School of Electrical Engineering, Computing and Mathematical Sciences

Extreme Risk Forecast for Quantitative Financial Risk Management

Lequn Zhang

0000-0001-8453-8454

**This thesis is presented for the Degree of
Doctor of Philosophy
of
Curtin University of Technology**

September 2022

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made. This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

Lequn Zhang

16/09/2022

"*Discover the truth through practice, and again through practice verify and develop the truth. Start from perceptual knowledge and actively develop it into rational knowledge; then start from rational knowledge and actively guide revolutionary practice to change both the subjective and the objective world. Practice, knowledge, again practice, and again knowledge. This form repeats itself in endless cycles, and with each cycle the content of practice and knowledge rises to a higher level. Such is the whole of the dialectical-materialist theory of knowledge, and such is the dialectical-materialist theory of the unity of knowing and doing.*"

— Mao, Zedong

Acknowledgements

I wish to express my deepest appreciation to Dr Quanxi Shao from Data61 CSIRO, who is not only a great supervisor but also a great mentor no matter from the study, work or life. Without his extensive guidance and support, I cannot finish my PhD study.

I would like to sincerely appreciate my supervisor Dr Honglei Xu from Curtin University, who gave me great support throughout my PhD study.

I would like to thank my CSIRO mentor Dr Brett McLindin, associate supervisors Dr Song Wang and Dr Grace AW for providing study guidance and advice.

I would like to thank my CSIRO colleagues Dr Ming Li, Dr Fang Huang, Dr Siyu Hu and Dr Yuan Mei for their discussions, conversations and advice during my study. Their help has had an immensely positive influence on my postgraduate study journey.

I thank Erica Jeanne Bell and Diane Josey for the professional editing of my thesis.

I would like to acknowledge and thank Data61 CSIRO for providing scholarships and the Australian government for providing research training support.

My deepest gratitude goes to my wife Huilan Zhao, my parents Qihua Zhang and Mingxin He, my parents-in-law Youda Zhao and Youzhu Xu, my brother-in-law Xuanyu Zhao, and my daughter Chuyang Zhang who accompanied, supported and encouraged me throughout this hard journey.

Abstract

Extreme risk in finance refers to financial events with small probability but significant consequences, which potentially lead to the defaults or bankruptcy of a financial institution. Risk measures, therefore, are employed to manage such risk quantitatively. One of the most important risk measures is Value at Risk (VaR), which quantifies the worst loss of an investment given a confidence level. Various methods have been developed to obtain an accurate and robust forecast of VaR.

Research results (Abad, Benito, & López, 2014; Calmon, Ferioli, Lettieri, Soares, & Pizzinga, 2021) show that methods based on the statistical extreme value theory (EVT) have the top performance in forecasting VaR. EVT proves that the tail of a random variable asymptotically belongs to generalised Pareto distribution with peak-over-threshold sampling method under some limiting conditions. Statistically speaking, VaR is a tail or extreme quantile of a statistical distribution. Hence, EVT is a natural tool for VaR forecast, as EVT focuses on modelling the tail. However, the limiting conditions of EVT are difficult to satisfy in practice. Therefore, the purpose of this thesis is to employ and develop extended extreme value approach to improve the VaR forecast.

The first contribution of the thesis is to promote an extended extreme value approach to forecast unconditional VaR, in which the extended Burr XII (EB) is a generalization of generalised Pareto (GP) distribution. The performance of the method was compared with three competing methods through a simulation study and an empirical study. The simulation study suggested that EB has the best distribution fittings and VaR estimate accuracy of all four candidate

distributions. In the empirical study, the VaR estimate was backtested through UC and MAPE in short-, medium-, and long-term windows with different VaR levels. The results indicate that EB outperforms other candidate distributions and is capable of modelling unconditional VaR well under a range of different scenarios.

The second contribution is that the extended extreme value approach is incorporated with the generalised autoregressive conditional heteroscedasticity (GARCH) model to forecast dynamic VaR, which is an effective tool for risk monitoring. The performance of the proposed method is examined through a large-scale empirical study with nearly 50 financial assets. The empirical research suggests that GARCH + EB outperforms existing GARCH + EVT approaches, and is robust and accurate in forecasting VaR through different backtesting procedures.

The third contribution is that we improve the forecast efficiency of the classic GARCH + EVT framework by incorporating a mixture distribution. The classic GARCH + EVT framework uses a two-step approach to forecast dynamic VaR. However, the two-step approach is not efficient. We propose a mixture density for GARCH innovation where the bulk part is non-parametric density and the tail is using some tail density. An algorithm is also developed to estimate the GARCH parameters and forecast VaR for the proposed model. The performance of the proposed method is evaluated through a simulation study and an empirical study, and compared to the classic GARCH-EVT framework. Both studies show that a GARCH-EVT approach with mixture density can forecast VaR accurately and efficiently.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Context

Risk management is an important concept in the financial industry. The purpose of financial risk management is to protect an investment from an adverse event arising from the market that could decrease the economic value of the investment. However, it was not until Markowitz (1952) that researchers and practitioners began to assess investment risk using quantitative methods. Markowitz (1952) used the variance of the return of a portfolio as a quantitative measure of risk.

