Department of Mathematics and Statistics

Existence of solutions to various classes of nonlinear elliptic differential equations and their applications

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Declaration

To the best of my Knowledge and belief this thesis contains no material previously published by any other person except where due acknowledge has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

Signature:

Date: October 2020

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Abstract

This thesis studies various classes of nonlinear elliptic differential equations with nonlocal terms, which arise from many disciplinary fields such as quantum electrodynamics, semiconductor theory and chemistry. By using the variational method and the topological method, the existence and nonexistence of solutions, multiple solutions and symmetric properties of solutions have been established for various classes of nonlinear elliptic partial differential equations with nonlocal terms.

The research includes two key parts. The first part, presented in Chapter 3 and Chapter 4, is concerned with a class of nonlinear elliptic differential equations with variational structure, which are the Euler-Lagrange equations of variational functionals. We prove the existence of a nontrivial solution and multiple solutions to elliptic equations with nonlocal terms by searching for the critical points of the related variational functional. By using the critical point theory, we establish the bounded sequence of the approximate solution to prove the existence of solutions to the corresponding equations. On the other hand, by developing the scaling technique, we establish the sufficient condition for breaking symmetry of the least energy solution of the elliptic equations with nonlocal terms. A sufficient condition for the non-existence of solutions is also obtained for the nonlinear elliptic equation with variational structure.

The second part, presented in Chapter 5, is concerned with a class of elliptic differential equations without variational structure, which cannot be regarded as an Euler-Lagrange equation of variational functional. By using the Leray-Schauder degree theory and the method of upper and lower solutions, we prove the existence of solutions to a class of elliptic equations with nonlocal terms. We also develop a new method to establish the non-existence of solution by a prior estimate to the solutions.

Keywords: Elliptic equation, nonlocal nonlinearity, variational methods, Leray-Schauder degree

List of Publications Related to This Thesis

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Contents

Chapter 1

Introduction

1.1 General

It is widely known that many phenomena in nature and industrial processes can be described by nonlinear partial differential equations. Various classes of nonlinear partial differential equations have attracted worldwide attention from many scholars because of their strong practical application, which also bring many challenging problems to mathematicians. Nonlinear partial differential equations have become an important bridge between nonlinear problems in pure mathematics and natural sciences. In addition, the study of nonlinear partial differential equations based on practical problems has become a very important research field in applied mathematics. For example, the gauged field equations, including the gauged Schrödinger equation in the Maxwell field, are the most important mathematical model to study the gravitational force, the strong force, the weak force and the electromagnetic interaction between objects in Physics. The variational method, the topological degree theory, the fixed point theorem and the upper and lower solutions method in nonlinear analysis are widely used in the study of these differential equations arising from physical problems. The existence of solutions, the multiplicity of solutions and symmetric property of solutions to the related nonlinear equations promote the development of a series of profound mathematical theories. Many mathematicians carried out in-depth research on the related mathematical theories. For example, Strauss [102] studied the compact embedding theory of Sobolev space with symmetric functions; Lions [13, 75, 76] developed the principle of concentration and compactness; Rabinowitz [88] established the global bifurcation theory with applications to the ordinary and partial differential equation; Gidas-Ni-Nirenberg [42] used the moving plane method to study the symmetric property of positive solutions to the scalar field equation, etc.

This thesis focuses on the research of the existence and properties of solutions to various classes of elliptic equations, which arise in the study of the solitary wave solution of the gauged Schrödinger equation in the Maxwell field. Under the Coulomb gauge condition, the Maxwell field can be expressed as the integral form of a wave function [15], which forms the standard nonlocal term of nonlinear elliptic equations for the study of the solitary wave solution of the gauged Schrödinger equation in the Maxwell field. Usually, these kinds of elliptic equations with nonlocal terms related to the Maxwell field are known as Schrödinger-Poisson equations [15], which is one of the non-local problems of partial differential equations that have attracted much attention in recent years. There had been many literatures discussing the topic of standing wave solutions to the Schrödinger-Poisson equations with various potential functions (see [2] and the references therein). Due to the opposite signs of non-local terms and local nonlinear terms, the variational functional corresponding to the Schrödinger-Poisson equation is rich in geometric structure in the general Sobolev space [91]. It is not easy to obtain a bounded approximation solution sequence by using the critical point theory directly. The existence of standing wave solutions becomes a challenging problem due to the special geometric structure of variational functional. Meanwhile, there exist new phenomena in the property of the standing wave solution owing to the appearance of standard nonlocal terms. For instance, there exist a priori estimates for the infinite modulus of the solution to the Schrödinger-Poisson equations [57]; the existence of non-radial symmetric least energy solution for the Schrödinger-Poisson equation was obtained in the bounded domain [92]. The study of the above mentioned new phenomena is mainly limited to the Schrödinger-Poisson equation with standard local and nonlocal nonlinear terms. However, it is inevitable to consider small error or perturbation on local and nonlocal nonlinear terms in some special applications such as seeking numerical solutions, in which case the methods established for solving the standard models are invalid for overcoming the difficulties caused by the nonstandard model with small error or perturbation. To study these nonlinear models with small error or perturbation, we have to consider a class of elliptic equations with nonlocal terms under general assumptions.

In the following, we briefly present the objectives and outline of this thesis.

1.2 Objectives

Our main objectives are to study the existence of solution, multiple solutions and symmetric properties of solutions to various classes of elliptic equations under general assumptions on local terms and general nonlocal terms. The specific objectives are listed as follows.

(1) Establish the existence of solution and multiple solutions to a class of elliptic equations with general local nonlinearity.

We aim to investigate the existence of ground state solution and multiple solutions to the Schrödinger-Poisson equations with general local nonlinearity. The Schrödinger-Poisson equations are equivalent to the following elliptic equation with a nonlocal term.

$$-\Delta u + V(x)u + u(x)\int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}dy = f(u),\ x \in \mathbb{R}^3, \tag{1.1}$$

where $V(x)$ is a potential function; $u(x)\int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}dy$ is the nonlocal term and $f(u)$ is the general local nonlinearity. The general assumptions in this thesis on the potential and nonlinearity cover the case of harmonic potential and the Slater approximation exchange term, which are relevant to various physics models. From this point of view, we verify the stability of the Slater approximation to the exchange term of the many-body system. Specifically, we assume that

$$V(x) \text{ is a continuous function with } \lim_{|x|\to+\infty} V(x) = +\infty;$$

and

$$|f(u)| \leq \tau(|u| + |u|^p) \text{ holds for some } \tau > 0 \text{ and } 1 < p < 2,$$

which cover a large class of perturbations to the Slater approximation exchange term $|u|^{2/3}u$. We aim to establish the existence of nontrivial solutions of (1.1) under general assumptions by showing the existence of critical points of the following variational functional

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 dx + \frac{1}{16\pi}\int_{\mathbb{R}^6} \frac{u^2(x)u^2(y)}{|x-y|} dxdy - \int_{\mathbb{R}^3} F(u)dx,$$

where $F(u) = \int_0^u f(s)ds$. To overcome the difficulties arising from the general assumption, we have to develop new methods to study the complex geometric structure of the above functional.

(2) Study the existence of solutions and symmetric properties of solutions to the elliptic equations with nonlocal nonlinearity by using the variational technique.

The term $u(x)\int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}dy$ in (1.1) is a standard nonlocal nonlinearity arising from the Maxwell field. When a small error or perturbation occurs to the standard nonlocal nonlinearity, many authors considered the following nonlinear elliptic equation with nonlocal term,

$$-\Delta u + V(x)u + \frac{\lambda}{4\pi}|u|^{q-2}u(x)\int_{\mathbb{R}^3} \frac{|u(y)|^q}{|x-y|}dy = |u|^{p-1}u, \; x \in \mathbb{R}^3, \quad (1.2)$$

where $V(x)$ is the potential function; $p, q > 1$ are the nonlinear exponents; $\lambda \in \mathbb{R}$ is a parameter. When $q = 2$, (1.2) is equivalent to the Schrödinger-Poisson equation (1.1). So far, very few literature has showed the symmetry of solutions to (1.2) with general nonlinear exponent $q > 1$. We aim to study the effects of changing values of the nonlinear exponents p, q and the parameter λ on the existence and symmetric properties of solutions to the general elliptic equation model (1.2). Various methods have been established to study the symmetric properties of solutions of elliptic equations. However, the opposite signs of nonlocal terms and local nonlinear terms in (1.2) make the classical method above not valid for the study of (1.2). We aim to develop the scaling technique to study the symmetric properties of the least energy solution of (1.2).

(3) Study the existence, nonexistence and a priori estimate of solutions to nonlinear elliptic equations without variational structure.

We consider the elliptic equation with more general nonlocal nonlinearity than (1.2).

$$-\Delta u + V(x)u + \lambda |u|^{r-2}u(x)\int_{\Omega} |u(y)|^q G(x,y)dy = |u|^{p-1}u,\ x \in \Omega. \quad (1.3)$$

where p, q, r are nonlinear exponents; λ is a parameter; $\Omega = \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$ is a bounded domain and $G(x,y)$ is the Green function defined in the domain Ω with homogeneous Dirichlet boundary conditions. When $\Omega = \mathbb{R}^3$, we get that $G(x,y) = \frac{1}{4\pi|x-y|}$; and it is clear to see that (1.3) with $r = q$ is equivalent to (1.2), which is the Euler-Lagrange equation in regard to the functional

$$\frac{1}{2}\int_{\Omega} |\nabla u|^2 + V(x)u^2 dx - \frac{1}{p}\int_{\Omega} |u|^{p+1}dx \\ + \frac{\lambda}{2q}\int_{\Omega\times\Omega} G(x,y)|u(x)|^q|u(y)|^q dxdy.$$

One could obtain the existence of non-trivial solution of (1.3) with $r = q$ by seeking the non-trivial critical points of the above functional. However, (1.3) with $r \neq q$ is a complex nonlinear problem without variational structure. The variational methods is invalid for the study of (1.3) with general nonlinear exponents p, q and r. One of our objectives is to develop topological methods such as the method of upper and lower solutions and the Leray-Schauder degree theory for studying the existence, nonexistence and a priori estimate of solutions to (1.3) with general nonlinear exponents.

1.3 Outline of the thesis

This thesis consists of 6 chapters. Chapter one presents the objectives of the research, the background and the scope of the thesis. Chapter two gives a review of literature, which discusses the current research status, methods and progress in studying the existence and properties of solutions to various classes of nonlinear elliptic equations. The key research results are presented in Chapter 3-5.

In Chapter 3, we consider the existence of a ground state solution, multiple solutions and non-existence of solution of the Schördinger-Poisson equation (1.1) with $V(x) = W(x) - \mu$, where $W(x)$ is a potential, μ is a parameter. If $V(x) = W(x) - \mu$, the equation (1.1) is equivalent to the following system,

$$\begin{cases} -\Delta u + (W(x) - \mu)u + \phi(x)u = f(u), \\ -\Delta \phi = u^2, \lim\limits_{|x|\to+\infty} \phi(x) = 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $f(u)$ is the nonlinearity. We consider the potential function and nonlinearity with general assumptions. Motivated by methods in [55, 59], we first prove that the variational functional of (1.4) satisfies the Palais-Smale compactness condition under general assumptions. Then, we obtain the existence of solution and multiple solutions to (1.4) by applying the variation method. Moreover, we also get the nonexistence of solution to (1.4).

In Chapter 4, we investigate the nonlinear elliptic equation (1.2) with $V(x) = |x|^2 - \mu$, where $|x|^2$ is a harmonic potential, μ is a parameter. The elliptic equation is equivalent to the following system,

$$\begin{cases} -\Delta u + (|x|^2 - \mu)u + \lambda\phi(x)|u|^{q-2}u = |u|^{p-1}u, \ x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^q, \ \lim_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{1.5}$$

where $p, q > 1$ are nonlinear exponents. Suppose that $1 < p < q < 5$ and $\lambda > 0$. For some $\mu \in \mathbb{R}$, it is proved that there exist multiple solutions of (1.5). Assume $q \in [2, 5)$, $p \in (\frac{8q-5}{7}, q)$. If the parameters $\lambda, \mu > 0$ are close to 0, we prove that the solution with the least energy of the problem (1.5) is not radially symmetric.

In Chapter 5, we consider the nonlinear elliptic equation (1.3), which is equivalent to the following system,

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi(x)|u|^{r-2}u = |u|^{p-1}u, \ x \in \Omega, \\ -\Delta\phi(x) = |u|^q, \ x \in \Omega, \\ \phi(x) = u(x) = 0 \text{ for } x \in \partial\Omega, \end{cases} \tag{1.6}$$

where λ is a parameter, $V(x)$ is a potential and $p, q, r > 1$. For $r, q \in (1, 5)$ and $p \in (1, \min\{r, q\})$, let

$$c_{p,r,q} = \frac{p(p-1)}{r(q-1)} \left(\frac{r-p}{r}\right)^{\frac{r-p}{p}} \left(\frac{q-p}{q-1}\right)^{\frac{(r-p)(q-p)}{p-1}} > 0.$$

If $V(x) = 1$, $\Omega \subset \mathbb{R}^3$ is a bounded domain and $\lambda \geq c_{p,r,q}$, we prove that the solution of (1.6) is trivial; if $\lambda > 0$ is small enough, we prove that (1.6) has a positive solution by using the Leray-Schauder degree method. By using the method of upper and lower solutions, we also obtain the sufficient condition for the existence of a positive solution to (1.6) for the case $\Omega = \mathbb{R}^3$.

In Chapter 6, we summarize the results and innovations of our thesis about the existence of solution, multiple solutions and symmetric properties of solutions to nonlinear elliptic equations with nonlocal terms.

Chapter 2

Literature Review

Nonlinear partial differential equations involve a very wide range of research fields. Many researchers have devoted to study the existence of solutions and multiple solutions of nonlinear elliptic equations. Many in-depth theories have been established in nonlinear analysis for solving the problems related to nonlinear elliptic equations [20, 60, 96, 97, 127]. In this thesis, we are interested in three kinds of nonlinear elliptic differential equations with nonlocal terms, which arise from studying the physical models such as the Hartree-Fock system and the gauged field theory [15, 82, 92].

In this chapter, we review the related literature by presenting the related mathematical models arising from Physics and the methods used in the study of the existence and symmetric properties of solutions to nonlinear elliptic equations.

2.1 Previous studies on nonlinear models arising from Physics

The gravitational force, the strong force, the weak force and the electromagnetic interaction between objects in physics are studied by various kinds of field equations including the following gauged Shrödinger equation [45, 46, 49–51]

$$iD_0\phi + V(x)\phi + \sum_{k=1}^{3} D_k D_k \phi = g(x, |\phi|)\phi, x \in \mathbb{R}^3, \tag{2.1}$$

where $V(x)$ is the potential function, $g(x,s)$ is the local nonlinearity, $i = \sqrt{-1}$, $\phi : \mathbb{R}^4 \to \mathbb{C}$ is the scalar field, and $D_k = \partial_k + iA_k$ is the covariant derivative with $A_k : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ the components of the gauge potential and $k = 0, 1, 2, 3$. Under the Coulomb gauge condition [109, 117], there is one kind of special Maxwell fields as follows,

$$A_1 = A_2 = A_3 = 0, \text{ and } A_0 = \lambda \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|} dy \text{ with parameter } \lambda > 0.$$

Let $\phi(t,x) = u(x)e^{-i\omega t}$ be the standing wave with $\omega > 0$ and $u : \mathbb{R}^3 \to \mathbb{R}$, Followed by (2.1), we have

$$-\Delta u + (V(x) - \omega)u + \lambda u(x) \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|} dy = g(x, |u|)u, x \in \mathbb{R}, \tag{2.2}$$

which is a nonlinear elliptic equation with nonlocal terms. If the parameter $\lambda = 0$, (2.2) is reduced to the semilinear elliptic equation below,

$$-\Delta u + (V(x) - \omega)u = g(x, |u|)u, x \in \mathbb{R}. \tag{2.3}$$

When $g(x, |u|) = |u|^{2/3}$, the equation (2.2) is the so called Schrödinger Poisson Slater equation, which was used by Slater to approximate the Hartree-Fock equations [40, 72, 77, 99],

$$\begin{aligned} -\Delta\psi_j(x) + (V(x) - E_j)\psi_j + \psi_j \int_{\mathbb{R}^3} \frac{|\rho(y)|^2}{4\pi|x-y|} dy \\ - \int_{\mathbb{R}^3} \frac{\sum_{k=1}^{N} \psi_k(x)\overline{\psi_k(y)}\psi_j(y)}{4\pi|x-y|} dy = 0, \end{aligned} \tag{2.4}$$

where $\rho = \frac{1}{N}\sum_{k=1}^{N} |\psi_k(x)|^2$ with the orthogonal set of $\{\psi_j : \mathbb{R}^3 \to \mathbb{C}, j = 1, 2, \cdots, N\}$. The Hartree-Fock equations are consistent with the Pauli exclusion principle appeared in quantum mechanics system with N particles. The most complex term $\int_{\mathbb{R}^3} \frac{\sum_{k=1}^{N} \psi_k(x)\overline{\psi_k(y)}\psi_j(y)}{4\pi|x-y|} dy$ is called the exchange term. Slater in [100] established a simple approximation of this term by

$$\int_{\mathbb{R}^3} \frac{\sum_{k=1}^{N} \psi_k(x)\overline{\psi_k(y)}\psi_j(y)}{4\pi|x-y|} dy \sim C_s \rho^{1/3}\psi_j(x), \text{ where } C > 0.$$

By the method of mean field approximation, the density $\rho(x)$ above can be estimated as $\rho(x) \sim u(x)$, where $u(x)$ is the solution of (2.2) with positive parameters.

In mathematics, many authors considered the nonlinear elliptic equation (2.2) as a class of elliptic equation. For example, if $g(x, |u|) = |u|^{p-2}$ and $V(x) - \omega = \omega$, D'Aprile and Mugnai in [32] studied (2.2) by the Schrödinger-Poisson equations below,

$$\begin{cases} -\Delta u + \omega u + \lambda \phi u = |u|^{p-2}u, \ x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, \ \lim_{|x| \to +\infty} \phi(x) = 0. \end{cases} \tag{2.5}$$

For $\omega > 0$ and $2 \leq 4 < 6$, D'Aprile and Mugnai in [32] obtained the existence of nontrivial solutions of (2.5) by using the variational method. Ruiz [91] considered the existence and nonexistence of solution to (2.5) with $\omega = 1$ for the case $\lambda > 0, p \in (1, 5)$. Ambrosetti and Ruiz showed in [4] the multiplicity results of (2.5) when $p \in (1, 5)$. Azzollini and Pomponio proved in [8] that the existence of a ground state solution of (2.5) for $p \in (2, 5)$. If $V(x)$ is non-constant and satisfies the following conditions

(I) $V : \mathbb{R} \to \mathbb{R}$ is measurable;

(II) $V(\infty) := \lim\limits_{|z| \to \infty} V(z) \geq V(x)$ and $V(x) \not\equiv V(\infty)$;

(III) there exists $\tilde{C} > 0$ such that $\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 dx \geq \tilde{C}(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx) > 0$ for any $u \in H^1(\mathbb{R}^3)$.

