School of Electrical Engineering, Computing and Mathematical Sciences, Curtin University

Numerical Methods for Option Pricing

Mikhail Dokuchaev

This thesis is presented for the Degree of
Doctor of Philosophy
of
Curtin University

May 2021

Declaration

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

Signature:

Date:

Numerical Methods for Option Pricing

by

Mikhail Dokuchaev

Submitted to School of Electrical Engineering, Computing and Mathematical Sciences

on May 2021, in fulfilment of the requirements for the Degree of Doctor of Philosophy

The thesis studies numerical method for solving partial differential equations arising in financial modelling. More precisely, the thesis is focused on methods of solutions of parabolic equations with state dependent coefficients describing the fair price for European options and American options with parameters that depend on the state price.

The thesis commences with a description of option pricing problems and related boundary value problems for parabolic partial differential equations. An approach representing a modification of the Galerkin method is presented in Chapter 2. This approach is developed for solutions of a Cauchy problem for linear parabolic partial differential equation for European option prices with state state dependent coefficients. The parabolic equation is approximated by systems of ordinary differential equations. Convergence of the method is established. The effectiveness of the method is analysed in numerical experiments, including experiments with state dependent volatility. This approach is extended in Chapter 3 on the case of a free boundary problem for parabolic equation for the price of American options. The approach in this case is based on the power penalty method; the impact of the presence of the moving boundary is imitated by inclusion of a non-linear penalty term in the equation. The solution of the corresponding non-linear equation approximates the solution of the free boundary problem (Stefan problem). The convergence of the Galerkin approximations is established. Numerical experiments show the effectiveness of the method.

List of keywords:

1. **Keywords for Chapter 1**:

   Galerkin Method, European Options, Parabolic Equations.

2. **Keywords for Chapter 2**:

   Galerkin's Method, American Options, Parabolic Equations, Power Penalty Method

Australian and New Zealand Standard Research Classifications (ANZSRC)

ANZSRC code:
010302 Numerical Solution of Differential and Integral Equations

Fields of Research (FoR) Classification

FoR code:
010302 Numerical Solution of Differential and Integral Equations

Publications and presentations during candidature

I. Peer-review papers:

- M. Dokuchaev, G. Zhou, S. Wang. (2021) "A modification of Galerkin's method for option pricing". *Journal of Industrial and Management Optimization*, online first published https://www.aimsciences.org/article/doi/10.3934/jimo.2021077.

  doi: 10.3934/jimo.2021077.

  This paper is derived from the work presented in Chapter 2 of this thesis.

II. Presentations:

- Milestone 3 presentation: Title: "A Modification of Galerkin's Method for Option Pricing". Presentation completed successfully on 29/04/2021. Location: Perth.

Acknowledgements

I am grateful to Professor Kok Lay Teo for being part of my thesis committee and always being ready to help.

I am grateful to Professor Song Wang for teaching me and making invaluable suggestions on how to improve my work.

I am grateful to Professor Guanglu Zhou for always timingly answering any query and being always positive about my work.

I am grateful to Curtin University for giving me the opportunity to grow as a mathematician and a teacher.

I gratefully acknowledge School of Electrical Engineering, Computing and Mathematical Sciences, Curtin University.

The provision of ICT support and computing resources by Curtin IT Services is gratefully acknowledged. Curtin Information Technology Services (CITS) provides Information and Communication Technology systems and services in support of Curtin's teaching, learning, research and administrative activities.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Background: financial modelling and option pricing

The financial sector is usually one of the largest sections in an economy, and its effectiveness is supported by the presence of options and other derivatives. In addition, different types of derivatives are used in energy trading, agriculture trading, and insurance. Options are a type of financial derivatives that let the user speculate about a stock price in the future. Buying an option means buying the right to buy stock at a specified price after a certain period of time or to sell stock at a specified price after a certain period of time. The options can be used to hedge the risks of market fluctuations. Speculators use options to take advantage from predicted market movements. Options are traded for stock, commodities, currencies, and even weather. Typically, option trading involves shifting risks among parties. Options are important instruments for financial risk management.

In this thesis, we assume consider options for the stock market.

1.1.1 The model for stock prices

Consider a risky asset (stock, foreign currency unit, etc.) with time depending price $S(t)$, for example, daily prices.

The simplest model of price evolution is such that

$$S(t) = S(0)e^{rt+\sigma w(t)-\frac{1}{2}\sigma^2 t}, \tag{1.1.1}$$

where $w(t)$ is a Brownian motion process, $r \geq 0$ is the risk free rate (inflation rate), $\sigma > 0$ is a parameter that is called the volatility coefficient, or the volatility. The process $w(t)$ is a Gaussian process such that

$$\mathbf{E}\{w(t+\Delta t)|w(\cdot)|_{[0,t]}\} = w(t),$$
$$\text{Var}\,[w(t+\Delta t) - w(t)] = \sigma^2 \cdot \Delta t \quad \forall t > 0,\ \Delta t > 0.$$

We denote by $\mathbf{E}\{\cdot|\cdot\}$ the conditional expectation.

The identity above for the conditional expectations of w given history of observations means that $w(t)$ is a martingale.

Equation (1.1.1) can be replaced by the equation

$$dS(t) = S(t)(rdt + \sigma dw(t)). \tag{1.1.2}$$

This equation is called stochastic differential equation. Its solution is such that the discounted process $S_D = e^{-rt}S(t)$ is a martingale, i.e., the following holds for the conditional expectations given history of observations:

$$\mathbf{E}\{S_D(t+\Delta t)|w(\cdot)|_{[0,t]}\} = S_D(t) \quad \forall t > 0,\ \Delta t > 0.$$

In this case,

$$\begin{aligned} &S(t+\Delta t) - S(t) \approx S(t)(r\Delta t + \sigma(w(t+\Delta t) - w(t))), \\ &S_D(t+\Delta t) - S_D(t) \approx rS_D(t)\sigma(w(t+\Delta t) - w(t)) \quad \forall t > 0,\ \Delta t > 0. \end{aligned}$$

In particular,

$$\mathrm{Var}\, \frac{S(t+\Delta) - S(t)}{S(t)} = \mathrm{Var}\, \frac{S_D(t+\Delta) - S_D(t)}{S_D(t)} \approx \sigma^2 \Delta t.$$

The coefficient σ is called the volatility of stock prices.

In more general model, the stock prices is described by the equation

$$dS(t) = S(t)(rdt + \sigma(S(t), t))dw(t)), \tag{1.1.3}$$

where $\sigma(x,t) : \mathbf{R}^2 \to \mathbf{R}$ is a bounded and continuous function such that $\sigma(x,t) > 0$ for all $x > 0$, $t > 0$, with some restrictions on the rate of decreasing of $\sigma(x,t)$ as $x \to 0+$. In this case, the discounted stock price process $S_D = e^{-rt}S(t)$ is a martingale again.

1.1.2 Background: Options

Options or derivatives are contracts traded on stock markets.

There are many different types of options. The most common options are so-called European and American options.

Buying an European option means buying the right to buy stock at a specified price after a certain period of time (call option) or to sell stock at a specified price after a certain period of time (put option).

An American option lets the option holder exercise the option at any time until the expiry date. There are also other types of options such as exotic options.

The fair option price is a mathematical concept. It is defined as the price that eliminates arbitrage (the risk-free gain) in option trading. By obtaining a theoretical price, the buyer and the seller can use it as a reference point in their dealings. Thus, it is important to develop numerical methods for pricing options accurately.

Let $(x)^+ = \max(x, 0)$.

Let an expiration time $T > 0$ be given.

The following types of options are the most common.

- European call option: the payoff is $f(S(T)) = (S(T) - K)^+$, where $K > 0$ is the strike price.

- European put option: the payoff is $f(S(T)) = (K - S(T))^+$, where $K > 0$ is the strike price.

- American call option: the payoff is $f(S(\tau)) = (S(\tau) - K)^+$, where $K > 0$ is the strike price, and where is an exercise time selected by the option holder such that $\tau \in [0, T]$.

- American put option the payoff is $f(S(\tau)) = (K - S(\tau))^+$, where $K > 0$ is the strike price, and where τ is an exercise time selected by the option holder such that $\tau \in [0, T]$.

The exercise time for American options has to be selected based on the historical observations of stock prices, without possibility to forecast future prices.

An American option lets the option holder exercise the option at any time until the expiry date, while a European option lets the option holder exercise the option only at the expiry date.

The key role in financial modelling belongs to a concept of the "fair price" of options. The fair option price eliminates arbitrage (or possibility of risk-free gain) in option trading. The model for obtaining the fair price is based on the concept of perfect hedging of the payoff by the option seller, using the price paid the option buyer as he initial wealth. In reality, the perfect hedging is not possible, and the actual price of option should be impacted by many factors, including transaction costs fo the hedging. Nevertheless, by obtaining a theoretical fair price price, the buyer and the seller can use it as a reference point in their dealings. Thus, it is important to develop numerical methods for pricing options accurately.

1.1.3 Pricing of European options

It was suggested in Bachelier (1900) that, for the European options with the payoff $f(x)$, the fair $P_E(0)$ price at time 0 is

$$P_E(0) = e^{-rT}\mathbf{E}f(S(T)). \tag{1.1.4}$$

The fair price at time $t \in [0, T]$ is defined as the conditional expectation

$$P_E(t) = e^{-rT}\mathbf{E}\{f(S(T))|S(t)\}. \tag{1.1.5}$$

We have that

$$e^{-r(T-t)}\mathbf{E}\{f(S(T))|S(t)\} = e^{-rT}\mathbf{E}\{f(S(t)e^{\eta_t})|S(t)\},$$

where

$$\eta_t = -\frac{\sigma^2(T-t)}{2} + \sigma[w(T) - w(t)].$$

We have that $\eta_t \sim N(-\sigma^2(T-t)/2, \sigma^2(T-t))$. It follows that

$$\eta_t = -\sigma^2(T-t)/2 + \sigma\sqrt{T-t}\,\xi,$$

where $\xi \sim N(0,1)$ is a Gaussian random variable,

Then the fair price of the option at time $t \in [0,T)$ is

$$e^{-r(T-t)}\mathbf{E}\{f(S(T))|S(t)\} = e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} f\left(\rho(S(t),x,t)\right) dx,$$

where

$$\rho(S(t),x,t) = S(t)\exp\left[r(T-t) - \frac{\sigma^2(T-t)}{2} + \sigma x\sqrt{T-t}\right].$$

In particular, it follows that:

(a) For $f(x) = (x-K)^+$, the price $P_E(t)$ converges to

$$\lim_{x\to 0} e^{-r(T-t)}f(x) = 0$$

as $S(t) \to 0+$.

(b) For $f(x) = (x-K)^+$, the ratio $\frac{P_E(t)}{S(t)}$ converges to

$$\lim_{x\to 0}\frac{f(x)}{x} = 1$$

as $S(t) \to +\infty$.

(c) For $f(x) = (K-x)^+$, the price $P_E(t)$ converges to

$$\lim_{x\to 0} e^{-r(T-t)}f(x) = e^{-r(T-t)}K$$

as $S(t) \to 0+$.

(d) For $f(x) = (K-x)^+$, the price $P_E(t)$ converges to

$$\lim_{x\to 0}\frac{f(x)}{x} = 0$$

as $S(t) \to +\infty$.

In fact, these conditions hold for a more general model of stock prices where $\sigma = \text{const}\, S(t)^{\alpha-1}$, where $\alpha \neq 1$. (See, e.g., Nicholls and Sward (2015)). It can be noted that these boundary conditions are defined by the economical meaning of the option contracts.

Black & Scholes (1973) established the fundamental principle that the same pricing formulae (1.1.4)-(1.1.6) give the fair price even if the stock price evolves as

$$dS(t) = S(t)(a(t)dt + \sigma dw(t)),$$

where $a(t)$ is an arbitrary random process that can be different from r; $a(t)$ is adapted to the $w(t)$.

The process $a(t)$ is called the appreciation rate. This pricing principle implies that the option price is the same for the cases where $a(t) > 0$ (the price has positive trend) or $a(t) < 0$ (the price is negative trend).

In addition, Black & Scholes (1973) showed that, for the stock price (1.1.2), the price of the European option given that $S(t) = x$ is $V(x, x)$, where where $V(x, t)$ is a solution of the boundary value problem for the following linear parabolic second-order degenerate partial differential equation parabolic equation:

$$\begin{aligned} &LV(x,t) = 0, \\ &V(x,T) = f(x), \quad x > 0,\ t \in [0,T]. \end{aligned} \tag{1.1.6}$$

Here

$$LV(x,t) = \frac{\partial V}{\partial t}(x,t) + \mathrm{A}V(x,t)$$

is the Black-Scholes differential operator, where

$$\mathrm{A}V(x,t) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(x,t) + rx\frac{\partial V}{\partial x}(x,t) - rV(x,t).$$

This is an elliptic differential operator with a singularity at $x = 0$, where the coefficient for the second order derivative vanishes.

In the model where the stock is paying dividends with the constant rate d, then the operator A has the form

$$\mathrm{A}v(x,t) = \frac{\partial V}{\partial t}(x,t) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(x,t) + x(r-d)\frac{\partial V}{\partial x}(x,t) - rV(x,t).$$

The Black-Scholes equation can then be solved analytically for the case of the constant r, d, σ. The solution is given by the so-called Black-Scholes formula.

Consider now the stock price model described by (1.2.2) with some function $\sigma(x)$ and with dividends being paid by the stock with the rate $d(S(t))$. For this model, the Black-Scholes differential operator has the form

$$\mathrm{A}V(x,t) = \frac{1}{2}\sigma(x)^2 x^2 \frac{\partial^2 V}{\partial x^2}(x,t) + x(r-d(x))\frac{\partial V}{\partial x}(x,t) - rV(x,t).$$

If the coefficients σ and d are constant, then the solutions of the Black-Scholes equation is given by an explicit formula for the price of European put and call options; is is so called Black-Scholes formula.

Assume that $d(x) \equiv 0$.

Let $K > 0$, $\sigma > 0$, $r \geq 0$, and $T > 0$ be given.

Let $C_{BS}(x)$ and $P_{BS}(x)$ denotes the fair prices at time $t = 0$ for call and put options respectively with the payoff functions $f(S(T))$ described above and under the assumption that $S(0) = x$. Then

$$C_{BS,c}(x, K, \sigma, T, r) = x\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

$$P_{BS,p}(x,K,\sigma,T,r) = C_{BS,c}(x,K,\sigma,T,r) - x + Ke^{-rT},$$

where

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds,$$

and where

$$\begin{aligned} d_+ &= \frac{\ln(x/K) + Tr}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}, \\ d_- &= d_+ - \sigma\sqrt{T}. \end{aligned}$$

If the coefficients are not constant, then the solutions of the Black-Scholes equation have to be found numerically.

1.1.4 Pricing of American options

The fair price $P_A(0)$ of an American option at time $t = 0$ is represented by the formula

$$P_A(0) = \sup_{\tau} \mathbf{E} e^{-r\tau} f(S(\tau)),$$

where the supremum is taken over the set of random exercise times such that $\tau \in [0,T]$ with probability 1 and that τ is calculated based on the currently available historical observations of $S(\cdot)$. More precisely, the option holder decides at a given time τ to exercise the option based only on the history $S(s)$, $s \leq \tau$, without possibility to forecast future price movements.

The fair price $P_A(t)$ of an American option at time $t \in [0,T]$ is defined as the supremum of the conditional expectations

$$P_A(t) = \sup_{\tau} \mathbf{E}\{e^{-r(\tau-t)} f(S(\tau)) | S(s)|_{s\in[0,t]}\},$$

where the supremum is taken over the set of random exercise times such that $\tau \in [t,T]$ with probability 1 and that the choice of the exercise time τ is calculated based on the historical observations of the prices $S(s)|_{s\in[0,\tau]}$.

It follows from the description of the European and American option contacts that, at any time t,

$$P_A(t) \geq f(S(t)), \quad P_A(t) \geq P_E(t).$$

The first inequality holds since, otherwise, the option holder would exercise the option immediately at the time of purchase t with an immediate risk-free positive profit, which would make the price unfair.

The second inequality holds since American options give more flexibility for the option holder than their European analogs.

It follows from the definition of the American options that:

(a) For $f(x) = (x - K)^+$, the price $P_A(t)$ converges to

$$\lim_{x \to 0} f(x) = 0$$

as $S(t) \to 0+$.

(b) For $f(x) = (x - K)^+$, the ratio $\frac{P_A(t)}{S(t)}$ converges to

$$\lim_{x \to 0} \frac{f(x)}{x} = 1$$

as $S(t) \to +\infty$.

(c) For $f(x) = (K - x)^+$, the price $P_A(t)$ converges to

$$\lim_{x \to 0} f(x) = K$$

as $S(t) \to 0+$.

(d) For $f(x) = (K - x)^+$, the price $P_E(t)$ converges to

$$\lim_{x \to 0} \frac{f(x)}{x} = 0$$

as $S(t) \to +\infty$.

It can be noted that condition (c) is different from the corresponding condition for the European put option. This is because of the nature of the American options: if $S(t) \approx 0$, then American option can be exercised instantly at time t with the payoff K, so the price cannot be smaller than K.

One of the most important results in financial modelling is that the price at time t can be represented as $V(S(t), t)$, where $V(x, t)$ is a function satisfying the following conditions:

$$\begin{aligned}
&V(x,t) \geq f(x), \\
&\text{If} \quad V(x,t) > f(x) \quad \text{then} \quad LV(x,t) = 0, \\
&V(x,T) = f(x), \quad x > 0,\ x \in [0,T].
\end{aligned} \tag{1.1.7}$$

This is so-called free boundary problem (Stefan problem). The solution of this free boundary problem cannot be found explicitly and has to be found numerically even for the case of constant coefficients.

Problem (1.1.7) can be reformulated as the following problem:

$$\begin{aligned}
&V(x,t) \geq f(x) \\
&LV(x,t) \leq 0, \\
&LV(x,t)(V(x,t) - f(x)) = 0, \\
&\quad V(x,T) = f(x), \quad x > 0,\ t \in [0,T].
\end{aligned}$$

In fact, the boundary value problem is not linear with respect to the input f; however, it is called a linear complementarity problem. The term is originated from the finite-dimensional setting where related linear complementarity problems (LCP) were widely studied; see. e.g., Ferris and Pang, (1997).

For partial differential equations, linear complementarity problems. were introduced as an approach for free boundary problems for elliptic equations; see Brandt and Cryer (1983).

For American options and parabolic equations, the linear complementarity problems were considered in Wilmott, Dewynne, and Howison (1993), Wang, Yang and Teo (2006), Angermann and Wang (2007), Oosterlee (2003).

1.2 Background: numerical methods for the Black-Scholes equations

Mathematical methods of European and American option pricing were widely studied; see the review of these methods in Wilmott, Dewynne and Howison (1993).

In particular, it is common to use Monte-Carlo simulation that allows to find the fair price in a particular state point without solving partial differential equations; see Broadie, Glasserman and Jain (1997), Rogers and Shi (1995). However, the approach based PDEs allows more precise estimation of the entire value function.

The first numerical approach to the Black-Scholes equations was the lattice technique proposed in Cox et al. (1979) and improved in Hull and White (1988). That approach is equivalent to an explicit time-stepping scheme.

Overcoming singularity of Black-Scholes equations

The Black-Scholes equation is an equation with a singularity at $x = 0$ since it has a non-constant higher order coefficient that vanishes at zero. This makes it difficult for many numerical methods.

The equation can be transformed into a non-degenerate diffusion equation using the change of variables $y = \log x$, $x \in (0, +\infty)$ (the Euler transformation). This allows to remove the singularity of the differential operator; see, e.g., Jaillet, Lamberton and Lapeyre (1990), Angermann and Wang (2007).

A major setback of this method is that the natural equidistant grid on the transformed domain corresponds to heavily deformed grid in the original domain $(0, +\infty)$ for the original Black-Scholes equation, with excessive concentration of grid points in the neighbourhood of $x = 0$.

There is an alternative approach: to exclude the point singularity $x = 0$ and consider the parabolic equation in truncated interval $[\varepsilon, L]$ for some $\varepsilon > 0$ and $X > \varepsilon$; see, for example, Vazquez (1998). There is also an approach based on a transformation technique which transforms the space interval $(0, L]$ into a semi-infinite interval (see, for example, Barles, Daher and Romano (1995), Barles (1997). These approaches do not resolve the singularity completely since the truncation on the left-side at the neighbourhood of the singularity point of the domain cause additional numerical errors. Note that the truncation at the right-hand side of the domain is unavoidable, but is not that damaging since the the diffusion coefficient is separated from zero therein as the Black-Scholes operator is regular for $x > 0$.

A different approach allowing to avoid transformation of the state space was accepted in Wang (2004), Wang, Yang and Teo (2006), Angermann and Wang (2007): a special functional Sobolev space is suggested for the solutions in the original state domain $(0, +\infty)$. A review of Sobolev spaces can be found in Kufner (1985) and Brenner and Scott (1994). This approach was further developed in Valkov (2014), where Black-Scholes equation was studied on the half line transformed into a finite interval $(0, 1)$.

More difficult problem is solution for the stock pricing model where the stock price does not have a log-normal distribution such as models with state dependent volatility. A more general version of this model is the Constant Elasticity of Variance (CEV) model, where equation for $S(t)$ is replaced by the equation

$$dS(t) = rS(t)dt + \sigma S(t)^{\alpha} dw(t) \tag{1.2.1}$$

This model was considered in Becker (1980), Nicholls and Sward (2015), Cox (1996), Hsu et al (2008), Wong and Zhao (2008).

It can be noted that equation (1.2.2) can be rewritten as an equation

$$dS(t) = S(t)(rS(t)dt + \sigma(S(t))dw(t)),$$

where $\sigma(x) = \text{const} \cdot x^{\alpha-1}$. This would be an example of state dependent volaltility considered iin this thesis.

Achdou (2005) considered the problem with state dependent volatility in a general form that does cover CEV model.

Numerical schemes

As was mentioned above, in the case of constant volatility, the Euler transformation transforms the Black-Scholes equation into a heat equation. Some numerical schemes based on classical finite difference methods applied to constant-coefficient heat equations have been developed; see, e.g. Barles (1997), Barles et al. (1995), Courtadon (1982), Hull & White (1996), Rogers & Talay (1997), Schwartz (1977), Vazquez (1998), Wilmott et al. (1993). A comprehensive review of finite difference methods can be found in Brenner and Scott (1994) and Haslinger et al. (1999)

To improve computational precision for European options, Wang (2004) considered numerical solution of parabolic Black-Scholes equation based on the so-called fitted finite volume method. This was a discretization method based on a finite volume formulation of the problem coupled with a fitted local approximation to the solution and on an implicit time-stepping technique. The local approximation is determined by a set of two-point boundary value problems; this fitting technique is based on the idea proposed by Allen and Southwell (1955), for convection-diffusion equations. This approach has been extended to triangular and tetrahedral elements by several authors; for example, by Miller and Wang (1994a,b), Wang (1997), Angermann and Wang (2003), Zhang et al (2009), Valkov (2014). An error analysis for the stationary case was given in Wang (2004) was based on the assumption that

$$3r - d - \sigma^2 \geq \beta$$

for a positive constant $\beta > 0$. This restriction was used for establishing the coercivity of the bilinear form in the finite element formulation of the stationary problem; however, this restrictions was removed in Wang and Angermann (2007).

Overall, the method in Wang (2004) represents a special case of Petrov-Galerkin's method and increases the accuracy of calculation of the price comparing with the straightforward finite-difference method. In related works published by Angermann and Wang (2003), Angermann and Wang (2007), Wang, Yang, and Teo (2006), Wang and Yang (2008), Zhang and Wang (2012), some regularity problems were solved.

Solution of the free boundary problem

Solution of the pricing problem for American option pricing via analysis of the PDEs is more complicated since it requires to solve free boundary problem, or linear complementarity problem involving the Black-Scholes differential operator and a constraint on the value of the option; see Wilmott, Dewynne, and Howison (1993).

It does not have an exact analytic solution even in the case of constant coefficients.

There are several methods based on PDEs for the numerical valuation of American options.

In Holtz and Kunoth (2004), higher-order splines were applied to a formulation of the American option valuation problem in terms of a variational inequality, where the resulting discrete problems are solved by a monotone multigrid method. However, the convergence of the full discretization is demonstrated by numerical examples, only.