Over the course of the development of the financial market, investors can assess financial products, such as leverage, to invest more efficiently. Leverage is an investment strategy wherein the investor borrows money in order to profit from risk capital. Unsurprisingly, banks are the largest investors to employ leverage. Households lend their savings to the bank in return for a given rate of interest; banks then invest these savings to generate a return. A wide range of financial institutions invest and trade financial products by employing leverage, including Morgan Stanley, CME, and ING.

Leverage is a double-edged sword. It increases the efficiency of capital operation and boosts the profits of risk capital. However, it also entails substantial risk both to the investment and to the financial system. Unforeseen extreme risk events cause the capital price to decrease suddenly and dramatically. In such cases, the lending institution may call in the loan, even though the capital price is lower than the initial amount borrowed. This may trigger a chain reaction in the financial market, leading to a sudden drop in liquidity. This mechanism is characteristic of modern financial crises, and has been observed in the 1997 Asian financial crisis, the 2008 global financial crisis (Jorion, 2007), and the early stages of the COVID-19 pandemic in 2020.

Extreme risks have a low probability, but a great consequence. The quantitative management tool employed by is not adequate for managing extreme risk (Malevergne & Sornette, 2006). An alternative measure, Value at Risk (VaR), which measures the worst loss of an investment in a given period and the confidence level, may provide a more effective tool for managing extreme risk arising from unforeseen risk events. The Basel Committee on Bank Supervision (BCBS) uses VaR as a risk measure to regulate the operations of the bank ("Basel Accord III", 2019), while J.P. Morgan uses VaR to monitor the risk of financial investment dynamically (Longerstaey & Spencer, 1996).

In this thesis, we research and develop advanced statistical methods and tools to forecast VaR.

1.2 Motivation and Thesis Structures

Researchers and practitioners have developed a range of quantitative tools to manage risk, which work adequately under normal market conditions. However, in a more volatile financial market, greater concern should be given to extreme risk scenarios, which could have catastrophic consequences for financial institutions. The overarching motivation behind this thesis is, therefore, to develop a more accurate and robust statistical method of forecasting risk under extreme market conditions.

The aims of this thesis are twofold. First, financial regulators often draw upon long-term average risk forecasts to develop the standards regulating financial firms. As such, this thesis aims to develop tools that could improve the unconditional VaR forecast for the purposes of financial regulation. Second, a key concern for investors is running risk on a daily basis. Therefore, it also seeks to develop robust and efficient dynamic VaR forecast tools.

Chapter 2 introduces the concept of the risk measure, which is defined mathematically as a function summarising a random variable with uncertain outcomes into a single number. We list commonly used risk measures and explain their interpretations in a risky scenario. We then discuss VaR, which may be the most important risk measure in quantitative financial risk management, since it answers the question "what might be the worst loss in an investment?". Methods of forecasting VaR in the existing literature are comprehensively reviewed. This chapter concludes by reviewing various backtesting procedures, which serve to assess the validity of a given VaR model.

Chapter 3 promotes an extended extreme value theory (EVT) to forecast unconditional Value at Risk. VaR measures the worst loss of an investment, which is a high quantile of distribution, whereas EVT focuses on modelling the probability of rare events, i.e. the tail of the distribution. Therefore, EVT is a natural tool for forecasting VaR. The chapter first introduces the classic extreme value theory with the block maxima sampling method, which results in the generalised extreme value (GEV) distribution as the limiting distribution. However, as the block maxima method does not employ sample data information efficiently, a peak-over-threshold (POT) method is developed to model extremes over a selected threshold. This results in the generalised Pareto (GP) distribution as the limiting distribution. Nevertheless, both GEV and GP are limiting distributions under the limiting condition of a sufficient sample size. In a VaR forecast application, this condition is difficult to satisfy, making VaR forecasts from GP sensitive to threshold selection and less robust. To address these shortcomings, we promote an Extended Burr XII (EB) distribution to forecast VaR. As an extension of the EVT, EB enjoys several valuable statistical properties.

The last section of this chapter validates the performance of EB in forecasting VaR through a simulation study and an empirical study.

Chapter 4 expands the extended EVT approach outlined in Chapter 3 to a conditional case by incorporating the generalised autoregressive conditional heteroskedasticity (GARCH) model to forecast dynamic VaR. Unconditional VaR assumes that a financial time series is independently and identically distributed (i.i.d.). However, this assumption limits the forecasting capability of dynamic VaR. Therefore, one possible approach is to extend EVT to the conditional case. In the first section of this chapter, we discuss the unfeasibility of extending EVT directly to the conditional case to forecast dynamic VaR. In the second section, we turn our focus to the GARCH model, which captures the volatility feature of a financial time series. We then incorporate GARCH with EB in a two-step process. In the last section, the performance of the proposed method is evaluated through a large-scale empirical study.