Azzollini and Pomponio [8] obtained the existence of a ground state solution of

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-1}u, \ x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, \ x \in \mathbb{R}^3, \end{cases} \tag{2.6}$$

for any $p \in (3, 5)$. Zhao [126] established the existence of a ground state solution to (2.6) for $p \in (2, 5)$ under hypotheses $(I) - (III)$. After that, some classes of general Schrödinger-Poisson equations were introduced as follows,

$$\begin{cases} -\Delta u + V(x)u + \lambda K(x)\phi u = f(x, u), \ x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)|u|^2, x \in \mathbb{R}^3. \end{cases} \tag{2.7}$$

Under some assumptions on the potential $V(x), K(x)$ and the nonlinearity $f(x,u)$, the existence of solutions of (2.7) were obtained in [2, 4, 8, 19, 23–26, 36, 57, 70, 104, 125, 126]. There are also some other kinds of general Schrödinger-Poisson equations such as the fractional Schrödinger-Poisson equations, for more details, the reader is referred to [5–7,16,39,47,63,64,78, 87,111,120,121,123] and the references therein.

In some special applications such as seeking numerical solutions, one has to consider the small error or perturbation on the nonlocal nonlinear terms of the Schrödinger-Poisson equation. Recently, various kinds of general Schrödinger-Poisson equations with perturbation on the nonlocal nonlinear terms have been treated via establishing the existence of solutions to the elliptic equations below,

$$-\Delta u+V(x)u+\lambda|u|^{q-2}u(x)\int_{\mathbb{R}^3}\frac{|u(y)|^q}{4\pi|x-y|}dy=|u|^{p-1}u,\ x\in\mathbb{R}^3, \quad (2.8)$$

which is the Euler-Lagrange equation of the functional

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+V(x)u^2dx+\frac{\lambda}{8q\pi}\int_{\mathbb{R}^6}\frac{|u(x)|^q|u(y)|^q}{|x-y|}dydx-\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx. \quad (2.9)$$

The authors in [66, 70, 85] studied the existence of solutions of (2.8) by applying the variational method. This thesis aims to study the existence, multiplicity and symmetric properties of solutions to various classes of nonlinear equations including (2.8) and the following general elliptic equation models,

$$\begin{cases} \Delta u+V(x)u+\lambda\phi(x)|u|^{r-2}u=|u|^{p-1}u,\ x\in\Omega, \\ -\Delta\phi(x)=|u|^q,\ x\in\Omega, \\ \phi(x)=u(x)=0 \text{ for } x\in\partial\Omega, \end{cases} \quad (2.10)$$

where $p,q,r>1$ are nonlinear exponents, λ is a parameter, $V(x)$ is a potential, $\Omega=\mathbb{R}^3$ or Ω is a bounded domain in $\mathbb{R}^3$. For the case $\Omega=\mathbb{R}^3$ and $q=r$, (2.10) is reduced to (2.8). For $q\neq r$, we get that (2.10) is not an Euler-Lagrange equation of a variational functional.

2.2 Methods for the existence of solutions to nonlinear elliptic equations

The variational method is a discipline founded at the end of the 17th century, which is a powerful tool in the research of nonlinear elliptic equations. Many meaningful results for the existence of positive solutions, sign-changing solutions and multiple solutions of nonlinear elliptic equations were obtained by the variational mehtod [20, 60, 96, 97, 127]. It is closely related to many mathematics branches, such as minimal surface theory, mathematical physics, integral equations, differential geometry, and Riemannian geometry, etc. The law of motion of all objects studied in the variational problem obeys the variational principle, that is, transforming the variational problem in nature into the extreme value problem or critical point problem of a certain functional under certain conditions. Thus, the key to study nonlinear elliptic equations is to find the critical points of the related functional. This is the basic idea of variational method and also one of the most important methods of nonlinear functional analysis. The Brachistochrone curve problem is the first variational problem in history, which was raised by Johann Bernoulli in 1696 ([41]). It is also a sign of the development of variational methods. Weierstrass's famous course had epochal impact on this theory ([44]), and it can be said that he was the first to put it on an indisputable basis. The 20th and 23rd Hilbert problems published in 1900 promoted the further development of this theory.

In recent decades, modern variational method has been continuously developed and improved with the efforts of many mathematicians [3, 38]. In 1973, Ambrosetti and Rabinowitz proposed the minimax method including the mountain pass lemma, which greatly promoted the development of modern variational methods [3]. The Minimax theory is one of the main theories of modern variational methods, and it is also a milestone in the development of critical point theory and nonlinear differential equation theory [89]. Based on these theories, lots of famous mathematicians have produced many pioneering results in the research of nonlinear elliptic problems and eigenvalue problems [3, 13, 71, 74–76].

Due to the opposite signs of non-local terms and local nonlinear terms in the Schrödinger-Poisson equation, the corresponding variational functional (2.9) with $q = 2$ in the Sobolev space is complex in geometric structure [91]. The special geometric structure of variational functional makes the application of variational method for studying the existence of solution as a challenging problem. Usually, it is not easy to get a bounded approximation solution sequence via the critical point theory. Many new techniques have been developed to establish bounded sequence of approximation solutions to the Schrödinger-Poisson equation. For example, the monotonicity method established in [52] has been used together with the Pohozaev-type identity to obtain the bounded Palais-Smale sequences [4,8, 126]; truncation techniques together with a priori estimate of solution were applied to establish a bounded sequence of approximation solutions [61]; the Nehari manifold is also useful for establishing bounded Palais-Smale sequences by the critical point theory [91,105]. The first step of successfully applying the methods mentioned above is that these kinds of Schrödinger-Poisson equations are the Euler-Lagrange equations of some variational functionals. However, the small error or perturbation on the nonlocal terms of the Schrödinger-Poisson equation reduces to a class of nonlinear elliptic equations that do not have a variational structure. The variational method is invalid to study the class of nonlinear elliptic equations with no variational structure. It is worth developing the topological method [83] for the study of the existence of solutions to nonlinear elliptic equations with general nonlocal terms.

Besides the variational method, some researchers examined the existence of solutions to nonlinear elliptic equations by the topological degree method, such as the Leray-Schauder degree, the upper and lower solutions method, as detailed in [4, 93] and the references therein.

2.3 Method for the symmetric property of solutions to nonlinear elliptic equations

There are new phenomena in the property of solutions to the Schrödinger-Poisson equations owing to the appearance of standard nonlocal terms. For instance, there is a priori estimate for the infinite modulus of the solution to the Schrödinger-Poisson equation [57]; the ground state solution of the equation is non-radically symmetric on the bounded domain [92]. The Coulomb-Sobolev function space [73, 85] and the corresponding embedding theorem [85,92] play a key role to study the symmetric property of the ground state solution to the Schrödinger-Poisson equations [55]. Recently, a class of elliptic equations with general nonlocal nonlinearity, which can be regarded as small error or perturbation to the nonlocal terms of the Schrödinger-Poisson equation, was widely studied by many authors and the reader is referred to [9,66,70,85] and the references therein for more details on the research of existence of solutions and multiple solutions. However, very few work has been done involving the symmetric properties and a priori estimate to the solutions of this class of elliptic equations with general nonlocal nonlinearity. It is interesting to study the property for this class of elliptic equations with general nonlocal nonlinearity.

Various methods have been established to study the symmetric properties of solutions to nonlinear elliptic equations. The method of moving planes was established to study the symmetric properties of positive solutions to nonlinear elliptic equations, as detailed in [17,27,42] for the semilinear elliptic equation and [28] for the integral equation. By the method of moving planes established in [28], Ma and Zhao [81] proved the radially symmetry of positive solutions to the stationary Choquard equation below.

$$\Delta u - u + 2u\int_{\mathbb{R}^3}\frac{u^2(y)}{4\pi|x-y|}dy = 0, u \in H^1(\mathbb{R}^3). \tag{2.11}$$

The equation above is equivalent to the integral equation

$$u = K_1 * F(u, K_2 * G(u)),$$

where K_1 and K_2 are kernels, F and G are positive nonlinear functions.

The Schwarz symmetrization [71] and the reflection method [80] were established to study the symmetric properties of minimizers of a class of variational problems related to the Laplacian equation with fractional powers, the generalized Choquard functional and the Davy-Stewartson equation. However, the methods mentioned above are invalid for a class of elliptic equations (2.8) and (2.9), in which the nonlocal terms and local nonlinear terms have opposite signs. It is worth studying the symmetric properties of this class of elliptic equations (2.8) and (2.9) with general nonlocal nonlinearities.

2.4 Preliminaries

For the complement of this thesis and the convenience of the following statement, we give some basics for the application of variational method and topological method. This part includes some useful inequalities, critical point theorems and the Leray-Schauder degree theory , etc.

1.4.1 Some useful inequalities

Theorem 2.4.1. *(Minkowski inequality, [95]) Let $\Omega \subset \mathbb{R}^N$ be a measurable set. $p \geq 1$, $f_1(x), f_2(x) \in L^p(\Omega)$, then*

$$\|f_1 + f_2\|_{L^p(\Omega)} \leq \|f_1\|_{L^p(\Omega)} + \|f_2\|_{L^p(\Omega)}.$$

That is,

$$\left(\int_\Omega |f_1 + f_2|^p dx\right)^{1/p} \leq \left(\int_\Omega |f_1|^p dx\right)^{1/p} + \left(\int_\Omega |f_2|^p dx\right)^{1/p}.$$

Theorem 2.4.2. *(Sobolev embedding theorem, Theorem 1.8, [116]) The following embedding*

$$H^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad 2 \leq q < \infty, n = 1, 2,$$

$$H^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \quad 2 \leq q < 2^*, N \geq 3,$$

$$D^{1,2}(\mathbb{R}^n) \subset L^{2^*}(\mathbb{R}^n), \quad 2 \leq q < \infty, n \geq 3,$$

is continuous. Specially, the following Sobolev inequality

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^n), |u|_{2^*}=1} |\nabla u|_2^2 > 0$$

holds.

1.4.2 Critical point theory

Definition 2.4.3. *($(PS)_c$ condition, [84]) Let $(X, \|\cdot\|_X)$ be a real Banach space, the dual space be $(X^*, \|\cdot\|_*)$, the variational functional $J \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. If*

$$J(u_n) \to c, \quad J'(u_n) \to 0 \quad in \quad X^*, \quad as \quad n \to +\infty,$$

where $J'(u_n)$ is the Fréchet derivative of $J(u_n)$. If any sequence $\{u_n\} \subset X$ has a convergent subsequence in X, then J is said to satisfy the Palais-Smale compact condition ($(PS)_c$ condition for short).

Theorem 2.4.4. *(Mountain pass lemma, [116]) Let X be a real Hilbert space, the functional $J \in C^1(X, \mathbb{R})$, if J satisfies the following geometry structure,*

(1) J(0)=0, there exist $r, \rho > 0$ such that $J(u) \geq \rho$ for any $u \in X$ and $\|u\|_X = r$;

(2) there exists $e \in X$ and $\|e\|_X > r$ such that $J(e) \leq 0$.

Let $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

If J satisfies the $(PS)_c$ condition, then c is the positive critical value of the functional J.

Let X be a real Banach space and Γ be the family of sets $B \subset X \setminus \{0\}$ such that B is closed and symmetric with regard to 0. For any $B \in \Gamma$, let

$$T = \left\{k : k = 1, 2, \cdots, \text{there is an odd map } \varphi \in C(B, \mathbb{R}^k \setminus \{0\})\right\}.$$

If $T \neq \emptyset$, the genus of B can be defined by $\gamma(B) = \min_{k \in \Theta} k$, otherwise $\gamma(B) = \infty$. Set $\gamma(\emptyset) = 0$.

Remark 2.4.5. *Let S^{n-1} be a unit sphere in $\mathbb{R}^n$. If $B \subset H^1(\mathbb{R}^3) \setminus \{0\}$ is homeomorphic to S^{n-1} by an odd map, then $\gamma(B) = n$.*

Then, we state the Clark's theorem as follows.

Theorem 2.4.6. [89, Theorem 9.1] *Let X be a real Banach space. Assume $J(u) \in C^1(X, R)$ is even, bounded below and satisfies the $(PS)_c$ condition, $J(0) = 0$. For any $n \in \mathbb{N}$, there exists $B \in \Gamma$ such that $\gamma(B) = n$ and $\sup_{u \in B} J(u) < 0$. Then $J(u)$ has at least j distinct pairs of critical points, and the associated critical values are*

$$c_i = \inf_{B \in \Gamma, \gamma(B) \geq j} \sup_{u \in B} J(u).$$

1.4.3 Topological degree theory

Definition 2.4.7. *([29,83]) Let X be a Banach space, $\Omega \subset X$ is an open bounded set, I is a identity operator, $g : \bar{\Omega} \to X$ is a compact mapping, $y \in \Omega$ and $y \notin (I - g)(\partial \Omega)$, the Leray-Schauder degree $deg_{LS}[I - g, \Omega, y]$ of $I - g$ in Ω over y is*

$$deg_{LS}[I - g, \Omega, y] = \sum_{a \in (I-g)^{-1}(y)} (-1)^{\sigma_j(a)},$$

where $\sigma_j(a)$ is the sum of the algebraic multiplicities of the eigenvalues of $f'(a)$ contained in $]1, \infty[$.

Theorem 2.4.8. *([65]) The properties of the Leray-Schauder degree are listed below.*

(I)(Additivity) Let $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are disjoint open sets, if $y \notin (I - g)(\partial \Omega_1) \cup (I - g)(\partial \Omega_2)$, then

$$deg_{LS}[I - g, \Omega, y] = deg_{LS}[I - g, \Omega_1, y] + deg_{LS}[I - g, \Omega_2, y].$$

(II)(Existence) If $deg_{LS}[I - g, \Omega, y] \neq 0$, then $y \in (I - g)(\Omega)$.

(III) (Homotopy invariance) Assume that $U \subset X \times [0, 1]$ is bounded and open, and $H : \bar{U} \to X$ is a compact mapping. If $h - H(h, \lambda) \neq y$ for each $(h, \lambda) \in \partial U$, then $deg_{LS}[I - H(\cdot, \lambda), U_\lambda, y]$ is not dependent on λ, where $U_\lambda = \{h \in X : (h, \lambda) \in U\}$.

Chapter 3

Schrödinger-Poisson equation with general nonlinearity

3.1 General

Nonlinear elliptic equation (1.1) with nonlocal terms arises from the Physical models such as gauged fields equation and Hartree-Fock system, for more details, the reader is referred to [14, 15, 69, 74, 77, 82, 92, 100, 122] and the references therein. If $V(x) = W(x) - \mu$, (1.1) is equivalent to the following Schrödinger-Poisson system,

$$\begin{cases} -\Delta u + (W(x) - \mu)u + \phi(x)u = f(u), \\ -\Delta \phi = u^2, \ \lim_{|x| \to +\infty} \phi(x) = 0, \quad x \in \mathbb{R}^3, \end{cases} \tag{3.1}$$

where μ is a parameter, $W(x)$ is a potential and $f(u)$ is the nonlinearity. The harmonic potential $W(x) = |x|^2$ and the Slater term $f(u) = |u|^{2/3}u$ are more relevant from the viewpoint of physics models [92, 100]. Here, we consider the potential function and nonlinearity of (3.1) with more general application via the following assumptions:

(W_1) $W(x)$ is a continuous function, and $\lim_{|x| \to +\infty} W(x) = +\infty$;

(H_1) there is a positive constant $\tau > 0$, such that

$$|f(u)| \leq \tau(|u| + |u|^p), 1 < p < 2;$$

(H_2) $f(t)$ is nonnegative for small $t > 0$;

(H_3) $f(t) = -f(-t)$ for $t \in \mathbb{R}$.

Obviously, $W(x) = |x|^2$ satisfies (W_1); and $f(u) = |u|^{2/3}u$ satisfies (H_1)-(H_3). We denote by

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 + W(x)u^2 dx < +\infty \right\}$$

the Hilbert space with norm

$$\|u\| = \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 + W(x)u^2 dx \right\}^{1/2}.$$

Under assumption (W_1), we see that the embedding from E to $L^p(\mathbb{R}^3)$ is compact for all $p \in [2, 6)$. Under assumptions (W_1) and (H_1), it is a standard process to prove that equation (3.1) is the Euler-Lagrange equation of a functional on E, which is defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + (W(x) - \mu)u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \int_{\mathbb{R}^3} F(u)dx, \tag{3.2}$$

where $\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy$, $F(u) = \int_0^u f(s)ds$.

Problem (3.1) has been studied under various conditions on the potential and nonlinearity, and for more details the reader is referred to [8, 32, 57, 59, 61, 62, 91, 115, 126] and the references therein. From the references mentioned above, the variable potentials V and nonlinearities $f(u)$ give different geometric properties of I. Indeed, as $f(u) = |u|^{p-1}u$ with $p \geq 3$, the function I satisfies the $(PS)_c$ condition, which means that any sequence $\{u_n\} \subset H$ such that $I'(u_n) \overset{n}{\to} 0$ and $\{I(u_n)\}$ is bounded, has a convergent subsequence, as shown in [32] for $W(x) - \mu \equiv 1$ and [8] for a nonconstant potential $W(x)$. If $f(u) = |u|^{p-1}u$ with $p \in (1, 3)$, the geometric properties of I are more complex than those for $p \in [3, 5)$. For example, we do not know whether I satisfies the $(PS)_c$ condition when $p \in (1, 3)$. To overcome this difficulty, a related Pohozaev type equality is used to get a bounded approximate sequence to the solution of (3.1) with $p \in (2, 3)$, as in [126]. For the case $f(u) = |u|^{p-1}u$ with $p \in (1, 2)$, the symmetric properties of functions are used to prove that the $(PS)_c$ condition of I holds when $W(x) - \mu = 1$; and therefore the existence of radially symmetric solution is obtained [91]. If there is a small parameter in the coefficient of the nonlocal term '$\phi_u u$' in (3.1), a truncated functional could be used to give a bounded sequence of approximate solutions to (3.1) without the radial symmetry constraint on the working functional space [61]. For non-constant positive potential $W(x) - \mu$ and $f(u) = |u|^{p-1}u$ with $p \in (1, 2)$, a priori estimate to the solution of (3.1) is given in [57] to prove the compactness of a sequence of approximate solutions to (3.1), so the existence of a ground state solution is proved.

