for simplicity of exposition, one possible approach is to multiply the first and second terms of (2) with positive scalars α and β, respectively, with the relative magnitudes of α and β capturing the relative time-scales of the two processes. The same can be done for Model 2, in (3). The clear distinguishing between assimilation of logical interdependence into the opinions, and the introspective process, as detailed in Remark 1, enables one to model relative time-scales as suggested. The analysis above indicates that for Model 2, the convergence condition in Theorems 5 continues to apply irrespective of α and β. In contrast, it is possible that a fast interpersonal influence process, viz. large α, can yield instability for Model 1 (see Theorem 1, and Corollary 1). We illustrate this with simulations below, and leave further investigations to future work.

5 Simulations

We now present a short simulation to illustrate some key results. We consider a network $\mathcal{G}$ of 8 nodes, with the associated Laplacian $\mathcal{L}$ is given in our arXiv extended version [20]. The coupling matrix is given by

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ -0.1 & 0.2 & 0.7 \\ 0.1 & -0.8 & 0.1 \end{bmatrix} \tag{19}$$

Each component of the initial condition vector $\boldsymbol{x}(0)$ is generated from independent uniform distributions in the interval $[-1, 1]$. Initially, we assume $b_i = 0, \forall i \in \mathcal{I}$.

The given $\mathcal{L}$ and $\boldsymbol{C}$ satisfy Assumptions 1, 2 and 3, and also satisfy the condition of (8) in Theorem 1. The dynamics of Model 1 and Model 2 are shown in Fig. 1 and 2, respectively. We see that the opinions for all 3 topics reach a consensus, and as discussed in Section 4.2, the final consensus value for both models is the same, though the transients differ. For the same $\mathcal{L}$ and $\boldsymbol{x}(0)$, Fig. 3 shows the case where the topics are uncoupled with $\boldsymbol{C} = \boldsymbol{I}_d$. With $\boldsymbol{C}$ as in (19), Topic 2 is coupled to Topics 1 and 3 by a negative and positive weight respectively. The coupling effect is clear: the consensus value of Topic 2 in Fig. 1 is further from the consensus value of Topic 1 and closer to the consensus value of Topic 3 when compared to Fig. 3. Next, we introduce stubbornness, with $\boldsymbol{b} = [0, 0.1, 0, 0.05, 0, 0.4, 0, 0.3]^\top$, and the same $\boldsymbol{x}(0)$, $\mathcal{L}$, and $\boldsymbol{C}$ as above. Consistent with Theorem 4, opinions converge to a persistent disagreement as shown in Fig. 4. Last, we return to $b_i = 0, \forall i \in \mathcal{I}$, and use the same $\boldsymbol{x}(0)$ and $\boldsymbol{C}$ (as in (19)), but each edge weight is multiplied by 3, i.e. the new Laplacian $\bar{\mathcal{L}}$ satisfies $\bar{\mathcal{L}} = 3\mathcal{L}$. Eq. (8) of Theorem 1 is not satisfied; we see from Fig. 5 that the opinions diverge for Model 1, but consensus is still achieved for Model 2, see Fig. 6.

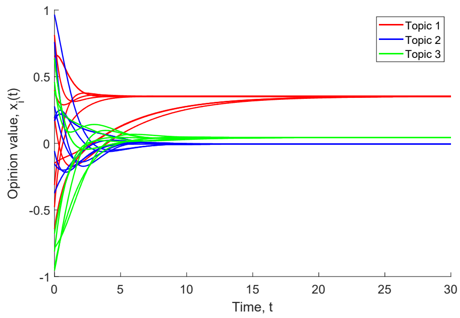

Fig. 1. Model 1: Consensus is reached on all 3 topics when (8) in Theorem 1 is satisfied.