The extended GARCH-EVT presented in **Chapter 4** falls within the general GARCH-EVT framework, which comprises a two-step process. However, this framework has a critical drawback, insofar as the EVT estimation is sensitive to the GARCH estimation. In particular, the quasi-maximum likelihood estimate (QMLE) of GARCH has less efficiency when the innovation is misspecified. Consequently, the VaR estimate resulting from this procedure is not efficient.

In **Chapter 5**, we propose a semi-parametric model, in which the innovation is approximated by a mixture density. The bulk part of the mixture uses a non-parametric density, while the tail part uses an extreme value distribution. We develop an algorithm to estimate the parameters and the dynamic VaR. In the last part of the chapter, the proposed method is evaluated through a simulation study and a large-scale empirical study.

Chapter 6 summarises the VaR methods proposed in this thesis, and situates them within the broader context of quantitative financial risk management. It then outlines the contribution of this thesis toward enriching the knowledge and tools of extreme risk forecasting. Finally, it discusses the limitation of this study and the potential for future research.

1.3 Publications from This Research

Most results from Chapters 3, 4 and 5 of this thesis are included in L. Zhang, Shao, and Xu (2022); L. Zhang, Shao, Xu, and Wang (2021).

Chapter 2

Background

As discussed in Chapter 1, the key to quantitative financial risk management is the measurement of risk in situations requiring risky decisions. Therefore, this chapter begins by introducing the mathematical concept of risk, which can be neatly summarised as a function of a random variable having uncertain outcomes (Rockafellar & Uryasev, 2013). Different functions correspond to different interpretations of the risk measure. This section focuses on Value at Risk (VaR), a risk measure which captures extreme risk, the main topic of this thesis. A comprehensive review of available VaR estimation methods is then provided, including their advantages and disadvantages. The final section of this chapter outlines existing backtesting methods as a formal procedure for evaluating the performance of a VaR forecast.

2.1 Risk and Risk Measure

Risk is a foundational concept in a wide range of fields, and pervades many research and application domains. Examples of risk in different practice areas include:

- Business risk: In business, risk arises from uncertain events, including changes in consumer preferences, new competitors in the market, changes in government regulations, and labour shortages, amongst others. Managers judge the level of risk of these events and make business decisions accordingly.

- Economic risk: Economic risk concerns uncertainty in the consumption, distribution, and production of goods and services. For example, interest rates will increase borrowing costs throughout the economy; changes to government regulations will have an impact that extends beyond the targeted policy area. Economists pay close attention to economic events to provide policy suggestions.

- Environmental risk: Uncertain environmental events, such as bushfires, earthquakes, floods, storms, and droughts impact the safety of both human beings and the environment. Environmental scientists monitor natural hazards in order to provide advance warnings and strategies to the general public.

- Financial risk: Finance concerns the operation and management of money, both within individual companies or within the macroeconomy. Financial risk often arises from uncertain events that impact financial markets, such as market change, credit default, a lack of liquidity, central bank changes to the interest rate, operational failures, and so on. Financial regulators and risk departments in financial institutions monitor these risks, and enforce relevant procedures to protect capital when necessary.

- Health risk: Health risk refers to uncertain events that could impact human health and wellbeing. Examples include disease and biological hazards. Health departments and organisations monitor potential issues and prepare health management measures.

- Information technology risk: Information risk arises from uncertain events relating to information disclosure and cyber attacks, which could have a significant impact upon a wide range of security concerns. IT departments monitor these events and develop policies to protect information.

- Insurance risk: Insurance risk refers to both the risk of the insured and the risk of the insurer. In uncertain events, a large number of claims may be submitted at once, and may lead to bankruptcy for insurance companies whose total claims exceed their ability to pay. Actuaries build statistical models to forecast these risks and recommend measures to protect insurers.

- Project risk: Project risk is concerned with conditions that could negatively impact a project. A project manager is responsible for identifying and addressing all potentially risky events to ensure timely project delivery.

The concept of risk appears across many different domains. Nevertheless, it is characterised by two core features: an event with an uncertain outcome, and a decision based upon projections of that outcome.

To develop a systematic and quantitative understanding of risk, we denote X as a random variable representing the outcome of a risk-related event. The outcomes of X are uncertain in the sense that some of the outcomes are desirable, while some are undesirable. The undesirable outcomes constitute the risk. Lower values of X are therefore preferable, since higher values of X imply greater loss, cost, or damage. For instance, in a financial context, negative outcomes of X represent a profit while positive outcomes of X represent a loss.

The uncertain outcome of a random variable is difficult to implement when formulating decisions and objectives. For example, some investment outcomes will produce a profit, while others will incur a loss; the numerous possibilities are difficult to factor into an investment decision. Summarising an uncertain outcome into a single number can simplify the formulation of the decision, making it easier to compare alternative investment options. Risk measure, denoted as R, serves this purpose. Mathematically, R is a function measuring hazard in a random variable by summarising the undesired outcome into a single number: $R(X)$.