Under the general assumptions (W_1), $W(x) - \mu$ is a sign-changing function for some μ. It is almost impossible to obtain a priori estimate as in [57] to the solution of (3.1), although the nonlinearity $f(u)$ has similar behavior as $|u|^{p-1}u$ with $p \in (1, 2)$. Since the coefficient of the nonlocal term in (3.1) is not a small parameter, it is not possible to construct a truncated functional as in [61] to give a bounded sequence of approximate solutions.

Motivated by the method in [55, 59], we first prove that the functional of (3.1) satisfies the $(PS)_c$ condition under assumptions (W_1) and (H_1) , then via the variational method, we establish the existence of a ground state solution, multiple solutions and nonexistence of solutions to (3.1). The main results together with the proofs are presented in Sections 3.2 and 3.3, which have been published in [56].

By assumptions (W_1) and (H_1), we get that $I \in C^1(E, \mathbb{R})$ by the standard method in [89]. For any $\varphi \in E$, we have the Fréchet derivative of I as

$$\langle I'(u), \varphi\rangle = \int_{\mathbb{R}^3} \nabla u \nabla \varphi + (W(x) - \mu)u\varphi dx + \int_{\mathbb{R}^3} \phi_u u \varphi dx - \int_{\mathbb{R}^3} f(u)\varphi dx. \tag{3.3}$$

If $I'(u) = 0$ in E^{-1}, it is known that u is a critical point of I and thus a weak solution of (3.1). To make these notations more clear, we give the following definitions to be used later.

Definition 3.1.1. *$u \in E$ is said to be a weak solution of (3.1) if*

$$I'(u)\varphi = \langle I'(u), \varphi \rangle = 0 \text{ for all } \varphi \in E. \tag{3.4}$$

Definition 3.1.2. *Let $\mathcal{G} = \{u \in E \setminus \{0\} : I'(u) = 0\}$. Then v is said to be a ground state solution to (3.1) if v is a weak solution of (3.1) and $I(v) = \inf_{u \in \mathcal{G}} I(u) > -\infty$.*

3.2 The $(PS)_c$ condition of variational functional

In this section, we prove that the functional I as defined in (3.2) satisfies the $(PS)_c$ condition under the assumptions (W_1) and (H_1) by developing Lemma 3.2.1 and 3.2.2 below. The following proof of the lemmas is based on the methods in [55, 59]. But there are some differences at certain points since we consider a more general case here. For convenience of the reader, we give a complete proof as follows.

Lemma 3.2.1. *Assume (W_1) and (H_1) hold. For any $\mu \in \mathbb{R}$, the sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ with $I(u_n) \leq C$ is bounded.*

Proof. It follows from $-\Delta\phi_{u_n} = u_n^2$ and the Young's inequality that

$$\frac{\sqrt{3}}{4}\int_{\mathbb{R}^3} |u_n|^3 dx \leq \frac{3}{8}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{8}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx. \tag{3.5}$$

Let $\tau > 0$ be given by (H_1). We have

$$\int_{\mathbb{R}^3} F(u_n)dx = \int_{\mathbb{R}^3}\int_0^{u_n} f(t)dtdx \leq \frac{\tau}{2}\int_{\mathbb{R}^3} u_n^2 dx + \frac{\tau}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} dx. \tag{3.6}$$

In the following, let $\theta > 0$ be a given number sufficiently large such that

$$\int_{\mathbb{R}^3} \frac{\sqrt{3}}{4}|u_n|^3 + \frac{\theta - \tau}{2}u_n^2 dx - \frac{\tau}{p+1}|u_n|^{p+1} dx \geq 0.$$

It follows from (3.5) and (3.6) that

$$
\begin{aligned}
J_\theta(u_n) &\triangleq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{\theta}{2}\int_{\mathbb{R}^3}u_n^2dx + \frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx - \int_{\mathbb{R}^3}F(u_n)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{\theta-\tau}{2}\int_{\mathbb{R}^3}u_n^2dx + \frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx \\
&\quad - \frac{\tau}{p+1}\int_{\mathbb{R}^3}|u_n|^{p+1}dx \\
&= \frac{1}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{3}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx \\
&\quad + \frac{\theta-\tau}{2}\int_{\mathbb{R}^3}u_n^2dx - \frac{\tau}{p+1}\int_{\mathbb{R}^3}|u_n|^{p+1}dx \\
&\geq \frac{\sqrt{3}}{4}\int_{\mathbb{R}^3}|u_n|^3dx + \frac{\theta-\tau}{2}\int_{\mathbb{R}^3}u_n^2dx - \frac{\tau}{p+1}\int_{\mathbb{R}^3}|u_n|^{p+1}dx \\
&\quad + \frac{1}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx \\
&= \int_{\mathbb{R}^3}\frac{\sqrt{3}}{4}|u_n|^3 + \frac{\theta-\tau}{2}u_n^2 - \frac{\tau}{p+1}|u_n|^{p+1}dx + \frac{1}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx \\
&\geq \frac{1}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx,
\end{aligned} \tag{3.7}
$$

Let $\eta = \mu + \theta$, by combining (3.2) and (3.7) we obtain that

$$
\begin{aligned}
I(u_n) &= J_\theta(u_n) + \frac{1}{2}\int_{\mathbb{R}^3}(W(x)-\mu-\theta)u_n^2dx + \frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx \\
&\geq \frac{1}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx + \frac{1}{2}\int_{\mathbb{R}^3}(W(x)-\eta)u_n^2dx.
\end{aligned} \tag{3.8}
$$

Let $\Omega_\eta = \{x \in \mathbb{R}^3 : W(x) < \eta\}$. If $\theta > 0$ is large enough, then $\eta > 0$ and Ω_η is a non-empty bounded set by assumption (W_1). Obviously, we see that $\Omega_\eta \subset \Omega_{2\eta}$. Since $I(u_n) \leq C$, it follows from (3.8) that

$$
\begin{aligned}
&\frac{1}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega_\eta}(W(x)-\eta)u_n^2dx + \frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx \\
&\leq C - \frac{1}{2}\int_{\Omega_\eta}(W(x)-\eta)|u_n|^2dx \\
&\leq C + (\eta + |\min_{x\in\mathbb{R}^3}W(x)|)\int_{\Omega_\eta}u_n^2dx.
\end{aligned} \tag{3.9}
$$

For $x \in \mathbb{R}^3 \setminus \Omega_{2\eta}$, $W(x) \geq 2\eta$ and then $\int_{\Omega_{2\eta}\setminus\Omega_\eta}(W(x)-\eta)u_n^2dx \geq 0$, and therefore,

$$
\int_{\mathbb{R}^3\setminus\Omega_\eta}(W(x)-\eta)u_n^2dx \geq \int_{\mathbb{R}^3\setminus\Omega_{2\eta}}(W(x)-\eta)u_n^2dx.
$$

This and (3.9) implies that

$$\frac{1}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx+\frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega_{2\eta}}(W(x)-\eta)u_n^2dx+\frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx \le C+(\eta+|\min_{x\in\mathbb{R}^3}W(x)|)\int_{\Omega_\eta}u_n^2dx. \tag{3.10}$$

We claim that $\{|u_n|_2\}$ is bounded. Otherwise, there exists a subsequence such that $|u_n|_2 \stackrel{n}{\to} +\infty$. Let $v_n = \frac{u_n}{|u_n|_2}$. By using (3.10) we have

$$\begin{aligned}&\frac{1}{8}\int_{\mathbb{R}^3}|\nabla v_n|^2dx+\frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega_{2\eta}}(W(x)-\eta)v_n^2dx+\frac{1}{8}|u_n|_2^2\int_{\mathbb{R}^3}\phi_{v_n}v_n^2dx\\ &\le \frac{C}{|u_n|_2^2}+(\eta+|\min_{x\in\mathbb{R}^3}W(x)|)\int_{\Omega_\eta}|v_n|^2dx\\ &\le \eta+|\min_{x\in\mathbb{R}^3}W(x)|+o(1), \quad \text{as } n\to+\infty. \end{aligned}\tag{3.11}$$

It follows that $\{|\nabla v_n|_2\}$ must be bounded and $\int_{\mathbb{R}^3}\phi_{v_n}v_n^2dx \stackrel{n}{\to} 0$. Hence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$. There exists $v_0 \in H^1(\mathbb{R}^3)$ and up to a subsequence of $\{v_n\}$ such that

$$v_n \stackrel{n}{\rightharpoonup} v_0 \text{ weakly in } H^1(\mathbb{R}^3)$$

and

$$\int_{\mathbb{R}^3}\phi_{v_0}v_0^2dx \le \liminf_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{v_n}v_n^2dx = 0.$$

Hence, $v_0(x)=0$ a.e. $x\in\mathbb{R}^3$ and the compact embedding theorem shows that $\int_{\Omega_{2\eta}}|v_n|^2dx \stackrel{n}{\to} 0$ ($\Omega_{2\eta}$ is bounded as mentioned above). Those facts together with (3.11) imply that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3\setminus\Omega_{2\eta}}(W(x)-\eta)v_n^2dx=0.$$

Therefore,

$$\eta|v_n|_2^2 \le \int_{\mathbb{R}^3\setminus\Omega_{2\eta}}(W(x)-\eta)|v_n|^2dx+\eta\int_{\Omega_{2\eta}}|v_n|^2dx \stackrel{n}{\to} 0,$$

which contradicts with $|v_n|_2 \equiv 1$. By using (3.10) and the assumption (W_1), we get

$$\frac{1}{8}\int_{\mathbb{R}^3}|\nabla u_n|^2dx+\frac{1}{8}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx \le C+(\eta+|\min_{x\in\mathbb{R}^3}W(x)|)|u_n|_2^2.$$

So $\{\|u_n\|\}$ is bounded. □

Lemma 3.2.2. *Assume (W_1) and (H_1) hold. Let $\mu \in \mathbb{R}$ and $\{u_n\} \subset E$ be a bounded (P.S.) sequence for the functional I, i.e., as $n \to +\infty$,*

$$I(u_n) \to c, \text{ and } I'(u_n) \to 0, \text{ in } E^{-1}.$$

Then $\{u_n\}$ has a convergent subsequence in E.

Proof. Under assumption (W_1), the embedding $E \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $q \in [2, 6)$, there exists $u \in L^q(\mathbb{R}^3)$ such that, up to a subsequence,

$$u_n \stackrel{n}{\to} u \text{ in } L^q(\mathbb{R}^3). \tag{3.12}$$

Let p be given by (H_1) and $n \to +\infty$, it follows from (3.12) that

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) dx \right| \\ \leq \; & |u_n - u|_2 |f(u_n) - f(u)|_2 \\ \leq \; & |u_n - u|_2 (|f(u_n)|_2 + |f(u)|_2) \\ \leq \; & \tau |u_n - u|_2 (|u_n|_2 + |u|_2 + |u_n|_{2p}^p + |u|_{2p}^p) \\ \to \; & 0. \end{aligned} \tag{3.13}$$

By using the Hölder's inequality, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) dx \right| \\ \leq \; & \left| \int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u) dx \right| + \left| \int_{\mathbb{R}^3} \phi_u u (u_n - u) dx \right| \\ \leq \; & \left(\left\{ \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \right\}^{\frac{1}{2}} |\phi_{u_n}|_6^{\frac{1}{2}} + \left\{ \int_{\mathbb{R}^3} \phi_u u^2 dx \right\}^{\frac{1}{2}} |\phi_u|_6^{\frac{1}{2}} \right) |u_n - u|_{\frac{12}{5}}. \end{aligned} \tag{3.14}$$

By $-\Delta \phi_{u_n} = u_n^2$ we see that $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = |\nabla \phi_{u_n}|_2^2 \leq |u_n|_{12/5}^4$. Moreover, Sobolev inequality implies that $|\phi_{u_n}|_6 \leq |\nabla \phi_{u_n}|_2$. As $n \to +\infty$, it follows from (3.12) and (3.14) that

$$\left| \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) dx \right| \leq C |u_n - u|_{\frac{12}{5}} \to 0. \tag{3.15}$$

On the other hand, it follows from (3.3) that

$$\begin{aligned} & \|u_n - u\|^2 = \langle I'(u_n) - I'(u), u_n - u \rangle + \mu \int_{\mathbb{R}^3} |u_n - u|^2 dx \\ & - \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) dx + \int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) dx. \end{aligned} \tag{3.16}$$

We also see that $\langle I'(u_n) - I'(u), u_n - u \rangle \stackrel{n}{\to} 0$ is obvious by assumption. Then we prove this lemma by combining (3.13) and (3.15). $\square$

3.3 Existence and nonexistence of solutions

In this section, we develop three theorems, establishing conditions for the existence of a ground state solution, multiple solutions, and the nonexistence of solutions to (3.1).

Before presenting our theorems, we state the following eigenvalue problem,

$$\begin{cases} -\Delta\psi + W(x)\psi = \mu\psi, \ x \in \mathbb{R}^3, \\ \psi \in E. \end{cases} \tag{3.17}$$

Under the assumption (W_1), the eigenvalue problem (3.17) has infinite eigenvalues. Throughout this section, we denote by $\{\psi_m\}_{m=1}^{\infty} \subset L^2(\mathbb{R}^3)$ the normalized eigenfunctions of (3.17) corresponding to the eigenvalues $\{\mu_m\}_{m=1}^{\infty}$, i.e.,$|\psi_m|_2 = 1$ for any $m \in \mathbb{N}$. It is well known that $\mu_i \to +\infty$ as $i \to +\infty$ and $\mu_1 < \mu_2 \leq \mu_3 \leq \cdots$.

Firstly, the following two theorems are established for the existence of a ground state solution and multiple solutions of equations (3.1).

Theorem 3.3.1. *Assume (W_1) and $(H_1) - (H_2)$ hold. If $\mu > \mu_1$, then (3.1) has a ground state solution $u \in E$ and $I(u) < 0$.*

Theorem 3.3.2. *Assume (W_1) and $(H_1) - (H_3)$ hold. If $\mu_m < \mu$ for some $m \in \mathbb{N}$, there exist at least m pairs of distinct solutions $\pm u_j$ to (3.1). Moreover, $I(\pm u_j) < 0$ for $1 \leq j \leq m$.*

Remark 3.3.3. *We note that the potential $W(x)$ under assumption (W_1) may be a sign-changing function. The Schrödinger-Poisson equation with some sign-changing potentials has been studied in [58] by using the method of upper and lower solutions. Theorems 3.3.1 and 3.3.2 are supplement to the previous research in [58].*

Remark 3.3.4. *From Theorems 3.3.1-3.3.2, we see that (3.1) has nontrivial solutions when the parameter μ is greater than μ_1, which is the first eigenvalue of problem (3.17). It is natural to ask whether there exist nontrivial solutions of (3.1) for some $\mu < \mu_1$. We partially answer this question in the following theorem, which need the assumption below,*

(W_2): $W(x)$ is nonnegative for $x \in \mathbb{R}^3$.

Roughly speaking, we have the estimate that $\mu_1 \geq 0$ under assumptions (W_1) and (W_2).

Theorem 3.3.5. *Assume $(W_1) - (W_2)$ and (H_1) hold. Let $\tau > 0$ be given by (H_1) and $C_p = 2^{\frac{p-1}{p-2}}(p-1)^{\frac{1}{2-p}}(2-p) > 0$. If $\mu < -\tau^{\frac{1}{2-p}}C_p - \tau$, then (3.1) has only trivial solution in E.*

The proofs of the three theorems are given below.

Proof of Theorem 3.3.1: Let μ_1 be the first eigenvalue of (3.17) and φ_1 be the corresponding eigenfunction. Then $\int_{\mathbb{R}^3} |\nabla\varphi_1|^2 + W(x)\varphi_1^2 dx = \mu_1 \int_{\mathbb{R}^3} \varphi_1^2 dx$. Let $h > 0$ be small enough, it follows from $\mu > \mu_1$ and (H_2) that

$$\begin{aligned} I(h\varphi_1) &= -\frac{\mu-\mu_1}{2}\int_{\mathbb{R}^3} h^2\varphi_1^2 dx + \frac{1}{4}h^4\int_{\mathbb{R}^3}\phi_{\varphi_1}\varphi_1^2 dx - \int_{\mathbb{R}^3} F(h\varphi_1)dx \\ &\leq -\frac{\mu-\mu_1}{2}\int_{\mathbb{R}^3} h^2\varphi_1^2 dx + \frac{1}{4}h^4\int_{\mathbb{R}^3}\phi_{\varphi_1}\varphi_1^2 dx < 0. \end{aligned} \tag{3.18}$$

Let $\mathcal{M} = \{u \in E, I(u) < 0\}$, $\mathcal{N} = \{u \in E, I(u) \geq 0\}$, then $E = \mathcal{M} \cup \mathcal{N}$ and $\mathcal{M} \subset E$ is bounded by Lemma 3.2.1. Since I is a continuous functional on E, we get

$$0 > \inf_{u\in E} I(u) = \inf_{u\in\mathcal{M}} I(u) > -\infty. \tag{3.19}$$

By assumption (H_1) and the compactness of embedding $E \hookrightarrow L^q(\mathbb{R}^3)$ for $q \in [2, 6)$, we get that I is lower semicontinuous on E. Hence, there exists $u_0 \in E$ such that

$$I(u_0) = \inf_{u\in E} I(u) < 0.$$

Therefore, u_0 is a nontrivial critical point of I with minimal energy and u_0 is a ground state of (3.1). □

Proof of Theorem 3.3.2. From (3.19), we see that I is bounded from below. By applying Lemmas 3.2.1 and 3.2.2, we obtain that I satisfies the (P.S.) condition. It follows from assumption (H_3) that the functional I is even. It is clear to see that $I(0) = 0$. To apply Theorem 2.4.6 with $X = E$ to the functional I in (3.2), we construct the following set $S_m = \{u \in \bigoplus_{j=1}^{m} \mathbb{R}\psi_i, |u|_2 = 1\}$, where ψ_i is the normalized eigenfunction of (3.17) related to the i-th eigenvalue μ_i. For any $u_m \in S_m$, we have

$$u_m = \sum_{i=1}^{m} c_i \psi_i \ \text{ with } \ \sum_{i=1}^{m} c_i^2 = 1.$$

It follows that

$$I(\sigma u_m) = \frac{\sigma^2}{2}(-\mu + \sum_{i=1}^{m} c_i^2 \mu_i) + \frac{\sigma^4}{4}\int_{\mathbb{R}^3} \phi_{u_m} u_m^2 dx - \int_{\mathbb{R}^3} F(\sigma u_m) dx, \quad (3.20)$$

where $\sigma > 0$. By the second equation in (3.1), we see that $\int_{\mathbb{R}^3} \phi_{u_m} u_m^2 dx \leq |u_m|_{12/5}^4$. Combining this with the Sobolev inequality implies that there exists $C > 0$ such that $\int_{\mathbb{R}^3} \phi_{u_m} u_m^2 dx \leq C\|u_m\|^4$. By assumption (H_2), we see that $F(\sigma u_m) \geq 0$ for small $\sigma > 0$. Based on these facts, it follows from (3.20) that

$$I(\sigma u_m) \leq \frac{\sigma^2}{2}(-\mu + \sum_{i=1}^{m} c_i^2 \mu_i) + C\sigma^4 \|u_m\|^4. \quad (3.21)$$