The approach developed in Allegretto, Lin and Yang (2001) was based on use of power penalty functions. The power penalty functions had been introduced in nonlinear programming; see for example Rubinov and Yang (2003), Yang and Huang (2001).

For the free-boundary problem, a penalty function method was used in Zvan, Forsyth and Vetzal (1998) for solving the discrete complementarity problem. A penalty function approach was used also in Forsyth and Vetzal (2002). The approach is the same as the linear penalty function proposed by several authors; see for example Bensoussan and Lions (1982), Glowinski (1984).

Zhang et al (2004), Wang, Yang and Teo (2006), and Angermann and Wang (2007), extended the fitted finite volume method combined with power penalty method to the pricing of American option. In Wang, Yang and Teo (2006) and Angermann and Wang (2007), power penalty functions were used to approximate the solution $\widetilde{V}$ of the the free boundary problem (Stefan problem) for a linear Black-Scholes parabolic equation with moving boundary by the solution $V_{\lambda,k}$ by solutions of nonlinear problems

$$LV_\lambda(x,t) - \lambda(f(x) - V_\lambda(x,t))_+^{1/k} = 0,$$

$$V(x,T) = f(x), \tag{1.2.2}$$

with fixed boundary with the nonlinear penalty term

$$\lambda(f(x) - V_{\lambda,k}(x,t))_+^{1/k} \tag{1.2.3}$$

representing the power penalty for violating the Stefan boundary condition. Here $k > 0$ is a preselected fixed parameter, and $\lambda > 0$ is the penalty parameter that is supposed to be variable.

The method also includes a Newton's iterative method for the nonlinear system and a smoothing technique for the nonlinear term based on the properties of M-matrices; see Ortega and Rheinboldt (1970).

When $[f(x) - V_\lambda(x,t)]_+ = 0$, the non-linear term vanishes, and the parabolic equation becomes the standard linear Black-Scholes equation.

For the case where $k = 1$, the penalty (3.1.1) is a piecewise linear function used in Forsyth and Vetzal (2002), Bensoussan and Lions (1982), Glowinski (1984). It was shown in Bensoussan and Lions (1982) that the error rate $\|\tilde{V} - V_{\lambda,k}\|$ is of order $O(\lambda^{-1/2})$.

This square root rate of convergence requires that λ is sufficiently large to achieve a given accuracy of the approximate solution. However, it was reported that large values of λ will cause computational problems in practice; see, e.g., for example Fletcher (1987).

It has been shown in Wang, Yang and Teo (2006), Theorem 5.2) that the matrices of the sequence of linear systems arising from Newton's method are M-matrices, and therefore the discretization scheme is monotone. Wang, Yang and Teo (2006) analysed convergence of the solutions of penalised Black-Scholes equation to the solution of Stefan problem defining the price of the American option. They showed that the error rate is $\|\tilde{V} - V_{\lambda,k}\|$ is of order $O(\lambda^{-k/2})$ for some Sobolev type norm. In Wang, Yang and Teo (2006) and Angermann and Wang (2007), solution of nonlinear penalised equations for parabolic equations with time-depending coefficients was studied based on the fitted finite volume method which is a modification of Petrov-Galerkin's method. It was shown therein that the method ensures convergence of discretized solution $V_{\lambda,N}$ to V_λ for any $\lambda > 0$, where V_N is the solution using N basis functions for Petrov-Galerkin's setting. Option value at any point along the state variable can be found by interpolating two basis functions (Wang, Yang and Teo (2006)).

Some regularity problems were solved in Angermann and Wang (2003), Angermann and Wang (2007), and related papers Wang, Yang and Teo (2006), Wang and Yang (2008), Zhang and Wang (2012), Lesmana and Wang (2015) Convergence of a fitted finite volume method for the penalized Black-Scholes equation requires to remove the degeneracy, as is done in most of the existing methods. This may cause computational errors. Very few papers deal with proper boundary conditions (see, e.g., Allegretto, Lin, Yang (2001), Han and Wu (2003)). We comment that the papers either lack investigation of convergence rates or assume unrealistic step-size restrictions. Another notable work is Benth, Karlsen and Reikvam (2004) in which the authors propose and analyze an upwind finite difference method of predictor-corrector type.

In Wang and Angermann (2007), an error analysis was provided for the full discretization method for (1.2.2) with $k = 1$. Stability and the convergence were established for the time-stepping scheme; it was shown that the interpolated error converges at $t = 0$ linearly in both space and time. In Zhang et al (2004), convergence was studied for two-asset American option. Lesmana and Wang (2015) considered pricing of American options under the transaction costs.

1.3 The contribution of the present work

This work develops a method of solution of Black-Scholes parabolic equations in a setting where the volatility coefficient and the dividend rate depend on the current stock price. The work suggests a modification of the Galerkin method.

The novelty of the approach is that basis functions ϕ_k such that $A\phi_k = 0$, where

$$A = \rho\frac{\partial^2}{\partial y^2} + \eta\frac{\partial}{\partial y},$$

and where $\rho(y) = \frac{1}{2}\sigma(y)^2$, $\eta(y) = r - d(y) - \frac{1}{2}\sigma^2(y)$. This operator is the operator that corresponds to the operator

$$\mathrm{A}V(x,t) = \frac{1}{2}\sigma(x)^2x^2\frac{\partial^2 V}{\partial x^2}(x,t) + rx\frac{\partial V}{\partial x}(x,t) - rV(x,t).$$

after the Euler transformation $x = \log y$.

These basis functions are convenient for the equations with state dependent volatility and dividends since they allow explicit calculation of the coefficients of the ODEs generated by Galerkin method for the case of state dependent volatility $\sigma(x)$ and state dependent dividend rate $d(x)$. This allow relatively easy approximation of the corresponding boundary value problems for parabolic equations describing prices of European and American options by systems of ordinary differential equations. The option price value at any point along the state variable can be found by interpolating two basis functions.

With these convenient basis functions, we were able to establish convergence of Galerkin method for the case of state-dependent volatility and dividend rate.

For the case of American options, we extended the power penalty approach Wang, Yang and Teo (2006) and Angermann and Wang (2007) on the case of state-dependent volatility and dividend rate. In this approach, the solution of free boundary problem is approximated by solution of a nonlinear problem with fixed boundary with a non-linear term non-homogeneous input term representing a power penalty.

An additional novelty of the present work is that we used a new non-linear term representing a power penalty.

$$\mathcal{N}_{\lambda,k}(x) = \lambda[x]_+^{1/k}, \quad x < C$$
$$\mathcal{N}_{\lambda,k}(x) = \lambda C^{1/k-1}x, \quad x > C,$$

where $\lambda > 1$, $k > 0$, $C > 0$ are penalty parameters. This function is piecewise continuous. If $k = 1$ this function is piecewise linear. For large $C > 0$, this function approximates penalty $\lambda [x]_+^{1/k}$, $x > 0$ that was used in Wang, Yang and Teo (2006) and Angermann and Wang (2007). In general, this power penalty is a modification of power penalty from Wang, Yang and Teo (2006) and Angermann and Wang (2007).

This modification allowed to establish convergence of penalised solutions where $k \leq 1$. It can be noted that Wang, Yang and Teo (2006) and Angermann and Wang (2007) considered the case where $k \geq 1$.

It can be noted that the convergence for the case where $k > 1$ is not covered by our method, since the proof based on our approach would require the Lipschitz continuity for $\mathcal{N}_{\lambda,k}$ which does not hold for $k > 1$.

Chapter 2

A modification of Galerkin's method for European option pricing

Abstract

We present a novel method for solving a complicated form of a partial differential equation called the Black-Scholes equation arising from pricing European options. The complication is that the terms of the equation, namely the volatility and dividend, as variables dependent on the state price. We develop a Galerkin finite element method to solve the problem. More specifically, we discretize the system along the state variable and build new basis functions which we use to approximate the solution. The novelty of the approach is that we are using a new special basis functions. We establish convergence of the proposed method and numerical results are reported to show the proposed method is accurate and efficient.

2.1 Introduction for Chapter 2

This chapter consider pricing of European options. As was mentioned above, there is a vast volume of literature for mathematical methods of option pricing; see the review of these methods in Wilmott, Dewynne and Howison (1993). However, there are still problems of convergence and computational precision of these methods for the case of non-constant coefficients. This chapter suggests a modification of the approach developed in Wang (2006), Angermann and Wang (2007), Angermann and Wang (2004), Wang, Yang and Teo (2006), Wang and Yang (2008), Zhang and Wang (2012) for improving the computational precision. The novelty of the result in this chapter that we consider the state-dependent functions for volatility and dividends. Another novelty of this chapter is that we construct basis functions such that $A\phi = 0$ locally, for the state-dependent functions for volatility and dividends, where A is the differential operator presented in the Black-Scholes equation after the log transformation of the state space. Each basis function can be derived using the given parameters. Option value at any point along the state variable can be found by interpolating two basis functions. The local approximation is determined by a set of two-point boundary value problems. Overall, this method represents a special case of Galerkin' method.

In addition, the converges is proven based on a different approach than in Wang (2006), Angermann and Wang (2007), Angermann and Wang (2004), Wang, Yang and Teo (2006), Wang and Yang (2008), Zhang and Wang (2012).

The chapter is organized as follows. In Section 2, we set up the problem by transforming the original Black-Scholes equation. In Section 3, we set up basis functions to apply Galerkin's method. In Section 4, we introduce weak formulation of the problem as well as the main theorem that is important for solving the problem. In Section 5, we prove the theorem. In Section 6, we review some methods of solution: exact method and Crank-Nicolson's method. In Section 7, we prove the convergence of the method. In Section 8, we test the method numerically.

2.2 Problem setting

We will consider a model of a stock price $S(t)$ described by the following stochastic Ito equation

$$dS(t) = S(t)(rdt + \sigma(S(t),t)dw(t)), \quad t > 0, \tag{2.2.1}$$

where $w(t)$ is a Wiener process, r is a risk-free rate, σ is the volatility of this stock. Assume that there are dividends $d(x)$ on the stock. The pricing problem can be formulated as follows: Let a random variable X represent a payoff of a financial option. If $X = f(S(T))$ for some function $f : \mathbf{R} \to \mathbf{R}$ then the option is said to be of a European type. In this case, the "fair" option price at time t is $P(t) = e^{-r(T-t)}E\{f(S(T)|S(t)\}$, i.e. it can be calculated as the conditional expectation.

We assume that one of the following conditions is satisfied:

1. $r = 0$

2. $f(x) = (x - K)^+$ or $f(x) = (K - x)^+$, where $K > 0$ is given, for call and put options respectively. Here we use the notation $(x)^+ = \max(x, 0)$.

It can be shown, using Ito's Lemma, that

$$P(t) = V(S(t), t),$$

where $V(x,t)$ is a solution of the boundary value problem for the following partial differential equation:

$$\begin{aligned}
&\frac{\partial V}{\partial t}(x,t) + \frac{1}{2}\sigma(x)^2 x^2 \frac{\partial^2 V}{\partial x^2}(x,t) + x(r - d(x))\frac{\partial V}{\partial x}(x,t) - rV(x,t) = 0,\\
&V(x,T) = f(x), \quad x > 0,\\
&\lim_{x\to\infty} \frac{V(x,t)}{f(x)} = R_0(t), \quad 0 < t < T,\\
&V(0,t) = R_0(t)f(0), \quad 0 < t < T.
\end{aligned} \tag{2.2.2}$$

Here, $R_0(t) \equiv 1$ for the call option and $R_0(t) \equiv e^{-r(T-t)}$ for the put option. Equation (2.2.2) is a so-called Black-Scholes equation which is a special case of a parabolic equation, see, e.g., Evans (2010). Here, $x \in (0, \infty)$, $t \in [0, T)$, $\sigma(x)$ represents the volatility coefficient, $d(x)$ represents the dividend rate, r represents the risk-free bank rate, $T > 0$ is the terminal time. We assume that $d(x)$ is a bounded function and that $\sigma(x)$ is a bounded function with a bounded first derivative. In this paper we consider volatility $\sigma(x)$ as a function of underlying stock price x rather than a constant.

We introduce a variable y such that

$$x = e^y, \quad V(x,t) = \hat{V}(\ln x, t). \tag{2.2.3}$$

Then $\frac{\partial V}{\partial x} = \frac{\partial \hat{V}}{\partial y} e^{-y}$. $\frac{\partial^2 V}{\partial x^2} = e^{-2y}(\frac{\partial^2 \hat{V}}{\partial y^2} - \frac{\partial V}{\partial y})$. Equation (2.2.2) takes the form

$$\begin{aligned}
&\frac{\partial \hat{V}}{\partial t}(y,t) + \frac{1}{2}\sigma(y)^2(\frac{\partial^2 \hat{V}}{\partial y^2}(y,t) - \frac{\partial \hat{V}}{\partial y}(y,t)) + (r - d(y))\frac{\partial \hat{V}}{\partial y}(y,t) - r\hat{V}(y,t) = 0,\\
&\hat{V}(y,T) = f(y), \quad x \in (-L, L),\\
&\hat{V}(L,t) = R_0(t)f(L), \quad \hat{V}(-L,t) = R_0(t)f(-L), \quad t \in [0,T)
\end{aligned} \tag{2.2.4}$$

Note that we write $f(e^y)$ and $\sigma(e^y)$ as $f(y)$ and $\sigma(y)$ respectively, to keep notations short. Next, we use the substitution:

$$\hat{v}(y,t) = e^{-r(t-T)}\hat{V}(y,t).$$

Let $R(t) := R_0(t)e^{-r(t-T)}$. By the definitions, $R(t) = 1$ for the call options.

Clearly,

$$\begin{aligned}
\frac{\partial \hat{V}}{\partial t}(y,t) \quad &= re^{r(t-T)}\hat{v}(y,t) + \frac{\partial \hat{v}}{\partial t}(y,t)e^{r(t-T)},\\
\hat{V}(y,T) \quad &= e^{r(T-T)}f(y) = f(y),
\end{aligned}$$

$$\begin{aligned}\hat{v}(L,t) &= R(t)f(L),\\ \hat{v}(-L,t) &= R(t)f(-L).\end{aligned}$$

Eq. (2.2.4) becomes

$$\begin{aligned}&re^{r(t-T)}\hat{v}(y,t) + \frac{\partial \hat{v}}{\partial t}(y,t)e^{r(t-T)} + \frac{1}{2}e^{r(t-T)}\sigma(y)^2(\frac{\partial^2 \hat{v}}{\partial y^2}(y,t) - \frac{\partial \hat{v}}{\partial y}(y,t))\\ &+e^{r(t-T)}(r-d(y))\frac{\partial \hat{v}}{\partial y}(y,t) - re^{r(t-T)}\hat{v}(y,t) = 0.\end{aligned}$$

Simplifying the above, we obtain that

$$\begin{aligned}&\frac{\partial \hat{v}}{\partial t}(y,t) + \frac{1}{2}\sigma(y)^2\frac{\partial^2 \hat{v}}{\partial y^2}(y,t) + (r - d(y) - \frac{1}{2}\sigma^2(y))\frac{\partial \hat{v}}{\partial y}(y,t) = 0,\\ &\hat{v}(y,T) = f(y), \quad \hat{v}(L,t) = R(t)f(L), \quad \hat{v}(-L,t) = R(t)f(-L). \quad (2.2.5)\end{aligned}$$

For brevity, we will rewrite this as

$$\begin{aligned}&\frac{\partial \hat{v}}{\partial t}(y,t) + \rho(y)\frac{\partial^2 \hat{v}}{\partial y^2}(y,t) + \eta(y)\frac{\partial \hat{v}}{\partial y}(y,t) = 0,\\ &\hat{v}(y,T) = f(y), \quad \hat{v}(L,t) = R(t)f(L), \quad \hat{v}(-L,t) = R(t)f(-L) \quad (2.2.6)\end{aligned}$$

Here, $\rho(y) = \frac{1}{2}\sigma(y)^2$, $\eta(y) = r - d(y) - \frac{1}{2}\sigma^2(y)$ and $R(t) = e^{r(t-T)}$. We assume that in (2.2.2), $|f(x)| \leq c(1+|x|)$, for some constant $c > 0$ and we look for solution $|\hat{v}(x,t)| \leq C_1(1+|x|)$. Let us introduce a differential operator A such that

$$A\hat{v} = \rho\frac{\partial^2 \hat{v}}{\partial y^2} + \eta\frac{\partial \hat{v}}{\partial y}. \quad (2.2.7)$$

We consider (2.2.5) for $y \in (-L, L)$, where $L > 0$ is a sufficiently large constant. We consider the boundary problem

$$\begin{aligned}&\frac{\partial \hat{v}}{\partial t} + A\hat{v} = 0, \quad \hat{v}(y,T) = f(y), \quad y \in \mathbf{R},\\ &\hat{v}(-L,t) = R(t)f(-L),\\ &\hat{v}(L,t) = R(t)f(L). \quad (2.2.8)\end{aligned}$$

We use the substitution

$$v(y,t) = \hat{v}(y,t) - f(y)R(t). \quad (2.2.9)$$

Then,

$$\frac{\partial \hat{v}}{\partial t} = \frac{\partial v}{\partial t} + f(y)\frac{\partial R}{\partial t}. \quad (2.2.10)$$

This leads to the following problem:

$$\begin{aligned}&\frac{dv}{dt} = -Av - AfR(t) - f(y)\frac{\partial R}{\partial t}, \quad (2.2.11)\\ &v(y,T) = 0,\\ &v(-L,t) = v(L,t) = 0.\end{aligned}$$

Here $y \in D$, $D = (-L, L)$.

2.3 Basis functions

Let $\{y_k\}_{k=0}^{N+1} \subset D$ be selected such that $-L = y_0 < y_1 < y_1 < ... < y_{N+1} = L$. Let us introduce the basis functions $\phi(y) = \phi_k(y), k = 0, .., N+1$ that satisfy the following conditions:

1. $\phi_k(y_k) = 1$, $\phi_k(y_{k-1}) = \phi_k(y_{k+1}) = 0$.
2. $\phi_k(y) \geq 0$.
3. $\phi_k(y) = 0$ for $y \notin (y_{k-1}, y_{k+1})$.
4. $\phi_y|_{[y_{k-1},y_k]} \in C^2([y_{k-1}, y_k])$; $\phi_y|_{[y_k,y_{k+1}]} \in C^2([y_k, y_{k+1}])$. Here, C^2 is the space of twice differentiable functions.
5. $\phi_k|_{[y_{k-1},y_{k+1}]} \in W^1_\infty([y_{k-1}, y_{k+1}])$. Here, W^1_∞ is a Sobolev space of functions with bounded first derivative.
6. $A\phi_k(y) = 0$ for $y \in (y_{k-1}, y_k) \cup (y_k, y_{k+1})$.
7. $\phi_k(y) + \phi_{k+1}(y) = 1$ for $y \in [y_k, y_{k+1}]$.

Diagrams of these functions form intersecting deformed triangles on the $\phi - x$ plane. To find $\phi' s$, we need to solve the following equation:

$$A\phi_k(y) = \rho(y)\frac{\partial^2\phi_k}{\partial y^2}(y) + \eta(y)\frac{\partial\phi_k}{\partial y}(y) = 0. \tag{2.3.1}$$

We consider two cases: (1) $y \in [y_{k-1}, y_k]$ and (2) $y \in [y_k, y_{k+1}]$. The boundary conditions are as follows:

$$\phi_k(y_{k-1}) = 0, \quad \phi_k(y_k) = 1, \quad \phi_k(y_{k+1}) = 0. \tag{2.3.2}$$

Let $\gamma(y) = \frac{\partial\phi_k}{\partial y}(y)$. The equation (2.3.1) becomes

$$\rho(y)\frac{\partial\gamma}{\partial y}(y) + \eta(y)\gamma(y) = 0.$$

This equation can be solved exactly using the integrating factor method. It can be written as

$$\frac{\partial\gamma}{\partial y}(y) + \frac{\eta(y)}{\rho(y)}\gamma(y) = 0.$$

Let

$$\omega_{k,-}(y) = \int_{y_{k-1}}^{y} \frac{\eta(x)}{\rho(x)}dx + c.$$

Let integrating factor be $\mu = C_1 e^{\omega_{k,-}(y)}$. Then

$$\gamma_k(y) = C_1 e^{-\omega_{k,-}(y)}.$$

To find ϕ_k, we integrate γ:

$$\phi_k(y) = \int_{y_{k-1}}^{y} \gamma(x)dx + C_2.$$

Next, we check the initial conditions (2.3.2):

$$\phi_k(y_{k-1}) = \int_{y_{k-1}}^{y_{k-1}} \gamma(x)dx + C_2 = 0,$$

$$C_2 = 0,$$

$$\phi_k(y_k) = \int_{y_{k-1}}^{y_k} \gamma(x)dx = C_1 \int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)}dx = 1.$$

Hence,

$$C_1 = \frac{1}{\int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)}dx}.$$

Thus, we determined $\phi_k(y)$ for $y \in (y_{k-1}, y_k)$, and

$$\phi_k(y) = \frac{1}{\int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)}dx} \int_{y_{k-1}}^{y} e^{-\omega_{k,-}(x)}dx. \tag{2.3.3}$$

When the discretization step is small enough, this function will be increasing from y_{k-1} to y_k.

Similarly, we find $\phi_k(y)$ for $y \in (y_k, y_{k+1})$. Let

$$\omega_{k,+}(y) = \int_{y_k}^{y} \frac{\eta(x)}{\rho(x)}dx + c.$$

Let $\mu = C_1 e^{\omega_{k,+}(y)}$ and $\gamma_k(y) = C_1 e^{-\omega_{k,+}(y)}$. To find $\phi_k(y)$, we integrate $\gamma(x)$:

$$\phi_k(y) = \int_{y_k}^{y} \gamma(x)dx + C_2.$$

Next, we match the initial conditions (2.3.2) to find C_1 and C_2 as follows:

$$\phi_k(y_k) = \int_{y_k}^{y_k} \gamma(x)dx + C_2 = 1,$$

$$C_2 = 1,$$

$$\phi_k(y_{k+1}) = \int_{y_k}^{y_{k+1}} \gamma(x)dx + 1 = C_1 \int_{y_k}^{y_{k+1}} e^{-\omega_{k,+}(x)}dx + 1 = 0,$$

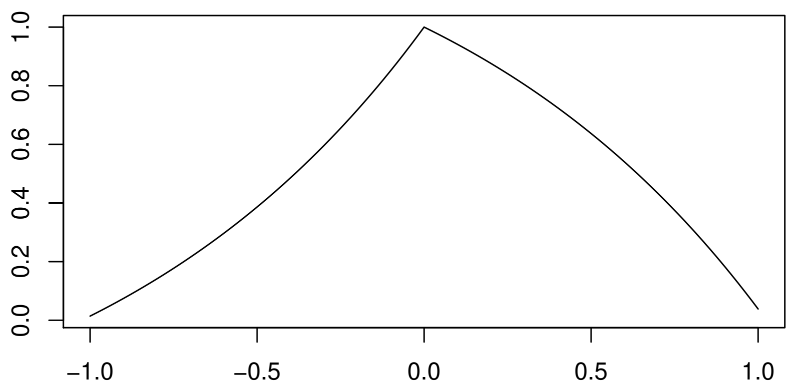

Figure 2.1: Basis function $\phi_k(y)$ for $y_{k-1} = -1$, $y_k = 0$, $y_{k+1} = 1$, $\rho = 0.045$, and $\eta = -0.045$

.

$$C_1 = \frac{-1}{\int_{y_k}^{y_{k+1}} e^{-\omega_{k,+}(x)} dx}.$$

Thus, $\phi_k(y)$ for $y \in (y_k, y_{k+1})$ is determined as follows:

$$\phi_k(y) = \frac{-1}{\int_{y_k}^{y_{k+1}} e^{-\omega(x)_{k,+}} dx} \int_{y_k}^{y} e^{-\omega_{k,+}(x)} dx + 1.$$

We will use these basis functions to discretize the system. At any point, $V(y)$ can be represented as a linear combination of two appropriate basis functions in the domain: $v_k(t)\phi_k(y) + v_{k+1}(t)\phi_{k+1}(y)$ if $y \in [y_k, y_{k+1}]$.

Figure 2.1 shows an example of such a basis function.