Defining risk measure allows the risk-related events listed above to be expressed as a mathematical statement. Given a constant, threshold, or decision boundary C, we would like to know if risk measure $R(X)$ is adequately less than C (Rockafellar & Uryasev, 2013):

$$\text{model "}X \text{ adequately} \leq C\text{ " by the inequality } R(X) \leq C.$$

The meaning of the above statement is as follows. While the nature of C depends upon individual risk preferences and upon the field of inquiry, uncertainty inescapably generates some outcomes of X that are greater than C. Identifying risk measure assists decision-makers to take the necessary steps when $R(X) > C$.

Risk measure R is a function summarising a random variable X into a single number. As such, there are many function choices for R. Amongst most common is the statistical expectation of X, i.e., $R(X) = E(X)$. In this scenario, a decision is made based on the average of X. If the average outcome of X is greater than C, then steps must be taken to address the risk. A tighter version of this is $R(X) = E(X) + \lambda\sigma(X)$, in which decisions are based on a safety margin, expressed as λ multiplied by the standard deviation of X above the expectation.

Another important risk measure is $Q(X)$, which measures the quantile of X with a given probability α, where Q is a quantile function. If the quantile associated with this given probability is greater than C, i.e., $Q_\alpha(X) > C$, then relevant decisions will be made. This risk measure is also known as Value at Risk. Superquantile $\bar{Q}(X)$ (expected shortfall) is closely related to $Q(X)$, since it measures the expectation of X over $Q(X)$. There are many other less commonly risk measures (Rockafellar & Uryasev, 2013).

Risk measures have a number of important mathematical properties, including monotonicity, sub-additivity, positive homogeneity, translation invariance, convexity and elicitability. These properties can help researchers and practitioners to select appropriate risk measures in various applications, and have been rigously studied by Artzner, Delbaen, Eber, and Heath (1999); Gneiting (2011); Rockafellar and Royset (2014); Rockafellar and Uryasev (2002); Rockafellar, Uryasev, and Zabarankin (2006).

2.2 Value at Risk

We define VaR mathematically in a financial context. Let P_t be a random variable describing the price of financial asset A at time t. X_t is the percentage loss of A at time t with an unknown distribution function F. The relationship between P_t and X_t is $X_t = -ln(P_t/P_{t-1})$. $\alpha \in (0, 1)$ is selected probability or confidence. The VaR with confidence level α is defined as

$$\begin{aligned} VaR^{\alpha}(X_t) &= \sup\{x \in \mathbf{R} : F(x) < (1-\alpha)\} \\ &= F^{-1}(1-\alpha), \end{aligned} \tag{2.1}$$

where $F^{-1}(\cdot)$ is the inverse distribution function or quantile function. This definition implies that the maximum loss of A at day t given probability α is the quantile of X at the cumulative probability $1 - \alpha$. Therefore, VaR can be interpreted as the worst loss of a financial asset with a given confidence.

Although the concept of VaR is simple, forecasting VaR is not a straightforward task[1]. The principal difficulties arise from the unknown F and the dependence of X_t. In order to address these challenges, a wide range of methods have been developed to forecast VaR, each with advantages and disadvantages. These methods can be classified into three categories, based upon their assumptions regarding the loss distribution F: parametric, non-parametric, and semi-parametric. Alternatively, they can be classified into unconditional (static) or conditional (dynamic) VaR, according to their assumptions regarding dependency and risk management. A selection of different methods are reviewed and summarised in Table 2.1.

[1]Researchers from finance prefer using "forecasting VaR" whereas researchers from statistics prefer using "estimating VaR". We use both terms interchangeably.

Table 2.1: Summary of VaR Forecast Methods

	Parametric	Non-parametric	Semi-parametric
Unconditional	Covariance Matrix	Historical Simulation, Kernel Density	Extreme Value Theory (EVT)
Conditional (Dynamic)	GARCH, GARCH Extensions	Filtered Historical Simulation (FHS)	Volatility Weighted Historical Simulation, Conditional Autoregressive Value at Risk (CAViaR), Time-Varying Higher Order Conditional Moment (HOM), GARCH + GPD

1. **Variance-covariance** The variance–covariance method is a simple and easily implemented parametric VaR method, and was used widely in the financial industry in the early nineteen's . However, it was not properly documented until the publication of a technical risk management document by JP Morgan (Longerstaey & Spencer, 1996). The variance–covariance method assumes the loss X_t is normally distributed and serially independent among observations. Under the normality assumption, percentiles are known multiples of the standard deviation. In other words, VaR_t^α can be obtained from

$$VaR_t^\alpha = \mu + \lambda\sigma,$$

where μ is the sample mean, σ is the sample standard deviation, and λ is a multiplier. Thus, the calculation only requires the estimation of μ and σ. The assumption of serial independence implies that the value of X_{t-k} does not impact the value of X_t. In other words, X_t is i.i.d.. Consequently, longer horizon standard deviations can be obtained by multiplying daily horizon standard deviations by the square root of the number of days in the longer horizon. The equally weighted moving average and the exponentially weighted moving average (Hendricks, 1996) are two variants of the variance–covariance method.