Under assumption that $\mu > \mu_m$, we have

$$-\mu + \sum_{i=1}^{m} c_i^2 \mu_i < -\mu + \mu_m < 0.$$

Let $\sigma_0 > 0$ be small enough, from (3.21) we obtain that

$$I(\sigma_0 u_m) \leq \frac{\sigma_0^2}{2}(-\mu + \mu_m) + C\sigma_0^4 \|u_m\|^4 < 0. \quad (3.22)$$

Let $S = \{\sigma_0 u : u \in S_m\}$. Remark 2.4.5 implies the genus $\gamma(S) = m$. By (3.22), we obtain that

$$\sup_{u \in S} I(u) \leq \frac{\sigma_0^2}{2}(-\mu + \mu_m) + C\sigma_0^4 < 0,$$

where we used the norm equivalence in finite dimensional space. By Theorem 2.4.6 with $B = S$ and $X = E$, we get that I owns at least m distinct pairs of critical points $\{\pm u_i\}_{i=1}^m$ with $I(\pm u_i) \leq \sup_{u \in S} I(u) < 0$. □

Proof of Theorem 3.3.5. Let $u \in E$ be a solution of (3.1), then

$$\int_{\mathbb{R}^3} |\nabla u|^2 + (W(x) - \mu)u^2 dx + \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} f(u) u dx = 0. \quad (3.23)$$

It follows from $-\Delta \phi_u = u^2$ and Hölder inequality that

$$2\int_{\mathbb{R}^3} |u|^3 dx \leq \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \phi_u u^2 dx. \quad (3.24)$$

By combining (3.23) and (3.24), we deduce that

$$\int_{\mathbb{R}^3} (W(x)-\mu)u^2 + 2|u|^3 dx - \int_{\mathbb{R}^3} f(u)u dx \leq 0. \tag{3.25}$$

On the other hand, by assumption (H_1) , there exists a constant $\tau > 0$ such that $|f(u)| \leq \tau(|u| + |u|^p)$ for $p \in (1,2)$. Combining this inequality with assumption (W_2) implies that

$$\begin{aligned} & \int_{\mathbb{R}^3} (W(x)-\mu)|u|^2 + 2|u|^3 dx - \int_{\mathbb{R}^3} f(u)u dx \\ \geq & \int_{\mathbb{R}^3} (W(x)-\mu)|u|^2 + 2|u|^3 - \tau|u|^2 - \tau|u|^{p+1} dx \\ \geq & \int_{\mathbb{R}^3} (-\mu-\tau)|u|^2 + 2|u|^3 - \tau|u|^{p+1} dx \end{aligned} \tag{3.26}$$

Combining (3.25) with (3.26), we get that

$$\int_{\mathbb{R}^3} (-\mu-\tau)|u|^2 + 2|u|^3 - \tau|u|^{p+1} dx \leq 0. \tag{3.27}$$

Let $h(t) := -\mu - \tau + 2t - \tau t^{p-1}$. By calculation, we get that $t_0 = (\frac{2}{\tau(p-1)})^{\frac{1}{p-2}}$ is the unique minimizer of $h(t)$ when $p \in (1,2)$. Let $C_p = (2/(p-1))^{\frac{p-1}{p-2}}(2-p) > 0$ for $p \in (1,2)$. If $\mu < -\tau^{\frac{1}{2-p}} C_p - \tau < 0$, then $h(t) \geq h(t_0) \geq 0$ holds for all $t \in \mathbb{R}$. Therefore, the inequality (3.27) implies that $u(x) = 0$ for almost everywhere $x \in \mathbb{R}^3$ when $\mu < -\tau^{\frac{1}{2-p}} C_p - \tau$. $\square$

Chapter 4

A class of elliptic equations with nonlocal nonlinearity

4.1 General

We investigate the nonlinear elliptic equation (1.2) with $V(x) = |x|^2 - \mu$, which is equivalent to the following system,

$$\begin{cases} -\Delta u + (|x|^2 - \mu)u + \lambda\phi(x)|u|^{q-2}u = |u|^{p-1}u, \ x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^q, \ \lim_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{4.1}$$

where $|x|^2$ is the harmonic potential, $p, q > 1$, $\lambda, \mu \in \mathbb{R}$. If $|u|^q$ is integrable, $\phi(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|u(y)|^q}{|x-y|}dy$ is the nonlocal nonlinearity. If $q = 2$ and u is a solution of (4.1), then, $\psi(x,t) = e^{-i\mu t}u(x)$ is the standing waves of the following Schrödinger system

$$\begin{cases} i\psi_t - \Delta\psi + |x|^2\psi + \lambda\phi(x,t)\psi = |\psi|^{p-1}\psi, x \in \mathbb{R}^3, t > 0 \\ -\Delta\phi(x,t) = |\psi|^2, \end{cases}$$

which arises in the study of quantum mechanics such as the Hartree-Fock equations and the Kohn-Sham equations [14, 15, 82, 91, 92, 94, 100].

Many authors have studied the model (4.1) under various conditions (see [9, 66, 70, 85]). For the model (4.1) on a bounded domain, the authors in [9] studied the existence of solutions of the system below,

$$\begin{cases} -\Delta u - \lambda|u|^{q-2}u\phi + |u|^{p-1}u = 0,\ x \in \Omega, \\ -q\Delta\phi = 2\lambda|u|^q, \quad x \in \Omega, \\ u = \phi = 0, \quad on\ \partial\Omega \end{cases}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $p \in (1,5), \lambda > 0$ and $q > 1$. They obtained that there are at least one nontrivial solution for $\lambda \in (0, \lambda_0)$ with $\lambda_0 > 0$ and $p \in (1,5)$. If the value of $V(x)$ is constant, and the nonlinear term $|u|^{p-1}u$ is replaced by the general nonlinearity $f(u)$, (4.1) is equivalent to

$$\begin{cases} -\Delta u + bu + \lambda\phi|u|^{q-2}u = f(u),\ x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^q, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.2}$$

where $b > 0$ and $q \in [2,5)$. The authors in [66] proved the existence of positive radially symmetric solutions of the system (4.2) for the subcritical case $q \in [2,5)$. They showed that there are at least one radially symmetric positive solution of (4.2) for $\lambda \in [0, \lambda_0)$ with $\lambda_0 > 0$. For the case $q = 5$, the existence of solution to (4.2) was considered in [70,85]. Moreover, some useful inequality related to (4.2) was established, and for more details the reader is referred to Theorem 1 and Theorem 4 in [85].

In particular, for $q = 2$, system (4.1) becomes the following form

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^{p-1}u,\ x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \lim\limits_{|x|\to\infty} \phi(x) = 0, x \in \mathbb{R}^3, \end{cases} \tag{4.3}$$

which is called the Schrödinger-Poisson system. This system arises from studying the Schrödinger equation combined with the Poisson equation. If $V(x) \equiv 1$ and $\lambda = 1$, the existence of a radial positive solution of (4.3) for $p \in (2,5)$ was obtained in [32,91]. If $\lambda > 0$ and $p \in (1,5)$, the existence of multiple solutions to this system can be proved [4,91,98]. When $V(x)$ is not a constant, this system was also considered by many other authors [57,67,86,125,126]. Recently, Jiang-Wang-Zhou [55] and Jiang-Wei-Wu [56] studied the existence of ground state solution, multiple solutions and the symmetry of solutions to some more general systems with nonlinearities.

Inspired by the works mentioned above, we consider the existence and radially symmetric property of solutions of (4.1) with $q \in [2,5)$, $\mu \in \mathbb{R}$ and $V(x) = |x|^2$. For more information about the harmonic potential $|x|^2$, we refer the interested reader to [30].

Before presenting the main results, we introduce some notations, definitions and recall some useful inequalities related to the second equation (Poisson equation) in (4.1). Let $||\cdot||$ and $|\cdot|_k$, respectively, be the standard norms of $H^1(\mathbb{R}^3)$ and $L^k(\mathbb{R}^3)$ with $k \in (1, +\infty)$. Denote

$$D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\},$$
$$H = \{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |x|^2 u^2 dx < \infty\}.$$

Clearly $H \subset H^1(\mathbb{R}^3)$ and H is a Hilbert space. The scalar product and the norm are given by

$$\langle u, v\rangle_H = \int_{\mathbb{R}^3} \nabla u \nabla v + |x|^2 uv dx \text{ and } \|u\|_H^2 = \langle u, u\rangle_H.$$

If $u \in L^{\frac{6q}{5}}(\mathbb{R}^3)$, by Lemma 2.1 of [91], we deduce that $-\Delta\phi = |u|^q$ with $q \in [2,5)$ has a unique solution in $D^{1,2}(\mathbb{R}^3)$ with the form

$$\phi(x) := \phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u|^q(y)}{|x-y|} dy, \text{ for any } u \in L^{\frac{6q}{5}}(\mathbb{R}^3), \tag{4.4}$$

and

$$\|\nabla \phi_u(x)\|_2 \leqslant C\|u\|_{6q/5}^q, \quad \int_{\mathbb{R}^3} \phi_u(x) u^2 dx \leqslant C\|u\|_{6q/5}^{2q}. \tag{4.5}$$

For $u \in H$, we define the variational functional of (4.1) as follows:

$$\begin{aligned} I(u) =& \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + (|x|^2 - \mu)u^2 dx \\ & - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} dx + \frac{\lambda}{2q} \int_{\mathbb{R}^3} \phi_u(x)|u|^q dx \end{aligned}$$

By (4.5), I is well defined and $I \in C^1(H, \mathbb{R})$ with

$$\begin{aligned} (I'(u), v) =& \int_{\mathbb{R}^3} \nabla u \nabla v + (|x|^2 - \mu)uv dx \\ & + \lambda \int_{\mathbb{R}^3} \phi_u(x)|u|^{q-2}uv dx - \int_{\mathbb{R}^3} |u|^{p-1} uv dx \end{aligned}$$

for all $v \in H$ with $p \in (1,5)$.

The rest of the chapter is organized as follows. Section 4.2 discusses the properties of the variational $I(u)$. Section 4.3 establishes various results for the existence and nonexistence of nontrivial solutions to (4.1). Section 4.4 studies the symmetric property of the ground state solutions, and Section 4.5 establishes the condition for the existence of multiple solutions of (4.1).

4.2 The property of variational functional

We find that I with $1 < p < q < 5$ is bounded in H from below, for all $\lambda > 0$, which also implies that I satisfies the $(PS)_c$ condition in H. We give conclusions via the lemmas below.

Lemma 4.2.1. *Assume that $1 < q < 5$. If ϕ_u is the solution of $-\Delta\phi = |u|^q$ with $u \in D^{1,2}(\mathbb{R}^3)$, then the following inequality holds*

$$\sqrt{\lambda/4q}\int_{\mathbb{R}^3}|u|^{q+1}dx \le \frac{1}{4}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{\lambda}{4q}\int_{\mathbb{R}^3}\phi_u|u|^q dx. \tag{4.6}$$

Proof. Let $\Omega_+ = \{x \in \mathbb{R}^3 : u(x) > 0\}$. By multiplying both sides of $-\Delta\phi = |u|^q$ with $u_+(x) = \max\{u(x), 0\}$, one has

$$\int_{\Omega_+}\nabla\phi_u\nabla u dx = \int_{\Omega_+}|u|^{q+1}dx.$$

Let $\Omega_- = \{x \in \mathbb{R}^3 : u(x) < 0\}$, similarly, we get

$$\int_{\Omega_-}\nabla\phi_u\nabla u dx = \int_{\Omega_-}|u|^{q+1}dx.$$

It follows that

$$\int_{\mathbb{R}^3}\nabla\phi_u\nabla u dx = \int_{\mathbb{R}^3}|u|^{q+1}dx.$$

The above equality together with the Young's inequality yields

$$\sqrt{\lambda/4q}\int_{\mathbb{R}^3}|u|^{q+1}dx \le \frac{1}{4}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{\lambda}{4q}\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx. \tag{4.7}$$

At the same time, we know that

$$\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx = \int_{\mathbb{R}^3}\phi_u|u|^q dx. \tag{4.8}$$

Then (4.6) holds by combining (4.7) and (4.8).

Lemma 4.2.2. *Assume* $1 < p < q < 5$. *For any* $\mu \in \mathbb{R}$ *and* $\lambda > 0$, *the sequence* $\{u_n\} \subset H$ *with* $I(u_n) \leq C$ *is bounded.*

Proof. Followed by (4.6), we have

$$\sqrt{\lambda/4q}\int_{\mathbb{R}^3}|u_n|^{q+1}dx \leq \frac{1}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{1}{4q}\lambda\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^q dx. \tag{4.9}$$

Let $c_{q,\lambda} = \sqrt{\lambda/4q}$ and $\theta = \frac{2}{p+1}[\frac{1}{(q+1)c_{q,\lambda}}]^{\frac{q-1}{q-p}}$, we see that

$$\frac{1}{p+1}|t|^{p+1} \leq \frac{\theta_\lambda}{2}t^2 + c_{q,\lambda}|t|^{q+1},\ \forall\, t \in \mathbb{R}. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\int_{\mathbb{R}^3}\left[\frac{1}{4}|\nabla u|^2 + \frac{\lambda}{4q}\phi_u|u|^q - \frac{1}{p+1}|u|^{p+1}\right]dx \geq \frac{-\theta_\lambda}{2}\int_{\mathbb{R}^3}u^2dx,\ \forall\, u \in H.$$

Therefore, for $\lambda > 0$, $p \in (1, q)$ and $u \in H$, we get

$$I(u) \geq \int_{\mathbb{R}^3}\left[\frac{1}{4}|\nabla u|^2 + \frac{1}{2}(|x|^2 - \theta_\lambda - \mu)u^2\right]dx + \frac{\lambda}{4q}\int_{\mathbb{R}^3}\phi_u|u|^q dx,$$

which implies that

$$\int_{\mathbb{R}^3}\left[\frac{1}{4}|\nabla u_n|^2 + \frac{1}{2}(|x|^2 - \theta_\lambda - \mu)u_n^2\right]dx + \frac{\lambda}{4q}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^q dx \leq C, \tag{4.11}$$

Now we prove that $\{|u_n|_2\}$ is bounded. If not, $|u_n|_2 \overset{n}{\to} +\infty$. Let $v_n = \frac{u_n}{|u_n|_2}$, and $|v_n|_2 = 1$. Then, multiplying (4.11) by $\frac{1}{|u_n|_2^2}$, we get

$$\begin{aligned}&\int_{\mathbb{R}^3}\left[\frac{1}{4}|\nabla v_n|^2 + \frac{|x|^2}{2}v_n^2\right]dx + \frac{\lambda}{4q}|u_n|_2^{2q-2}\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^q dx\\ &\leq \frac{\theta_\lambda + \mu}{2} + \frac{C}{|u_n|_2^2}.\end{aligned} \tag{4.12}$$

Thus, $\{v_n\}$ is bounded in H and has a convergent subsequence. Suppose that $v_n \overset{n}{\rightharpoonup} v$ weakly in H, then we have

$$\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^q dx \overset{n}{\to} \int_{\mathbb{R}^3}\phi_v|v|^q dx.$$

By (4.12) and $|u_n|_2 \overset{n}{\to} +\infty$, we get that $\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^q dx \overset{n}{\to} 0$. Then

$$0 = \lim_{n\to+\infty}\int_{\mathbb{R}^3}\phi_{v_n}|v_n|^q dx = \int_{\mathbb{R}^3}\phi_v|v|^q dx = \int_{\mathbb{R}^3}|\nabla\phi_v|^2dx,$$

which indicates that $v = 0$ a.e. in $\mathbb{R}^3$. The result contradicts the fact that $|v|_2 = \lim_{n\to+\infty}|v_n|_2 = 1$, where we have used the compactness of embedding from H to $L^2(\mathbb{R}^3)$. Because $\{u_n\}$ is bounded in $L^2(\mathbb{R}^3)$, $\{u_n\}$ is bounded in H by (4.11). □

Lemma 4.2.3. *Assume that $1 < p < q < 5$. For any $\mu \in \mathbb{R}$ and $\lambda > 0$, $I(u) \in H$ is bounded from below.*

Proof. We set $H = \{u \in H : I(u) < 0\} \cup \{u \in H : I(u) \geq 0\}$. Obviously, $\{u \in H : I(u) < 0\} \subset H$ is a bounded set. Followed by the definition of I and $H \hookrightarrow L^{p+1}(\mathbb{R}^3)$, we have that there is a constant $C_1 > 0$ such that

$$\begin{aligned}\inf_{u\in H} I(u) = \inf_{\{u\in H:I(u)<0\}} I(u) &\geq \inf_{\{u\in H:I(u)<0\}} \frac{-1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx \\ &\geq \inf_{\{u\in H:I(u)<0\}} \frac{-C_1}{p+1}\|u\|^{p+1} > -\infty.\end{aligned}$$

□

4.3 Existence of nontrivial solutions

In this section, three theorems are developed, which establish the conditions for the existence and nonexistence of the nontrivial solutions of (4.1) upon the value of parameters μ and λ.

Before presenting the theorems, we first state the following eigenvalue problem

$$-\Delta u + |x|^2 u = \mu u, x \in \mathbb{R}^3. \tag{4.13}$$

There exist eigenvalues $\{\mu_i\}_{i=1}^{+\infty}$ of problem (4.13) such that $\mu_i \xrightarrow{i} +\infty$. Obviously, $\mu_1 > 0$. For each $i \in \mathbb{N}$, we denote by φ_i the eigenfunction corresponding to μ_i.

Theorem 4.3.1. *Suppose that $1 < p < q < 5$. For $\mu \geq \mu_1$, there exists a ground state solution u_0 of (4.1) for all $\lambda > 0$. Moreover, $I(u_0) = c_0 < 0$.*

For the case $\mu < \mu_1$, (4.1) has a non-trivial solution under the constraint that $\lambda > 0$ is small; (4.1) has only trivial solution as $\lambda > 0$ is large, as detailed in the following two theorems.