2.4 ODE implied by the Galerkin Method

Using ϕ_m from the previous section we can approximate V in (2.2.12) as $V_N(y,t) = \sum_{k=1}^{N} v_k(t)\phi_k(y)$. The boundary conditions is ensured if we select $v_0(t) = 0$ and $v_{N+1}(t) = 0$. Let S_N be the span of $\{\phi_k\}_{k=1,\dots,N}$. Let us consider a bilinear mapping $a : H_0^1(D) \times H_0^1(D) \to \mathbf{R}$ such that $(Au, w)_{L_2(D)} = a(u, w)$ for all $u, w \in H_0^1(D) \cap W_2^2(D)$. In a weak form the equation (2.2.12) is

$$\left(\frac{du}{dt}, w\right)_{L_2(D)} = -a(u, w) - R(t)a(f, w) - R'(t)(f, w)_{L_2(D)} \qquad (2.4.1)$$

for all $w \in H_0^1(D)$, where $H_0^1(D)$ is the space of functions belonging to $L_2(D)$ together with their first derivatives and such that they vanish at ∂D. Following the Galerkin

Method, we look for v_k such that

$$(V_N', w)_{L_2(D)} = -a(V_N, w) - R(t)a(f_N, w) - R'(t)(f_N, w)_{L_2(D)}, \tag{2.4.2}$$

for all $w \in S_N$, $m = 1, .., N$. Formally, equation (2.4.2) can be presented as

$$\sum_{k=0}^{N+1} v_k'(t)\phi_k(y) = -\sum_{k=0}^{N+1} v_k(t)A\phi_k(y) - R(t)\sum_{k=0}^{N+1} \xi_k A\phi_k(y) - R'(t)\sum_{k=0}^{N+1} \xi_k\phi_k(y).$$

Here, $\xi_k = f(y_k)$. Since $v_0 = v_{N+1} = 0$, we can rewrite the above as

$$\sum_{k=1}^{N} v_k'(t)\phi_k(y) = -\sum_{k=1}^{N} v_k(t)A\phi_k(y) - R(t)\sum_{k=0}^{N+1} \xi_k A\phi_k(y) - R'(t)\sum_{k=0}^{N+1} \xi_k\phi_k(y).$$

Multiplying by ϕ_m, for $m = 1, ..., N$ and integration by dy gives

$$\begin{aligned} \int_{\mathbb{R}} \quad \sum_{k=1}^{N} v_k'(t)\phi_k(y)\phi_m(y)dy &= -\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t)A\phi_k(y)\phi_m(y)dy \\ -R(t)\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A\phi_k(y)\phi_m(y)dy &- R'(t)\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k\phi_k(y)\phi_m(y)dy. \end{aligned} \tag{2.4.3}$$

Using the above we can construct the following system.

Theorem 2.4.1

$$Mv'(t) = Bv(t) + \varphi, \quad v(T) = 0, \tag{2.4.4}$$

where

$$\varphi = R(t)(B\xi + \zeta) - R'(t)(M\xi + \kappa). \tag{2.4.5}$$

Here,

$$v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_{N-1}(t) \\ v_N(t) \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{N-1} \\ \xi_N \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_1 \\ 0 \\ \vdots \\ 0 \\ \zeta_N \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_1 \\ 0 \\ \vdots \\ 0 \\ \kappa_N \end{pmatrix}, \tag{2.4.6}$$

$$M = \begin{pmatrix} \mu_{1,1} & \mu_{1,2} & 0 & 0 & \dots & 0 \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & 0 & \dots & 0 \\ 0 & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_{N-1,N-2} & \mu_{N-1,N-1} & \mu_{N-1,N} \\ 0 & 0 & \dots & 0 & \mu_{N-1,N} & \mu_{N,N} \end{pmatrix}, \tag{2.4.7}$$

$$B = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & 0 & 0 & \dots & 0 \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} & 0 & \dots & 0 \\ 0 & \beta_{3,2} & \beta_{3,3} & \beta_{3,4} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_{N-1,N-2} & \beta_{N-1,N-1} & \beta_{N-1,N} \\ 0 & 0 & \dots & 0 & \beta_{N-1,N} & \beta_{N,N} \end{pmatrix}, \tag{2.4.8}$$

The components of M are

$$\begin{aligned} \mu_{m,m-1} &= \int_{y_{m-1}}^{y_m} \phi_{m-1}(y)\phi_m(y)dy, \\ \mu_{m,m} &= \int_{y_{m-1}}^{y_{m+1}} \phi_m^2(y)dy, \\ \mu_{m,m+1} &= \int_{y_m}^{y_{m+1}} \phi_{m+1}(y)\phi_m(y)dy. \end{aligned} \tag{2.4.9}$$

The components of B are

$$\begin{aligned} \beta_{m,m-1} &= \frac{\sigma^2(y_m)}{2}(\phi'_{m-1}(y_m - 0), \\ \beta_{m,m} &= -\frac{\sigma^2(y_m)}{2}(\phi'_m(y_m + 0) - \phi'_m(y_m - 0)), \\ \beta_{m,m+1} &= -\frac{\sigma^2(y_m)}{2}(\phi'_{m+1}(y_m + 0)). \end{aligned} \tag{2.4.10}$$

The components ξ, ζ and κ are

$$\begin{aligned} &\xi_k = f(y_k), \\ &\zeta_1(t) = \beta_{1,0}\xi_0 = \frac{\sigma^2(y_1)}{2}\phi'_0(y_1 - 0))\xi_0, \\ &\zeta_N(t) = \beta_{N,N+1}\xi_{N+1} = \frac{\sigma^2(y_N)}{2}\phi'_{N+1}(y_N + 0)\xi_{N+1}. \\ &\kappa_1(t) = \xi_0(t)\int_{y_0}^{y_1} \phi_0(y)\phi_1(y)dy = \mu_{1,0}\xi_0. \\ &\kappa_N(t) = \xi_{N+1}(t)\int_{y_N}^{y_{N+1}} \phi_{N+1}(y)\phi_N(y)dy = \mu_{1,0}\xi_{N+1}. \end{aligned} \tag{2.4.11}$$

2.5 Proof of the Theorem (2.4.1)

2.5.1 Finding the components for M in (2.4.4)

Let us look at the LHS of (2.4.3). If $k < m-1$ or $k > m+1$ then $\phi_k\phi_m = 0$. Therefore, for $m = 2, ..., N-1$, we have that

$$\int_{\mathbb{R}} \sum_{k=1}^{N} v'_k(t)\phi_k(y)\phi_m dy$$

$$
\begin{aligned}
&= \frac{dv_{m-1}(t)}{dt}\int_{y_{m-1}}^{y_m}\phi_{m-1}(y)\phi_m(y)dy + \frac{dv_m(t)}{dt}\int_{y_{m-1}}^{y_{m+1}}\phi_m^2(y)dy \\
&\quad + \frac{dv_{m+1}(t)}{dt}\int_{y_m}^{y_{m+1}}\phi_{m+1}(y)\phi_m(y)dy \\
&= \mu_{m-1}\frac{dv_{m-1}(t)}{dt} + \mu_m\frac{dv_m(t)}{dt} + \mu_{m+1}\frac{dv_{m+1}(t)}{dt}.
\end{aligned} \tag{2.5.1}
$$

Let us consider boundary cases $m = 1$ and $m = N$. For the case where $m = 1$,

$$
\begin{aligned}
&\frac{dv_0}{dt}\int_{y_0}^{y_1}\phi_0(y)\phi_1(y)dy + \frac{dv_1}{dt}\int_{y_0}^{y_2}\phi_1^2(y)dy + \frac{dv_2}{dt}\int_{y_1}^{y_2}\phi_2(y)\phi_1(y)dy \\
&= 0 + \mu_1\frac{dv_1}{dt} + \mu_2\frac{dv_2}{dt},
\end{aligned} \tag{2.5.2}
$$

since $\frac{dv_0}{dt} = 0$. Similarly, for the case where $m = N$ we have that

$$
\begin{aligned}
&\frac{dv_{N-1}}{dt}\int_{y_{N-1}}^{y_N}\phi_{N-1}(y)\phi_N(y)dy + \frac{dv_N}{dt}\int_{y_{N-1}}^{y_{N+1}}\phi_N^2(y)dy \\
&\quad + \frac{dv_{N+1}}{dt}\int_{y_N}^{y_{N+1}}\phi_{N+1}(y)\phi_N(y)dy \\
&= \mu_{N-1}\frac{dv_{N-1}}{dt} + \mu_N\frac{dv_N}{dt} + 0.
\end{aligned} \tag{2.5.3}
$$

Thus, we found the components for M in the theorem.

2.5.2 Finding the components for $M\xi + \kappa$ in (2.4.4)

We represent f in (2.2.12) by ξ_k for which $f_N(y) = \sum_{k=0}^{N+1}\xi_k\phi_k(y)$. We need to consider $\int_{-L}^{L}\sum_{k=0}^{N+1}\xi_k\phi_k(y)\phi_m(y)dy$. We found $\int_{-L}^{L}\sum_{k=1}^{N}\phi_k(y)\phi_m(y)dy$ in the previous section for $y_1, y_2, ...y_N$. The result was M which is applicable to $f_N(y)$. However, we consider boundary cases differently than in the previous section since ξ_0 and ξ_{N+1} are not necessarily zeros. When $m = 1$,

$$
\begin{aligned}
\int_{\mathbb{R}}\sum_{k=0}^{N+1}\xi_k\phi_k(y)\phi_1(y)dy \quad &= \xi_0(t)\int_{y_0}^{y_1}\phi_0(y)\phi_1(y)dy + \xi_1(t)\int_{y_0}^{y_2}\phi_1^2(y)dy \\
&\quad + \xi_2(t)\int_{y_1}^{y_2}\phi_2(y)\phi_1(y)dy.
\end{aligned}
$$

When $m = N$, we have

$$
\begin{aligned}
\int_{\mathbb{R}}\sum_{k=0}^{N+1}\xi_k\phi_k(y)\phi_N(y)dy = \xi_{N-1}(t)\int_{y_{N-1}}^{N}\phi_{N-1}(y)\phi_N(y)dy + \xi_N(t)\int_{y_{N-1}}^{y_{N+1}}\phi_N^2(y)dy \\
+\xi_{N+1}(t)\int_{y_N}^{y_{N+1}}\phi_{N+1}(y)\phi_N(y)dy
\end{aligned}
$$

To keep M in (2.4.8) applicable to $\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k \phi_k(y)\phi_m(y)dy$ we introduce $\kappa = (\kappa_1, .., \kappa_N)$, such that

$$\kappa_1 = (\phi_0, \phi_1)_{L_2(D)}\xi_0 = \xi_0(t) \int_{y_0}^{y_1} \phi_0(y)\phi_1(y)dy = \mu_{1,0}\xi_0.$$

$$\kappa_N = (A\phi_{N+1}, \phi_N)_{L_2(D)}\xi_{N+1} = \xi_{N+1}(t) \int_{y_N}^{y_{N+1}} \phi_{N+1}(y)\phi_N(y)dy = \mu_{1,0}\xi_{N+1}.$$

$$\kappa_k = 0$$

for $k = 2, .., N-1$.

2.5.3 Finding the coefficients for B in (2.4.4).

We consider the term in (2.4.3)

$$R'(t) \int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) A\phi_k(y)\phi_m(y)dy = R'(t)v_k(t) \int_{\mathbb{R}} \sum_{k=1}^{N} A\phi_k(y)\phi_m(y)dy.$$

Let

$$(A\phi_k, \phi_m)_{L_2(D)} = \int_{-L}^{L} A\phi_k(y)\phi_m(y)dy,$$

and let

$$\rho(y) = \frac{\sigma(y)^2}{2}, \quad \eta(y) = (r - d(y) - \frac{1}{2}\sigma(y)^2).$$

We have that

$$\begin{aligned} A\phi_k(y) = \frac{1}{2}\sigma(y)^2\phi_k''(y) + (r - d(y) - \frac{1}{2}\sigma(y)^2)\phi_k'(y) &= \rho(y)\phi_k''(y) + \eta(y)\phi_k'(y) \\ &= (\rho(y)\phi'(y))' - \rho(y)'\phi_k'(y) + \eta(y)\phi_k'(y). \end{aligned}$$

By the definitions,

$$\begin{aligned} (A\phi_k, \phi_m)_{L_2(D)} \quad &= \int_{-L}^{L} (\rho(y)\phi_k'(y))'\phi_m(y)dy - \int_{-L}^{L} \rho(y)'\phi_k'(y)\phi_m(y)dy \\ &\quad + \int_{-L}^{L} \eta(y)\phi_k'(y)\phi_m(y)dy \\ &= \rho(y)\phi_k'(y)\phi_m(y)\Big|_{-L}^{L} - \int_{-L}^{L} \rho(y)\phi_k'(y)\phi_m'(y)dy \\ &\quad - \int_{-L}^{L} \rho(y)'\phi_k'(y)\phi_m(y)dy + \int_{-L}^{L} \eta(y)\phi_k'(y)\phi_m(y)dy \\ &= 0 - \int_{-L}^{L} \rho(y)\phi_k'(y)\phi_m'(y)dy - \int_{-L}^{L} \rho(y)'\phi_k'(y)\phi_m(y)dy \end{aligned}$$

$$+\int_{-L}^{L}\eta(y)\phi_k'(y)\phi_m(y)dy.$$

For $k \le m-2$ or $k \ge m+2$ we have $(A\phi_k, \phi_m)_{L_2(D)} = 0$, since for $y \le y_{m-1}$ or $y \ge y_{m+1}$, $\phi_m(y) = \phi_m'(y) = 0$ and for $y \le y_{k-1}$ or $y \ge y_{k+1}$, $\phi_k(y) = \phi_k'(y) = 0$ by definitions. It follows that we need to consider three cases: $k = m-1$, $k = m$, and $k = m+1$. First, let us consider $k = m-1$:

$$\begin{aligned}(A\phi_{m-1}, \phi_m)_{L_2(D)} &= -\int_{-L}^{L}\rho(y)\phi_{m-1}'(y)\phi_m'(y)dy - \int_{-L}^{L}\rho(y)'\phi_{m-1}'(y)\phi_m(y)dy \\ &\quad + \int_{-L}^{L}\eta(y)\phi_{m-1}'(y)\phi_m(y)dy.\end{aligned}$$

Since the $\phi_{m-1}(y)$ and $\phi_m(y)$ are multiplied inside the integrals, the boundaries for integrals will be (y_{m-1}, y_m) since for $y \le y_{m-1}$, $\phi_m(y) = \phi_m'(y) = 0$ and for $y \ge y_m$, $\phi_{m-1}(y) = \phi_{m-1}'(y) = 0$. Let

$$J = \int_{y_{m-1}}^{y_m}\rho(y)\phi_{m-1}'(y)\phi_m'(y)dy.$$

Then

$$\begin{aligned}(A\phi_{m-1}, \phi_m)_{L_2(D)} = -J - \int_{y_{m-1}}^{y_m}\rho(y)'\phi_{m-1}'(y)\phi_m(y)dy \\ + \int_{y_{m-1}}^{y_m}\eta(y)\phi_{m-1}'(y)\phi_m(y)dy.\end{aligned}$$

We have that

$$\begin{aligned}J &= \int_{y_{m-1}}^{y_m}\rho(y)\phi_{m-1}'(y)\phi_m'(y)dy \\ &= \phi_m(y)\rho(y)\phi_{m-1}'(y)\Big|_{y_{m-1}}^{y_m} - \int_{y_{m-1}}^{y_m}(\rho(y)\phi_{m-1}(y)')'\phi_m(y)dy \\ &= \rho(y_m)\phi_{m-1}'(y_m - 0) - \int_{y_{m-1}}^{y_m}(\rho(y)\phi_{m-1}'(y))'\phi_m(y)dy.\end{aligned}$$

Substituting J back in, we obtain that

$$\begin{aligned}(A\phi_{m-1}, \phi_m)_{L_2(D)} &= -\rho(y_m)\phi_{m-1}'(y_m - 0) + \int_{y_{m-1}}^{y_m}(\rho(y)\phi_{m-1}'(y))'\phi_m(y)dy \\ &\quad - \int_{y_{m-1}}^{y_m}\rho(y)'\phi_{m-1}'(y)\phi_m(y)dy \\ &\quad + \int_{y_{m-1}}^{y_m}\eta(y)\phi_{m-1}'(y)\phi_m(y)dy \\ &= -\rho(y_m)\phi_{m-1}'(y_m - 0)\end{aligned}$$

$$+\int_{y_{m-1}}^{y_m}[(\rho(y)\phi'_{m-1}(y))'\phi_m(y)-\rho(y)'\phi'_{m-1}(y)\phi_m(y)$$
$$+\eta(y)\phi'_{m-1}(y)\phi_m(y)]dy$$
$$=-\rho(y_m)\phi'_{m-1}(y_m-0)+\int_{y_{m-1}}^{y_m}[(\rho(y)\phi'_{m-1}(y))'$$
$$-\rho(y)'\phi'_{m-1}(y)+\eta(y)\phi'_{m-1}(y)]\phi_m(y)dy.$$

Since $A\phi_{m-1}=(\rho(y)\phi'_{m-1})'-\rho(y)'\phi'_{m-1}+\eta(y)\phi'_{m-1}=0$ as defined in Section 3, we obtain that

$$(A\phi_{m-1},\phi_m)_{L_2(D)}=-\rho(y)(y_m)\phi'_{m-1}(y_m-0)=-\beta_{m,m-1}. \tag{2.5.4}$$

$\beta_{m,m-1}$ is the component to the left of the main diagonal in B. Now, let us consider $k=m$:

$$(A\phi_m,\phi_m)_{L_2(D)} \quad =-\int_{-L}^{L}\rho(y)\phi'_m(y)\phi'_m(y)dy-\int_{-L}^{L}\rho(y)'\phi'_m(y)\phi_m(y)dy$$
$$+\int_{-L}^{L}\eta(y)\phi'_m(y)\phi_m(y)dy.$$

The boundaries for the integrals are (y_{m-1},y_{m+1}) since for $y\le y_{m-1}$, $\phi_m(y)=\phi'_m(y)=0$ and for $y\ge y_{m+1}$, $\phi_m(y)=\phi'_m(y)=0$.. We will need to consider the following:

$$(A\phi_m,\phi_m)_{L_2(D)} \quad =-\int_{y_{m-1}}^{y_m}\rho(y)\phi'_m(y)\phi'_m(y)dy-\int_{y_{m-1}}^{y_m}\rho(y)'\phi'_m(y)\phi_m(y)dy$$
$$+\int_{y_{m-1}}^{y_m}\eta(y)\phi'_m(y)\phi_m(y)dy-\int_{y_m}^{y_{m+1}}\rho(y)\phi'_m(y)\phi'_m(y)dy$$
$$-\int_{y_m}^{y_{m+1}}\rho(y)'\phi'_m(y)\phi_m(y)dy+\int_{y_m}^{y_{m+1}}\eta(y)\phi'_m(y)\phi_m(y)dy.$$

Let

$$(A\phi_m,\phi_m)_{L_2(D)}=Q_1+Q_2,$$

such that

$$Q_1 \quad =-\int_{y_{m-1}}^{y_m}\rho(y)\phi'_m(y)\phi'_m(y)dy-\int_{y_{m-1}}^{y_m}\rho(y)'\phi'_m(y)\phi_m(y)dy$$
$$+\int_{y_{m-1}}^{y_m}\eta(y)\phi'_m(y)\phi_m(y)dy$$

and

$$Q_2 \quad =-\int_{y_m}^{y_{m+1}}\rho(y)\phi'_m(y)\phi'_m(y)dy-\int_{y_m}^{y_{m+1}}\rho(y)'\phi'_m(y)\phi_m(y)dy$$

$$+\int_{y_m}^{y_{m+1}} \eta(y)\phi'_m(y)\phi_m(y)dy.$$

Let us consider Q_1 first. Let

$$J_1 = \int_{y_{m-1}}^{y_m} \rho(y)\phi'_m(y)\phi'_m(y)dy.$$

Then

$$Q_1 = -J_1 - \int_{y_{m-1}}^{y_m} \rho(y)'\phi'_m(y)\phi_m(y)dy + \int_{y_{m-1}}^{y_m} \eta(y)\phi'_m(y)\phi_m(y)dy,$$

We have that

$$\begin{aligned} J_1 \quad &= \int_{y_{m-1}}^{y_m} \rho(y)\phi'_m(y)\phi'_m(y)dy \\ &= \phi_m(y)\rho(y)\phi'_m(y)\Big|_{y_{m-1}}^{y_m} - \int_{y_{m-1}}^{y_m} (\rho(y)\phi'_m(y))'\phi_m(y)dy \\ &= \rho(y_m)\phi'_m(y_m-0) - \int_{y_{m-1}}^{y_m} (\rho(y)\phi'_m(y))'\phi_m(y)dy. \end{aligned}$$

Substituting J_1 back in, we obtain that

$$\begin{aligned} Q_1 \quad &= -\rho(y_m)\phi'_m(y_m-0) + \int_{y_{m-1}}^{y_m} (\rho(y)\phi'_m(y))'\phi_m(y)dy \\ &\quad - \int_{y_{m-1}}^{y_m} \rho(y)'\phi'_m(y)\phi_m(y)dy + \int_{y_{m-1}}^{y_m} \eta(y)\phi'_m(y)\phi_m(y)dy \\ &= -\rho(y_m)\phi'_m(y_m-0) + \int_{y_{m-1}}^{y_m} [(\rho(y)\phi'_m(y))'\phi_m(y) \\ &\quad -\rho(y)'\phi'_m(y)\phi_m(y) + \eta(y)\phi'_m(y)\phi_m(y)]dy \\ &= -\rho(y_m)\phi'_m(y_m-0) \\ &\quad + \int_{y_{m-1}}^{y_m} [(\rho(y)\phi'_m(y))' - \rho(y)'\phi'_m(y) + \eta(y)\phi'_m(y)]\phi_m(y)]dy. \end{aligned}$$

Since $A\phi_m(y) = (\rho(y)\phi'_m(y))' - \rho(y)'\phi'_m(y) + \eta(y)\phi'_m(y) = 0$ as defined in Section 3, we obtain that

$$Q_1 = -\rho(y_m)\phi'_m(y_m-0).$$

Now, let us consider Q_2. Let

$$J_2 = \int_{y_m}^{y_{m+1}} \rho(y)\phi'_m(y)\phi'_m(y)dy.$$

Then

$$Q_2 = -J_2 - \int_{y_m}^{y_{m+1}} \rho(y)'\phi'_m(y)\phi_m(y)dy + \int_{y_m}^{y_{m+1}} \eta(y)\phi'_m(y)\phi_m(y)dy.$$

We have that

$$\begin{aligned}
J_2 \quad &= \int_{y_m}^{y_{m+1}} \rho(y)\phi_m'(y)\phi_m'(y)dy \\
&= \phi_m(y)\rho(y)\phi_m'(y)\Big|_{y_m}^{y_{m+1}} - \int_{y_m}^{y_{m+1}} (\rho(y)\phi_m'(y))'\phi_m(y)dy \\
&= -\rho(y_m)\phi_m'(y_m+0) - \int_{y_m}^{y_{m+1}} (\rho(y)\phi_m'(y))'\phi_m(y)dy.
\end{aligned}$$

Substituting J_2 back in, we obtain that

$$\begin{aligned}
Q_2 \quad &= \rho(y_m)\phi_m'(y_m+0) + \int_{y_m}^{y_{m+1}} (\rho(y)\phi_m'(y))'\phi_m(y)dy \\
&\quad - \int_{y_m}^{y_{m+1}} \rho(y)'\phi_m'(y)\phi_m(y)dy + \int_{y_m}^{y_{m+1}} \eta(y)\phi_m'(y)\phi_m(y)dy \\
&= \rho(y_m)\phi_m'(y_m+0) + \int_{y_m}^{y_{m+1}} [(\rho(y)\phi_m'(y))'\phi_m(y) - \rho(y)'\phi_m'(y)\phi_m(y) \\
&\quad + \eta(y)\phi_m'(y)\phi_m(y)]dy \\
&= \rho(y_m)\phi_m'(y_m+0) + \int_{y_m}^{y_{m+1}} [[(\rho(y)\phi_m'(y))' - \rho(y)'\phi_m'(y) + \eta(y)\phi_m'(y)]\phi_m(y)]dy.
\end{aligned}$$

Since $A\phi_m = (\rho(y)\phi_m')' - \rho(y)'\phi_m'(y) + \eta(y)\phi_m'(y) = 0$, we obtain that

$$Q_2 = \rho(y_m)\phi_m'(y_m+0).$$

Substituting Q_1 and Q_2 back in we have that

$$(A\phi_m, \phi_m)_{L_2(D)} = -\rho(y_m)\phi_m'(y_m-0) + \rho(y_m)\phi_m'(y_m+0) = -\beta_{m,m}. \tag{2.5.5}$$