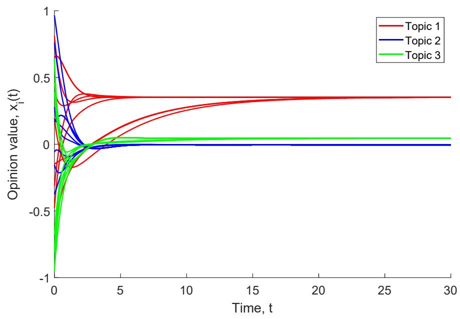

Fig. 2. Model 2: Consensus is reached when $\boldsymbol{C}$ and $\mathcal{G}$ separately satisfy Assumption 1 and 2.

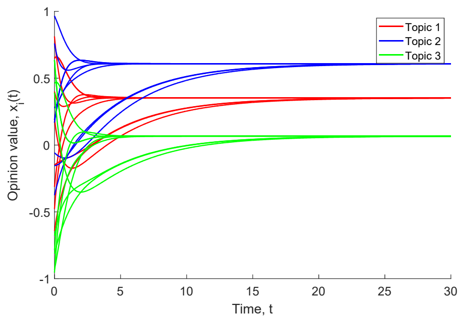

Fig. 3. Model 1 and 2: When the topics are uncoupled, $\boldsymbol{C} = \boldsymbol{I}_d$, consensus is reached but the final consensus values are different due to the lack of logic coupling.

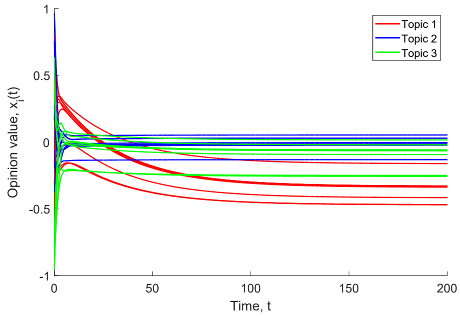

Fig. 4. Model 1: In the presence of stubborn individuals, a state of persistent disagreement is achieved when the conditions in Theorem 4 are met.

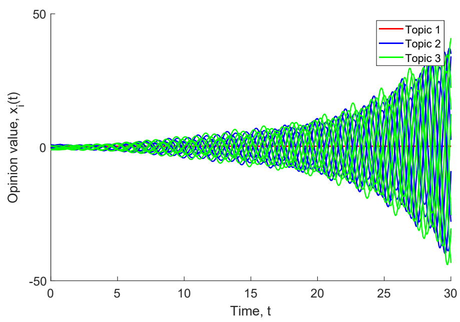

Fig. 5. Model 1: Increasing the interpersonal interaction strengths means (8) of Theorem 1 is not satisfied, and the opinion system becomes unstable.

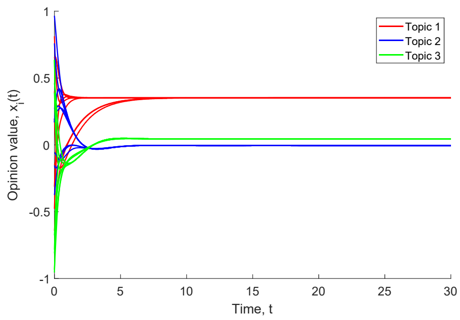

Fig. 6. Model 2: Increasing the interpersonal interaction strengths changes the transient, but consensus is still achieved.

6 Conclusions

In this paper, we have proposed two continuous-time opinion dynamics model for a social network discussing opinions on multiple logically interdependent topics. When there are no stubborn individuals in the network, separate necessary and sufficient conditions are derived for networks to achieve a consensus of opinions in both models. The condition for Model 1 depends on the interplay between the logic coupling matrix and the graph topology, which is in contrast to Model 2, where separate conditions on the logic matrix and graph Laplacian matrix need to be satisfied. Further sufficient conditions for consensus were obtained for Model 1, to better understand the role of the logic matrix and graph topology. Networks with stubborn individuals were studied for both models, with sufficient conditions obtained for the opinions to converge to a limit. Future work will involve further analysis of networks with stubborn individuals in Model 1, and to consider heterogeneous logic matrices among the individuals.

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