Theorem 4.3.2. *Suppose that $1 < p < q < 5$ and $\lambda > 0$. For $\mu < \mu_1$, there exists $\Lambda > 0$ such that problem (4.1) has trivial solution only if $\lambda \geq \Lambda$.*

Theorem 4.3.3. *Suppose that $1 < p < q < 5$ and $\mu < \mu_1$. Then there is a $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, (4.1) has a ground state solution u_0 with $I(u_0) < 0$. Moreover, (4.1) has a solution w of mountain pass type with $I(w) > 0$.*

The proofs of the three theorems are given below.

Proof of Theorem 4.3.1. Recalling that φ_1 is the eigenfunction corresponding to the first eigenvalue of the operator $-\Delta+|x|^2$. The assumption that $1<p<q<5$ implies that $p+1<q+1<2q$. For $\mu\geq\mu_1$, if $l>0$ is small enough,

$$\begin{aligned}I(l\varphi_1)&=\frac{l^2}{2}\int_{\mathbb{R}^3}|\nabla\varphi_1|^2+(|x|^2-\mu)\varphi_1^2dx+\frac{\lambda l^{2q}}{2q}\int_{\mathbb{R}^3}\phi_{\varphi_1}|\varphi_1|^q dx\\&\quad-\frac{l^{p+1}}{q+1}\int_{\mathbb{R}^3}|\varphi_1|^{p+1}dx,\\&=(\mu_1-\mu)\frac{l^2}{2}\int_{\mathbb{R}^3}\varphi_1^2dx+\frac{\lambda l^{2q}}{2q}\int_{\mathbb{R}^3}\phi_{\varphi_1}|\varphi_1|^q dx\\&\quad-\frac{l^{p+1}}{p+1}\int_{\mathbb{R}^3}|\varphi_1|^{p+1}dx<0.\end{aligned}$$

By applying Lemma 4.2.3, we get that $I(u)$ is bounded from below. Hence we have

$$c_0=\min_{u\in H}I(u)<0.$$

By using Lemma 4.2.3 again, we see that $I(u)$ is coercive. So there exists a point $u_0\in H$ such that $I(u_0)=c_0<0$ and $I'(u_0)=0$. □

Proof of Theorem 4.3.2. For any $\epsilon\in[0,1)$, let

$$F_\epsilon(u)=(1-\epsilon)\int|\nabla u|^2dx+\int|x|^2u^2dx.$$

Then there exists $\varphi_\epsilon\in H$ such that $\mu_\epsilon=F_\epsilon(\varphi_\epsilon)=\inf_{|\varphi|_2=1}F_\epsilon(\varphi)$ and

$$\mu_\epsilon\leq F_\epsilon(e_1)\leq F_0(e_1)=\mu_1.$$

It follows that $\int|\nabla\varphi_\epsilon|^2dx\leq\frac{\mu_1}{1-\epsilon}$ and hence

$$\begin{aligned}0\leq\mu_1-\mu_\epsilon&=\inf_{|u|_2=1}F_0(u)-\inf_{|u|_2=1}F_\epsilon(u)\leq F_0(\varphi_\epsilon)-F_\epsilon(\varphi_\epsilon)\\&=\varepsilon\int|\nabla\varphi_\epsilon|^2dx\\&\leq\frac{\epsilon\mu_1}{1-\epsilon}.\end{aligned}$$

That is, $\lim_{\epsilon\to0}\mu_\epsilon=\mu_1$. For $\delta_0\in(0,\frac{\mu_1-\mu}{2})$ small, there exists a small $\varepsilon_0>0$ such that $\mu_1-\delta_0<\varepsilon_0$. Hence,

$$(\mu_1-\delta_0)\int_{\mathbb{R}^3}u^2dx<\inf_{|u|_2=1}(1-\varepsilon_0)\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}|x|^2u^2dx.$$

As a result of (4.4), we have

$$\begin{aligned}
I'(u)u = &\int_{\mathbb{R}^3} (1-\varepsilon_0)|\nabla u|^2 + (|x|^2 - \mu_1 + \delta_0)u^2 dx + \varepsilon_0 \int_{\mathbb{R}^3} |\nabla u|^2 dx \\
&+ (\mu_1 - \mu - \delta_0) \int_{\mathbb{R}^3} |u|^2 dx + \lambda \int_{\mathbb{R}^3} \phi_u |u|^q dx - \int_{\mathbb{R}^3} |u|^{p+1} dx. \\
&\geq \varepsilon_0 \int_{\mathbb{R}^3} |\nabla u|^2 dx + (\mu_1 - \mu - \delta_0) \int_{\mathbb{R}^3} |u|^2 dx \\
&+ \lambda \int_{\mathbb{R}^3} \phi_u |u|^q dx - \int_{\mathbb{R}^3} |u|^{p+1} dx \\
&\geq \int_{\mathbb{R}^3} (\mu_1 - \mu - \delta_0)|u|^2 + 2\sqrt{\varepsilon_0 \lambda}|u|^{p+1} - |u|^{q+1} dx.
\end{aligned}$$

Because of $\mu_1 - \mu - \delta_0 > 0$ and $2 < p+1 < q+1$, there is a large constant $\Lambda > 0$ such that $I'(u)u > 0$ for all $u \in H^1 \setminus \{0\}$ when $\lambda \geq \Lambda$. □

Proof of Theorem 4.3.3. If $\lambda = 0$, it is easy to know that $\inf I(u) = -\infty$. That is, there exists $\varphi_1 \in H$ such that $I(\varphi_1) < 0$. On the other hand, we observe that $\lim_{\lambda \to 0} I(\varphi_1) = I_0(\varphi_1) < 0$. Hence, $\inf I(u) < 0$ if $\lambda > 0$ is small enough. By using Lemma 4.2.3, we have $\inf I(u) > -\infty$ when $\lambda > 0$. Since $I(u)$ is coercive, we get that there exists $u_0 \in H$ such that $I(u_0) = c_0 < 0$ and $I'(u_0) = 0$.

On the other hand, it is shown that $I(0) = 0$. If $\lambda = 0$, there exists $\rho, \sigma > 0$ satisfying that

$$\inf_{\|u\|=\rho} I(u) > \sigma.$$

By the continuity of $I(u)$, for each small $\lambda > 0$ there exist $\sigma_\lambda > 0$ and $\rho_\lambda \in (0, \|u_0\|)$ such that

$$\inf_{\|u\|=\rho_\lambda} I(u) > \sigma_\lambda.$$

By the mountain pass lemma (see Theorem 2.4.4 in Chapter 2), we get a constant $c_1 > 0$ and a sequence $\{u_n\} \subset H$ such that

$$I(u_n) \to c_1 > 0 \text{ and } I'(u_n) \to 0 \text{ in } H, \text{ as } n \to +\infty.$$

It is showed that $\{u_n\}$ is bounded by Lemma 4.2.3. So there exists $u_0 \in H$ such that $u_n \overset{n}{\rightharpoonup} u_0$ weakly in H. Noting the compactness of embedding $H \hookrightarrow L^k$ for $2 < k < 6$, it is easy to see that there exists a subsequence $\{u_n\}$ such that $u_n \overset{n}{\to} w$ strongly in H. Thus, $I'(\omega) = 0$, $I(\omega) > 0$ and ω is a mountain pass type solution of (4.1). □

4.4 The symmetry breaking for the solutions with least energy

From Theorems 4.3.1 and 4.3.3, we see that (4.1) with $\mu > 0$ has ground state solutions. It is interesting to consider the symmetric property of those ground state solutions.

In this section, we focus on the effects of external potential function on the symmetry of the solutions with least energy. We show that solutions with least energy of (4.1) for $p \in (\frac{8q-5}{7}, q)$ are not radially symmetric under the effects of harmonic potential.

Firstly, we give some geometric properties of the functional I as follows. For any $a > 0$, let $v(x) = a^{\frac{1}{2(q-p)}} u(a^{\frac{p-1}{4(q-p)}} x)$, then we have

$$a^{\frac{p-5}{4(q-p)}} \int_{\mathbb{R}^3} |\nabla v|^2 - a^{\frac{p-1}{2(q-p)}} \mu v^2 dx = \int_{\mathbb{R}^3} |\nabla u|^2 - \mu u^2 dx,$$

$$\int_{\mathbb{R}^3} |x|^2 v^2 dx = \int_{\mathbb{R}^3} |x|^2 a^{\frac{1}{(q-p)}} u^2(a^{\frac{p-1}{4(q-p)}} x) dx = a^{\frac{9-5p}{4(q-p)}} \int_{\mathbb{R}^3} |x|^2 u^2 dx,$$

$$\int_{\mathbb{R}^3} |v|^{p+1} dx = a^{\frac{5-p}{4(q-p)}} \int_{\mathbb{R}^3} |u|^{p+1} dx,$$

and

$$\int_{\mathbb{R}^3} \frac{|v(x)|^q |v(y)|^q}{|x-y|} dx = a^{1+\frac{5-p}{4(q-p)}} \int_{\mathbb{R}^3} \frac{|u(x)|^q |u(y)|^q}{|x-y|} dx.$$

The above equalities imply that

$$\begin{aligned} I(u) &= \frac{1}{2} \int |\nabla u|^2 + (|x|^2 - \mu)|u|^2 dx + \frac{\lambda}{2q} \int \phi_u |u|^q dx - \frac{1}{p+1} \int |u|^{p+1} dx \\ &= a^{\frac{p-5}{4(q-p)}} \left(\frac{1}{2} \int |\nabla v|^2 + \left[a^{\frac{p-1}{q-p}} |x|^2 - a^{\frac{p-1}{2(q-p)}} \mu \right] |v|^2 dx \right) \\ &\quad + a^{\frac{p-5}{4(q-p)}} \left(\frac{\lambda a^{-1}}{2q} \int \phi_v |v|^q dx - \frac{1}{p+1} \int |v|^{p+1} dx \right). \end{aligned}$$

Let $a = \lambda$, we introduce the following functional.

$$\begin{aligned} J = &\frac{1}{2} \int |\nabla v|^2 + \frac{1}{2q} \int \phi_v |v|^q dx - \frac{1}{p+1} \int |v|^{p+1} dx \\ &+ \lambda^{\frac{p-1}{2(q-p)}} \int_{\mathbb{R}^3} (\lambda^{\frac{p-1}{2(q-p)}} |x|^2 - \mu)|v|^2 dx. \end{aligned}$$

Then,

$$I(u) = \lambda^{\frac{p-5}{4(q-p)}} J(v). \tag{4.14}$$

Let $H_r = \{u \in H(\mathbb{R}^3) : u(x) = u(|x|)\}$, we get that

$$\lambda^{\frac{p-5}{4(q-p)}} \inf_{u\in H_r} J(u) = \inf_{u\in H_r} I(u). \tag{4.15}$$

In fact, for any $u \in H_r$, we have $\lambda^{\frac{1}{2(q-p)}} u(\lambda^{\frac{p-1}{4(q-p)}} x) \in H_r$. Thus,

$$I(u) = \lambda^{\frac{p-5}{4(q-p)}} J(\lambda^{\frac{1}{2(q-p)}} u(\lambda^{\frac{p-1}{4(q-p)}} x)) \geq \lambda^{\frac{p-5}{4(q-p)}} \inf_{u\in H_r} J(u),$$

which implies that

$$\inf_{u\in H_r} I(u) \geq \lambda^{\frac{p-5}{4(q-p)}} \inf_{u\in H_r} J(u).$$

For any $u \in H_r$, let $w = \lambda^{-\frac{1}{2(q-p)}} u(\lambda^{-\frac{p-1}{4(q-p)}} x)$, we have $w \in H_r$ and $\lambda^{\frac{1}{2(q-p)}} w(\lambda^{\frac{p-1}{4(q-p)}} x) = u(x)$. It follows from (4.14) that

$$\lambda^{\frac{p-5}{4(q-p)}} J(u) = \lambda^{\frac{p-5}{4(q-p)}} J(\lambda^{\frac{1}{2(q-p)}} w(\lambda^{\frac{p-1}{4(q-p)}} x)) = I(w) \geq \inf_{v\in H_r} I(v),$$

which implies that $\inf_{u\in H_r} I(u) \geq \lambda^{\frac{p-5}{4(q-p)}} \inf_{u\in H_r} J(u)$. Thus (4.15) holds. Similarly, we also have

$$\lambda^{\frac{p-5}{4(q-p)}} \inf_{u\in H} J(u) = \inf_{u\in H} I(u). \tag{4.16}$$

Then we have the following lemma.

Lemma 4.4.1. *Assume that $p \in (\frac{8q-5}{7}, q)$ and a positive constant $\mu < \frac{1}{2S}\sqrt[6]{\frac{3}{\pi}}$. There exists a constant $C > 0$ (independent of $\lambda > 0$) such that $\inf_{u\in H_r^1} J(u) > -C$.*

Proof. Let $\Omega_\mu = \{x \in \mathbb{R}^3 : \lambda^{\frac{p-1}{2(q-p)}} |x|^2 < \mu\}$. By using the Cauchy inequality and the Sobolev inequality, we get

$$\begin{aligned}\left|\int_{\Omega_\mu} (\lambda^{\frac{p-1}{2(q-p)}} |x|^2 - \mu) v^2 dx\right| &\leq S^2 |\nabla v|_2^2 \left(\int_{\Omega_\mu} |\lambda^{\frac{p-1}{2(q-p)}} |x|^2 - \mu|^{\frac{3}{2}} dx\right)^{2/3} \\ &= \lambda^{\frac{-(p-1)}{2(q-p)}} S^2 |\nabla v|_2^2 \left(\int_{|x|^2<\mu} ||x|^2 - \mu|^{\frac{3}{2}} dx\right)^{2/3}.\end{aligned}$$

By the Cauchy inequality, we get that

$$\begin{aligned}&\left(\int_{|x|^2<\mu} ||x|^2 - \mu|^{\frac{3}{2}} dx\right)^{2/3} \\ &\leq \left(\int_{|x|^2<\mu} dx\right)^{1/3} \left(\int_{|x|^2<\mu} ||x|^2 - \mu|^3 dx\right)^{1/3} \\ &< 2\mu^2.\end{aligned}$$

It follows that

$$\left|\int_{\Omega_\mu}(\lambda^{\frac{p-1}{2(q-p)}}|x|^2-\mu)v^2dx\right|\leq 2\lambda^{\frac{-(p-1)}{2(q-p)}}S^2\mu^2\int|\nabla v|^2dx.$$

Let $\beta=1-4S^2\mu^2\sqrt[3]{\frac{\pi}{3}}$. Since $q<p$, we get that $\beta>0$ as $\mu<\frac{1}{2S}\sqrt[6]{\frac{3}{\pi}}$.

$$J(v)\geq\frac{\beta}{2}\int|\nabla v|^2dx+\frac{1}{2q}\int\phi_v|v|^qdx-\frac{1}{q+1}\int|v|^{q+1}dx\doteq\Phi(v).$$

Now we prove that Φ is coercive in H_r. For any $v\in H_r$, define the functionals $A,M:H_r\to\mathbb{R}$ as follows,

$$A(u)=\int_{\mathbb{R}^3}\phi_u|u|^qdx,$$

$$M(u)=\beta\int_{\mathbb{R}^3}|\nabla u|^2dx+A(u).$$

Clearly, $M(u)>0$ for any $u\in H_r\backslash\{0\}$. Let

$$\tau=[M(u)]^{\frac{q-1}{q-5}},\qquad w(x)=\tau^{\frac{2}{q-1}}u(\tau x).$$

Then,

$$\int_{\mathbb{R}^3}|u|^{p+1}dx=\tau^{3-\frac{2(p+1)}{q-1}}\int_{\mathbb{R}^3}|w|^{p+1}dx,\tag{4.17}$$

and

$$M(w)=\tau^{\frac{5-q}{q-1}}M(u)=1.$$

It follows from $p\in(\frac{8q-5}{7},q)$ and Theorem 4 in [85] that,

$$|u|_{p+1}^2\leq K\left[\beta\int_{\mathbb{R}^3}|\nabla u|^2dx+(A(u))^{\frac{1}{q}}\right],\ \forall\,u\in H_r$$

where $K>0$ is a constant. Since $M(w)=1$ and $\max\limits_{t\geq0}(t^{\frac{1}{q}}-t)=\frac{q-1}{q}q^{q-1}$, we get that

$$\begin{aligned}|w|_{p+1}^2&\leq K[\beta\int_{\mathbb{R}^3}|\nabla w|^2dx+(A(w))^{\frac{1}{q}}]\\&=K[M(w)+(A(w))^{\frac{1}{q}}-A(w)]\\&\leq K(1+\frac{q-1}{q}q^{q-1}).\end{aligned}\tag{4.18}$$

Followed by (4.17) and (4.18), we have that, for any $u\in H_r\backslash\{0\}$,

$$\begin{aligned}\Phi(u)&\geq\frac{1}{4}M(u)-\frac{1}{q+1}\int_{\mathbb{R}^3}|u|^{q+1}dx\\&=\frac{1}{4}M(u)-\frac{1}{q+1}[M(u)]^{\frac{2q-1}{3}}\int_{\mathbb{R}^3}|w|^{q+1}dx\\&\geq\frac{1}{4}M(u)-K_1[M(u)]^{\frac{2q-1}{3}},\end{aligned}\tag{4.19}$$

where $K_1 = \frac{1}{q+1}(\frac{5}{4}K)^{\frac{q+1}{2}}$. Note that $\frac{2(5-q)}{5-p} - 1 = 1 - \frac{2(q-p)}{5-q} \in (0,1)$ for $p \in (\frac{8q-5}{7}, q)$, it follows from (4.19) that

$$\frac{\beta}{2}\int |\nabla v|^2 dx + \frac{1}{2q}\int \phi_v |v|^q dx - \frac{1}{q+1}\int |v|^{q+1} dx \geq -C > \infty$$

holds for some constant C. □

Let $S = \inf_{u\in H^1(\mathbb{R}^3)} \frac{|\nabla u|_2}{|u|_6}$. We can then establish the following theorem which shows that the ground state solutions of (4.1) are nonradial if $\lambda, \mu > 0$ are small.

Theorem 4.4.2. *Assume $q \in [2,5)$, $p \in (\frac{8q-5}{7}, q)$ and $\mu \in (0, \frac{1}{2S}\sqrt[6]{\frac{3}{\pi}})$. Then there exists $\lambda_1 > 0$ such that any ground state of problem (4.1) must be nonradial for $0 < \lambda < \lambda_1$.*

Remark 4.4.3. *The symmetry breaking property of the ground states to problem (4.1) has been investigated in [92] (or [55]) specially for $q = 2$. Theorem 4.4.2 is more general than the results in [92] (or [55]).*

The proof of the theorem is given below.