The Monte Carlo simulation is a simulation-based variance–covariance method that assumes that P_t follows a random walk. Various price paths are simulated, and the distribution of loss is computed through these paths. The VaR is then obtained. Beder (1995) has compared the performance of the Monte Carlo simulation method with other variance–covariance methods, and has argued that their performance is similar.

A key drawback of variance–covariance methods is their strong assumptions regarding the distribution function F. Real data may not have a normal distribution.

2. **Historical simulation** Historical simulation (HS), a non-parametric method, addresses the shortcoming of variance-covariance method by using the empirical distribution function to approximate F. VaR can then be obtained by

$$VaR_t^{\alpha} = \hat{F_n}^{-1}(1-\alpha),$$

where $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$ and n is the total number of the sample. For example, the 1% VaR of 100 samples is the sixth-largest loss observed in the sample.

The HS method has several variants. Time-weighted historical simulation (Richardson, Boudoukh, & Whitelaw, 1998) applies exponentially declining weights to past returns, before identifying the appropriate percentile of this time-weighted empirical distribution. Filtered historical simulation (FHS) (Barone-Adesi, Giannopoulos, & Vosper, 1999, 2002) considers the volatility of the markets to filter the observations, and then resamples the data. Conditional historical simulation (CHS) (Hull & White, 1998; Žiković & Aktan, 2011) proposes a optimisation to determine the optimal decay factor for the BRW method. Bootstrap historical simulation (BHS) obtains bootstrap samples from past observations, and develops a VaR forecast from the bootstrap sample quantile (Pritsker, 2006).

The advantage of HS is that it makes no assumptions about the data. HS is typically accurate for sufficiently large datasets, but lacks extrapolative capability for small sample sizes. In addition, it has high variance at extreme quantile, as past observations in the tail are limited (Pritsker, 2006). Furthermore, HS does not easily accommodate translations between multiple percentiles and holding periods (Hendricks, 1996).

3. **GARCH and its extension** The assumption of independence in variance-covariance and HS methods works well for unconditional VaR forecasts. However, this assumption omits the volatility clustering effect frequently

found in financial time series, in which large loss tends to follow large loss and small loss tends to follow a small loss. Engle (1982) identified this feature in inflation in the United Kingdom, and modelled it using autoregressive conditional heteroscedasticity (ARCH). Bollerslev (1986) generalised ARCH model into GARCH model, which will be discussed in greater detail in section 4.2. The GARCH model and its extensions have been employed to forecast dynamic VaR (see for instance GARCH with student's t (Hansen, 1994), GARCH with generalised error distribution (Fan, Zhang, Tsai, & Wei, 2008), GARCH with skewed student's t (W.-H. Cheng & Hung, 2011), GARCH with skewed error distribution (Theodossiou, 1998), SGT-GARCH (Bali & Theodossiou, 2007), Markov switching GARCH (Ardia, Bluteau, Boudt, & Catania, 2018) and GARCH with two-sided Weibull (Q. Chen & Gerlach, 2013)).

The advantage of the GARCH model and its extensions is that it successfully captures the volatility clustering effect. However, while leptokurtosis[2] in the GARCH model is reduced, it is not eliminated. Consequently, the GARCH model typically underestimates VaR. Moreover, it is subject to model selection bias.

4. **Extreme value theory** The extreme-based method builds upon extreme value theory (EVT). EVT states that, under some general conditions, a random variable at its tail is asymptotically distributed as the extreme value distribution. This method enjoys two critical advantages. First, the extreme-based method avoids a choice of distributions, which may lead to model selection bias. Second, it has a parametric form that can extrapolate high quantile robustly. Theoretically, the extreme-based method is an appropriate tool for estimating VaR. It will be covered in greater detail in Chapter 3.

   A number of authors have applied the extreme-based method to VaR fore-

[2] The statistical property of having a greater kurtosis than a normal distribution.

casts. Notable examples include generalised extreme value (GEV) based on block maximum (Bekiros & Georgoutsos, 2005; Byström, 2004; Longin, 2000), Generalised Pareto (GP) based on peak-over-threshold (Bali, 2003; Cifter, 2011; Fong Chan & Gray, 2006; Mcaleer & da Veiga, 2008; Youssef, Belkacem, & Mokni, 2015) and Weibull (WB) based on peak-over-threshold (Gebizlioglu, Şenoğlu, & Kantar, 2011; Wang, Chen, & Gerlach, 2019). McNeil and Frey (2000) has applied the extreme-based method to standarised residuals to forecast dynamic VaR..

Other distributions have also been proposed in the risk analysis, such as T-distribution (W.-H. Cheng & Hung, 2011; Hansen, 1994) and Laplace distribution (Taylor, 2019). In terms of extreme risk forecasting, t-distribution and distribution underperform methods such as extreme value theory (Bali & Weinbaum, 2007), CAViaR (Engle & Manganelli, 2004) and Filtered Historical Simulation (Pritsker, 2006). Abad and Benito (2013); Abad et al. (2014) has provided a detailed review of VaR methods, showing that extreme-based methods outperform other methods. More recently, Calmon et al. (2021) conducted an extensive comparison of well-established VaR methods found that extreme-based methods enjoy top performance. Brooks, Clare, Dalle Molle, and Persand (2005) and Nolde and Zhou (2021) also provided an excellent review of existing extreme-based methods for VaR forecasts. Therefore, in this thesis, we do not compare our proposed methods with T-distribution and Laplace distribution, and focus the development of extreme-based methods for risk forecasting.