$\beta_{m,m}$ is the diagonal component in B. Finally, let us consider $k = m+1$.

$$\begin{aligned}
(A\phi_{m+1}, \phi_m)_{L_2(D)} \quad &= -\int_{-L}^{L} \rho(y)\phi_{m+1}'(y)\phi_m'(y)dy - \int_{-L}^{L} \rho(y)'\phi_{m+1}'(y)\phi_m(y)dy \\
&\quad + \int_{-L}^{L} \eta(y)\phi_{m+1}'(y)\phi_m(y)dy
\end{aligned}$$

Since the $\phi_{m+1}(y)$ and $\phi_m(y)$ are multiplied inside the integral, the boundaries for the integrals will be (y_m, y_{m+1}) since for $y \le y_m$, $\phi_{m+1}(y) = \phi_{m+1}'(y) = 0$ and for $y \ge y_{m+1}$, $\phi_m(y) = \phi_m'(y) = 0$. Let

$$J = \int_{y_m}^{y_{m+1}} \rho(y)\phi_{m+1}'(y)\phi_m'(y)dy.$$

$$(A\phi_{m+1}, \phi_m)_{L_2(D)} \quad = -J - \int_{y_m}^{y_{m+1}} \rho(y)'\phi_{m+1}'(y)\phi_m(y)dy$$

$$+\int_{y_m}^{y_{m+1}} \eta(y)\phi'_{m+1}(y)\phi_m(y)dy.$$

We have that

$$\begin{aligned} J &= \int_{y_m}^{y_{m+1}} \rho(y)\phi'_{m+1}(y)\phi'_m(y)dy \\ &= \phi_m(y)\rho(y)\phi'_{m+1}(y)\Big|_{y_m}^{y_{m+1}} - \int_{y_m}^{y_{m+1}} (\rho(y)\phi'_{m+1}(y))'\phi_m(y)dy \\ &= -\rho(y_m)\phi'_{m+1}(y_m+0) - \int_{y_m}^{y_{m+1}} (\rho(y)\phi'_{m+1}(y))'\phi_m(y)dy. \end{aligned}$$

Substituting J back in, we obtain that

$$\begin{aligned} (A\phi_{m+1}, \phi_m)_{L_2(D)} &= \rho(y_{m+1})\phi'_{m+1}(y_m+0) + \int_{y_m}^{y_{m+1}} (\rho(y)\phi'_{m+1}(y))'\phi_m(y)dy \\ &\quad - \int_{y_m}^{y_{m+1}} \rho(y)'\phi'_{m+1}(y)\phi_m(y)dy + \int_{y_m}^{y_{m+1}} \eta(y)\phi'_{m+1}(y)\phi_m(y)dy \\ &= \rho(y_m)\phi'_{m+1}(y_m+0) + \int_{y_m}^{y_{m+1}} [(\rho(y)\phi'_{m+1}(y))'\phi_m(y) \\ &\quad -\rho(y)'\phi'_{m+1}(y)\phi_m(y) + \eta(y)\phi'_{m+1}(y)\phi_m(y)]dy \\ &= \rho(y_m)\phi'_{m+1}(y_m+0) \\ &\quad + \int_{y_m}^{y_{m+1}} [(\rho(y)\phi'_{m+1}(y))' - \rho(y)'\phi'_{m+1}(y) \\ &\quad +\eta(y)\phi'_{m+1}(y)]\phi_m(y)]dy. \end{aligned}$$

Since $A\phi_{m+1} = (\rho(y)\phi'_{m+1}(y))' - \rho(y)'\phi'_{m+1}(y) + \eta(y)\phi'_{m+1}(y) = 0$, as defined in Section 3 we obtain that

$$(A\phi_{m+1}, \phi_m)_{L_2(D)} = \rho(y_m)\phi'_{m+1}(y_m+0) = -\beta_{m,m+1}. \tag{2.5.6}$$

$\beta_{m,m+1}$ is the component to the right of the main diagonal in B.

Combining our findings and remembering that the term had a minus in the beginning, we obtain that

$$\begin{aligned} -\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t)A\phi_k(y)\phi_m(y)dy &= \rho(y_m)\phi'_{m-1}(y_m-0)v_{m-1}(t) \\ &\quad +\rho(y_m)(\phi'_m(y_m-0) - \phi'_m(y_m+0))v_m(t) \\ &\quad -\rho(y_m)\phi'_{m+1}(y_m+0)v_{m+1}(t) \\ &= \beta_{m,m-1}v_{m-1}(t) + \beta_{m,m}v_m(t) \\ &\quad +\beta_{m,m+1}v_{m+1}(t). \end{aligned} \tag{2.5.7}$$

Thus, we found the right side components for B. Let us consider the boundary cases: $m = 1$ and $m = N$. When $m = 1$, if we refer to (2.4.3), we will not consider the case when $k = m - 1 = 0$ since $v_0 = 0$. Thus,

$$\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) A\phi_k(y)\phi_1(y)dy = (-\rho(y_1)\phi_1'(y_1 - 0) + \rho(y_1)\phi_1'(y_1 + 0))v_1(t)$$
$$+\rho(y_1)\phi_2'(y_1 + 0)v_2(t). \quad (2.5.8)$$

Similarly, when $m = N$, we will not consider the case when $k = m+1 = N+1$ since $v_{N+1} = 0$. Thus

$$\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) A\phi_k(y)\phi_N(y)dy = -\rho(y_N)\phi_{N-1}'(y_N - 0)v_{N-1}(t)$$
$$+(-\rho(y_N)\phi_N'(y_N - 0) - \rho(y_N)\phi_N'(y_N + 0))v_N(t). \quad (2.5.9)$$

Thus, we found the components for the matrix B.

2.5.4 Alternative way of finding the coefficients for B in (2.4.4).

Let us show an alternative shorter way of calculation of B. This approach involves so called delta functions.

Delta functions: some background facts

Consider a mapping $F : C(\mathbf{R}) \to \mathbf{R}$ such that $F(g(\cdot)) = g(a)$. We have that

$$F(g(\cdot)) = g(a) = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \delta_\varepsilon(t-a)g(t)dt \qquad \forall g(\cdot) \in C(\mathbf{R}),$$

where

$$\delta_\varepsilon(t-a) = 0, \quad |t-a| > \varepsilon,$$
$$\delta_\varepsilon(t-a) = \frac{1}{2\varepsilon}, \quad |t-a| \le \varepsilon.$$

Usually, the limit here is denoted symbolically as

$$\int_{-\infty}^{+\infty} (t-a)g(t)dt,$$

where (t) is called *delta-function*, or *point source*. In fact, (t) is not a function of t, it is a so-called *generalized function*, and the integral of $(t-a)g(t)$ is not an integral, it is just a symbol.

Let $PC^1(\mathbf{R})$ be the set of continuous functions on $\mathbf{R}$ that are piecewise continuously differentiable and have with finite support.

Let a function $h(t)$ be continuously differentiable for $t < a$ and $t > a$, and such that $h(a-0) \neq h(a+0)$ and has bounded support. Let the function $\tilde{h}(x)$ be defined as

$$\tilde{h}(t) = h'(t), \quad t \neq a,$$

$$\tilde{h}(t) = 0, \quad t = a.$$

Then integration by parts gives that

$$\begin{aligned} -\int_{-\infty}^{\infty} h(t)g'(t)dt \quad &= \int_{-\infty}^{a} h'(t)g(t)dt + \int_{a}^{\infty} h'(t)g(t)dt + [h(a+0) - h(a-0)]g(t) \\ &= \int_{-\infty}^{\infty} \tilde{h}(t)g(t)dt + [h(a+0) - h(a-0)]g(t) \end{aligned}$$

for functions $g \in PC^1(\mathbf{R})$.

We interpret this fact as

$$\int_{-\infty}^{\infty} h'(t)g(t)dt = \int_{-\infty}^{\infty} \tilde{h}(t)g(t)dt + [h(a+0) - h(a-0)]g(t),$$

i.e.

$$h'(t) = [h(a+0) - h(a-0)]\dot{(}t-a) + \tilde{h}(t)$$

as equality of operators on $PC^1(\mathbf{R})$. Here this symbol for the derivative of h is understood as an operator $h'(\cdot) : PC^1(\mathbf{R}) \to \mathbf{R}$.

Furthermore, the functions ϕ_k are such that their first derivatives are defined on $\mathbf{R}\backslash\{y_k\}_{m=k-1,k,k+1}$ and have jumps that these points. Hence,

$$\begin{aligned} &\phi_k''(y) \\ &= \rho(y_{k-1})(\phi'(y_{k-1}+0) - 0)\delta(y - y_{k-1}) + \rho(y_k)(\phi'(y_k+0) - \phi'(y_k-0))\delta(y - y_k) \\ &+\rho(y_{k+1}(0 - \phi'(y_{k+1}-0))\delta(y - y_{k+1}) \end{aligned}$$

and

$$\begin{aligned} &\rho(y)\phi_k''(y) \\ &= \rho(y_{k-1})(\phi'(y_{k-1}+0) - 0)\delta(y - y_{k-1}) + \rho(y_k)(\phi'(y_k+0) - \phi'(y_k-0))\delta(y - y_k) \\ &+\rho(y_{k+1}(0 - \phi'(y_{k+1}-0))\delta(y - y_{k+1}) \end{aligned}$$

By the choice of ϕ_k, we have that

$$A\phi_k(y) = 0, \quad y \in (-\infty, y_{k-1}) \cup (y_{k-1}, y_k) \cup (y_k, y_{k+1}) \cup (y_{k+1}, \infty).$$

Thus, we can write, in the sense of equality for delta-functions, that

$$\begin{aligned} A\phi_k(y) = &\rho(y_{k-1})(\phi'(y_{k-1}+0) - 0)\delta(y - y_{k-1}) \\ &+\rho(y_k)(\phi'(y_k+0) - \phi'(y_k-0))\delta(y - y_k) \\ &+\rho(y_{k+1}(0 - \phi'(y_{k+1}-0))\delta(y - y_{k+1}). \end{aligned}$$

We need to find v_k such that, in the sense of equality for delta-functions,

$$\sum_{k=1}^{N} v_k'(t)\phi_k(y) = -\sum_{k=1}^{N} v_k(t)A\phi_k(y) + \sum_{k=1}^{N} \xi_k(t)A\phi_k(y).$$

Let us estimate

$$\sum_{k=1}^{N} \int_{\mathbb{R}} v_k(t) A\phi_k(y)\phi_m(y)dy.$$

We have that

$$\int_{\mathbb{R}} A\phi_k(y)\phi_m(y)dy = \int_{\mathbb{R}} A\phi_{m-1}(y)\phi_m(y)dy + \int_{\mathbb{R}} A\phi_m(y)\phi_m(y)dy + \int_{\mathbb{R}} A\phi_{m+1}(y)\phi_m(y)dy$$

since for $y \leq y_{m-1}$ and $y \geq y_{m+1}$, $\phi_m(y) = 0$.

Let us consider the above three terms separately.

We have that

$$\begin{aligned}\int_{\mathbb{R}} A\phi_m(y)\phi_m(y)dy &= \int_{\mathbb{R}} \rho(y_{k-1})\delta(y - y_{m-1})(\phi'_m(y_{m-1} + 0))\phi_m(y)dy \\ &+ \int_{\mathbb{R}} \rho(y_m)\delta(y - y_m)(\phi'(y_m + 0) - \phi'_m(y_m - 0)\phi_m(y)dy \\ &+ \int_{\mathbb{R}} \rho(y_{m+1})\delta(y - y_{m+1})(-\phi'_m(y_{m+1} - 0))\phi_m(y)dy.\end{aligned}$$

Since $\phi_m(y_{m-1}) = 0$ and $\phi_m(y_{m+1}) = 0$ then the above expression will be:

$$\begin{aligned}&\int_{\mathbb{R}} A\phi_m(y)\phi_m(y)dy \\ &= \int_{\mathbb{R}} \rho(y_m)\delta(y - y_m)(\phi'_m(y_m + 0) - \phi'(y_m - 0)\phi_m(y_m)dy \\ &= \rho(y_m)(\phi'_m(y_m + 0) - \phi'(y_m - 0))\phi_m(y_m) \\ &= \rho(y_m)(\phi'_m(y_m + 0) - \phi'(y_m - 0)).\end{aligned}$$

Similarly we find $\int_{\mathbb{R}} A\phi_{m-1}(y)\phi_m dy$ and $\int_{\mathbb{R}} A\phi_{m+1}(y)\phi_m dy$.

$$\begin{aligned}&\int_{\mathbb{R}} A\phi_{m-1}(y)\phi_m dy \\ &= \int_{\mathbb{R}} \rho(y_{m-2})\delta(y - y_{m-2})(\phi'_{m-1}(y_{m-2} + 0) - 0)\phi_m(y)dy \\ &+ \int_{\mathbb{R}} \rho(y_{m-1})\delta(y - y_{m-1})(\phi'_{m-1}(y_{m-1} + 0) - \phi'_{m-1}(y_{m-1} - 0))\phi_m(y)dy \\ &+ \int \rho(y_m)\delta(y - y_m)(0 - \phi'_{m-1}(y_m - 0))\phi_m(y)dy.\end{aligned}$$

Notice that, $\phi_m(y_{m-2}) = 0$ and $\phi_m(y_{m-1}) = 0$. So the above becomes:

$$
\begin{aligned}
&\int_{\mathbb{R}} A\phi_{m-1}(y)\phi_m(y)dy \\
&= \int_{\mathbb{R}} \rho(y_m)\delta(y-y_m)(0-\phi'_{m-1}(y_m-0))\phi_m(y)dy \\
&= \rho(y_m)(0-\phi'_{m-1}(y_m-0))\phi_m(y_m) \\
&= \rho(y_m)(-\phi'_{m-1}(y_m-0)).
\end{aligned}
$$

For $\int_{\mathbb{R}} A\phi_{m+1}(y)\phi_m(y)dy$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} A\phi_{m+1}(y)\phi_m(y)dy \\
&= \int_{\mathbb{R}} \rho(y_m)\delta(y-y_m)(\phi'_{m+1}(y_m+0)-0)\phi_m(y)dy \\
&+ \int_{\mathbb{R}} \rho(y_{m+1})\delta(y-y_{m+1})(\phi'_{m+1}(y_{m+1}+0)-\phi'_{m+1}(y_{m+1}-0))\phi_m(y)dy \\
&+ \int_{\mathbb{R}} \rho(y_{m+2})\delta(y-y_{m+2})(\phi'_{m+1}(0-(y_{m+2}-0)))\phi_m(y)dy.
\end{aligned}
$$

Notice that, $\phi_m(y_{m+1}) = 0$ and $\phi_m(y_{m+2}) = 0$. So the above becomes:

$$
\begin{aligned}
&\int_{\mathbb{R}} A\phi_{m+1}(y)\phi_m(y) \\
&= \int_{\mathbb{R}} \frac{\sigma^2}{2}\delta(y-y_m)(\phi'_{m+1}(0-y_m))\phi_m(y)dy \\
&= \rho(y_m)(\phi'_{m+1}(y_m+0)-0)\phi_m(y_m) \\
&= \rho(y_m)(\phi'_{m+1}(y_m+0)) \times 1.
\end{aligned}
$$

We combine the equations to see that

$$
\begin{aligned}
-\int_{\mathbb{R}} \sum_{k=1}^{N} Av_k(t)\phi_k(y)\phi_m(y)dy &= -v_{m-1}(t)\frac{\sigma^2(y_m)}{2}(-(\phi'(y_m-0))) \\
-v_{m+1}(t)\rho(y_m)(\phi'_{m+1}(y_m+0)) &- v_m(t)\rho(y_m)(\phi'(y_m+0)-\phi'(y_m-0)).
\end{aligned}
$$

we can rewrite the above as

$$
\beta_{m,m-1}v_{m-1}(t) + \beta_{m,m}v_m + \beta_{m+1,m}v_{m+1}(t),
$$

where

$$
\begin{aligned}
\beta_{m,m-1} &= -\rho(y_m)(\phi'(y_m+0), \\
\beta_{m,m} &= -\rho(y_m)(\phi'(y_m+0)-\phi'(y_m-0)), \\
\beta_{m,m+1} &= -\rho(y_m)(-\phi'(y_m-0)).
\end{aligned}
$$

Thus, we found the same coefficients in previous subsection.

2.5.5 Finding the components for $B\xi + \zeta$ in (2.4.4)

We represent f in (2.2.12) by $f_N(y) = \sum_{k=0}^{N+1} \xi_k \phi_k(y)$. We have found $(A\phi_k, \phi_m)_{L_2(D)}$ in the previous section for $y_1, y_2, ...y_N$. We have to consider boundary cases differently than in the previous section since ξ_0 and ξ_{N+1} are not necessarily zeros. For $m = 1$, we have

$$\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A\phi_k(y)\phi_1(y)dy \qquad = -\rho(y_1)\phi_0'(y_1 - 0)\xi_0 + (\rho(y_1)\phi_1'(y_1 - 0) \\ -\rho(y_1)\phi_1'(y_1 + 0))\xi_1 + \rho(y_1)\phi_2'(y_1 + 0)\xi_2.$$

For $m = N$, we have

$$\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A\phi_k(y)\phi_N(y)dy \qquad = -\rho(y_N)\phi_{N-1}'(y_N - 0)\xi_{N-1} + (-\rho(y_N)\phi_N'(y_N - 0) \\ +\rho(y_N)\phi_N'(y_N + 0))\xi_N + \rho(y_1)\phi_2'(y_1 + 0)\xi_{N+1}.$$

To keep B in (2.4.8) applicable to $\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A\phi_k(y)\phi_m(y)dy$ we introduce $\zeta = (\zeta_1, .., \zeta_N)$, such that

$$\begin{aligned} \zeta_1 \quad &= (A\phi_0, \phi_1)_{L_2(D)}\xi_0 \\ &= -\rho(y_1)\phi_0'(y_1 - 0) \\ &= -\beta_{1,0}\xi_0. \\ \zeta_N \quad &= (A\phi_{N+1}, \phi_N)_{L_2(D)}\xi_{N+1} \\ &= \rho(y_N)\phi_{N+1}'(y_N + 0) \\ &= -\beta_{N,N+1}\xi_{N+1}. \\ \zeta_k \quad &= 0, \quad for \quad k = 2, .., N - 1. \end{aligned}$$

2.5.6 Proof of Theorem 2.4.1: conclusion

Combining the statements above we obtain the statement (2.4.4). This concludes the proof of Theorem 2.4.1.

2.6 Solution of the ODE (2.4.4)

2.6.1 Exact solution of equation (2.4.4)

To solve the resulting system, we use the exponential matrix method: From the properties of linear ODEs, we obtain that

$$V(t) = -\int_t^T e^{E(t-s)} M^{-1} \varphi ds.$$

In particular,

$$V(0) = -\int_0^T e^{E(-s)} M^{-1} \varphi ds$$

will give us the option value at the present time.

$$V(t_k) = e^{Et_k} \int_{t_k}^{T} -e^{E(-s)} M^{-1} \varphi ds.$$

Let $E = M^{-1}B$. Note that since matrix M is tridiagonal, finding its inverse is numerically feasible.

2.6.2 Crank-Nicolson method

We can also solve system (2.4.4) using Crank-Nicolson method backwards as

$$\frac{u^{t+\Delta t} - u^t}{\Delta t} = M^{-1}\frac{1}{2}[(Bu^{t+\Delta t} + \varphi(t + \Delta t)) + (Bu^t + \varphi(t))]. \tag{2.6.1}$$

This gives

$$u^t = (M + \Delta t\frac{1}{2}B)^{-1}[Mu^{t+\Delta t} - \Delta t\frac{1}{2}Bu^{t+\Delta t} - \Delta t(\varphi(t) + \varphi(t + \Delta t))]. \tag{2.6.2}$$

2.6.3 Backwards substitution

After solving the system we obtain the vector v_k, $k = 1..N$. To solve the original equation, we have to reverse the substitutions (2.2.3), (2.2.5), and (2.2.9). The last substitution made was (2.2.9). We transform the answer:

$$\hat{v}_k(t) := v_k(t) + \xi_k, \quad k = 1..N.$$

Consider (2.2.5). To reverse it:

$$\hat{V}_k(t) := \hat{v}_k(t)e^{r(t-T)}.$$

Now, consider (2.2.3). Clearly, $y = \ln x$. Therefore, finding V(x) from the original Black-Scholes equation is equivalent to finding $V_N(y) = V_N(\ln x)$. To find V(x,t) for particular x,

$$V(x,t) = \hat{V}_k(t)\phi_k(\log x) + \hat{V}_{k+1}(t)\phi_{k+1}(\log x), \quad y_k \leq \log x \leq y_{k+1},$$

2.7 Convergence

To show convergence we will apply Theorem 7.1 from Douglas and Dupont's paper [20]. In what follows, we will show all the conditions for convergence in Douglas and Dupont's result are satisfied by our method.