Proof of Theorem 4.4.2. Since $p < q$, by Theorem 1.5 in [92], there exists $\{u_n\} \subset C_0^\infty(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} |u_n|^{p+1} dx \overset{n\to+\infty}{\to} +\infty$ and

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4q\pi}\int_{\mathbb{R}^6} \frac{|u_n|^q(x)|u_n|^q(y)}{|x-y|} dxdy \leq 1.$$

Let C be as given by Lemma 4.4.1 and choose large $n_0 \in \mathbb{N}$ such that $\int_{\mathbb{R}^3} |u_{n_0}|^{q+1} dx > (q+1)(C+2)$. Since $u_{n_0} \in C_0^\infty(\mathbb{R}^3)$, we may assume that $supp(u_{n_0}) \subset B(0,R)$ for some $R > 0$. Then,

$$\int_{\mathbb{R}^3} |x|^2 |u_{n_0}|^2 dx \leq R^2 \int_{\mathbb{R}^3} |u_{n_0}|^2 dx,$$

There exists a small constant $\lambda_1 > 0$ such that

$$\lambda^{\frac{p-1}{q-p}} \frac{1}{2}\int_{\mathbb{R}^3} |x|^2 |u_{n_0}|^2 dx \leq \lambda^{\frac{p-1}{q-p}} \frac{R}{2}\int_{\mathbb{R}^3} |u_{n_0}|^2 dx < 1, \text{ for } 0 < \lambda < \lambda_1.$$

For $0 < \lambda < \lambda_1$, it follows that

$$\begin{aligned} J(u) &\leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4\pi q}\int_{\mathbb{R}^6} \frac{|u_n|^q(x)|u_n|^q(y)}{|x-y|} dxdy \\ &+ \lambda^{\frac{p-1}{q-p}} \frac{1}{2}\int_{\mathbb{R}^3} |x|^2 |u_{n_0}|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} dx \\ &< -C. \end{aligned}$$

We get

$$\inf_{u\in H} J(u) \le J_\lambda(u_{n_0}) < -C.$$

This inequality together with Lemma 4.4.1 implies that

$$\inf_{u\in H} J(u) < \inf_{u\in H_r} J(u) \tag{4.20}$$

holds for $0 < \lambda < \lambda_1$. From (4.15), (4.16) and (4.20) we get

$$\inf_{u\in H} I(u) < \inf_{u\in H_r} I(u) \text{ for small } \lambda \in (0, \lambda_1),$$

which implies that the ground state solutions are non-radially symmetric.

4.5 Existence of multiple solutions

In this section, we establish and prove the existence of multiple solutions to (4.1) by using the Clark's theorem and index theory.

Theorem 4.5.1. *Assume* $1 < p < q < 5$, $\lambda > 0$. *For* $\mu > \mu_i$ *with* $i = 2, 3, \cdots$, *problem (4.1) has i pairs of solutions* $\pm u_i$.

Proof. By Lemmas 4.2.3, we obtain that I is bounded below, and satisfies the $(PS)_c$ condition. Let φ_k be the eigenfunction of (4.13) corresponding to μ_k. Define

$$S_k = \{u \in \bigoplus_{j=1}^{k} \mathbb{R}\varphi_i, |u|_2 = 1\}.$$

It follows that, for any $u_k \in S_k$,

$$u_k = \sum_{i=1}^{k} c_i\varphi_i \text{ with } \sum_{i=1}^{k} c_i^2 = 1.$$

If $\mu > \mu_k$, we have

$$-\mu + \sum_{i=1}^{k} c_i^2\mu_i < \mu_k - \mu < 0.$$

Thus, for $l > 0$ small,

$$\begin{aligned} I(lu_k) &= \frac{1}{2}(-\mu + \sum_{i=1}^{k} c_i^2\mu_i)l^2 + \frac{l^{2q}\lambda}{2q}\int_{\mathbb{R}^3} \phi_{u_k}|u_k|^q dx - \frac{l^{p+1}}{4}\int_{\mathbb{R}^3} |u_k|^{p+1}dx \\ &\le \frac{1}{2}(\mu_k - \mu)l^2 + Cl^{2q}\lambda\|u_k\|^{2q} - \frac{l^{p+1}}{4}\int_{\mathbb{R}^3} |u_k|^{p+1}dx \\ &\le \frac{1}{2}(\mu_k - \mu)l^2 + Cl^{2q}\lambda < 0, \end{aligned}$$

which indicates that

$$\sup_{u \in \mathbb{S}} I(u) \leq \frac{1}{2}(\omega + \mu_k)l^2 + Cl^4\lambda < 0,$$

where $\mathbb{S} = \{lu : u \in S_k\}$. Remark 2.4.5 implies that $\gamma(\mathbb{S}) = k$. Setting $B = \mathbb{S}, E = H^1(\mathbb{R}^3)$ in Theorem 2.4.6, we get that I has at least k distinct pairs of critical points $\{\pm u_i\}_{i=1}^k$ with $I(\pm u_i) = c_i < 0$. $\square$

Chapter 5

A class of nonlinear elliptic equations without variational structure

5.1 General

We consider the nonlinear elliptic equation (1.3), which is equivalent to the system below,

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi(x)|u|^{r-2}u = |u|^{p-1}u,\ x \in \Omega, \\ -\Delta\phi(x) = |u|^q,\ x \in \Omega, \\ \phi(x) = u(x) = 0 \text{ for } x \in \partial\Omega, \end{cases} \tag{5.1}$$

where $p, q, r > 1$ are nonlinear exponents, λ is a parameter, $V(x)$ is a potential and $\Omega = \mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$ is bounded. For the case $\Omega = \mathbb{R}^3$, $\phi(x) = u(x) = 0$ for $x \in \partial\Omega$ in this chapter means that $\lim\limits_{|x|\to+\infty} \phi(x) = \lim\limits_{|x|\to+\infty} u(x) = 0$. When $q = r = 2$, (5.1) was introduced to study the standing waves of the Schrödinger-Poisson equation, which apppears in the mean field theory for the Hartree Fock model, and the reader is referred to [2, 58, 82, 94, 100] for details.

When $q = r \in [2, 5)$, the elliptic system (5.1) is the Euler-Lagrange equation in regard to the functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$,

$$\begin{aligned} I_\lambda(u) =& \frac{1}{2}\int_\Omega |\nabla u|^2 + V(x)u^2 dx - \frac{1}{p}\int_\Omega |u|^{p+1} dx \\ &+ \frac{\lambda}{2q}\int_{\Omega\times\Omega} G(x,y)|u(x)|^q|u(y)|^q dxdy \end{aligned} \tag{5.2}$$

where $G(x, y)$ is the Green function for the Laplace operator, which was defined in the domain Ω with homogeneous Dirichlet boundary conditions [9, 32, 66, 93]. For the special case $q = r \in [2, 5)$, one could obtain the existence of nontrivial solutions by seeking the nontrivial critical points of the functional I_λ in (5.2), for details see [9, 93] for the case when $\Omega \subset \mathbb{R}^3$ is bounded, and [4,8,66,85,91,115,126] for the case when $\Omega = \mathbb{R}^3$. In general, the existence of nontrivial solution is related to the value of the parameter λ in the case $q = r$. For $q = r = 2$, (5.1) has a positive solution for $p \in (1, 2)$, when $\lambda > 0$ is small. However, (5.1) with $p \in (1, 2)$ has only trivial solution when $\lambda > 0$ is large, as shown in [93]. Similar conclusions for (5.1) with $q = r \in [2, 5)$ were obtained in [9]. When $q = r$, the functional I_λ in (5.2) and inequalities such as

$$\int_{\mathbb{R}^3} |u|^{q+1} dx \le \frac{l^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2l^2}\int_{\Omega\times\Omega} G(x,y)|u(x)|^q|u(y)|^q dxdy, l > 0. \tag{5.3}$$

play a key role in the study of the existence and nonexistence of nontrivial solutions of (5.1) [9, 66, 93]. When $q \neq r$, we do not know the variational functional of (5.1) as well as the related inequalities as (5.3). In the case $q \neq r$, the variational method seems invalid in studying the existence and nonexistence of nontrivial solution to (5.1).

In this chapter, we study system (5.1) for two cases, that is $\Omega = \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^3$ is bounded. The new contribution of this chapter include two aspects. First, by establishing a priori estimates to the solution and applying the degree theory, we obtain results for nonexistence and existence of solutions to (5.1) with general condition including $q \neq r$ in the bounded domain Ω, which are given respectively in Sections 5.2 and 5.3. The other contribution is the establishment of the existence of nontrivial solutions in the case $\Omega = \mathbb{R}^3$, which is presented in Section 5.4.

5.2 A priori estimate and nonexistence of solutions to elliptic equations

In Section 5.2 and Section 5.3, we study (5.1) on the bounded domain $\Omega \subset \mathbb{R}^3$. For the sake of simplicity, we restrict $V(x) = 1$. Therefore, we investigate the following system

$$\begin{cases} -\Delta u + u + \lambda\phi(x)|u|^{r-2}u = |u|^{p-1}u, \ x \in \Omega, \\ -\Delta\phi(x) = |u|^q, \ x \in \Omega, \\ \phi(x) = u(x) = 0 \text{ for } x \in \partial\Omega, \end{cases} \tag{5.4}$$

Denote by $H_0^1(\Omega)$ the Sobolev space, and

$$H_0^1(\Omega) = \left\{u(x) \in L^2(\Omega) : |\nabla u(x)| \in L^2(\Omega)\right\}.$$

As in [43], we also denote the collection of continuous function in Ω by $C(\Omega)$. For $\alpha \in (0,1)$ and $m \in \mathbb{N}$, let

$$C^m(\Omega) = \left\{u \in C(\Omega) : \partial^l u \in C(\Omega) \text{ for } l = 1, 2, \cdots, m\right\},$$

$$C^{m,\alpha}(\Omega) = \left\{u \in C^m(\Omega) : \sup_{x,y\in\Omega} \frac{|\partial^l u(x) - \partial^l u(y)|}{|x-y|^\alpha} < \infty, l = 1, 2, \cdots, m\right\}.$$

The pair $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is called the weak solution of (5.4), if

$$\int_\Omega \nabla\phi\nabla v dx = \int_\Omega |u|^q v dx$$

and

$$\int_\Omega \nabla u\nabla v dx + \int_\Omega uv dx + \lambda\int_\Omega \phi(x)|u|^{r-2}uv dx - \int_\Omega |u|^{p-1}uv dx = 0,$$

hold for $v \in H_0^1(\Omega)$. If (u, ϕ) of (5.4) belongs to $C^2(\Omega) \times C^2(\Omega)$, then (u, ϕ) is called a classical solution of (5.4).

Motivated by Lemma 3.1 of [57], we first establish the a priori estimate to the weak solution of (5.4) in Lemma 5.2.1. Then, a theorem is established for the nonexistence of solution to (5.4), followed by proofs of the Lemmas and the theorems.

Lemma 5.2.1. *Let the domain $\Omega \subset \mathbb{R}^3$ be bounded and $\lambda > 0$. Assume $r, q \in (1, 5)$ and $p \in (1, \min\{r, q\})$. If $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a weak solution of (5.4), it follows that*

$$|u(x)| \leqslant M_{p,q,r,\lambda}, \text{ a.e. in } x \in \Omega,$$

where $M_{p,q,r,\lambda} = \frac{r-p}{r}\left[\frac{p(p-1)}{\lambda r(q-1)}\right]^{\frac{p}{r-p}}\left(\frac{q-p}{q-1}\right)^{\frac{p(q-p)}{(r-p)(p-1)}}$.

Theorem 5.2.2. *For $r, q \in (1, 5)$ and $p \in (1, \min\{r, q\})$, let*

$$c_{p,r,q} = \frac{p(p-1)}{r(q-1)}\left(\frac{r-p}{r}\right)^{\frac{r-p}{p}}\left(\frac{q-p}{q-1}\right)^{\frac{q-p}{p-1}} > 0.$$

Suppose $\Omega \subset \mathbb{R}^3$ is bounded. If $\lambda \geq c_{p,r,q}$, we get that the weak solution of (5.4) is trivial.

Remark 5.2.3. *Let (u, ϕ) be a weak solution of (5.4). For the special case $q = r = 2$, by the inequality (5.3) with $l^2 = q = 2$, it is obvious that*

$$\int_\Omega (u^2 - |u|^{p+1} + |u|^3)dx \leq 0$$

holds for $\lambda \geq 1/4$ and $p \in (1, 2)$, which implies $u = \phi = 0$ (see Theorem 4.1 in [91] for details). However, it is not clear whether the inequality (5.3) holds when $r \neq q$. Hence, the method established in [91] can not be applied directly to study the nonexistence of solution of (5.4) without conditon $r = q$. At the same time, for $q = r = 2$, we have $p \in (1, 2)$ and $c_{p,2,2} = \frac{p}{2}(p-1)(\frac{2-p}{2})^{\frac{2-p}{p}}(2-p)^{\frac{2-p}{p-1}}$ by Theorem 5.2.2. If $p \in (1, 2)$ is close to 1, then $c_{p,2,2} < \frac{1}{4}$, which shows that Theorem 5.2.2 improves Theorem 4.1 in [91].

Proof of Lemma *5.2.1:* Since $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a weak solution of (5.4), one has

$$\int_\Omega \nabla u \nabla v dx + \int_\Omega uv dx + \lambda \int_\Omega \phi(x)|u|^{r-2}uv dx - \int_\Omega |u|^{p-1}uv dx = 0, \tag{5.5}$$

$$\int_\Omega \nabla\phi\nabla v dx = \int_\Omega |u|^q v dx, \tag{5.6}$$

holds for any $v \in H_0^1(\Omega)$. Let $\rho_{p,q} = \frac{p-1}{q-1}\left(\frac{q-p}{q-1}\right)^{\frac{q-p}{p-1}}$, multiplying both sides of (5.6) by $\rho_{p,q}$ and using (5.6), we have

$$\begin{aligned}\int_\Omega \nabla u \nabla v dx + \int_\Omega [u + c_{p,q}|u|^q - |u|^{p-1}u]v dx + \lambda \int_\Omega \phi(x)|u|^{r-2}uv dx \\ = \rho_{p,q}\int_\Omega \nabla\phi\nabla v dx, \quad \text{for any } v \in H_0^1(\Omega).\end{aligned} \tag{5.7}$$

On the basis of the above equalities, let

$$v_1(x) = (u(x) - \rho_{p,q}\phi(x))^+ \text{ and } \Omega_1 = \{x \in \Omega : v_1(x) > 0\}, \tag{5.8}$$

then we have $v_1 \in H_0^1(\Omega)$ and $u(x) \geqslant \rho_{p,q}\phi(x) > 0$ for all $x \in \Omega_1$. Taking $v(x) = v_1(x)$ in (5.7), we get

$$\int_{\Omega_1} \nabla u \nabla v_1 dx + \int_{\Omega_1} [u + \rho_{p,q}|u|^q - |u|^{p-1}u]v_1 dx \leqslant \rho_{p,q} \int_{\Omega_1} \nabla\phi\nabla v_1 dx. \tag{5.9}$$

If $p \in (1, q)$, we have $s + \rho_{p,q}s^q - s^p \geqslant 0$ for all $s \geqslant 0$. Then, (5.9) implies that

$$\int_{\Omega_1} \nabla u \nabla v_1 dx - \rho_{p,q} \int_{\Omega_1} \nabla\phi\nabla v_1 dx \leqslant 0.$$

That is,

$$\int_{\Omega_1} \nabla(u - \rho_{p,q}\phi)\nabla v_1 dx = \int_{\Omega_1} |\nabla v_1|^2 dx = 0. \tag{5.10}$$

Hence, $|\Omega_1| = 0$ or $v_1|_{\Omega_1} \equiv$ constant. According to the definition of Ω_1 and $v_1 \equiv 0$ in $\Omega \setminus \Omega_1$, we get that $u(x) \leqslant \rho_{p,q}\phi(x)$ a.e. in Ω. In the same way, we replace u by $-u$ and repeat the above procedure, then we get that $-u(x) \leqslant \rho_{p,q}\phi(x)$. Hence,

$$|u(x)| \leqslant \rho_{p,q}\phi(x) \text{ a.e. in } x \in \Omega. \tag{5.11}$$

In the following, we prove that

$$|u(x)| \leqslant M_{p,q,r,\lambda} \triangleq \frac{r-p}{r}\left[\frac{p(p-1)}{\lambda r(q-1)}\right]^{\frac{p}{r-p}}\left(\frac{q-p}{q-1}\right)^{\frac{p(q-p)}{(r-p)(p-1)}}$$

a.e.$x \in \Omega$. Let

$$v_2(x) = (u(x) - M_{p,q,r,\lambda})^+ \text{ and } \Omega_2 = \{x \in \Omega : v_2(x) > 0\}, \tag{5.12}$$

then $v_2 \in H_0^1(\Omega)$ and $u(x) \geqslant M_{p,q,r,\lambda} > 0$ for $x \in \Omega_2$. It is easy to get that $\max\limits_{s\geqslant 0}\{s^p - \frac{\lambda}{\rho_{p,q}}s^r\} = M_{p,q,r,\lambda}$ if $p \in (1, r)$. It follows from (5.5) and (5.11) that

$$\int_{\Omega_2} \nabla u\nabla v_2 + uv_2 dx \leqslant \int_{\Omega_2} u^p v_2 - \frac{\lambda}{\rho_{p,q}}u^r v_2 dx \leqslant \int_{\Omega_2} M_{p,q,r,\lambda}v_2 dx,$$

The above gives

$$\begin{aligned}&\int_{\Omega_2} |\nabla(u - M_{p,q,r,\lambda})^+|^2 + |(u - M_{p,q,r,\lambda})^+|^2 dx\\ &= \int_{\Omega_2} \nabla[u - M_{p,q,r,\lambda}]\nabla v_2 + [u - M_{p,q,r,\lambda}]v_2 dx \leqslant 0,\end{aligned}$$

which shows that $|\Omega_2| = 0$. Hence $u(x) \leqslant M_{p,q,r,\lambda}$ a.e in $x \in \Omega$. Similarly, $-u(x) \leqslant M_{p,q,r,\lambda}$. Hence

$$|u(x)| \leqslant M_{p,q,r,\lambda}, \text{ a.e. in } x \in \Omega. \tag{5.13}$$

***Proof of Theorem** 5.2.2:* Let $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be a weak solution of (5.4), then a standard argument shows that $u \in C^2(\Omega)$. We get

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 dx + \lambda \int_\Omega \phi(x)|u|^r dx = \int_\Omega |u|^{p+1} dx.$$

If u is nontrivial, it follows from $\lambda > 0$ that

$$\int_\Omega u^2 dx < \int_\Omega |u|^{p+1} dx,$$

which implies that

$$\|u\|_{L^\infty(\Omega)} > 1 \text{ for } p > 1. \tag{5.14}$$

By Lemma 5.2.1, we get

$$|u(x)| \leqslant \frac{r-p}{r}\left[\frac{p(p-1)}{\lambda r(q-1)}\right]^{\frac{p}{r-p}}\left(\frac{q-p}{q-1}\right)^{\frac{p(q-p)}{(r-p)(p-1)}}, \quad \text{for all } x \in \Omega. \tag{5.15}$$

Combining (5.14) with (5.15), we have

$$\lambda < c_{p,r,q} \triangleq \frac{p(p-1)}{r(q-1)}\left(\frac{r-p}{r}\right)^{\frac{r-p}{p}}\left(\frac{q-p}{q-1}\right)^{\frac{q-p}{p-1}}.$$

Hence, (u, ϕ) must be trivial when $\lambda \geq c_{p,r,q}$. □

5.3 Existence of solution to an elliptic equation

From Theorem 5.2.2, we see that the small value constraint of parameter $\lambda > 0$ is necessary for the existence of nontrivial solution of (5.4) with general nonlinear exponents p, q, r.