2.3 Backtesting

Backtesting is a standard and systematic procedure employed to evaluate the forecast capability of a VaR model. A VaR model can be evaluated through its capability in forecasting VaR in terms of probability, number of occurrences, and magnitude. **Table 2.2** identifies major backtesting methods in the literature, which will be reviewed in detail in this section.

Table 2.2: Summary of Backtesting Methods

Category	Backtesting Measures	Abbreviation
Accuracy	Unconditional Coverage	UC
	Conditional Coverage	CC
	Dynamic Quantile	DQ
Occurrence	Actual over Expected Ratio	AE
	Absolute Percentage Error	APE
Magnitude	Absolute deviation	AD
	Quantile Loss	QL

2.3.1 VaR Accuracy

The probability associated with a loss event is a key factor in VaR models, since VaR measures the worst loss of a risky event in a period with a given confidence level. It is important to ensure the hypothetical confidence level is equal to the realised probability. Several backtesting procedures are available for this purpose. To simplify the exposition, we first define a hit variable $h_t = I\{x_t \leq VaR_t^{\alpha}\}$ with a value of one when the loss is over the forecasted VaR, and zero in all other cases.

The most widely used probability test is the unconditional coverage (UC) test (Kupiec, 1995), a likelihood ratio statistic testing the null hypothesis $E(h_t) = \alpha$ against the alternative $E(h_t) \neq \alpha$. This test assumes that the hit variable is serial independent and follows a binomial distribution with success rate α; that is, the hypothetical VaR level. The UC test can then be formulated as

$$UC_{\alpha} = -2\ln\left[(1-\alpha)^{n-m}\,\alpha^{m}\right] + 2\ln[(1-\frac{m}{n})^{n-m}(\frac{m}{n})^{m}],$$

where n is the total number of observations and m is the number of violations. The test statistic has an asymptotic chi-square distribution with the degree of freedom one.

However, the serial independence assumption of the UC test is not always satisfied in practice. In fact, there is strong autocorrelation among financial time series. Therefore, the conditional coverage test (CC) (Christoffersen, 1998), a joint test of independence and coverage, was developed to overcome the weakness of the UC test. The CC test assess independence against the first-order Markov chain with a Markov transition probability matrix:

$$\Pi_1 = \begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix},$$

where π_{01} and π_{11} are transition probabilities from state zero to one and one to one respectively. The corresponding likelihood is

$$L\left(\Pi_1\right) = \left(1-\pi_{01}\right)^{n_{00}} \pi_{01}^{n_{01}} \left(1-\pi_{11}\right)^{n_{10}} \pi_{11}^{n_{11}},$$

where n_{ij} is the number of the observations from state i to j and $i, j \in 0, 1$. Alternatively, if the series is independent, the transitional probability from state zero to one is same as from one to one, i.e.

$$\Pi_2 = \begin{bmatrix} 1-\pi_2 & \pi_2 \\ 1-\pi_2 & \pi_2 \end{bmatrix}.$$

The likelihood of the Null becomes

$$L\left(\Pi_2\right) = \left(1-\pi_2\right)^{n_{00}+n_{01}} \pi_2^{n_{01}+n_{11}}.$$

Therefore, the likelihood ratio of the independence is as follows

$$LR_{ind} = 2\ln \frac{L\left(p; I_1, I_2, \ldots, I_n\right)}{L\left(\widehat{\pi}_1; I_1, I_2, \ldots, I_n\right)} \ \chi^2\left((s-1)^2\right) \ \chi^2\left(1\right).$$

The combined likelihood ratio of both independence and coverage is

$$LR_{cc} = 2\ln \frac{L(p; I_1, I_2, \ldots, I_n)}{L(\widehat{\pi}_1; I_1, I_2, \ldots, I_n)} \ \chi^2\left(s\left(s-1\right)\right) \ \chi^2(2),$$

which is asymptotically χ^2 distributed with degree of $s(s-1)$.

Though the CC test detects the serial correlation of the hit series, it provides only a necessary (but not sufficient) condition to assess a quantile model. The dynamic quantile (DQ) (Engle & Manganelli, 2004) is a test of the joint hypothesis that the expectation of the violation process is zero and that the hit variables are distributed independently. Under the correct model specification, unconditionally and conditionally, the hit variable has a zero mean and is serially uncorrelated. The DQ test is the traditional Wald test of the joint nullity of all coefficients in the following linear regression. Like Engle and Manganelli (2004), we choose $L = 4$ lags. Under the null hypothesis of correct unconditional and conditional coverages, DQ is asymptotically chi-square distributed with $L + 2$ degree of freedom.