Theorem 2.7.1 *(Theorem 7.1 [20]) There exist constants C and δ which depend on T,n,D,K_0,C_0 and C_1, such that for v and V_N, solutions to (2.4.1) and (2.4.2), respectively, and any function $\tilde{u}$ of the form $\sum_{i=1}^{N} \alpha_i \tilde{u}_i$*

$$\sup_{0\leq t\leq T} \|u(\cdot,t) - V_N(\cdot,t)\|_{L^2(D)} + \delta \int_0^T \|u(\cdot,t) - V_N(.,t)\|^2_{H_0^1(D)} dt$$

$$\leq C[\sup_{0\leq t\leq T}\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{L^2(D)}+\int_0^T\|u(\cdot,t)-\tilde{u}\|^2_{H^1_0(D)}dt$$
$$+\|\frac{\partial}{\partial t}(u-\tilde{u})\|^2_{L^2(D\times(0,T))}]$$

First, we want to define our problem in the terms used in [20]. Note that we will refer to the following notations from [20] in a special way to avoid confusion: $a(u,w)$ as $\widetilde{a}(u,w)$, A as $\widetilde{A}$, (f,w) as $(\widetilde{f},w)$ and η as $\widetilde{\eta}$. Also, we will use y instead of x from [20]. We need to rewrite (4.1) in our paper as (7.2) in [20]:

$$\langle\frac{\partial u}{\partial t},w\rangle+\widetilde{a}(u,w)=\langle f(u),w\rangle.$$

Here,

$$\widetilde{a}(u,w)=\int_D\widetilde{A}(y,t,u(y,t),\frac{\partial w}{\partial y}(y,t))dy$$

and

$$\widetilde{f}(u)=\widetilde{f}(y,t,u(y,t),\frac{\partial u}{\partial y}(y,t))$$

for some measurable functions $\widetilde{A}(y,t,u,p):D\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\widetilde{f}(y,t,u,p):D\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Note that in this paper p is not a vector. We will need to find the functions $\widetilde{A}$ and $\widetilde{f}$ to obtain (7.4) in [20]: Let us consider $a(u,w)$ and $R(t)a(f,w)$ in (2.4.1).

$$\begin{aligned}a(u,w)\quad&=(Au,w)_{L_2(D)}=(\rho\phi''_{yy},w)_{L_2(D)}+(\eta\phi'_y,w)_{L_2(D)}\\&=((\rho u'_y)'_y,w)_{L_2(D)}-(\rho'_yu'_y,w)_{L_2(D)}+(\eta u'_y,w)_{L_2(D)}\\&=\int_{-L}^L(\rho u'_y)'_yw\,dy-\int_{-L}^L\rho'_yu'_yw\,dy+\int_{-L}^L\eta u'_yw\,dy\\&=\rho u'_yw\Big|_{-L}^L-\int_{-L}^L\rho u'_yw'_y\,dy-\int_{-L}^L\rho'_yu'_yw\,dy+\int_{-L}^L\eta u'_yw\,dy\\&=0-\int_{-L}^L\rho u'_yw'_y\,dy-\int_{-L}^L\rho'_yu'_yw\,dy+\int_{-L}^L\eta u'_yw\,dy.\end{aligned}\tag{2.7.1}$$

Similarly,

$$\begin{aligned}R(t)a(f,w)\quad&=R(t)(Af,w)_{L_2(D)}\\&=R(t)((\rho f'_y)'_y,w)_{L_2(D)}-R(t)(\rho'_yf'_y,w)_{L_2(D)}+R(t)(\eta f'_y,w)_{L_2(D)}\\&=0-R(t)\int_{-L}^L\rho f'_yw'_y\,dy-R(t)\int_{-L}^L\rho'_yf'_yw\,dy\\&\quad+R(t)\int_{-L}^L\eta f'_yw\,dy\end{aligned}\tag{2.7.2}$$

Now, let us consider $R'(t)(f,w)_{L_2(D)}$ in (2.4.1).

$$R'(t)(f,w)_{L_2(D)} = R'(t)\int_{-L}^{L} fwdy. \tag{2.7.3}$$

We combine the terms with w'_y of the equations (2.7.1) and (2.7.2) to obtain $\widetilde{a}(u,w)$ from 7.2 in [20].

$$\widetilde{a}(u,w) = \int_{-L}^{L}(-\rho u'_y w'_y - R(t)\rho f'_y w'_y)dy.$$

So,

$$\widetilde{A}(y,t,u,u'_y) = -\rho u'_y - R(t)\rho f'_y.$$

This gives

$$\widetilde{A}(y,t,u,p) = -\rho p - R(t)\rho f'_y.$$

Next, we combine the terms with w of the equations (2.7.1),(2.7.2) and (2.7.3) to obtain $(\widetilde{f},w)$ from [20].

$$(\widetilde{f},w) = \int_{-L}^{L}\rho'_y u'_y w\,dy - \int_{-L}^{L}\eta u'_y w\,dy + R(t)\int_{-L}^{L}\rho'_y f'_y w\,dy$$
$$-R(t)\int_{-L}^{L}\eta f'_y w\,dy + R'(t)\int_{-L}^{L} fwdy.$$

So,

$$\widetilde{f}(x,t,u,u'_y) = \rho'_y u'_y - \eta u'_y + \rho'_y f'_y - R(t)\eta f'_y + R'(t)f.$$

This gives

$$\widetilde{f}(x,t,u,p) = \rho'_y p - \eta p + \rho'_y f'_y - R(t)\eta f'_y + R'(t)f.$$

Next, we check the conditions from [20] on $\widetilde{A}$ and $\widetilde{f}$ in order for the theorem to hold. First, it is clear that both functions are measurable. Second, $\frac{\partial a}{\partial u} = -\rho$ is such that $C_0 \leq C_1$, $0 \leq C_0 \leq C_1$ for some positive C_0 and C_1. Next, we check the condition (7.1) in [20]:

$$\|\widetilde{A}(y,t,w(y),p(y)\|^2_{L^2(D)} + \|\widetilde{f}(y,t,w(y),p(y)\|^2_{L^2(D)}$$
$$\leq C[\|w(y)\|^2_{L^2(D)} + \|p(y)\|^2_{L^2(D)} + 1]. \tag{2.7.4}$$

Observe that,

$$\|\widetilde{A}(y,t,w(y),p(y)\|^2_{L^2(D)} \quad = \| -\rho p - \rho f'_y\|^2_{L^2(D)}$$
$$\leq C(\|p\|^2_{L^2(D)} + 1)$$

and

$$\|\widetilde{f}(y,t,w(y),p(y)^2\|_{L^2(D)} \quad = \|\rho'_y p - \eta p + \rho'_y f'_y - R(t)\eta f' + R'(t)f\|^2_{L^2(D)}$$
$$\leq C(\|p\|^2_{L^2(D)} + 1).$$

Since ρ, ρ'_y and η are bounded and f'_y is bounded, and R(t) and R'(t) are bounded, the condition (2.7.4) holds.

Next, we check the conditions (7.11a-7.11e) in [20] for our $\widetilde{A}(x,t,u,u'_y)$ and $\widetilde{f}(x,t,u,u'_y)$ Suppose that $\widetilde{f} = \widetilde{f}_1 + \widetilde{f}_2$ and that there exists K_0 such that for $(y,t) \in D \times (0,T)$, r and s in $\mathbb{R}$, and p and q in $\mathbb{R}$.
Let us check the condition (7.11a)[20]:

$$|\widetilde{A}(y,t,r,u'_y(y,t)) - \widetilde{A}(y,t,s,u'_y(y,t))| \leq K_0|r-s|.$$

This holds since

$$|\widetilde{A}(y,t,r,u'_y) - \widetilde{A}(y,t,s,u'_y)| = -\rho u'_y - \rho f'_y - (-\rho u'_y - \rho(y) f'_y) = 0.$$

Let us check the condition (7.11b)[20]:

$$|\widetilde{f}_1(y,t,u(y,t),p) - \widetilde{f}_1(y,t,u(y,t),q)| \leq K_0|p-q|.$$

This holds since

$$\begin{aligned}
&|\widetilde{f}_1(y,t,u(y,t),p) - \widetilde{f}_1(x,t,u(y,t),q)| \\
&= |\rho'_y p - \eta p + \rho'_y f'_y - R(t)\eta f'_y + R'(t)f - (\rho'_y q - \eta q + \rho'_y f'_y - R(t)\eta f'_y + R'(t)f)| \\
&= |(\rho'_y - \eta)(p-q)| \\
&\leq K_0|p-q|.
\end{aligned}$$

and $\rho(y)$, $\rho'(y)$ and $\eta(y)$ are bounded. Let us check the condition (7.11c)[20]:

$$|\widetilde{f}_1(y,t,r,p) - \widetilde{f}_1(y,t,s,p) \leq K_0|r-s|.$$

This holds since

$$|\widetilde{f}_1(y,t,r,p) - \widetilde{f}_1(y,t,s,p)| = 0.$$

Let us check the condition (7.11d)[20]:

$$|\widetilde{f}_2(y,t,r,u'(y,t)) - \widetilde{f}_2(y,t,s,u'(y,t))| \leq K_0|r-s|.$$

This holds since

$$|\widetilde{f}_2(y,t,r,u'_y(y,t)) - \widetilde{f}_2(y,t,s,u'_y(y,t))| = 0.$$

Let us check the condition (7.11e)[20]:

$$|\widetilde{f}_2(y,t,r,p) - \widetilde{f}_2(y,t,r,q)| \leq K_0|p-q|.$$

This holds since

$$
\begin{aligned}
&|\widetilde{f}_2(y,t,r,p) - \widetilde{f}_2(y,t,r,q)| \\
&= \rho'_y p - \eta p + \rho'_y f'_y - R(t)\eta f'_y + R'(t) f \\
&\quad -(\rho'_y q - \eta q + \rho'_y f'_y - \eta f'_y - R(t)\eta f'_y + R'(t) f) \\
&= (\rho'_y - \eta)(p-q) \le K_0 |p-q|
\end{aligned}
$$

and $\rho'(y)$ and $\eta(y)$ are bounded. Theorem 2.7.1 follows from Theorem 7.1 in [20] which shows our method is strongly convergent.

2.8 Numerical experiments

2.8.1 Matching the Black-Scholes price

In these experiments we calculated the price for a put option in the classical Black-Scholes model using our method. We considered the payoff $(K - S(T))^+$. We found the solution of the equation (2.2.4) with $f(x) = (K - x)^+$ using our method and compared it with the solution given by the Black-Scholes formula. Table (2.1) shows the error

$$E = \sup_x |V(x,0) - V_{BS}(x,0)|. \tag{2.8.1}$$

Here, V is the solution of (2.2.4) obtained by our method and V_{BS} is the exact solution of (2.2.4) given by the Black-Scholes formula. In addition, the table shows where the maximum (2.8.1) is achieved. In this table, N is the parameter introduced in Section 3 representing the rate of discretization in x. N_t is the number of points along the time axis. We used $\sigma = 0.3$, $T = 0.1$, $L = 10$, $r = 0$, $d = 0$ and $K = 1$. For r=0.025 It is shown that the error is decreasing as we use more points along the

Table 2.1: Error of calculation of the put option for r=0.

N,N_t	E
20, 20	0.003902114
40, 40	0.006529201
80, 80	0.007018533
160, 160	0.003593509
320, 320	0.0003050627
640, 640	7.043937e-05

state price and time axes. Note that the largest error mostly occurs near the point where the final condition becomes zero. Figure **??** shows the comparison between the Black-Scholes and numerical solutions.

Table 2.2: Error of calculation of the put option for r=0.025

N,N_t	E
20, 20	0.003319792
40, 40	0.005971542
80, 80	0.006657621
160, 160	0.003484265
320, 320	0.001151376
640, 640	0.0003031325

Table 2.3: Error of calculation of the put option for r=0.05

N,N_t	E
20, 20	0.002830843
40, 40	0.00547276
80, 80	0.006323301
160, 160	0.003382576
320, 320	0.001133573
640, 640	0.0003015444

2.8.2 Experiments for state-dependent volatility

In these experiments we approximate a certain function using our approach. We will select the function $U(y,t)$, $d(y)$, and $\sigma(y)$, to satisfy the equation (2.2.5) exactly, and will estimate the error for our numerical method. Let us select

$$\begin{aligned}
&U(y,t) = e^{t-T}(L^2 - y^2),\\
&f(y) = L^2 - y^2,\\
&R(t) = R_0(t) = 1,\\
&\sigma(y) = \sqrt{(L^2 + 0.9(\sin(y))^2 y)},\\
&d(y) = -(0.9(\sin(y))^2 + y)/2 + (\sigma(y))^2/2,\\
&r = 0, \quad L = 1, \quad T = 1.
\end{aligned}$$

It can be verified directly that (2.2.5) is satisfied. Table (2.4) shows the error

$$E = \sup_y |V(y,0) - U(y,0)|.$$

Here, V is the solution of (2.2.4) obtained by our method and $U(y,0)$ is obtained by putting in the appropriate values into the chosen $U(y,t)$. In this table, N is the parameter introduced in Section 3 representing the rate of discretization in y and N_t is the parameter introduced similarly to the previous section. Again, it is shown that the maximum error is decreasing as we use more points along the state price and time axes. Note that the biggest error mostly occurs near the boundary point. Figure 2.3 shows the comparison between the exact and numerical solutions.

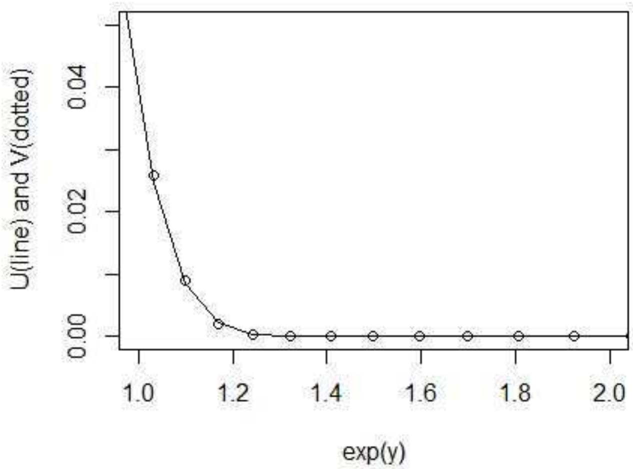

Figure 2.2: Comparison of exact solution U and numerical solution V

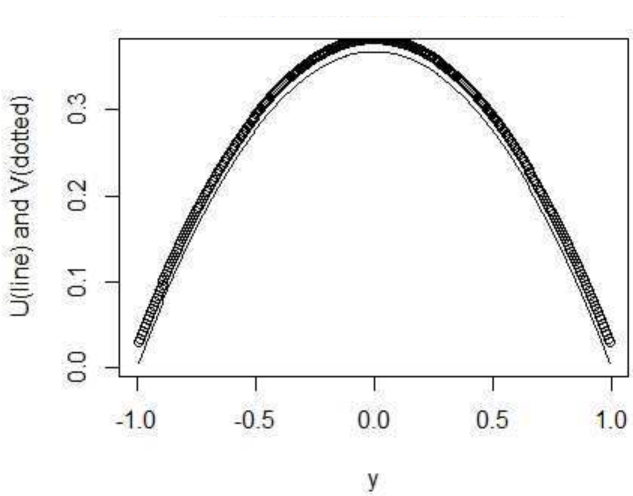

Figure 2.3: Comparison of exact function U and numerical solution V for the case of non-constant σ

Table 2.4: Error of calculation of the case of state-dependent volatility.

N,N_t	E
20, 20	8.60202
40, 40	0.1838133
80, 80	0.09596427
160, 160	0.04898591
320, 320	0.02474154
640, 640	0.01243255

2.9 Conclusion

In this chapter we presented a method for solving the Black-Scholes equation considering the volatility is a function rather than a constant. We used a Galerkin's method to discretize the boundary value problem with respect to the sate variable. As a result, the boundary value problem for the Black-Scholes equation is reduced to a vector ordinary differential equation. ODE. We suggested two methods of solving the ODE: the exact method and Crank-Nicolson method. Various numerical experiments were conducted which suggest the method's convergence with the solution.

Chapter 3

A modification of Galerkin's method for American option pricing

Abstract

We study the problem of numerical solutions of nonlinear parabolic equations used for pricing of American options by power penalty method. This method allow to approximate the solution of the Stefan problem for linear Black-Scholes parabolic equation with moving boundary by solutions of nonlinear problems with equation with fixed boundary. The nonlinear term represent the penalty for violating the Stefan boundary condition. We consider the case of state dependent coefficients the parabolic equation, including the volatility. We develop a Galerkin's finite element method to solve the arising problem. More specifically, we discretize the system and build new special basis functions to approximate the solution. We establish convergence of the proposed method and numerical results are reported to show the proposed method is accurate and efficient.

3.1 Introduction for Chapter 3

We study the problem of numerical solutions of nonlinear parabolic equations used for pricing of American options.

Mathematical methods of European and American option pricing were widely studied; see the review of these methods in Wilmott, Dewynne, and Howison (1993).

There are several competing methods in the literature. In particular, it is common to use Monte-Carlo simulation that allows to find a price in a particular state point without solving partial differential equations (see Broadie, Glasserman, and Jain (1997), Rogers, Shi (1995). However, the approach based PDEs allows more precise estimation of the entire value function.

For European options, Wang (2004) considered numerical solution of parabolic Black-Scholes equation based on a so-called fitted finite volume method. This was a discretization method based on a finite volume formulation of the problem coupled with a fitted local approximation to the solution and on an implicit time-stepping technique. The local approximation is determined by a set of two-point boundary value problems; this fitting technique is based on the idea proposed by Allen and Southwell (1955) for convection-diffusion equations. Overall, this method represents a special case of Petrov-Galerkin's method and increases the accuracy of calculation of the price comparing with the straightforward finite-difference method. In Angermann and Wang (2004), Angermann and Wang (2007), and related papers Wang, Yang, and Teo (2006), Wang and Yang (2008), Zhang and Wang (2012), some regularity problems were solved.

The pricing of American options is more challenging since it requires solution of a Stefan problem for parabolic Black-Scholes equations with moving boundaries.

Wang, Yang, and Teo (2006) and Angermann and Wang (2007) extended the fitted finite volume method to American options. Their approach allowed to approximate the solution $\tilde{V}$ of the Stefan problem for a linear Black-Scholes parabolic equation with moving boundary by the solution $V_{\lambda,k}$ by solutions of nonlinear problems with equation with fixed boundary with the nonlinear penalty term

$$\lambda(\tilde{V} - V_{\lambda,k})_+^{-1/k} \tag{3.1.1}$$

representing the penalty for violating the Stefan boundary condition. We use the notation $(x)^+ = \max(x, 0)$.

For the case where $k = 1$, the penalty (3.1.1) is a piecewise linear function used in Forsyth and Vetzal (2002), Bensoussan and Lions (1982), Glowinski (1984). It was shown in Bensoussan and Lions (1982) that the error rate is $\|\tilde{V} - V_{\lambda,k}\|$ is of order $O(\lambda^{-1/2})$.

Wang, Yang, and Teo (2006) analysed convergence of the solutions of penalised Black-Scholes equation to the solution of Stefan problem defining the price of the American option. They showed that the error rate is $\|\tilde{V} - V_{\lambda,k}\|$ is of order $O(\lambda^{-k/2})$ for some Sobolev type norm. In Wang, Yang, and Teo (2006) and Angermann and Wang (2007) for solution of nonlinear penalised equations for parabolic equations with time-depending coefficients was studied based on the fitted finite volume method which is a modification of Petrov-Galerkin's method. It was shown therein that the method ensures convergence of discretized solution $V_{\lambda,N}$ to V_λ for any $\lambda > 0$, where V_N is the solution using N basis functions for Petrov-Galerkin's setting.

In this chapter, we develop this approach and propose a Galerkin's finite element method for numerical solution of nonlinear penalised equations used for approximation of the Stefan problem for the option prices for the case of state dependent coefficients of the parabolic equation, including the volatility. More specifically, we discretize the system and build new special basis functions to approximate the solution. We establish convergence of the proposed method for the case where $k \leq 1$. Numerical results are show that the proposed method is accurate and efficient.

The novelty is to use the Galerkin's method for the nonlinear penalised parabolic equation using special basis functions using the state-dependent functions for volatility and dividends. Each basis function can be derived using the given parameters. Option value at any point along the state variable can be found by interpolating two basis functions. For linear equations, similar approach was used in Dokuchaev, Zhou, and Wang (2021) (See Chapter 2).

The chaper is organized as follows. In Section 2 we set up the problem by transforming the original Black-Scholes equation. In Section 3 we set up basis functions to apply Galerkin's method. In Section 4 we introduce weak formulation of the problem as well as the main theorem that is important for solving the problem. In Section 5 we prove the theorem. In Section 6 we review some methods of solution: exact method and Crank-Nicolson's method. In Section 7 we prove the convergence of the method. In Section 8 we test the method numerically.

3.2 Problem setting

We will consider a model of a stock price $S(t)$ described by the following stochastic Ito equation

$$dS(t) = S(t)(rdt + \sigma(S(t), t)dw(t)), \quad t > 0, \tag{3.2.1}$$

where $w(t)$ is a Wiener process, r is a risk-free rate, σ is the volatility of this stock. Assume that there are dividends $D(S(t), t)$ on the stock. The pricing problem can be formulated as follows: Let a random variable X represent a payoff of a financial option. If $X = f(S(T))$ for some function $f : \mathbf{R} \to \mathbf{R}$ then the option is said to be of a European type. In this case, the fair option price at time t is $P(t) = e^{-r(T-t)}E\{f(S(T)|S(t)\}$, i.e. it can be calculated as the conditional expectation.

We assume that one of the following conditions is satisfied:

1. $r = 0$

2. $f(x) = (x - K)^+$ or $f(x) = (K - x)^+$, where $K > 0$ is given, for call and put options respectively.

It can be shown, using Ito's Lemma, that

$$P(t) = V(S(t), t),$$

where $V(x,t)$ is a solution of the boundary value problem for the following parabolic equation:

$$\begin{aligned}
&\frac{\partial V}{\partial t}(x,t)+\frac{1}{2}\sigma(x)^2x^2\frac{\partial^2 V}{\partial x^2}(x,t)+x(r-d(x))\frac{\partial V}{\partial x}(x,t)-rV(x,t)=0,:\\
&V(x,T)=f(x), \quad x>0,\\
&\lim_{x\to\infty}\frac{V(x,t)}{f(x)}=1, \quad 0<t<T,\\
&V(0,t)=f(0), \quad 0<t<T. \qquad (3.2.2)
\end{aligned}$$

The equations of this type are also called Black-Scholes equations. Here, $x\in(0,\infty)$, $t\in[0,T)$, $\sigma(x)$ represents the volatility coefficient, $d(x)$ represents the dividend rate, r represents the risk-free bank rate, $T>0$ is the terminal time. We assume that $d(x)$ is a bounded function and that $\sigma(x)$ is a bounded function with a bounded first derivative. In this paper we consider volatility $\sigma(x)$ as a function of underlying stock price x rather than a constant.

In this paper our focus is on American options rather than European. We consider the following linear complementarity problem (the Stefan problem):

$$LV(x,t):=-\frac{\partial V}{\partial t}-\frac{1}{2}\sigma^2(x)x^2\frac{\partial^2 V}{\partial x^2}-x\left(r-d(x)\right)\frac{\partial V}{\partial x}+r(t)V\geq 0, \qquad (3.2.3)$$

$$V(x,t)-f(x)\geq 0, \qquad (3.2.4)$$

$$LV(x,t)\cdot[V(x,t)-f(x)]=0, \quad V(x,T)=f(x). \qquad (3.2.5)$$

To solve the arising problem we propose to use the so-called power penalty method where the Stefan problem is replaced by the problem

$$\begin{aligned}
&LV(x,t)+\mathcal{N}_{\lambda,k}(f(x)-V(x,t))\\
&=-\frac{\partial V}{\partial t}-\frac{1}{2}\sigma^2(x)x^2\frac{\partial^2 V}{\partial x^2}-x(r-d(x))\frac{\partial V}{\partial x}+r(t)V+\mathcal{N}_{\lambda,k}(f(x)-V(x,t))\\
&=0,\\
&x\in(0,\infty), \quad t\in[0,T)\\
&V(x,T)=f(x), \quad x>0,\\
&\lim_{x\to\infty}\frac{V(x,t)}{f(x)}=R_0(t), \quad 0<t<T,\\
&V(0,t)=f(0), \quad 0<t<T. \qquad (3.2.6)
\end{aligned}$$

Here

$$\begin{aligned}
&\mathcal{N}_{\lambda,k}(x)=\lambda[x]_+^{1/k}, \quad x<C\\
&\mathcal{N}_{\lambda,k}(x)=\lambda C^{1/k-1}x, \quad x>C,
\end{aligned}$$

where $\lambda>1$, $k>0$, $C>0$ are penalty parameters. This function is piecewise continuous. If $k=1$ this function is piecewise linear. For large $C>0$, this function approximates penalty $\mathcal{N}_{\lambda,k}(x)\equiv\lambda[x]_+^{1/k}$, $x>0$.