The existence of solutions of (5.4) is investigated below by using the Leray-Schauder degree. For this reason, we consider (5.4) as a small perturbation of the nonlinear equation,

$$-\Delta u(x) + u(x) = |u|^{p-1}u(x), x \in \Omega. \tag{5.16}$$

Let $X = C^{0,\alpha}(\Omega)$ and $X_0 = \{u \in X : u(x) = 0 \text{ for } x \in \partial\Omega\}$ be the Banach spaces equipped with the norm

$$\|u\| = \sup_{x\in\Omega} |u(x)| + \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

For each $f \in X_0$, we get that $-\Delta\phi = f(x)$ possesses a solution

$$\phi(x) = \int_\Omega G(x, y)f(y)dy,$$

and $G(x, y)$ is the Green function of the Laplace operator in the bounded domain Ω, which has Dirichlet homogeneous boundary condition. We get a well-defined compact operator $\Delta^{-1} : X \to X_0$ by

$$\Delta^{-1}(f) = \int_\Omega G(x, y)|f(y)dy.$$

Let $u^+ = \max\{u, 0\}$. For each $p, q > 1$, $r \geq 2$ and $\lambda \in \mathbb{R}$, we define the bounded continuous operator as

$$N_\lambda : X \to X \text{ by } N_\lambda(u) = (u^+)^p - u - \lambda\Delta^{-1}(|u|^q)(u^+)^{r-1}.$$

By using N_λ, we establish a compact perturbation of the identity map $Q_\lambda : X \to X$ by

$$Q_\lambda(u) = u - \Delta^{-1} \circ N_\lambda(u).$$

It follows that the zeroes of Q_λ are the solutions of (5.4). To apply the Leray-Schauder degree theory, we calculate the derivative of Q_λ at $u \in X$ by the following linear operator $Q'_\lambda(u) : X \to X$,

$$\begin{aligned} &Q'_\lambda(u)[v] \\ &= v - \Delta^{-1}[p(u^+)^{p-1}v - v - \lambda(r-1)\Delta^{-1}(|u|^q)(u^+)^{r-2}v \\ &\quad -\lambda q\Delta^{-1}(|u|^{q-2}uv)(u^+)^{r-1}]. \end{aligned}$$

For $q, r \geq 2$, $p > 1$ and $\lambda \in \mathbb{R}$, we see that the derivative $Q'_\lambda(u)$ with $u \in X$ is a well defined bounded linear operator in X.

Now we can establish the following lemmas.

Lemma 5.3.1. *Let $p \in (1, 5)$, $q > 1$ and $r \geq 2$. Then there is a $R > 0$ and $\lambda_0 > 0$ such that*

$$deg_{LS}[Q_\lambda, B(0, R), 0] = 0,$$

holds for all $\lambda \in (0, \lambda_0)$.

Proof: Since $p \in (1, 5)$, by Theorem 1.1 of [34] and the Schauder estimate, we get that the solution u of (5.16) has a priori estimate $\|u\| < R$ for a positive constant R. The zeroes of Q_0 are the solutions of (5.16). This implies that $\{u \in X : Q_0(u) = 0\} \subset B(0, R) \subset X$. A standard argument (see Theorem 2.1 in [34]) implies that

$$deg_{LS}[Q_0, B(0, R), 0] = 0. \tag{5.17}$$

By the definition of Q_0, there is $\delta > 0$ such that $\|Q_0(u)\| > \delta$ for all $\|u\| = R$. We get that

$$\begin{aligned} &Q_\lambda(u) - Q_0(u) \\ &= -\lambda\Delta^{-1}[(r-1)\Delta^{-1}(|u|^q)(u^+)^{r-2}v + q\Delta^{-1}(|u|^{q-2}uv)(u^+)^{r-1}], \end{aligned}$$

then there is $\lambda_0 > 0$ such that $\|Q_\lambda(u) - Q_0(u)\| < \delta$ for any $0 \le |\lambda| \le \lambda_0$ and $\|u\| \le R$. Therefore, $Q_\lambda(u) \ne 0$ for all $\|u\| = R$. By using the homotopy invariance of the Leray-Schauder degree (see Theorem 2.4.8 in Chapter 2 for detail), we get that

$$deg_{LS}[Q_\lambda, B(0, R), 0] = deg_{LS}[Q_0, B(0, R), 0]) = 0$$

holds for all $|\lambda| \in (0, \lambda_0)$. □

Lemma 5.3.2. *Assume $\lambda \in \mathbb{R}$, $q, r \ge 2$ and $p > 1$. Let $R > 0$ be given by Lemma 5.3.1. Then there is a small $\varepsilon \in (0, R)$ such that*

$$deg_{LS}[Q_\lambda, B(0, \varepsilon), 0] = deg_{LS}[Q'_\lambda(0), B(0, \varepsilon), 0] = 1.$$

Proof: We know that $Q_\lambda(0) = 0$ and $Q'_\lambda(0)$ is the identity operator on X. By the inverse function theorem, we get that there is a small $\varepsilon \in (0, R)$ such that $u = 0$ is the unique solution of $Q_\lambda(u) = 0$ in $B(0, \varepsilon) = \{u \in X : \|u\| < \varepsilon\}$. By the definition of Leray-Schauder degree, as stated in Definition 2.4.7 of Chapter 2, we see that

$$deg_{LS}[Q_\lambda, B(0, \varepsilon), 0] = 1$$

and

$$deg_{LS}[Q'_\lambda(0), B(0, \varepsilon), 0] = 1.$$

□

The following theorem is then developed, establishing the condition for the existence of a positive solution of (5.4) when λ is small.

Theorem 5.3.3. *Suppose $\Omega \subset \mathbb{R}^3$ is bounded . For any $q, r \geq 2$ and $1 < p < 5$, there exists a small $\lambda_0 > 0$ such that (5.4) has a positive solution when $|\lambda| \in (0, \lambda_0)$.*

Proof: Let λ_0 and R be the number as given by Lemma 5.3.1. Let ε be the number as given by Lemma 5.3.2. For $|\lambda| \in (0, \lambda_0)$, the additivity property of the Leray-Schauder degree (Theorem 2.4.8 in Chapter 2) implies that

$$\begin{aligned} & deg_{LS}[Q_\lambda, B(0,R) \setminus B(0,\varepsilon), 0] \\ & = deg_{LS}[Q_\lambda, B(0,R), 0] - deg_{LS}[Q_\lambda, B(0,\varepsilon), 0] \\ & = -1. \end{aligned}$$

That is, there is $v \in \{u \in X : \varepsilon \leq \|u\| < R\}$ such that $Q_\lambda(v) = 0$. The standard regularity theory shows that $v \in C_0^2(\Omega)$ is a classical solution of

$$-\Delta v(x) + v(x) + \lambda(v^+(x))^{r-1} \int_\Omega G(x,y)|v(y)|^q dy = (v^+(x))^p,$$

where $|\lambda| \in (0, \lambda_0)$. Followed by the Maximum principle, we obtain that $v(x)$ is positive.

5.4 Existence of solution to an elliptic equation with nonlocal nonlinearity

In this section, we consider system (5.1) for the case $\Omega = \mathbb{R}^3$. If $\Omega = \mathbb{R}^3$, then $G(x,y) = \frac{1}{4\pi|x-y|}$. For simplicity, we let $V(x) = P(x) + \mu$ and $\lambda = 1$, where μ is a parameter, then (5.1) is equivalent to the following elliptic equation with nonlocal term,

$$-\Delta u + (P(x) + \mu)u + |u|^{r-2}u \int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|} dy = |u|^{p-1}u, x \in \mathbb{R}^3 \quad (5.18)$$

where $p, q > 1$, $r \geq 2$. Here, we give more constraint on the potential function as follows.

(P_1) $P(x) \in C^{0,\alpha}_{loc}(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$, $\alpha \in (0, 1)$.

We introduce some notation and recall some basic results for the elliptic equations and the Schröinger operators. We denote the norm of $L^k(\mathbb{R}^3)$ space by $\|\cdot\|_{L^k}$, the norm of $W^{2,k}(\mathbb{R}^3)$ by $\|\cdot\|_{W^{2,k}}$, and the norm of $D^{2,k}(\mathbb{R}^3)$ by $\|\cdot\|_{D^{2,k}}$ for $1 \leqslant k \leqslant +\infty$. Let

$$\cap_{k=2}^{+\infty} W^{2,k}(\mathbb{R}^3) = \{\, u \mid u \in W^{2,k}(\mathbb{R}^3) \text{ for any } 2 \leqslant k < +\infty\}.$$

We define the operator S: $W^{1,2}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by

$$Su = -\Delta u + Pu \tag{5.19}$$

where $P \in L^\infty(\mathbb{R}^3)$. As is well known, the operator S is self-adjoint in $L^2(\mathbb{R}^3)$ and bounded below ([1], Theorem 1.1). In particular, setting $\Lambda = \inf(\sigma(S))$, where $\sigma(S)$ denotes the spectrum of S, then we have $\Lambda \geqslant \inf_{x\in\mathbb{R}^3} P(x) > -\infty$. In the following, we assume that

(P_2) $\inf_{x\in\mathbb{R}^3} P(x) < \liminf_{|x|\to+\infty} P(x)$ and the first eigenvalue Λ of S is negative.

(P_3) There exist $\tau > 0$, $l > \frac{3}{4}$ such that $P(x) \geqslant \tau^2 H(\tau|x|)$, where $H(r) = \frac{2l((2l-1)r^2-3)}{(r^2+1)^2}$.

Remark 5.4.1. *We see that $H(r)$ is a sign-changing function. Meanwhile, we can check that $P(x) = \tau^2 H(\tau|x|)$ satisfies the assumptions* (P_2) *and* (P_3).

Remark 5.4.2. *Under assumptions* (P_1) *and* (P_2)*, we get that Λ is a simple eigenvalue of operator S, and the first eigenfunction φ has the following properties*

$$\begin{cases} \varphi \in C^2(\mathbb{R}^3) \cap (\cap_{k=1}^{+\infty} W^{2,k}(\mathbb{R}^3)), \\ S\varphi = \Lambda\varphi, \\ \varphi(x) > 0 \quad \textit{for all} \quad x \in \mathbb{R}^3, \\ \lim_{|x|\to\infty} \varphi(x) e^{\gamma|x|} = 0 \quad \textit{for all} \quad \gamma < \sqrt{-\Lambda}. \end{cases} \tag{5.20}$$

Motivated by [37], by the upper and lower solutions method, we study the existence of a positive solution of (5.18).

Since there is a nonlocal term in (5.18) , the main difficulty in the application of the upper and lower solutions method is finding the appropriate lower-solutions and upper-solutions to (5.18).

The rest of this section is structured as follows. First, the definitions of the lower-solution and upper-solution for the problem are presented, then various lemmas are introduced as preliminaries for establishing the existence results, followed by a theorem establishing the condition for the existence of nontrivial solutions.

Definition 5.4.3. *An upper-solution of (5.18) is a positive function $\psi(x) \in C^2(\mathbb{R}^3)$ which satisfies*

$$-\Delta\psi + P(x)\psi + \mu\psi + |u|^{r-2}\psi \int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|} dy \geqslant \psi^p \quad x \in \mathbb{R}^3,$$

where $u(x) \in \{v \in W^{2,2}(\mathbb{R}^3) : 0 < u(x) \leqslant \psi(x)\ a.e\ x \in \mathbb{R}^3\}$.

Definition 5.4.4. *A lower-solution of (5.18), comparing with the upper-solution $\psi(x)$, is a positive function $\varphi(x) \in C^2(\mathbb{R}^3)$ which satisfies*

$$-\Delta\varphi + P(x)\varphi + \mu\varphi + |u|^{r-2}\varphi \int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|} dy \leqslant \varphi^p \quad x \in \mathbb{R}^3,$$

where $u(x) \in \{v \in W^{2,2}(\mathbb{R}^3) : 0 < u(x) \leqslant \psi(x)\ a.e\ x \in \mathbb{R}^3\}$.

To establish the upper-solutions and lower-solutions for problem (5.18), we establish the L^∞ estimation of the potential $\int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|} dy$ in the lemma below.

Lemma 5.4.5. *Let $\phi(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|} dy$, where u is a measurable function. Assume that $0 \leqslant u \leqslant \psi(x) = \frac{\beta}{(1+d^2|x|^2)^l}$, where β is a positive parameter, and the constants $d \neq 0, l > \frac{1}{2}$. Then $\phi(x) \in L^\infty(\mathbb{R}^3)$ and*

$$\phi(x) \leqslant \beta^2\left(\frac{1}{(2l-1)d^{4l}} + \frac{5}{6}\right), \text{ for any } x \in \mathbb{R}^3$$

.

Proof. Since $0 \leqslant u \leqslant \psi(x)$, we get

$$\begin{aligned} 4\pi\phi(x) &= \int_{\mathbb{R}^3} |y|^{-1}|u(x-y)|^p dy \\ &\leqslant \beta^q \int_{|y|\leqslant 1} |y|^{-1}\left(1+d^2|x-y|^2\right)^{-ql} dy \\ &\quad + \beta^q \int_{|y|>1} |y|^{-1}\left(1+d^2|x-y|^2\right)^{-ql} dy. \end{aligned} \tag{5.21}$$

A simple calculation shows that

$$\beta^q \int_{|y|\leqslant 1} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy \leqslant \beta^q \int_{|y|\leqslant 1} |y|^{-1}dy = 2\pi\beta^q, \quad (5.22)$$

and

$$\begin{aligned} &\beta^q \int_{|y|>1} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy \\ &= \beta^q \int_{\substack{|y|>1 \\ |x-y|>|y|}} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy \\ &+\beta^q \int_{\substack{|y|>1 \\ |x-y|\leqslant|y|}} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy. \end{aligned} \quad (5.23)$$

$$\begin{aligned} &\beta^q \int_{\substack{|y|>1 \\ |x-y|>|y|}} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy \\ &\leqslant \beta^q \int_{|y|>1} d^{-2ql}|y|^{-2ql-1}dy \\ &= \frac{2}{(2ql-1)}\pi\beta^q d^{-2ql}. \end{aligned} \quad (5.24)$$

$$\begin{aligned} &\beta^q \int_{\substack{|y|>1 \\ |x-y|\leqslant|y|}} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy \\ = \; &\beta^q \int_{1<|x-y|\leqslant|y|} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy \\ &+\beta^2 \int_{\substack{|y|>1 \\ |x-y|\leqslant 1}} |y|^{-1}(1+d^2|x-y|^2)^{-2l}dy. \end{aligned} \quad (5.25)$$

$$\begin{aligned} &\beta^q \int_{1<|x-y|\leqslant|y|} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy \\ &\leqslant \beta^q \int_{1<|x-y|} d^{-2ql}|x-y|^{-2ql-1}dy \\ &= \frac{2}{(2ql-1)}\pi\beta^q d^{-2ql}. \end{aligned} \quad (5.26)$$

$$\begin{aligned} &\beta^q \int_{\substack{|y|>1 \\ |x-y|\leqslant 1}} |y|^{-1}(1+d^2|x-y|^2)^{-ql}dy \\ &\leqslant \beta^q \int_{|x-y|\leqslant 1} (1+d^{2ql}|x-y|^{2ql})^{-1}dy \\ &\leqslant \frac{4\pi\beta^q}{3}. \end{aligned} \quad (5.27)$$

From (5.21)-(5.27), we get that

$$|\phi(x)| \leqslant \beta^q(\frac{1}{(2ql-1)d^{2ql}} + \frac{5}{6})$$

for any $x \in \mathbb{R}^3$. □

Based on the estimates above, we establish the lower solutions and upper solutions of problem (5.18) under the assumptions $(\mathrm{P}_1) - (\mathrm{P}_3)$.

Lemma 5.4.6. *Assume that $P(x)$ satisfies (P_1), (P_2) and (P_3). Let d and l be given in (P_3). Then there is a constant $\bar{\mu} > 0$. If $\mu \in (0, \bar{\mu}]$, we have $\psi(x) = \frac{\beta}{(1+d^2|x|^2)^l}$ with $\beta \in (0, \mu^{\frac{1}{p-1}}]$ is an upper-solution of (5.18). Let $\beta < 1$, $\epsilon > 0$ and $\varphi(x) > 0$ is the first eigenfunction of the operator S defined in (5.19), then $\epsilon\varphi(x)$ is a lower-solution of (5.18) comparing with the upper-solution $\psi(x)$.*

Proof: Let $\psi(x) = \frac{\beta}{(1+d^2|x|^2)^l}$ with $\beta > 0$. A simple calculation implies that $\Delta\psi(x) = d^2H(d|x|)\psi(x)$ with $H(|x|) = \frac{2l((2l-1)|x|^2-3)}{(|x|^2+1)^2}$. If follows from (P_3) and $\beta \in (0, \mu^{\frac{1}{p-1}}]$ that

$$\begin{aligned}
\Delta\psi + \psi^p &= d^2H(d|x|)\psi + \psi^p \\
&\leqslant P(x)\psi + \psi^p \\
&\leqslant P(x)\psi + \beta^{p-1}\psi \\
&\leqslant P(x)\psi + \mu\psi + |u|^{r-2}\psi \int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|}dy,
\end{aligned}$$

holds for all $u(x) \in W^{2,2}(\mathbb{R}^3)$ with $0 < u(x) \leqslant \psi(x)$. That is, $\psi(x) = \frac{\beta}{(1+d^2|x|^2)^l}$ with $\beta \in (0, \mu^{\frac{1}{p-1}}]$ and $d > 0$ is an upper-solution of (5.18).