2.3.2 VaR Occurrence

VaR occurrence compares the absolute number of failures of the incident in which VaR fails to predict the risk. As the name suggests, the actual over expected exceedance ratio (AE) compares the confirmed instances of loss exceeding VaR to the expected instances, as shown in the equation

$$AE = \frac{\sum_{i=1}^{n} I\left(x_i < VaR_i^{\propto}\right)}{n \times \alpha}. \tag{2.2}$$

The VaR model, with an AE value close to one, indicates a better VaR forecast, as the confirmed exceedance is close to the expected exceedance.

Absolute percentage error (APE) (Bali, 2007) refers to the absolute difference between the actual exceedance and the expected exceedance over the expected exceedance, and serves as another statistical tool to evaluate VaR model performance. Optimal performance is indicated by an APE value of zero, which occurs when the actual exceedance equals the expected exceedance. The APE represents the percentage of over- or under-estimated exceedance. An APE with a value of

1% means that 1% of VaR were over- or under-estimated, as shown in following equation:

$$APE = \frac{|\sum_{i=1}^{n} I(x_i < VaR_i^{\alpha}) - n \times \alpha|}{n \times \alpha}. \tag{2.3}$$

2.3.3 VaR Magnitude

Some authors have argued that the VaR amount, in addition to the occurrence, is important. For instance, if one loss instance is over forecasted VaR, the VaR is still likely to fall within an acceptable range. This is where VaR amount backtesting measures can help. Absolute deviation (AD) (Mcaleer & da Veiga, 2008) is defined as the absolute difference between the observation and the VaR value weighted by an indication function, as follows:

$$AD = \frac{\sum_{i=1}^{n} |x_i - VaR_i| \times I(x_i < VaR_i^{\alpha})}{\sum_{i=1}^{N} I(x_i < VaR_i^{\alpha})}.$$

This function weights differences as one when observation exceeds VaR, and zero when observation falls within the expected range. Thus, mean and maximum AD are defined as the average and maximum values of absolute deviation. The lower both the mean and maximum AD are, the better the performance of the VaR model is.

Quantile Loss (QL) (González-Rivera, Lee, & Mishra, 2004) described by the equation,

$$Q \equiv n^{-1} \sum_{i=1}^{n} [\alpha - I(x_i < VaR_i^{\alpha})](x_i - VaR_i^{\alpha}).$$

This is a VaR amount backtesting measure. Unlike AD, which only considers the loss amount over VaR, it considers both the amount over and under VaR. However, the amount under VaR is penalised with a lower weight, while the amount over VaR is penalised with a higher weight. The value is then averaged over the total number of samples. The smaller the value of QL, the better the performance of the VaR method.

Chapter 3

Forecasting Unconditional VaR using an Extended Extreme Value Approach

Financial regulatory authorities, such as the Basel Banking Supervision Committee, employ unconditional VaR to supervise financial institutions in order to keep financial markets stable under extreme market conditions. From a statistical perspective, unconditional VaR is a high or tail quantile of a statistical distribution, which measures the long-term average of the market risk. Methods based on EVT provide accurate unconditional VaR forecasts, since EVT focuses on modelling tails and extremes. However, the limiting conditions of EVT are not easy to satisfy in practice. This causes a sensitive performance in unconditional VaR forecasts. This thesis aims to improve the performance of the unconditional VaR forecast.

This chapter begins by introducing the classic EVT based on the block maxima method, with the generalised extreme value (GEV) distribution as the limiting distribution. In order to address the the block maxima method's inefficiency in using all available data, the modern EVT based on the threshold (Section 3.2) method was developed with the generalised Pareto (GP) distribution as the limiting distribution. However, GP is sensitive to the threshold selection in the threshold model, which causes a sensitive VaR forecast. Section 3.3 addresses this challenge by promoting the use of EB distribution in the threshold method to forecast unconditional VaR. Section 3.4 conducts a simulation study to compare the unconditional VaR forecasting capability of the candidate distributions under various simulated scenarios, demonstrating the improvement of our proposed method. In Section 3.5, we apply our proposed method in an empirical study.

3.1 Block Maxima

Consider a collection of random variables $\{X_1, X_2, \cdots, X_n\}$ representing random losses of a portfolio on a percentage scale[1] indexed by the integer-valued time[2]. Assume $\{X_1, X_2, \cdots, X_n\}$ are independent and have a common cumulative distribution (CDF) F. Let M_n be the maximum of the collection $\{X_1, X_2, \cdots, X_n\}$, i.e., $M_n = max\{X_1, X_2, \cdots, X_n\}$, which can be interpreted as the worst loss of a portfolio. We are particularly interested in the distribution of M_n, as it directly relates to VaR. The CDF of M_n is derived from the CDF of $X_1, X_2, \cdots, X_n$ by

$$\begin{aligned} Pr\{M_n \leq x\} &= Pr\{X_1 \leq x, X_2 \leq x, \cdots, X_n \leq x\} \\ &= \Pi_{i=1}^n Pr\{X_i \leq x\} \\ &= \Pi_{i=1}^n F(x) \\ &= F^n(x). \end{aligned}$$

The derivation of the above probability model is straightforward but has a noticeable weakness. A small discrepancy in the distribution function F leads to

[1]The market data of the portfolio are typically the closing prices. If we denote the closing price as P_t, the loss of the portfolio is simply the negative logarithmic return, i.e., $X_t = -\ln(P_t/P_{t-1})$.