We change variables such that

$$x = e^y, \quad V(x,t) = \hat{V}(\ln x, t). \tag{3.2.7}$$

Then

$$\frac{\partial V}{\partial x} = \frac{\partial \hat{V}}{\partial y} e^{-y}, \quad \frac{\partial^2 V}{\partial x^2} = e^{-2y}\left(\frac{\partial^2 \hat{V}}{\partial y^2} - \frac{\partial V}{\partial y}\right).$$

Equation (3.2.6) takes the form

$$\begin{aligned}
&\frac{\partial \hat{V}}{\partial t}(y,t) + \frac{1}{2}\sigma(y)^2\Big(\frac{\partial^2 \hat{V}}{\partial y^2}(y,t) \\
&\quad -\frac{\partial \hat{V}}{\partial y}(y,t)\Big) + (r - d(x))\frac{\partial \hat{V}}{\partial y}(y,t) - r\hat{V}(y,t) + \mathcal{N}_{\lambda,k}[f(y) - \hat{V}(y,t)] = 0, \\
&\hat{V}(y,T) = f(y), \quad x \in (-L, L), \\
&\hat{V}(L,t) = f(L), \quad \hat{V}(-L,t) = f(-L), \quad t \in [0,T).
\end{aligned} \tag{3.2.8}$$

Note that we replaced $f(e^y)$ and $\sigma(e^y)$ with $f(y)$ and $\sigma(y)$ respectively, to keep notations short. Next, we use the substitution:

$$\hat{v}(y,t) = e^{-r(t-T)}\hat{V}(y,t). \tag{3.2.9}$$

Let $R(t) := e^{r(t-T)}$. Clearly,

$$\begin{aligned}
\frac{\partial \hat{V}}{\partial t}(y,t) &= re^{r(t-T)}\hat{v}(y,t) + \frac{\partial \hat{v}}{\partial t}(y,t)e^{r(t-T)}, \\
\hat{V}(y,T) &= e^{r(T-T)}f(y) = f(y), \\
\hat{v}(L,t) &= R(t)f(L), \\
\hat{v}(-L,t) &= R(t)f(-L).
\end{aligned}$$

Hence (3.2.8) becomes

$$\begin{aligned}
re^{r(t-T)}\hat{v}(y,t) + \frac{\partial \hat{v}}{\partial t}(y,t)e^{r(t-T)} + \frac{1}{2}e^{r(t-T)}\sigma(y)^2\Big(\frac{\partial^2 \hat{v}}{\partial y^2}(y,t) - \frac{\partial \hat{v}}{\partial y}(y,t)\Big) \\
+e^{r(t-T)}(r - d(y))\frac{\partial \hat{v}}{\partial y}(y,t) - re^{r(t-T)}\hat{v}(y,t) + \mathcal{N}_{\lambda,k}[f(x) - e^{r(t-T)}\hat{v}(y,t)] = 0.
\end{aligned}$$

Simplifying the above, we obtain that

$$\begin{aligned}
&\frac{\partial \hat{v}}{\partial t}(y,t) + \frac{1}{2}\sigma(y)^2\frac{\partial^2 \hat{v}}{\partial y^2}(y,t) + (r - d(y) - \frac{1}{2}\sigma^2(y))\frac{\partial \hat{v}}{\partial y}(y,t) \\
&\quad +e^{-r(t-T)}\mathcal{N}_{\lambda,k}[f(x) - e^{r(t-T)}\hat{v}(y,t)] = 0, \\
&\hat{v}(y,T) = f(y), \quad \hat{v}(L,t) = R(t)f(L), \quad \hat{v}(-L,t) = R(t)f(-L).
\end{aligned} \tag{3.2.10}$$

For brevity, we will rewrite this as

$$\frac{\partial \hat{v}}{\partial t}(y,t) + \rho(y)\frac{\partial^2 \hat{v}}{\partial y^2}(y,t) + \eta(y)\frac{\partial \hat{v}}{\partial y}(y,t) + e^{-r(t-T)}\mathcal{N}_{\lambda,k}[f(y) - e^{r(t-T)}\hat{v}(y,t))] = 0,$$

$$\hat{v}(y,T) = f(y), \quad \hat{v}(L,t) = R(t)f(L), \quad \hat{v}(-L,t) = R(t)f(-L). \tag{3.2.11}$$

Here,

$$\rho(y) = \frac{1}{2}\sigma(y)^2, \quad \eta(y) = r - d(y) - \frac{1}{2}\sigma^2(y).$$

We assume that in (3.2.6), $|f(x)| \leq c(1+|x|)$, for some constant $c > 0$ and we look for solution $|\hat{v}(x,t)| \leq C_1(1+|x|)$. Let us introduce a differential operator A such that

$$A\hat{v} = \rho\frac{\partial^2 \hat{v}}{\partial y^2} + \eta\frac{\partial \hat{v}}{\partial y}.$$

Let

$$G(\hat{v},y,t) = e^{-r(t-T)}\mathcal{N}_{\lambda,k}\left(f(y) - e^{r(t-T)}\hat{v}(y,t)\right).$$

We consider (3.2.10) for $y \in (-L,L)$, where $L > 0$ is a sufficiently large constant. We consider the boundary problem

$$\begin{aligned}
&\frac{\partial \hat{v}}{\partial t} + A\hat{v} + G(\hat{v},y,t) = 0, \quad \hat{v}(y,T) = f(y), \quad y \in \mathbf{R},\\
&\hat{v}(-L,t) = R(t)f(-L),\\
&\hat{v}(L,t) = R(t)f(L).
\end{aligned}$$

We use the substitution

$$v(y,t) = \hat{v}(y,t) - R(t)f(y). \tag{3.2.12}$$

Then,

$$\frac{\partial \hat{v}}{\partial t} = \frac{\partial v}{\partial t} + f(y)\frac{dR}{dt}. \tag{3.2.13}$$

This gives the following problem:

$$\begin{aligned}
&\frac{\partial v}{\partial t} = -Av - R(t)Af - \frac{dR}{dt}(t)f(y) - G(v+Rf,y,t),\\
&v(y,T) = 0,\\
&v(-L,t) = v(L,t) = 0.
\end{aligned} \tag{3.2.14}$$

Here $y \in D$, $D = (-L,L)$,

$$\begin{aligned}
G(v+R(t)f,y,t) \quad &= e^{-r(t-T)}\mathcal{N}_{\lambda,k}\left(f(y) - e^{r(t-T)}[R(t)f(y)+v(y,t)]\right)\\
&= e^{-r(t-T)}\mathcal{N}_{\lambda,k}\left(f(y) - f(y) - e^{r(t-T)}v(y,t)\right)\\
&= e^{-r(t-T)}\mathcal{N}_{\lambda,k}\left(-e^{r(t-T)}v(y,t)\right)_+^{1/k}.
\end{aligned}$$

3.3 Basis functions

Let $\{y_k\}_{k=0}^{N+1} \subset D$ be selected such that $-L = y_0 < y_1 < y_k < ... < y_{N+1} = L$. Let us introduce the basis functions $\phi(y) = \phi_k(y)$, for $k = 0, .., N+1$ that satisfy $A\phi = 0$ on $(y_{k-1}, y_k) \cup (y_k, y_{k+1})$ and the following conditions:

1. $\phi_k(y_k) = 1$, $\phi_k(y_{k-1}) = \phi_k(y_{k+1}) = 0$.
2. $\phi_k(y) \geq 0$.
3. $\phi_k(y) = 0$ for $y \notin (y_{k-1}, y_{k+1})$.
4. $\phi_y|_{[y_{k-1},y_k]} \in C^2([y_{k-1}, y_k])$; $\phi_y|_{[y_k,y_{k+1}]} \in C^2([y_k, y_{k+1}])$. Here, C^2 is the space of twice differentiable functions.
5. $\phi_k|_{[y_{k-1},y_{k+1}]} \in W^1_\infty([y_{k-1}, y_{k+1}])$. Here, W^1_∞ is a Sobolev space of functions with bounded first derivative.
6. $A\phi_k(y) = 0$ for $y \in (y_{k-1}, y_k) \cup (y_k, y_{k+1})$.
7. $\phi_k(y) + \phi_{k+1}(y) = 1$ for $y \in [y_k, y_{k+1}]$.

Diagrams of these functions form intersecting deformed triangles on the $\phi - x$ plane. To find $\phi' s$, we need to solve the following equation:

$$A\phi_k(y) = \rho(y)\frac{\partial^2 \phi_k}{\partial y^2}(y) + \eta(y)\frac{\partial \phi_k}{\partial y}(y) = 0. \tag{3.3.1}$$

We consider two cases: (1) when $y \in [y_{k-1}, y_k]$ and (2) $y \in [y_k, y_{k+1}]$. The boundary conditions are as follows:

$$\phi_k(y_{k-1}) = 0, \quad \phi_k(y_k) = 1, \quad \phi_k(y_{k+1}) = 0. \tag{3.3.2}$$

Let $\gamma(y) = \frac{\partial \phi_k}{\partial y}(y)$. The equation (3.3.1) becomes

$$\rho(y)\frac{\partial \gamma}{\partial y}(y) + \eta(y)\gamma(y) = 0.$$

This equation can be solved exactly using the integrating factor method. It can be written as

$$\frac{\partial \gamma}{\partial y}(y) + \frac{\eta(y)}{\rho(y)}\gamma(y) = 0.$$

Let

$$\omega_{k,-}(y) = \int_{y_{k-1}}^{y} \frac{\eta(x)}{\rho(x)} dx + c.$$

Let integrating factor be $\mu = C_1 e^{\omega_{k,-}(y)}$. Then

$$\gamma_k(y) = C_1 e^{-\omega_{k,-}(y)}.$$

To find ϕ_k we integrate γ:

$$\phi_k(y) = \int_{y_{k-1}}^{y} \gamma(x)dx + C_2.$$

Next, we check the initial conditions (3.3.2):

$$\phi_k(y_{k-1}) = \int_{y_{k-1}}^{y_{k-1}} \gamma(x)dx + C_2 = 0,$$

$$C_2 = 0,$$

$$\phi_k(y_k) = \int_{y_{k-1}}^{y_k} \gamma(x)dx = C_1 \int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)}dx = 1.$$

Hence,

$$C_1 = \frac{1}{\int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)}dx}.$$

Thus, we determined $\phi_k(y)$ for $y \in (y_{k-1}, y_k)$, and

$$\phi_k(y) = \frac{1}{\int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)}dx} \int_{y_{k-1}}^{y} e^{-\omega_{k,-}(x)}dx.$$

When the discretization step is small enough, this function will be increasing from y_{k-1} to y_k. Similarly, we find $\phi_k(y)$ for $y \in (y_k, y_{k+1})$. Let

$$\omega_{k,+}(y) = \int_{y_k}^{y} \frac{\eta(x)}{\rho(x)}dx + c.$$

Let $\mu = C_1 e^{\omega_{k,+}(y)}$ and $\gamma_k(y) = C_1 e^{-\omega_{k,+}(y)}$. To find $\phi_k(y)$, we integrate $\gamma(x)$:

$$\phi_k(y) = \int_{y_k}^{y} \gamma(x)dx + C_2.$$

Next, we match the initial conditions (3.3.2) to find C_1 and C_2 as follows::

$$\phi_k(y_k) = \int_{y_k}^{y_k} \gamma(x)dx + C_2 = 1,$$

$$C_2 = 1,$$

$$\phi_k(y_{k+1}) = \int_{y_k}^{y_{k+1}} \gamma(x)dx + 1 = C_1 \int_{y_k}^{y_{k+1}} e^{-\omega_{k,+}(x)}dx + 1 = 0,$$

$$C_1 = \frac{-1}{\int_{y_k}^{y_{k+1}} e^{-\omega_{k,+}(x)}dx}.$$

Thus, we determined $\phi_k(y)$ for $y \in (y_k, y_{k+1})$, and

$$\phi_k(y) = \frac{-1}{\int_{y_k}^{y_{k+1}} e^{-\omega(x)_{k,+}}dx} \int_{y_k}^{y} e^{-\omega_{k,+}(x)}dx + 1.$$

We will use these basis functions to discretize the system. Any point $V(y)$ can be represented as a linear combination of two appropriate basis functions in the domain: $v_k(t)\phi_k(y) + v_{k+1}(t)\phi_{k+1}(y)$ if $y \in [y_k, y_{k+1}]$.

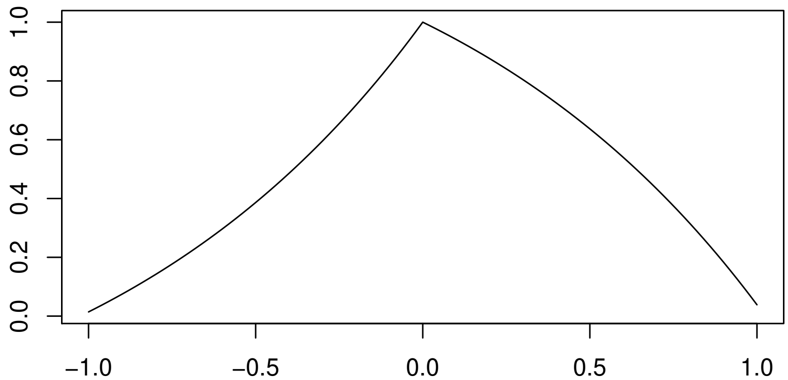

Figure 3.1: Basis function $\phi_k(y)$ for $y_{k-1} = -1$, $y_k = 0$, $y_{k+1} = 1$, $\rho = 0.045$, and $\eta = -0.045$

.

3.4 ODE implied by the Galerkin Method

Using ϕ_m from the previous section we can approximate v in (3.2.14) as $V_N(y,t) = \sum_{k=1}^{N} v_k(t)\phi_k(y)$. The boundary conditions can be interpreted as $v_0(t) = 0$ and $v_{N+1}(t) = 0$. Let S_N be the span of $\{\phi_k\}_{k=1,\dots,N}$. Let us consider a bilinear mapping $a : H_0^1(D) \times H_0^1(D) \to \mathbf{R}$ such that $(Au, w)_{L_2(D)} = a(u, w)$ for all $u, w \in H_0^1(D) \cap W_2^2(D)$. In a weak form the equation (3.2.14) is

$$(\frac{du}{dt}, w)_{L_2(D)} = -a(u, w) - a(f, w) - (G, w)_{L_2(D)} \qquad (3.4.1)$$

for all $w \in H_0^1(D)$, where $H_0^1(D)$ is the space of functions belonging to $L_2(D)$ together with their first derivatives and such that they vanish at ∂D. Following the Galerkin Method, we look for v_k such that

$$(V_N', w)_{L_2(D)} = -a(V_N, w) - a(f_N, w) - (G_N, w)_{L_2(D)} \qquad (3.4.2)$$

for all $w \in S_N$, $m = 1, .., N$. Formally, equation (3.2.14) can be presented as

$$\sum_{k=0}^{N+1} v_k'(t)\phi_k(y) = -\sum_{k=0}^{N+1} v_k(t)A\phi_k(y) - \sum_{k=0}^{N+1} \xi_k A\phi_k(y)$$
$$-e^{-r(t-T)}\mathcal{N}_{\lambda,k}[-e^{r(t-T)}\sum_{k=0}^{N+1} v_k\phi_k(y)].$$

Here, $\xi_k = f(y_k)$.

Since $v_0 = v_{N+1} = 0$, we can rewrite the above as

$$\sum_{k=1}^{N} v'_k(t)\phi_k(y) = -\sum_{k=1}^{N} v_k(t)A\phi_k(y) - \sum_{k=0}^{N+1} \xi_k A\phi_k(y)$$
$$-e^{-r(t-T)}\mathcal{N}_{\lambda,k}[\ -e^{r(t-T)}\sum_{k=0}^{N+1} v_k\phi_k(y)].$$

Multiplying by ϕ_m, for $m = 1, ..., N$ and integration by dy gives

$$\int_{\mathbb{R}} \sum_{k=1}^{N} v'_k(t)\phi_k(y)\phi_m(y)dy = -\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t)A\phi_k(y)\phi_m(y)dy$$
$$-\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A\phi_k(y)\phi_m(y)dy$$
$$-\int_{\mathbb{R}} e^{-r(t-T)}\mathcal{N}_{\lambda,k}[-e^{r(t-T)}\sum_{k=0}^{N+1} v_k\phi_k(y)]\phi_m(y)dy. \tag{3.4.3}$$

Using the above we can construct the following system.

Theorem 3.4.1

$$Mv'(t) = Bv(t) + B\xi + \zeta + \kappa, \tag{3.4.4}$$

where

$$v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_{N-1}(t) \\ v_N(t) \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n-1} \\ \xi_n \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_1 \\ 0 \\ \vdots \\ 0 \\ \zeta_N \end{pmatrix}, \kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \vdots \\ \kappa_{N-1} \\ \kappa_N \end{pmatrix}, \tag{3.4.5}$$

$$M = \begin{pmatrix} \mu_{1,1} & \mu_{1,2} & 0 & 0 & \dots & 0 \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & 0 & \dots & 0 \\ 0 & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_{N-1,N-2} & \mu_{N-1,N-1} & \mu_{N-1,N} \\ 0 & 0 & \dots & 0 & \mu_{N-1,N} & \mu_{N,N} \end{pmatrix}, \tag{3.4.6}$$

$$B = \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & 0 & 0 & \dots & 0 \\ \beta_{2,1} & \beta_{2,2} & \beta_{2,3} & 0 & \dots & 0 \\ 0 & \beta_{3,2} & \beta_{3,3} & \beta_{3,4} & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_{N-1,N-2} & \beta_{N-1,N-1} & \beta_{N-1,N} \\ 0 & 0 & \dots & 0 & \beta_{N-1,N} & \beta_{N,N} \end{pmatrix}, \tag{3.4.7}$$

The components of M are

$$\mu_{m,m-1} = \int_{y_{m-1}}^{y_m} \phi_{m-1}(y)\phi_m(y)dy,$$
$$\mu_{m,m} = \int_{y_{m-1}}^{y_{m+1}} \phi_m^2(y)dy,$$
$$\mu_{m,m+1} = \int_{y_m}^{y_{m+1}} \phi_{m+1}(y)\phi_m(y)dy. \tag{3.4.8}$$

The components of B are

$$\beta_{m,m-1} = \frac{\sigma^2(y_m)}{2}(\phi'_{m-1}(y_m-0),$$
$$\beta_{m,m} = -\frac{\sigma^2(y_m)}{2}(\phi'_m(y_m+0)-\phi'_m(y_m-0)),$$
$$\beta_{m,m+1} = -\frac{\sigma^2(y_m)}{2}(\phi'_{m+1}(y_m+0)). \tag{3.4.9}$$

The components ξ, ζ and κ are

$$\xi_k = f(y_k),$$
$$\zeta_1(t) = \beta_{1,0}\xi_0 = \frac{\sigma^2(y_1)}{2}\phi'_0(y_1-0))\xi_0,$$
$$\zeta_N(t) = \beta_{N,N+1}\xi_{N+1} = \frac{\sigma^2(y_N)}{2}\phi'_{N+1}(y_N+0)\xi_{N+1},$$
$$\kappa_m = -e^{-r(t-T)}\int_{y_{m-1}}^{y_{m+1}} \mathcal{N}_{\lambda,k}[-e^{r(t-T)}\sum_{k=m-1}^{m+1} v_k\phi_k(y)]\phi_m(y)dy. \tag{3.4.10}$$

3.5 Proof of Theorem 3.4.1

3.5.1 Finding the components for M in (3.4.4)

Let us look at the LHS of (3.4.3). If $k < m-1$ or $k > m+1$ then $\phi_k\phi_m = 0$. Therefore, for $m = 2, ..., N-1$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}}\sum_{k=1}^{N} v'_k(t)\phi_k(y)\phi_m dy \\
&= \frac{dv_{m-1}(t)}{dt}\int_{y_{m-1}}^{y_m}\phi_{m-1}(y)\phi_m(y)dy + \frac{dv_m(t)}{dt}\int_{y_{m-1}}^{y_{m+1}}\phi_m^2(y)dy \\
&\quad + \frac{dv_{m+1}(t)}{dt}\int_{y_m}^{y_{m+1}}\phi_{m+1}(y)\phi_m(y)dy \\
&= \mu_{m-1}\frac{dv_{m-1}(t)}{dt} + \mu_m\frac{dv_m(t)}{dt} + \mu_{m+1}\frac{dv_{m+1}(t)}{dt}.
\end{aligned} \tag{3.5.1}$$

Let us consider boundary cases $m = 1$ and $m = N$. For the case where $m = 1$,

$$\frac{dv_0}{dt}\int_{y_0}^{y_1}\phi_0(y)\phi_1(y)dy + \frac{dv_1}{dt}\int_{y_0}^{y_2}\phi_1^2(y)dy + \frac{dv_2}{dt}\int_{y_1}^{y_2}\phi_2(y)\phi_1(y)dy$$
$$= 0 + \mu_1\frac{dv_1}{dt} + \mu_2\frac{dv_2}{dt}, \quad (3.5.2)$$

since $\frac{dv_0}{dt} = 0$. Similarly, for the case where $m = N$ we have that

$$\frac{dv_{N-1}}{dt}\int_{y_{N-1}}^{y_N}\phi_{N-1}(y)\phi_N(y)dy + \frac{dv_N}{dt}\int_{y_{N-1}}^{y_{N+1}}\phi_N^2(y)dy$$
$$+\frac{dv_{N+1}}{dt}\int_{y_N}^{y_{N+1}}\phi_{N+1}(y)\phi_N(y)dy$$
$$= \mu_{N-1}\frac{dv_{N-1}}{dt} + \mu_N\frac{dv_N}{dt} + 0. \quad (3.5.3)$$

Thus, we found the components for M in the theorem.

3.5.2 Finding the components of B in (3.4.4).

We consider the term

$$\int_{-L}^{L}\sum_{k=1}^{N} v_k(t)A\phi_k(y)\phi_m(y)dy = \sum_{k=1}^{N} v_k\int_{-L}^{L}A\phi_k(y)\phi_m(y)dy$$

in (3.4.3). Let

$$(A\phi_k, \phi_m)_{L_2(D)} = \int_{-L}^{L}A\phi_k(y)\phi_m(y)dy.$$

Let

$$\rho(y) = \frac{\sigma(y)^2}{2}, \quad \eta(y) = (r - d(y) - \frac{1}{2}\sigma(y)^2).$$

We have that

$$\begin{aligned} A\phi_k(y) &= \frac{1}{2}\sigma(y)^2\phi_k''(y) + (r - d(y) - \frac{1}{2}\sigma(y)^2)\phi_k'(y) \\ &= \rho(y)\phi_k''(y) + \eta(y)\phi_k'(y) \\ &= (\rho(y)\phi_k'(y))' - \rho(y)'\phi_k'(y) + \eta(y)\phi_k'(y). \end{aligned}$$

By the definitions,

$$\begin{aligned} (A\phi_k, \phi_m)_{L_2(D)} &= \int_{-L}^{L}(\rho(y)\phi_k'(y))'\phi_m(y)dy - \int_{-L}^{L}\rho(y)'\phi_k'(y)\phi_m(y)dy \\ &\quad + \int_{-L}^{L}\eta(y)\phi_k'(y)\phi_m(y)dy \end{aligned}$$

$$= \rho(y)\phi_k'(y)\phi_m(y)\Big|_{-L}^{L} - \int_{-L}^{L} \rho(y)\phi_k'(y)\phi_m'(y)dy$$

$$- \int_{-L}^{L} \rho(y)'\phi_k'(y)\phi_m(y)dy + \int_{-L}^{L} \eta(y)\phi_k'(y)\phi_m(y)dy$$

$$= 0 - \int_{-L}^{L} \rho(y)\phi_k'(y)\phi_m'(y)dy - \int_{-L}^{L} \rho(y)'\phi_k'(y)\phi_m(y)dy$$

$$+ \int_{-L}^{L} \eta(y)\phi_k'(y)\phi_m(y)dy.$$

For $k \le m-2$ or $k \ge m+2$ we have $(A\phi_k, \phi_m)_{L_2(D)} = 0$, since for $y \le y_{m-1}$ or $y \ge y_{m+1}$, $\phi_m(y) = \phi_m'(y) = 0$ and for $y \le y_{k-1}$ or $y \ge y_{k+1}$, $\phi_k(y) = \phi_k'(y) = 0$ by definitions. It follows that we need to consider three cases: $k = m-1$, $k = m$, and $k = m+1$. First, let us consider $k = m-1$:

$$(A\phi_{m-1}, \phi_m)_{L_2(D)} = - \int_{-L}^{L} \rho(y)\phi_{m-1}'(y)\phi_m'(y)dy - \int_{-L}^{L} \rho(y)'\phi_{m-1}'(y)\phi_m(y)dy + \int_{-L}^{L} \eta(y)\phi_{m-1}'(y)\phi_m(y)dy.$$

Since the $\phi_{m-1}(y)$ and $\phi_m(y)$ are multiplied inside the integrals, the boundaries for integrals will be (y_{m-1}, y_m) since for $y \le y_{m-1}$, $\phi_m(y) = \phi_m'(y) = 0$ and for $y \ge y_m$, $\phi_{m-1}(y) = \phi_{m-1}'(y) = 0$. Let

$$J = \int_{y_{m-1}}^{y_m} \rho(y)\phi_{m-1}'(y)\phi_m'(y)dy.$$

Then

$$(A\phi_{m-1}, \phi_m)_{L_2(D)} = -J - \int_{y_{m-1}}^{y_m} \rho(y)'\phi_{m-1}'(y)\phi_m(y)dy + \int_{y_{m-1}}^{y_m} \eta(y)\phi_{m-1}'(y)\phi_m(y)dy.$$

We have that

$$\begin{aligned} J &= \int_{y_{m-1}}^{y_m} \rho(y)\phi_{m-1}'(y)\phi_m'(y)dy \\ &= \phi_m(y)\rho(y)\phi_{m-1}'(y)\Big|_{y_{m-1}}^{y_m} - \int_{y_{m-1}}^{y_m} (\rho(y)\phi_{m-1}(y)')'\phi_m(y)dy \\ &= \rho(y_m)\phi_{m-1}'(y_m - 0) - \int_{y_{m-1}}^{y_m} (\rho(y)\phi_{m-1}'(y))'\phi_m(y)dy. \end{aligned}$$

Substituting J back in, we obtain that

$$(A\phi_{m-1}, \phi_m)_{L_2(D)} = -\rho(y_m)\phi_{m-1}'(y_m - 0) + \int_{y_{m-1}}^{y_m} (\rho(y)\phi_{m-1}'(y))'\phi_m(y)dy$$

$$
\begin{aligned}
&-\int_{y_{m-1}}^{y_m} \rho(y)'\phi'_{m-1}(y)\phi_m(y)dy \\
&+\int_{y_{m-1}}^{y_m} \eta(y)\phi'_{m-1}(y)\phi_m(y)dy \\
=& -\rho(y_m)\phi'_{m-1}(y_m-0) \\
&+\int_{y_{m-1}}^{y_m} [(\rho(y)\phi'_{m-1}(y))'\phi_m(y) - \rho(y)'\phi'_{m-1}(y)\phi_m(y) \\
&+\eta(y)\phi'_{m-1}(y)\phi_m(y)]dy \\
=& -\rho(y_m)\phi'_{m-1}(y_m-0) + \int_{y_{m-1}}^{y_m} [(\rho(y)\phi'_{m-1}(y))' \\
&-\rho(y)'\phi'_{m-1}(y) + \eta(y)\phi'_{m-1}(y)]\phi_m(y)dy.
\end{aligned}
$$