If $u(x) \in W^{2,2}(\mathbb{R}^3)$ with $0 < u(x) \leqslant \psi(x)$, by applying Lemma 5.4.5 with $\beta \in (0, \mu^{\frac{1}{p-1}}]$, we get that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|}dy + \mu &\leqslant \mu + \beta^q(\frac{1}{(2ql-1)d^{2ql}} + \frac{5}{6}) \\
&\leqslant \mu + \mu^{\frac{q}{p-1}}(\frac{1}{(2ql-1)d^{2ql}} + \frac{5}{6}). \qquad (5.28)
\end{aligned}$$

Let $f(\mu) = \mu + \mu^{\frac{q}{p-1}}(\frac{1}{(2l-1)d^{4l}} + \frac{5}{6})$. For $p, q > 1$, we know that $f(\mu)$ is increasing in $[0, +\infty)$ and $f(0) = 0$. If $\Lambda < 0$, there exists $\bar{\mu} \in (0, +\infty)$ such that

$$-f(\mu) \geq \Lambda, \ \ \forall \mu \in (0, \bar{\mu}]. \qquad (5.29)$$

Then (5.28) and (5.29) imply that

$$-\int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|}dy - \mu \geq \Lambda$$

holds for all $\mu \in (0, \bar{\mu})$ and $u(x) \in W^{2,2}(\mathbb{R}^3)$ with $0 < u(x) \leqslant \psi(x)$. Let Λ be given by (P_2). Since φ is the first eigenfunction of the operator S, we have

$$-\Delta\varepsilon\varphi + P(x)\varepsilon\varphi = \Lambda\varepsilon\varphi \leq -\mu\varepsilon\varphi - \varepsilon\varphi\int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|}dy, \quad x \in \mathbb{R}^3,$$

If $\beta \in (0,1)$, then $0 < u(x) \leqslant \psi(x) < 1$ implies that $|u|^{r-2} < 1$. It follows that

$$\begin{aligned} &-\Delta\varepsilon\varphi + P(x)\varepsilon\varphi = \Lambda\varepsilon\varphi \\ &\leq -\mu\varepsilon\varphi - |u|^{r-2}\varepsilon\varphi\int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|}dy + (\varepsilon\varphi)^p, \quad x \in \mathbb{R}^3, \end{aligned} \tag{5.30}$$

holds for any $\varepsilon > 0$. By Definition 5.4.4 and (5.30), we see that $\varepsilon\varphi(x)$ with $\varepsilon > 0$ is a lower-solution of (5.18) comparing with the upper-solution $\psi(x)$. □

To develop and prove the theorem for the existence of solutions, the following classical conclusions are needed.

Lemma 5.4.7 (Proposition 27 in [33]; Proposition 3 in [101])**.** *Let $c > 0$, then we have*

(i) For each $f \in L^k(\mathbb{R}^3)$, $1 \leqslant k \leqslant \infty$, there is a unique solution $u = \mathcal{T}f \in L^k(\mathbb{R}^3)$ such that $-\Delta u(x) + cu(x) = f$ on $\mathbb{R}^3$ in the distribution sense.

(ii)For $f \in L^k(\mathbb{R}^3)$, $1 < k < +\infty$, we have $\mathcal{T}f \in W^{2,k}(\mathbb{R}^{N3})$; and there exists $C(k,c)$ such that $\|\mathcal{T}f\|_{W^{2,k}} \leqslant C(k,c)\|f\|_{L^k}$

Lemma 5.4.8 (Proposition 4.4, [103])**.** *Assume that $P(x) \in L^\infty(\mathbb{R}^3)$, and Λ is the first eigenvalue of S, $u \in ker(S - \Lambda I)$. If $\Lambda < P_\infty = \liminf_{|x|\to\infty} P(x)$, then $|u(x)| \leqslant C\|u\|_{L^\infty}e^{-\mu|x|}$ for all $x \in \mathbb{R}^3$, where $\mu \in (0, \sqrt{P_\infty - \Lambda})$ and C is a constant.*

Let $\psi(x)$ and $\varphi(x)$ be given by Lemma 5.4.6. By Lemma 5.4.8, we know that $|\varphi(x)| \leqslant C\|\varphi\|_{L^\infty}e^{-\mu|x|}$ for all $x \in \mathbb{R}^3$, where $\mu \in (0, \sqrt{P_\infty - \Lambda})$. Then by the decay of $\varphi(x)$ and $\psi(x)$, if ϵ is small enough, we have

$$\epsilon\varphi(x) < \psi(x) \text{ for all } x \in \mathbb{R}^3. \tag{5.31}$$

To apply the iterative methods, the following lemma is needed.

Lemma 5.4.9. *Assume that $K > 0$ is a constant, the functions w and v are defined on $\mathbb{R}^3$. Consider the equation below,*

$$- \Delta u(x) + Ku(x) = h(x, w, v), \qquad x \in \mathbb{R}^3, \tag{5.32}$$

where h is a function on $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$. $\psi(x)$ and $\varphi(x)$ are defined in Lemma 5.4.6. For $k > 1$, $\alpha \in (0, 1)$ and small $\epsilon > 0$, suppose that h, v and w satisfy

(h_1) *if $v, w \in C^{2,\alpha}_{loc}(\mathbb{R}^3) \cap W^{2,k}(\mathbb{R}^3)$ and $\epsilon\varphi(x) \leqslant u(x) \leqslant \psi(x)$, then*

$$h(x, w, v) \in C^{0,\alpha}_{loc}(\mathbb{R}^3) \cap L^k(\mathbb{R}^3);$$

(h_2) *if $u \in C^{2,\alpha}_{loc}(\mathbb{R}^3) \cap W^{2,k}(\mathbb{R}^3)$ and $\epsilon\varphi(x) \leqslant u(x) \leqslant \psi(x)$, then*

$$\begin{aligned} h(x, u(x), \epsilon\varphi(x)) &\leqslant h(x, u(x), u(x)) \\ &\leqslant h(x, u(x), \psi(x)); \end{aligned}$$

(h_3) *if $u \in C^{2,\alpha}_{loc}(\mathbb{R}^3) \cap W^{2,k}(\mathbb{R}^3)$ and $\epsilon\varphi(x) \leqslant u(x) \leqslant \psi(x)$, then*

$$-\Delta\epsilon\varphi(x) + K\epsilon\varphi(x) \leqslant h(x, u(x), \epsilon\varphi(x)),$$

$$-\Delta\psi(x) + K\psi(x) \geqslant h(x, u(x), \psi(x)).$$

Then there is a sequence $\{u_n\} \subset C^{2,\alpha}_{loc}(\mathbb{R}^3) \cap W^{2,k}(\mathbb{R}^3)$ such that

$$-\Delta u_{n+1}(x) + Ku_{n+1}(x) = h(x, u_n, u_n)$$

and

$$\epsilon\varphi(x) \leqslant u_n(x) \leqslant \psi(x).$$

Lemma 5.4.9 is similar to Lemma 3.2 in [58]. The proof is omitted here.

Now, we develop and prove the existence theorem.

Theorem 5.4.10. *Suppose that $P(x)$ satisfies (P_1), (P_2) and (P_3), then there is $\bar{\mu} \in (0, +\infty)$ such that for any $\mu \in (0, \bar{\mu}]$, (5.18) has a positive solution $u_0 \in C^{2,\alpha}_{loc}(\mathbb{R}^3) \cap (\cap_{k=2}^{+\infty} W^{2,k}(\mathbb{R}^3))$*

Proof. Let $\bar{\mu}$ be given by Lemma 5.4.6 and $K > 0$. For v, $w \in C^{2,\alpha}_{loc}(\mathbb{R}^3) \cap (\cap_{k=2}^{+\infty} W^{2,k}(\mathbb{R}^3))$ and $\mu \in (0, \bar{\mu}]$, let

$$g(x, w, v) = v^p + \left(K - \mu - P(x) - |w|^{r-2} \int_{\mathbb{R}^3} \frac{|w(y)|^q}{4\pi|x-y|} dy\right) v. \tag{5.33}$$

Obviously, the function g satisfies the assumption (h_1) of Lemma 5.4.9. Equation (5.18) is equivalent to

$$-\Delta u(x) + K u(x) = g(x, u, u). \tag{5.34}$$

Let ψ be as given in Lemma 5.4.6. By Lemma 5.4.5, we get that $\int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|} dy$ is bounded for all $0 < u \leqslant \psi$. So, under the assumption (P_1), there is $K > 0$ large enough such that

$$K - \mu - P(x) - |u|^{r-2} \int_{\mathbb{R}^3} \frac{|u(y)|^q}{4\pi|x-y|} dy > 0, \ for\ any\ x \in \mathbb{R}^3, \tag{5.35}$$

holds for all $0 < u \leqslant \psi$. Let $\varphi(x)$ and ϵ be given by Lemma 5.4.6. Followed by (5.33) and (5.35), we have

$$\begin{aligned} g(x, u(x), \epsilon\varphi(x)) &\leq g(x, u(x), u(x)) \\ &\leq g(x, u(x), \psi(x)) \end{aligned}$$

holds for $\epsilon\varphi(x) \leqslant u(x) \leqslant \psi(x)$. That is, the function g satisfies the assumption (h_2) of Lemma 5.4.9. For small $\varepsilon > 0$, by Lemma 5.4.6 we see that $\varepsilon\varphi$ and ψ are the lower-solution and upper-solution of (5.18). By the definitions of lower-solution and upper-solution, we see that g satisfies the assumption (h_3) of Lemma 5.4.9.

Now we can apply Lemma 5.4.9 to establish a sequence of $\{u_n\}_{n+1}^{+\infty}$. Indeed, let $u_1 = \epsilon\varphi(x)$, where ϵ is small enough such that (5.31) is valid. It is easy to see that $u_1 \in C^{2,\alpha}_{loc}(\mathbb{R}^3) \cap (\cap_{k=2}^{+\infty} W^{2,k}(\mathbb{R}^3))$ by (5.20). By Lemma 5.4.9, we get $\{u_n\}_{n=1}^{+\infty} \subset C^{2,\alpha}_{loc}(\mathbb{R}^3) \cap (\cap_{k=2}^{+\infty} W^{2,k}(\mathbb{R}^3))$ such that $\epsilon\varphi(x) \leqslant u_n(x) \leqslant \psi(x)$ and

$$-\Delta u_{n+1}(x) + K u_{n+1}(x) = g(x, u_n, u_n), \qquad x \in \mathbb{R}^3, \tag{5.36}$$

It is clear that $\psi \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Followed by (5.33), we get

$$\|g(x, u_n, u_n)\|_{L^k} \leqslant C\|\psi\|_{L^k}, \tag{5.37}$$

holds for any $n \geqslant 1$, $2 \leqslant k < +\infty$, where $C > 0$ is independent on n. Followed by (5.36), (5.37) and Proposition 5.4.7, we get

$$\|u_n\|_{W^{2,k}} \leqslant C\|\psi\|_{L^k}, \tag{5.38}$$

for all $n \in \mathbb{N}$ and $k \geq 2$. There is a subsequence of $\{u_n\}_{n=1}^{+\infty}$, still denoted by $\{u_n\}_{n=1}^{+\infty}$ and $u_0 \in \cap_{k=2}^{+\infty} W^{2,k}(\mathbb{R}^3)$, such that

$$u_n \rightharpoonup u_0 \ \text{ weakly in any } W^{2,k}(\mathbb{R}^3), \ \text{ as } n \to +\infty$$

where $2 \leqslant k < +\infty$. It is obvious that

$$\begin{aligned} & \lim_{n\to\infty} \int_{\mathbb{R}^3} u_n(x)[-\Delta\eta(x) + K\eta(x)]dx \\ = & \int_{\mathbb{R}^3} \lim_{n\to\infty} u_n(x)[-\Delta\eta(x) + K\eta(x)]dx \\ = & \int_{\mathbb{R}^3} u_0(x)[-\Delta\eta(x) + K\eta(x)]dx, \end{aligned} \tag{5.39}$$

and

$$\begin{aligned} & \lim_{n\to\infty} \int_{\mathbb{R}^3} u_n(x)[K - \mu - P(x) - u_n^{p-1}]\eta(x)dx \\ = & \int_{\mathbb{R}^3} u_0(x)[K - \mu - P(x) - u_0^{p-1}]\eta(x)dx \end{aligned} \tag{5.40}$$

holds for all $\eta(x) \in C_0^\infty(\mathbb{R}^3)$. By (5.37) and the lower semi-continuous property of the norm, we have

$$\|u_0\|_{W^{2,k}} \leqslant C\|\psi\|_{L^k}$$

holds for all $2 \leqslant k < +\infty$. Moreover we have $\epsilon\varphi(x) \leqslant u_0(x) \leqslant \psi(x)$ since $\epsilon\varphi(x) \leqslant u_n(x) \leqslant \psi(x)$ for any $n \geqslant 1$. By using the compactness results of the embedding theorem and the decay of $u_n(x)$ uniformly as $|x| \to +\infty$, we deduce that $u_n(x) \to u_0(x)$ uniformly on $x \in \mathbb{R}^3$ as $n \to +\infty$. It follows from the Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \|u_n - u_0\|_{L^k} = 0, \ \text{for all } k \geq 2.$$

Followed by the Hardy-Littlewood-Sobolev inequality, we get

$$\left| \int_{\mathbb{R}^6} \frac{|u_n(y)|^q |u_n|^{r-2} u_n(x)\eta(x)}{4\pi|x-y|} dydx - \int_{\mathbb{R}^6} \frac{|u_0(y)|^q |u_0|^{r-2} u_0(x)\eta(x)}{4\pi|x-y|} dydx \right| \overset{n\to\infty}{\to} 0 \tag{5.41}$$

holds for all $\eta(x) \in C_0^\infty(\mathbb{R}^3)$. Combining (5.33), (5.40) and (5.41), we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \{g(x, u_n, u_n) - g(x, u_0, u_0)\}\eta(x)dx = 0. \tag{5.42}$$

By (5.36), we have

$$\int_{\mathbb{R}^3} \{-\Delta u_{n+1}(x) + Ku_{n+1}(x)\}\eta(x)dx = \int_{\mathbb{R}^3} g(x, u_n, u_n)\eta(x)dx \tag{5.43}$$

holds for $\eta(x) \in C_0^\infty(\mathbb{R}^3)$. Followed by (5.39), (5.40), (5.42) and (5.43), we have

$$\int_{\mathbb{R}^3} \{-\Delta u_0(x) + Ku_0(x)\}\eta(x)dx = \int_{\mathbb{R}^3} g(x, u_0, u_0)\eta(x)dx. \tag{5.44}$$

Since $u_0 \in \cap_{q=2}^{+\infty} W^{2,q}(\mathbb{R}^3)$, we have $u_0 \in C^{1,\alpha}(\mathbb{R}^3)$ for some $\alpha \in (0, 1)$. By (5.33) we have $g(x, u_0, u_0) \in C_{loc}^{0,\alpha}(\mathbb{R}^3)$. Then (3.17) and Theorem 9.19 of [43] show that $u_0 \in C_{loc}^{2,\alpha}(\mathbb{R}^3)$ is a positive solution of (5.18).

□

Chapter 6

Summary and further research

6.1 Summary

This thesis are mainly devoted to the existence of solution, multiple solutions and symmetric properties of solutions to various classes of nonlinear elliptic equations with nonlocal terms. The main results and innovations are summarized in the following.

1. We proved the existence of a ground state solution and multiple standing waves of Schrödinger-Poisson equations with general local nonlinearity and unbounded potential. We established sufficient conditions for the existence and nonexistence of solutions to equation (1.1). The assumptions on the nonlinearity $f(u)$ cover the case of Slater approximation exchange term $|u|^{2/3}u$. From this point of view, we verify the stability of the method of Slater approximation to the exchange term of Hartree-Fork system.

2. We studied a class of nonlinear nonlocal elliptic equations, which are the Euler-Lagrange equations of the related variational functional. By using the variational method, we proved the existence of solution and multiple solutions of the nonlinear elliptic equations with nonlocal terms. Our theorems showed the effects of changing value of parameters μ and λ to the existence and nonexistence of solutions of (1.2) with harmonic potential.

We proved the non-radial symmetric property of the ground state solution of (1.2) for $1 < p < q < 5$. Ruiz [92] investigated the symmetry breaking property of the ground states to problem (1.2) with $q = 2$ on the bounded domain. We obtained a more general result.

3. We studied a class of nonlinear elliptic equations on the bounded domain and the unbounded domain, which are not the Euler-Lagrange equations of some variational functional. Without the variational structure, it is not possible to obtain the existence of solution by using the variational method. For $q \neq r$, we obtained the sufficient condition for the existence and nonexistence of solutions to (1.3), comparing to research of (1.3) under the assumption that $q = r$ in [2]. By using the topological method, we developed a framework for the study of the existence of solution to the equation (1.3) without variational structure.

6.2 Further research

This thesis has achieved some results in the existence and properties of solutions to a class of nonlinear elliptic equations with nonlocal terms, which arise from the study of gauged field equations. Further research may be carried out from the following aspects.

1. We will consider new estimates to the singular integral induced by the gauged fields arising from the Chern-Simons field. We have studied the Schrödinger equations in the Maxwell field in this thesis. However, there are still some related problems in the Chern-Simons field to be considered. When we investigate the variational functional, we need to solve problems caused by the singular integral. We are going to estimate the upper bound by using the Hardy-Littlewood-Sobolev inequality and consider the new estimates for the singular integral in further research.

2. We will solve the nonlinear eigenvalue problem by developing some new nonlinear analysis techniques. There is a series of eigenvalue problems in analysing standing wave solutions in the gauged fields. Many scholars studied the nonlinear eigenvalue problems by using the nonlinear analy-

sis theories. For example, Rabinowitz [88] considered the eigenvalues for some compact operators by the degree theory and established the global bifurcation theory. After that, Stuart [103] investigated the eigenvalue of the Hartree equation by the global bifurcation theory. We will generalize the related techniques in the gauged fields to investigate some nonlinear eigenvalue problems in further research.

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Appendix 1. Statement of Candidate's Contributions to Joint-Authored published

To Whom It May Concern,

I, Na Wei, made major contributions to the design of the research work, development of theories, analysis of results, and drafting of the paper entitled 'Multiple solutions for a Schrödinger-Poisson equation with a general nonlinearity, Acta Math Sci, accepted for publication.'

Na Wei

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Yongsheng Jiang

Yonghong Wu

Appendix 2. Statement of Candidate's Contributions to Joint-Authored published

To Whom It May Concern,

I, Na Wei, made major contributions to the design of the research work, development of theories, analysis of results, and drafting of the paper entitled 'Standing waves for a Schrödinger-Poisson type system with harmonic potential'.

Na Wei

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Yonghong Wu

Benchawan Wiwatanapataphee

Appendix 3. Statement of Candidate's Contributions to Joint-Authored published

To Whom It May Concern,

I, Na Wei, made major contributions to the design of the research work, development of theories, analysis of results, and drafting of the paper entitled 'A note on an elliptic equation with nonlocal nonlinearity'.

Na Wei

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

Yongsheng Jiang

Yonghong Wu

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