Since $A\phi_{m-1} = (\rho(y)\phi'_{m-1})' - \rho(y)'\phi'_{m-1} + \eta(y)\phi'_{m-1} = 0$ as defined in Section 3, we obtain that

$$
(A\phi_{m-1}, \phi_m)_{L_2(D)} = -\rho(y)(y_m)\phi'_{m-1}(y_m-0) = -\beta_{m,m-1}. \tag{3.5.4}
$$

$\beta_{m,m-1}$ is the component to the left of the main diagonal in B. Now, let us consider $k = m$:

$$
\begin{aligned}
(A\phi_m, \phi_m)_{L_2(D)} \quad &= -\int_{-L}^{L} \rho(y)\phi'_m(y)\phi'_m(y)dy - \int_{-L}^{L} \rho(y)'\phi'_m(y)\phi_m(y)dy \\
&+\int_{-L}^{L} \eta(y)\phi'_m(y)\phi_m(y)dy.
\end{aligned}
$$

The boundaries for the integrals are (y_{m-1}, y_{m+1}) since for $y \leq y_{m-1}$, $\phi_m(y) = \phi'_m(y) = 0$ and for $y \geq y_{m+1}$, $\phi_m(y) = \phi'_m(y) = 0$.. We will need to consider the following:

$$
\begin{aligned}
(A\phi_m, \phi_m)_{L_2(D)} \quad &= -\int_{y_{m-1}}^{y_m} \rho(y)\phi'_m(y)\phi'_m(y)dy - \int_{y_{m-1}}^{y_m} \rho(y)'\phi'_m(y)\phi_m(y)dy \\
&+\int_{y_{m-1}}^{y_m} \eta(y)\phi'_m(y)\phi_m(y)dy - \int_{y_m}^{y_{m+1}} \rho(y)\phi'_m(y)\phi'_m(y)dy \\
&-\int_{y_m}^{y_{m+1}} \rho(y)'\phi'_m(y)\phi_m(y)dy + \int_{y_m}^{y_{m+1}} \eta(y)\phi'_m(y)\phi_m(y)dy.
\end{aligned}
$$

Let

$$
(A\phi_m, \phi_m)_{L_2(D)} = Q_1 + Q_2,
$$

such that

$$
\begin{aligned}
Q_1 \quad &= -\int_{y_{m-1}}^{y_m} \rho(y)\phi'_m(y)\phi'_m(y)dy - \int_{y_{m-1}}^{y_m} \rho(y)'\phi'_m(y)\phi_m(y)dy \\
&+\int_{y_{m-1}}^{y_m} \eta(y)\phi'_m(y)\phi_m(y)dy
\end{aligned}
$$

and

$$Q_2 \quad = -\int_{y_m}^{y_{m+1}} \rho(y)\phi_m'(y)\phi_m'(y)dy - \int_{y_m}^{y_{m+1}} \rho(y)'\phi_m'(y)\phi_m(y)dy$$
$$+\int_{y_m}^{y_{m+1}} \eta(y)\phi_m'(y)\phi_m(y)dy.$$

Let us consider R_1 first. Let

$$J_1 = \int_{y_{m-1}}^{y_m} \rho(y)\phi_m'(y)\phi_m'(y)dy.$$

Then

$$Q_1 = -J_1 - \int_{y_{m-1}}^{y_m} \rho(y)'\phi_m'(y)\phi_m(y)dy + \int_{y_{m-1}}^{y_m} \eta(y)\phi_m'(y)\phi_m(y)dy,$$

We have that

$$\begin{aligned} J_1 \quad &= \int_{y_{m-1}}^{y_m} \rho(y)\phi_m'(y)\phi_m'(y)dy \\ &= \phi_m(y)\rho(y)\phi_m'(y)\Big|_{y_{m-1}}^{y_m} - \int_{y_{m-1}}^{y_m} (\rho(y)\phi_m'(y))'\phi_m(y)dy \\ &= \rho(y_m)\phi_m'(y_m - 0) - \int_{y_{m-1}}^{y_m} (\rho(y)\phi_m'(y))'\phi_m(y)dy. \end{aligned}$$

Substituting J_1 back in, we obtain that

$$\begin{aligned} Q_1 \quad &= -\rho(y_m)\phi_m'(y_m - 0) + \int_{y_{m-1}}^{y_m} (\rho(y)\phi_m'(y))'\phi_m(y)dy \\ &\quad - \int_{y_{m-1}}^{y_m} \rho(y)'\phi_m'(y)\phi_m(y)dy + \int_{y_{m-1}}^{y_m} \eta(y)\phi_m'(y)\phi_m(y)dy \\ &= -\rho(y_m)\phi_m'(y_m - 0) + \int_{y_{m-1}}^{y_m} [(\rho(y)\phi_m'(y))'\phi_m(y) \\ &\quad -\rho(y)'\phi_m'(y)\phi_m(y) + \eta(y)\phi_m'(y)\phi_m(y)]dy \\ &= -\rho(y_m)\phi_m'(y_m - 0) \\ &\quad + \int_{y_{m-1}}^{y_m} [(\rho(y)\phi_m'(y))' - \rho(y)'\phi_m'(y) + \eta(y)\phi_m'(y)]\phi_m(y)]dy. \end{aligned}$$

Since $A\phi_m(y) = (\rho(y)\phi_m'(y))' - \rho(y)'\phi_m'(y) + \eta(y)\phi_m'(y) = 0$ as defined in Section 3, we obtain that

$$Q_1 = -\rho(y_m)\phi_m'(y_m - 0).$$

Now, let us consider Q_2. Let

$$J_2 = \int_{y_m}^{y_{m+1}} \rho(y)\phi_m'(y)\phi_m'(y)dy.$$

Then

$$Q_2 = -J_2 - \int_{y_m}^{y_{m+1}} \rho(y)'\phi_m'(y)\phi_m(y)dy + \int_{y_m}^{y_{m+1}} \eta(y)\phi_m'(y)\phi_m(y)dy.$$

We have that

$$\begin{aligned} J_2 &= \int_{y_m}^{y_{m+1}} \rho(y)\phi_m'(y)\phi_m'(y)dy \\ &= \phi_m(y)\rho(y)\phi_m'(y)\Big|_{y_m}^{y_{m+1}} - \int_{y_m}^{y_{m+1}} (\rho(y)\phi_m'(y))'\phi_m(y)dy \\ &= -\rho(y_m)\phi_m'(y_m+0) - \int_{y_m}^{y_{m+1}} (\rho(y)\phi_m'(y))'\phi_m(y)dy. \end{aligned}$$

Substituting J_2 back in, we obtain that

$$\begin{aligned} Q_2 &= \rho(y_m)\phi_m'(y_m+0) + \int_{y_m}^{y_{m+1}} (\rho(y)\phi_m'(y))'\phi_m(y)dy \\ &\quad - \int_{y_m}^{y_{m+1}} \rho(y)'\phi_m'(y)\phi_m(y)dy + \int_{y_m}^{y_{m+1}} \eta(y)\phi_m'(y)\phi_m(y)dy \\ &= \rho(y_m)\phi_m'(y_m+0) + \int_{y_m}^{y_{m+1}} [(\rho(y)\phi_m'(y))'\phi_m(y) - \rho(y)'\phi_m'(y)\phi_m(y) \\ &\quad +\eta(y)\phi_m'(y)\phi_m(y)]dy \\ &= \rho(y_m)\phi_m'(y_m+0) + \int_{y_m}^{y_{m+1}} [[(\rho(y)\phi_m'(y))' - \rho(y)'\phi_m'(y) + \eta(y)\phi_m'(y)]\phi_m(y)]dy. \end{aligned}$$

Since $A\phi_m = (\rho(y)\phi_m')' - \rho(y)'\phi_m'(y) + \eta(y)\phi_m'(y) = 0$, we obtain that

$$Q_2 = \rho(y_m)\phi_m'(y_m+0).$$

Substituting Q_1 and Q_2 back in we have that

$$(A\phi_m, \phi_m)_{L_2(D)} = -\rho(y_m)\phi_m'(y_m-0) + \rho(y_m)\phi_m'(y_m+0) = -\beta_{m,m}. \tag{3.5.5}$$

$\beta_{m,m}$ is the diagonal component in B. Finally, let us consider $k = m+1$.

$$\begin{aligned} (A\phi_{m+1}, \phi_m)_{L_2(D)} &= -\int_{-L}^{L} \rho(y)\phi_{m+1}'(y)\phi_m'(y)dy - \int_{-L}^{L} \rho(y)'\phi_{m+1}'(y)\phi_m(y)dy \\ &\quad + \int_{-L}^{L} \eta(y)\phi_{m+1}'(y)\phi_m(y)dy. \end{aligned}$$

Since the $\phi_{m+1}(y)$ and $\phi_m(y)$ are multiplied inside the integral, the boundaries for the integrals will be (y_m, y_{m+1}) since for $y \leq y_m$, $\phi_{m+1}(y) = \phi_{m+1}'(y) = 0$ and for $y \geq y_{m+1}$, $\phi_m(y) = \phi_m'(y) = 0$. Let

$$J = \int_{y_m}^{y_{m+1}} \rho(y)\phi_{m+1}'(y)\phi_m'(y)dy.$$

$$
\begin{aligned}
(A\phi_{m+1}, \phi_m)_{L_2(D)} \qquad &= -J - \int_{y_m}^{y_{m+1}} \rho(y)'\phi'_{m+1}(y)\phi_m(y)dy \\
&\quad + \int_{y_m}^{y_{m+1}} \eta(y)\phi'_{m+1}(y)\phi_m(y)dy.
\end{aligned}
$$

We have that

$$
\begin{aligned}
J \quad &= \int_{y_m}^{y_{m+1}} \rho(y)\phi'_{m+1}(y)\phi'_m(y)dy \\
&= \phi_m(y)\rho(y)\phi'_{m+1}(y)\Big|_{y_m}^{y_{m+1}} - \int_{y_m}^{y_{m+1}} (\rho(y)\phi'_{m+1}(y))'\phi_m(y)dy \\
&= -\rho(y_m)\phi'_{m+1}(y_m+0) - \int_{y_m}^{y_{m+1}} (\rho(y)\phi'_{m+1}(y))'\phi_m(y)dy.
\end{aligned}
$$

Substituting J back in, we obtain that

$$
\begin{aligned}
(A\phi_{m+1}, \phi_m)_{L_2(D)} \qquad &= \rho(y_{m+1})\phi'_{m+1}(y_m+0) + \int_{y_m}^{y_{m+1}} (\rho(y)\phi'_{m+1}(y))'\phi_m(y)dy \\
&\quad - \int_{y_m}^{y_{m+1}} \rho(y)'\phi'_{m+1}(y)\phi_m(y)dy + \int_{y_m}^{y_{m+1}} \eta(y)\phi'_{m+1}(y)\phi_m(y)dy \\
&= \rho(y_m)\phi'_{m+1}(y_m+0) + \int_{y_m}^{y_{m+1}} [(\rho(y)\phi'_{m+1}(y))'\phi_m(y) \\
&\quad -\rho(y)'\phi'_{m+1}(y)\phi_m(y) + \eta(y)\phi'_{m+1}(y)\phi_m(y)]dy \\
&= \rho(y_m)\phi'_{m+1}(y_m+0) \\
&\quad + \int_{y_m}^{y_{m+1}} [(\rho(y)\phi'_{m+1}(y))' - \rho(y)'\phi'_{m+1}(y) \\
&\quad +\eta(y)\phi'_{m+1}(y)]\phi_m(y)]dy.
\end{aligned}
$$

Since $A\phi_{m+1} = (\rho(y)\phi'_{m+1}(y))' - \rho(y)'\phi'_{m+1}(y) + \eta(y)\phi'_{m+1}(y) = 0$, as defined in Section 3 we obtain that

$$
(A\phi_{m+1}, \phi_m)_{L_2(D)} = \rho(y_m)\phi'_{m+1}(y_m+0) = -\beta_{m,m+1}. \tag{3.5.6}
$$

$\beta_{m,m+1}$ is the component to the right of the main diagonal in B.

Combining our findings and remembering that the term had a minus in the beginning, we obtain that

$$
\begin{aligned}
-\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t)A\phi_k(y)\phi_m(y)dy \qquad &= \rho(y_m)\phi'_{m-1}(y_m-0)v_{m-1}(t) \\
&\quad +\rho(y_m)(\phi'_m(y_m-0) - \phi'_m(y_m+0))v_m(t) \\
&\quad -\rho(y_m)\phi'_{m+1}(y_m+0)v_{m+1}(t) \\
&= \beta_{m,m-1}v_{m-1}(t) + \beta_{m,m}v_m(t) \\
&\quad +\beta_{m,m+1}v_{m+1}(t).
\end{aligned} \tag{3.5.7}
$$

Thus, we found the right side components for B. Let us consider the boundary cases: $m = 1$ and $m = N$. When $m = 1$, if we refer to (3.4.3), we will not consider the case when $k = m - 1 = 0$ since $v_0 = 0$. Thus,

$$\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) A\phi_k(y)\phi_1(y)dy \quad = (-\rho(y_1)\phi_1'(y_1 - 0) + \rho(y_1)\phi_1'(y_1 + 0))v_1(t) \\ +\rho(y_1)\phi_2'(y_1 + 0)v_2(t). \quad (3.5.8)$$

Similarly, when $m = N$, we will not consider the case when $k = m + 1 = N + 1$ since $v_{N+1} = 0$. Thus,

$$\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) A\phi_k(y)\phi_N(y)dy \quad (3.5.9)$$

$$= -\rho(y_N)\phi_{N-1}'(y_N - 0)v_{N-1}(t) \\ +(-\rho(y_N)\phi_N'(y_N - 0) - \rho(y_N)\phi_N'(y_N + 0))v_N(t). \quad (3.5.10)$$

Thus, we found the components for the matrix B.

3.5.3 Finding the components for $B\xi + \zeta$ in (3.4.4)

We represent f in (3.2.14) by ξ_k for which $f_N(y) = \sum_{k=0}^{N+1} \xi_k\phi_k(y)$. We need to consider $\xi_k(A\phi_k(y), \phi_m(y))_{L_2(D)}$. We found $(A\phi_k, \phi_m)_{L_2(D)}$ in the previous section for $y_1, y_2, ...y_N$. The result was B which is applicable to $f_N(y)$. However, we consider boundary cases differently than in the previous section since ξ_0 and ξ_{N+1} are not necessarily zeros. When $m = 1$,

$$\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A\phi_k(y)\phi_1(y)dy \quad = -\rho(y_1)\phi_0'(y_1 - 0)\xi_0 + (\rho(y_1)\phi_1'(y_1 - 0) \\ -\rho(y_1)\phi_1'(y_1 + 0))\xi_1 + \rho(y_1)\phi_2'(y_1 + 0)\xi_2.$$

When $m = N$,

$$\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A\phi_k(y)\phi_N(y)dy \quad = -\rho(y_N)\phi_{N-1}'(y_N - 0)\xi_{N-1} + (-\rho(y_N)\phi_N'(y_N - 0) \\ +\rho(y_N)\phi_N'(y_N + 0))\xi_N + \rho(y_1)\phi_2'(y_1 + 0)\xi_{N+1}.$$

To keep B in (3.4.7) applicable to $\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A\phi_k(y)\phi_m(y)dy$ we introduce $\zeta = (\zeta_1, .., \zeta_N)$, such that

$$\begin{aligned} \zeta_1 \quad &= (A\phi_0, \phi_1)_{L_2(D)}\xi_0 \\ &= -\rho(y_1)\phi_0'(y_1 - 0) \\ &= -\beta_{1,0}\xi_0. \\ \zeta_N \quad &= (A\phi_{N+1}, \phi_N)_{L_2(D)}\xi_{N+1} \\ &= \rho(y_N)\phi_{N+1}'(y_N + 0) \\ &= -\beta_{N,N+1}\xi_{N+1}, \\ \zeta_k \quad &= 0, \quad for \quad k = 2, .., N - 1. \end{aligned}$$

3.5.4 Finding the components for κ in (3.4.4)

Let us consider the last term of the above expression (3.4.3):

$$\int_{\mathbb{R}} e^{-r(t-T)} \mathcal{N}_{\lambda,k}[-e^{r(t-T)} \sum_{k=0}^{N+1} (v_k \phi_k(y) - R(t)\xi_k \phi_k(y))] \phi_m(y) dy$$
$$= e^{-r(t-T)} \int_{y_{m-1}}^{y_{m+1}} \mathcal{N}_{\lambda,k}[-e^{r(t-T)} \sum_{k=m-1}^{m+1} v_k \phi_k(y)] \phi_m(y) dy,$$

since $\phi_m(y)$ is supported on $y \in (y_{m-1}, y_{m+1})$.

3.5.5 Proof of Theorem 3.4.1: conclusion

Combining the statements above we obtain the statement (3.4.4). This concludes the proof of Theorem 3.4.1.

3.6 Solution of the ODE (3.4.4)

3.6.1 Backward Euler method

The resulting ODE in (3.4.4) can be solved numerically using backwards Euler method, i.e. using disretization

$$\frac{u_k^{t+1} - u_k^t}{\Delta t} = M^{-1}[(B u_k^{t+1} + B\xi + \zeta + \kappa)]. \tag{3.6.1}$$

Respectively,

$$u_k^t = M^{-1}(M u_k^{t+1} - \Delta t(B u_k^{t+1} + B\xi + \zeta + \kappa). \tag{3.6.2}$$

3.6.2 Backwards substitutions

After solving the system we obtain the vector v_k, $k = 1..N$. To solve the original equation, we have to reverse the substitutions (3.2.7), (3.2.9), and (3.2.12). The last substitution made was (3.2.12). We transform the answer:

$$v_k := v_k + \xi_k, k = 1..N.$$

Consider (3.2.9). To reverse it:

$$v_k := v_k e^{r(t-T)}.$$

Now, consider (3.2.7). Clearly, $y = \ln x$. Therefore, finding V(x) from the original Black-Scholes equation is equivalent to finding $V_N(y) = V_N(\ln x)$. To find V(x) for particular x, where $y_k \leq \log x \leq y_{k+1}$,

$$V(x) = v_k \phi_k(\log x) + v_{k+1} \phi_{k+1}(\log x).$$

3.7 Convergence for $0 < k \leq 1$

The following convergence result is based on Theorem 7.1 from Douglas and Dupont's paper [20].

Theorem 3.7.1 *Let* $0 < k \leq 1$. *There exist constants* C *and* δ *which depend on* T,n,D,K_0,C_0 *and* C_1, *such that for* v *and* V_N, *solutions to (3.4.1) and (3.4.2), respectively, and any function* $\tilde{u}$ *of the form* $\sum_{i=1}^{N} \alpha_i \tilde{u}_i$

$$\begin{aligned}
&\sup_{0\leq t\leq T} \|u(\cdot,t) - V_N(\cdot,t)\|_{L^2(D)} + \delta \int_0^T \|u(\cdot,t) - V_N(.,t)\|^2_{H_0^1(D)} dt \\
&\leq C[\sup_{0\leq t\leq T} \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^2(D)} + \int_0^T \|u(\cdot,t) - \tilde{u}\|^2_{H_0^1(D)} dt \\
&\quad + \|\frac{\partial}{\partial t}(u - \tilde{u})\|^2_{L^2(D\times(0,T))}].
\end{aligned}$$

Let us adjust our setting to the notations of [20]. Note that we will refer to the following notations from [20] in a special way to avoid confusion: $a(u,w)$ as $\widetilde{a}(u,w)$, A as $\widetilde{A}$, (f,w) as $(\widetilde{f},w)$ and η as $\widetilde{\eta}$. Also, we will use y instead of x from [20]. We need to rewrite (4.1) in our paper as (7.2) in [20]:

$$\langle \frac{\partial u}{\partial t}, w\rangle + \widetilde{a}(u,w) = \langle f(u), w\rangle.$$

Here,

$$\widetilde{a}(u,w) = \int_D \widetilde{A}(y,t,u(y,t),\frac{\partial w}{\partial y}(y,t))dy$$

and

$$\widetilde{f}(u) = \widetilde{f}(y,t,u(y,t),\frac{\partial u}{\partial y}(y,t))$$

for some measurable functions $\widetilde{A}(y,t,u,p) : D\times[0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\widetilde{f}(y,t,u,p) : D\times[0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$. Note that in this paper p is not a vector. We will need to find the functions $\widetilde{A}$ and $\widetilde{f}$ to obtain (7.4) in [20]: Let us consider $a(u,w)$ and $R(t)a(f,w)$ in (3.4.1).

$$\begin{aligned}
a(u,w) &= (Au,w)_{L_2(D)} = (\rho\phi''_{yy},w)_{L_2(D)} + (\eta\phi'_y,w)_{L_2(D)} \\
&= ((\rho u'_y)'_y,w)_{L_2(D)} - (\rho'_y u'_y,w)_{L_2(D)} + (\eta u'_y,w)_{L_2(D)} \\
&= \int_{-L}^{L} (\rho u'_y)'_y w\,dy - \int_{-L}^{L} \rho'_y u'_y w\,dy + \int_{-L}^{L} \eta u'_y w\,dy \\
&= \rho u'_y w\Big|_{-L}^{L} - \int_{-L}^{L} \rho u'_y w'_y\,dy - \int_{-L}^{L} \rho'_y u'_y w\,dy + \int_{-L}^{L} \eta u'_y w\,dy \\
&= 0 - \int_{-L}^{L} \rho u'_y w'_y\,dy - \int_{-L}^{L} \rho'_y u'_y w\,dy + \int_{-L}^{L} \eta u'_y w\,dy. \qquad (3.7.1)
\end{aligned}$$

Similarly,

$$\begin{aligned} R(t)a(f,w) \quad &= R(t)(Af,w)_{L_2(D)} \\ &= R(t)((\rho f'_y)'_y,w)_{L_2(D)} - R(t)(\rho'_y f'_y,w)_{L_2(D)} + R(t)(\eta f'_y,w)_{L_2(D)} \\ &= 0 - R(t)\int_{-L}^{L} \rho f'_y w'_y \, dy - R(t)\int_{-L}^{L} \rho'_y f'_y w \, dy \qquad (3.7.2) \\ &\quad + R(t)\int_{-L}^{L} \eta f'_y w \, dy. \qquad (3.7.3) \end{aligned}$$

Now, let us consider $R'(t)(f,w)_{L_2(D)}$ in (3.4.1).

$$R'(t)(f,w)_{L_2(D)} = R'(t)\int_{-L}^{L} fwdy. \tag{3.7.4}$$

Finally, let us consider $(G_N,w)_{L_2(D)}$ in (3.4.1).

$$(G_N,w)_{L_2(D)} = \frac{1}{R(t)}\int_{-L}^{L} \mathcal{N}_{\lambda,k}[-R(t)u]\, w \, dy. \tag{3.7.5}$$

We combine the terms with w'_y of the equations (3.7.1) and (3.7.3) to obtain $\widetilde{a}(u,w)$ from 7.2 in [20].

$$\widetilde{a}(u,w) = \int_{-L}^{L} (-\rho u'_y w'_y - R(t)\rho f'_y w'_y) dy.$$

So,

$$\widetilde{A}(y,t,u,u'_y) = -\rho u'_y - R(t)\rho f'_y.$$

This gives

$$\widetilde{A}(y,t,u,p) = -\rho p - R(t)\rho f'_y.$$

Next, we combine the terms with w of the equations (3.7.1),(3.7.3),(3.7.4) and (3.7.4) to obtain $(\widetilde{f},w)$ from [20].

$$\begin{aligned} (\widetilde{f},w) \quad &= \int_{-L}^{L} \rho'_y u'_y w \, dy - \int_{-L}^{L} \eta u'_y w \, dy + R(t)\int_{-L}^{L} \rho'_y f'_y w \, dy \\ &\quad - R(t)\int_{-L}^{L} \eta f'_y w \, dy + R'(t)\int_{-L}^{L} fwdy - \frac{1}{R(t)}\int_{-L}^{L} \mathcal{N}_{\lambda,k}[-R(t)u]\, w \, dy. \end{aligned}$$

So,

$$\widetilde{f}(x,t,u,u'_y) = \rho'_y u'_y - \eta u'_y + \rho'_y f'_y - R(t)\eta f'_y + R'(t)f - \frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)u].$$

This gives

$$\widetilde{f}(x,t,u,p) = \rho'_y p - \eta p + \rho'_y f'_y - R(t)\eta f'_y + R'(t)f - \frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)u].$$

Next, we check the conditions from [20] on $\widetilde{A}$ and $\widetilde{f}$ in order for the theorem to hold. First, it is clear that both functions are measurable. Second, $\frac{\partial a}{\partial u} = -\rho$ is such that $C_0 \leq C_1$, $0 \leq C_0 \leq C_1$ for some positive C_0 and C_1. Next, we check the (7.1) in [20]: (7.1)

$$\begin{aligned} &\|\widetilde{A}(y,t,w(y),p(y)\|^2_{L^2(D)} + \|\widetilde{f}(y,t,w(y),p(y)\|^2_{L^2(D)} \\ &\qquad \leq C[\|w(y)\|^2_{L^2(D)} + \|p(y)\|^2_{L^2(D)} + 1]. \end{aligned} \tag{3.7.6}$$

Observe that,

$$\|\widetilde{A}(y,t,w(y),p(y)\|^2_{L^2(D)} = \|-\rho p - \rho f'_y\|^2_{L^2(D)} \leq C(\|p\|^2_{L^2(D)} + 1)$$

and

$$\begin{aligned} &\|\widetilde{f}(y,t,w(y),p(y)\|^2_{L^2(D)} \\ &= \|\rho'_y p - \eta p + \rho'_y f'_y - R(t)\eta f' + R'(t) f - \frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)p]\|^2_{L^2(D)} \\ &\leq C(\|p\|^2_{L^2(D)} + 1). \end{aligned}$$

Since ρ, ρ'_y and η are bounded and f'_y is bounded, and R(t) and R'(t) are bounded, the condition (3.7.6) holds. : .

Next, we check the conditions (7.11a-7.11e) in [20] for our $\widetilde{A}(x,t,u,u'_y)$ and $\widetilde{f}(x,t,u,u'_y)$ Suppose that $\widetilde{f} = \widetilde{f}_1 + \widetilde{f}_2$ and that there exists K_0 such that for $(y,t) \in D \times (0,T)$, r and s in $\mathbb{R}$, and p and q in $\mathbb{R}$.

Let us check the condition (7.11a)[20]:

$$|\widetilde{A}(y,t,r,u'_y(y,t)) - \widetilde{A}(y,t,s,u'_y(y,t))| \leq K_0|r-s|.$$

This holds since

$$|\widetilde{A}(y,t,r,u'_y) - \widetilde{A}(y,t,s,u'_y)| = -\rho u'_y - \rho f'_y - (-\rho u'_y - \rho(y) f'_y) = 0.$$

Let us check the condition (7.11b)[20]:

$$|\widetilde{f}_1(y,t,u(y,t),p) - \widetilde{f}_1(y,t,u(y,t),q)| \leq K_0|p-q|.$$

This holds since

$$\begin{aligned} &|\widetilde{f}_1(y,t,u(y,t),p) - \widetilde{f}_1(x,t,u(y,t),q)| \\ &= |\rho'_y p - \eta p + \rho'_y f'_y - R(t)\eta f'_y + R'(t) f - \frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)u] \\ &\quad -(\rho'_y q - \eta q + \rho'_y f'_y - R(t)\eta f'_y + R'(t) f - \frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)u])| \\ &= |(\rho'_y - \eta)(p-q)| \\ &\leq K_0|p-q|. \end{aligned}$$

and $\rho(y)$, $\rho'(y)$ and $\eta(y)$ are bounded. Let us check the condition (7.11c)[20]:

$$|\widetilde{f}_1(y,t,r,p) - \widetilde{f}_1(y,t,s,p) \leq K_0|r-s|.$$

This holds if $k \geq 1$, since

$$\begin{aligned} |\widetilde{f}_1(y,t,r,p) - \widetilde{f}_1(y,t,s,p)| \quad &= |-\frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)r] - (-\frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)s]| \\ &\leq K_0|r-s|. \end{aligned}$$

Let us check the condition (7.11d)[20]:

$$|\widetilde{f}_2(y,t,r,u'(y,t)) - \widetilde{f}_2(y,t,s,u'(y,t))| \leq K_0|r-s|.$$

This holds since

$$|\widetilde{f}_2(y,t,r,u'_y(y,t)) - \widetilde{f}_2(y,t,s,u'_y(y,t))| = 0.$$

Let us check the condition (7.11e)[20]:

$$|\widetilde{f}_2(y,t,r,p) - \widetilde{f}_2(y,t,r,q)| \leq K_0|p-q|.$$

This holds since

$$\begin{aligned} &|\widetilde{f}_2(y,t,r,p) - \widetilde{f}_2(y,t,r,q)| \\ &= \rho'_y p - \eta p + \rho'_y f'_y - R(t)\eta f'_y + R'(t)f - \frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)u] \\ &\quad -(\rho'_y q - \eta q + \rho'_y f'_y - \eta f'_y - R(t)\eta f'_y + R'(t)f - \frac{1}{R(t)}\mathcal{N}_{\lambda,k}[-R(t)u]) \\ &= (\rho'_y - \eta)(p-q) \\ &\leq K_0|p-q| \end{aligned}$$

and $\rho'(y)$ and $\eta(y)$ are bounded. Theorem 3.7.1 follows from Theorem 7.1 in [20] which shows our method is convergent for $k \leq 1$.

3.8 Numerical experiments

In these experiments we calculated the price for a put option using our method. We considered the payoff $(K - S(T))^+$. We found the solution of the equation (3.2.8) with $f(x) = (K-x)^+$ using our method and compared it with the solution given by a standard method for constant coefficients. Table (3.1) shows the average error

$$E = \sum \frac{|V(x_i,0) - \tilde{V}(x_i,0)|}{N}. \tag{3.8.1}$$

Here, V is the solution of (3.2.8) obtained by our method and $\tilde{V}$ is the exact solution of (3.2.8) given by the Black-Scholes formula. In this table, N is the parameter introduced in Section 3 representing the rate of discretization in x. N_t is the number of points along the time axis. We used fixed $k = 1$, $\lambda = 999$, $\sigma = 0.3$, $T = 1$, $L = 10$, $r = 0.2$, $d = 0.04$ and $K = 1$, and we consider variable N and N_t. In additions, we presumed that $C > L$ in our experiments; in this case, the actual choice of C does not impact the results.

We found that the choice of the truncation parameter $L = 10$ is near optimal; variations of L do not improve the result.

Variations of k with $k \neq 1$ did not improve the results.

Table 3.1: Error of calculation of the put option.

N, N_t	E
250, 250	0.07469794
250, 500	0.001745843
500, 250	0.000354912
500, 500	0.0003082416
500, 1000	0.000289123
1000, 500	0.0001507493
1000, 1000	0.0001690803

For the computation, we wrote a computer code in *R*. The computations were on a standard personal PC.

To increase precision we can discretize in two directions: N and/or N_t. Note, It is shown that the average error is decreasing as we use more points along the state price and time axes. We found the increasing of the grid for the sate variable is more effective than he increasing of the grid for the time variable. It can be seen from comparing the lines 2 and 3, and lines 5 and 6 in Table 3.1.

We found that the largest error mostly occurs near the point where the final condition becomes zero.

Figure **??** shows the comparison between the Black-Scholes and numerical solutions. More accurate approximations can be found by tweaking the parameters. In particular, the choice of λ and k could be tweaked to lead to better precision.

For example, in the problem above by increasing the parameters $N = 2000$, $N_t = 2000$,$L = 30$, the average error becomes 9.854932e-05. These calculations took a very long time.

There are some computational restrictions defined by the computer code and the capacity of computer; we were unable to run calculations with $N, N_t > 2000$ and $L > 30$.

Possibly, and preciseness improvement can be achieved with replacement of the truncation interval $[-L, L]$ by an interval $[-L_1, L_2]$ with $L_1 \neq L_2$.

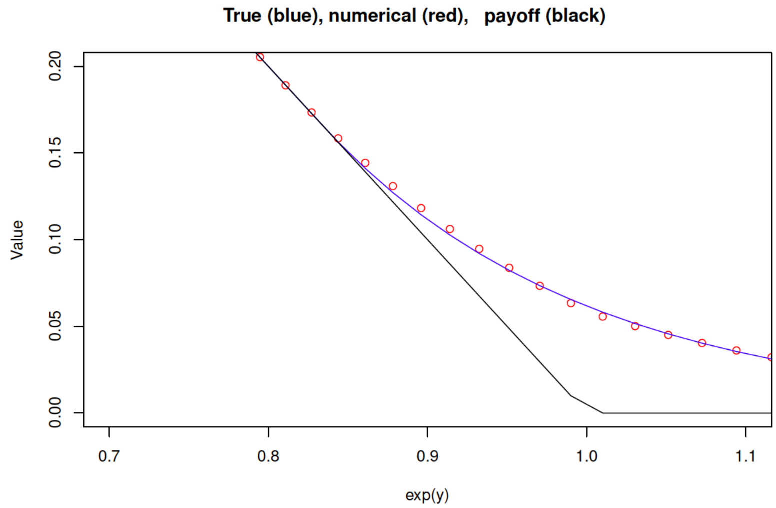

Figure 3.2: Comparison of standard solution and numerical solution V for American options

3.9 Conclusion

In this chapter, we presented a method for solving the free boundary problem for the American option where the volatility is a function of the current stock price rather than a constant. We approximated the free boundary problem by a problem with fixed boundary and with a non-linear penalty term. We used a Galerkin's method to discretize the problem and to reduce it to an ODE. This ODE can be solved by the backward Euler method. Various numerical experiments were conducted which suggest the method's convergence with the standard method solution.

Chapter 4

Conclusion

The thesis studied numerical solving boundary value problems for partial differential equations arising in financial modelling with state dependent coefficients.

The thesis commences with a description of the mathematical background of option pricing and related boundary value problems in Chapter 1. A review of numerical methods for boundary value problems for parabolic Black-Scholes equations is given. The Cauchy problem for European options and free boundary problem for American options are discussed therein.

In Chapter 2, we propose a numerical method of solutions for Black-Scholes parabolic equations in a setting where the volatility coefficient and the dividends rate depend on the current stock price. The work develops a modification of the Galerkin method based on a new special basis functions ϕ_k that are selected such that $A\phi_k = 0$, where A is the second order driving operator for the Black-Scholes equation for the parabolic equation after the Euler transformation $x = \log y$. These basis functions are convenient for the equations with state dependent volatility and dividends since they allow explicit calculation of the coefficients of the ODEs generated by Galerkin method. Each basis function can be derived explicitly using the given state dependent volatility and state dependent dividends. The parabolic equation is approximated by a linear ordinary differential equations for the coefficients for the basis functions. The coefficients of this equations are derived explicitly. The option price value at any point along the state variable can be found by interpolating two basis functions. Convergence of the solution is established. The effectiveness of the method is confirmed in numerical experiments. In particular, we found an an example of an exact solution of a Black-Scholes equation with state-dependent volatility, and established convergence of our numerical solution to this exact solution.

In Chapter 3, we extended the approach developed in Chapter 2 on the case of a free boundary problem for Black-Scholes equation for the price of American options. The approach in this case is based on the power penalty method; the impact of the presence of the moving boundary is imitated by inclusion of a nonlinear free term in the equation. The power penalty used in Chapter 3 is a new modification of power penalty from Angermann and Wang (2007); this modification was rather technical; it was needed to ensure convergence of the Galerkin method.

The nonlinear penalised parabolic equation is approximated again by an ordinary differential equations for the coefficients for the basis functions. In this case, the ordinary differential equation contains a nonlinear term corresponding to the penalty function. The coefficients of this equations are derived explicitly. The convergence of the Galerkin approximations is established. Numerical experiments for the price of put options show the effectiveness of the method.

Bibliography

[1] Achdou, Y. (2005). An inverse problem for a parabolic variational inequality arising in volatility calibration with American options. *SIAM Journal on Control and Optimization, 43*(5), 1583-1615. https://doi.org/10.1137/S0363012903424423

[2] Allegretto, W., Lin, Y. & Yang, H. (2001). Finite element error estimates for a nonlocal problem in American option valuation. *SIAM J. Numer. Anal., 39*(3), 834-857 (electronic). https://doi.org/10.1137/S0036142900370137

[3] Allen, D. N. De G., & Southwell, R. V. (1955). Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder. *The Quarterly Journal of Mechanics and Applied Mathematics, 8*, 129-145. https://doi.org/10.1093/qjmam/8.2.129

[4] Angermann, L., & Wang, S. (2003). Three-dimensional exponentially fitted conforming tetrahedral finite elements for the semiconductor continuity equations. *Applied Numerical Mathematics, 46*, 19-43. https://doi.org/10.1016/S0168-9274(02)00224-6

[5] Angermann, L., & Wang, S. (2007). Convergence of a fitted finite volume method for the penalized Black–Scholes equation governing European and American Option pricing. Numer. Math. (2007) 106:1–40.

[6] Bachelier, L. (1900). Theorie de la speculation. *Annales Scientifiques de l'École Normale Supérieure, 17*, 21-86. https://doi.org/10.24033/asens.476

[7] Barles, G. (1997). Convergence of Numerical Schemes for Degenerate Parabolic Equations Arising in Finance Theory. In L. Rogers & D. Talay (Eds.), *Numerical Methods in Finance* (Publications of the Newton Institute, pp. 1-21). Cambridge: Cambridge University Press. doi:10.1017/CBO9781139173056.002

[8] Barles, G., Daher, Ch. & Romano, M. (1995). Convergence of numerical schemes for problems arising in finance theory. *Mathematical Models and Methods in Applied Sciences, 5*, 125-143. https://doi.org/10.1142/S0218202595000085

[9] Becker, S. (1980). The constant elasticity of variance model and its implications for option pricing. *The Journal of Finance 35*(3), 661-673, https://doi.org/10.2307/2327490

[10] Benth, F. E., Karlsen, K. H., & Reikvam, K. (2004). A semilinear Black and Scholes partial differential equation for valuing American options: approximate solutions and convergence. *Interfaces and Free Boundaries 6*(4), 379-404. https://doi.org/10.4171/IFB/106

[11] Bensoussan, A., and Lions, J. L.(1982). Applications of Variational Inequalities in Stochastic Control, North-Holland, Amsterdam, Holland, 1982.

[12] Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy, 81*, 637-659. https://doi.org/10.1086/260062

[13] Brandt, A., & Cryer, C. W. (1983). Multigrid algorithms for the solution of linear complementarity problems arising from free boundary problems. *SIAM Journal on Scientific Computing, 4*, 655-684. https://doi.org/10.1137/0904046

[14] Brenner, S. C., & Scott, L. R. (1994). *The Mathematical Theory of Finite Element Methods.* New York: Springer. https://doi.org/10.1007/978-0-387-75934-0

[15] Broadie, M., Glasserman, P., and Jain, G. (1997). Enhanced Monte Carlo estimates for American option prices." The Journal of Derivatives 5, no. 1, 25-44.

[16] Courtadon, G. (1982). A more accurate finite difference approximation for the valuation of options. *Journal of Financial and Quantitative Analysis, 17*, 697-703. https://doi.org/10.2307/2330857

[17] Cox, J. (1996). The constant elasticity of variance option pricing model. *Journal of Portfolio Management, 23*(5) 15-17. https://doi.org/10.3905/jpm.1996.015

[18] Cox, J. C., Ross, S., & Rubenstein, M. (1979). Option pricing: a simplified approach. *Journal of Financial Economics, 7*, 229-264. https://doi.org/10.1016/0304-405X(79)90015-1

[19] Dokuchaev, M., Zhou, G., Wang, S. (2021) “A modification of Galerkin’s method for option pricing” *Journal of Industrial and Management Optimization*, online first published https://www.aimsciences.org/article/doi/10.3934/jimo.2021077. doi: 10.3934/jimo.2021077.

[20] Douglas, J. Jr., Dupont, T. (1979) Galerkin Methods for Parabolic Equations. SIAM Journal on Numerical Analysis, Vol. 7, No. 4, 575-626.

[21] Ferris, M. C., Pang, & J. S. (1997). Engineering and economic applications of complementarity problems, *SIAM Review, 39*, 669-713. https://doi.org/10.1137/S0036144595285963

[22] Forsyth, P.A. and Vetzal, K.R. (2002). Quadratic Convergence for Valuing American Options Using a Penalty Method SIAM J. Sci. Comput., 23(6), 2095–2122.

[23] Glowinski, R. (1984). *Numerical Methods for Nonlinear Variational Problems.* Springer, Berlin Heidelberg New York. https://doi.org/10.1002/zamm.19850650812

[24] Han, H., & Wu, X. (2003). A fast numerical method for the Black-Scholes equation of American options. *SIAM Journal of Numerical Analysis, 41*(6), 2081-2095. https://doi.org/10.1137/S0036142901390238

[25] Haslinger, J., Miettinen, M., & Panagiotopoulos, P. D. (1999). *Finite element method for hemivariational inequalities.* Springer US. https://www.springer.com/gp/book/9780792359517

[26] Holtz, M., & Kunoth, A. (2007). B-spline-based monotone multigrid methods. *SIAM journal of Numerical Analysis, 45*(3), 1175-1199 https://doi.org/10.1137/050642575

[27] Hsu, Y. L., Lin, T. I., & Lee, C. F. (2008) Constant elasticity of variance (CEV) option pricing model: integration and detailed derivation. *Mathematics and Computers in Simulation, 79*(1), Pages 60-71, https://doi.org/10.1016/j.matcom.2007.09.012

[28] Hull, J. C., & White, A. (1988) The use of control variate technique in option pricing. *Journal of Financial and Quantitative Analysis, 23*, 237-251. https://doi.org/10.2307/2331065

[29] Hull, J. & White, A. (1996) *Hull-White on Derivatives*. London: Risk Publications. https://doi.org/10.2307/2331065

[30] Jaillet, P., Lamberton, D., & Lapeyre, B. (1990). Variational inequalities and the pricing of American options. *Acta Applicandae Mathematicae, 21*(3), 263-289. https://doi.org/10.1007/BF00047211

[31] Kinderlehrer, D., & Stampacchia, G. (1980). *An Introduction to Variational Inequalities and Their Applications.* Academic New York. https://doi.org/10.1137/1.9780898719451

[32] Kufner, A. (1985). *Weighted Sobolev spaces.* Wiley, New York. https://www.worldcat.org/title/weighted-sobolev-spaces/oclc/10018405

[33] Lesmana, D. and Wang, S. (2015). Penalty approach to a nonlinear obstacle problem governing American put option valuation under transaction costs. Applied Mathematics and Computation. 251: pp. 318-330. https://doi.org/10.1016/j.amc.2014.11.060

[34] Miller, J. J. H., & Wang, S. (1994a). A new non-conforming Petrov-Galerkin method with triangular elements for a singularly perturbed advection-diffusion problem. *IMA Journal of Numerical Analysis, 14*, 257-276. https://doi.org/10.1093/imanum/14.2.257

[35] Miller, J. J. H., & Wang, S. (1994b). An exponentially fitted finite element volume method for the numerical solution of 2D unsteady incompressible flow problems. *Journal of Computational Physics, 115*, 56-64. https://doi.org/10.1006/jcph.1994.1178

[36] Nicholls, D., & Sward, A. (2015). A Discontinuous Galerkin Method for Pricing American Options Under the Constant Elasticity of Variance Model. *Communications in Computational Physics, 17*, 761-778. https://doi.org/10.4208/cicp.190513.131114a

[37] Oosterlee, C. W. (2003). On multigrid for linear complementarity problems with application to American-style options. *Electronic Transactions on Numerical Analysis 15*, 165-185. http://eudml.org/doc/123267

[38] Ortega, J.M., & Rheinboldt, W.C. (1970). *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York. https://doi.org/10.1016/C2013-0-11263-9

[39] Rogers, L.C.G. and Shi, Z. (1995) The value of an Asian Option. Journal of Applied Probability, 32, 1077-1088. https://doi.org/10.1017/S0021900200103559

[40] Rubinov, A. M., Yang, X. Q. (2003). Lagrange-type functions in constrained non-convex optimization. Boston, MA: Springer.

[41] Rogers, L. C. G. & Talay, D. (1997). *Numerical Methods in Finance*. Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9781139173056

[42] Schwartz, E. (1977). The valuation of warrants: implementing a new approach. *Journal of Financial Economics, 13*, 79-93. https://doi.org/10.1016/0304-405X(77)90037-X

[43] Valkov, R. (2014). Fitted finite volume method for a generalized Black-Scholes equation transformed on finite interval. *Numerical Algorithmns, 65*, 195-220. https://doi.org/10.1007/s11075-013-9701-3

[44] Varga, R. S. (1962). *Matrix Iterative Analysis*. Englewood Cliffs, NJ: Prentice-Hall. https://doi.org/10.1007/978-3-642-05156-2

[45] Vazquez, C. (1998). An upwind numerical approach for an American and European option pricing model. *Applied Mathematics and Computation, 97*, 273-286. https://doi.org/10.1016/S0096-3003(97)10122-9

[46] Wang, S. (1997). A novel exponentially fitted triangular finite element method for an advectiondiffusion problem with boundary layers. *Journal of Computational Physics, 134*, 253-260. https://doi.org/10.1006/jcph.1997.5691

[47] Wang, S. (2004). A novel fitted finite volume method for the Black-Scholes equation governing option pricing. IMA J. Numer. Anal. 24, 699-720. https://doi.org/10.1093/imanum/24.4.699

[48] Wang, S., Yang, X. Q., & Teo, K. L. (2006). A power penalty method for a linear complementarity problem arising from American option valuation. *Journal of Optimization Theory and Applications volume, 129*(2), 227-254. https://doi.org/10.1007/s10957-006-9062-3

[49] Wang S., Yang X., (2008). A power penalty method for linear complementarity problems, Oper. Res. Lett., 36, 211-214.

[50] Wilmott, P., Dewynne, J., & Howison, S. (1993). *Option Pricing: Mathematical Models and Computation.* Oxford: Oxford Financial Press. `https://books.google.com.au/books/about/Option_Pricing.html?id=cGuGcgAACAAJ&redir_esc=y`

[51] Wong, H. Y. and Zhao, & J. (2008). An artificial boundary method for American option pricing under the CEV model. *SIAM Journal on Numerical Analysis, 46*(4), 2183-2209, https://doi.org/10.1137/060671541

[52] Yang, X. Q., & Huang, X. X. (2001). A Nonlinear Lagrangian Approach to Constrained Optimization Problems, *SIAM Journal on Optimization, 14*, 1119-1144, https://doi.org/10.1137/S1052623400371806

[53] Zhang, K., Wang, S., Yang, X. Q., & Teo, K. L. (2009). A power penalty approach to numerical solutions of two-asset american options. *Numerical Mathematics: Theory, Methods and Applications, 2*(2), 202-223. http://gdxxjssxxb-e.periodicals.net.cn/default.html

[54] Zhang, K, Wang, S. (2012). Pricing American bond options using a penalty method. Automatica 48, 472-479.

[55] Zvan, R., Forsyth, P. A., Vetzal, K. R. (1998): Penalty methods for American options with stochastic volatility. *Journal of Computational and Applied Mathematics, 91*(2), 199-218. https://doi.org/10.1016/S0377-0427(98)00037-5

Written statements on co-authored peer reviewed papers

To Whom It May Concern

I, Mikhail Dokuchaev, contributed

1. majority of theoretical conception.

2. vast majority of mathematical framework construction.

3. vast majority of literature review and connecting the current work to extant literature.

4. vast majority of numerical algorithm development.

5. vast majority of programming implementation of the mathematical framework.

6. vast majority of interpretation of results and connection of results to extant literature, and

8. vast majority writing and editing of manuscript to be submitted for the peer reviewed publications, and preparation of responses to the panel of referee,

to the paper entitled M. Dokuchaev, G. Zhou, S. Wang. (2021) "A modification of Galerkin's method for option pricing", *Journal of Industrial and Management Optimization*, online first published https://www.aimsciences.org/article/doi/10.3934/jimo.2021077.

Mikhail Dokuchaev

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.

I, as a Co-Author, endorse that this level of contribution by the candidate indicated above is appropriate.