[2]Integer-valued time can appear on a daily, weekly, monthly, or yearly scale, depending on a VaR forecast application.

a large discrepancy in F^n as n increases. This weakness causes infeasibility in finding the distribution function of M_n from the best estimate of F in a VaR forecast application. To address this, we aim to find an approximating distribution of F^n, which may only require information from the extremes.

To find the approximating distribution, we consider the behaviour of $F^n(x) \to 0$ as $n \to 0$. The value of $F^n(x)$ is affected by the value of x. Assuming x_+ is the upper endpoint of F, then F^n degenerates to a point of mass on x_+. This degenerated distribution problem can be avoided by a linear re-normalisation of variable M_n through M_n^* by $M_n^* = (M_n - b_n)/a_n$ with a sequence of constant $a_n > 0$ and b_n. By selecting appropriate values for a_n and b_n, M_n^* is stabilised. As such, seeking the distribution of M_n^* is the key in EVT.

The limiting distribution of M_n^* is given by the Fisher-Tippett theorem (Fisher & Tippett, 1928), also known as Fisher-Tippett-Gnedenko theorem (Gnedenko, 1943). This theorem states that if there exists sequences of constants $a_n > 0$ and $b_n > 0$ such that

$$Pr(\frac{M_n - b_n}{a_n} \leq x) \to G(x) \qquad as \qquad n \to \infty,$$

where G is a non-degenerate distribution function, then G belongs to one of the following families

$$I : G(x) = exp{-}exp\left[-\frac{x-b}{a}\right], \; -\infty < x < \infty;$$

$$II : G(x) = \begin{cases} 0, & x \leq b, \\ exp{-}(\frac{x-b}{a})^{-a}, & x > b; \end{cases}$$

$$III : G(x) = \begin{cases} exp(\frac{x-b}{a})^{-a}, & x < b, \\ 1, & x \geq b. \end{cases}$$

To model extreme losses, we apply the Fisher-Tippett theorem using one of these families of distribution. This strategy has an important weakness: the distribution must be selected before data are analysed, which prevents uncertainty analysis of the selected distribution. To address this problem, the generalised extreme value (GEV) distribution (Jenkinson, 1955) was proposed to unify the Gumbel, Fréchet, and Weibull distributions by introducing a shape parameter ξ. The GEV distribution is described by the formula

$$G(x) = exp\{-[1 + \xi(\frac{z - \mu}{\sigma})]^{-1/\xi}\},$$

where μ is the location parameter and σ is the scale parameter. When $\xi < 0$, GEV corresponds to the Gumbell distribution, which has a rapidly decreasing tail. When $\xi = 0$, GEV corresponds to the Weibull distribution in which the tail diminishes after certain point. When $\xi > 0$, GEV corresponds to the Fréchet distribution whose tail decreases slowly.

This generalisation simplifies the statistical implementation of modelling extremes significantly. Introducing the shape parameter ξ allows the parameter estimation procedure to determine the appropriate tail distribution, thereby facilitating uncertainty analysis of the extreme distribution selection.

Employing a GEV distribution requires determining the appropriate normalising constants a_n and b_n. These normalising constants cannot be determined in advance; however, the Fisher-Tippett theorem indicates that

$$Pr(M_n^* = \frac{M_n - b_n}{a_n} \leq x) \approx G(x).$$

This is equivalent to

$$\begin{aligned} Pr(M_n \leq x) &\approx G(\frac{x - b_n}{a_n}) \\ &= G^*(x), \end{aligned}$$

where G^* is also a member of GEV family distribution (Coles, 2001). The above approximation ensures that M_n can, in practice, be modelled using GEV distribution without the normalising constants a_n and b_n.

The VaR application method is as follows. A series of n loss observations $\{X_1, X_2, \cdots, X_n\}$ from a financial instrument are divided into m blocks. M_j is the maximum loss of each block with $j = 1, \cdots, m$. M_j is then fitted to the GEV distribution. Since GEV is the limiting result, the block size must be large enough to ensure that the fit is sufficiently exact. VaR is then estimated by

$$VaR^{\alpha} = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\xi}}\{1 - [-\frac{n}{m}\ln(1-\alpha)]^{-\xi}\},$$

where $\hat{\mu}, \hat{\sigma}, \hat{\xi}$ are estimated parameters of GEV (for further detail, see Longin (2000)).

Danielsson and de Vries (1997) has compared the predictive performance of various VaR methods for simulated portfolios of seven US stocks. The results show that GEV becomes more accurate as tails become more extreme, whereas the variance–covariance method and the historical simulation method under- and over-predict losses, respectively. Similar results have been obtained by Longin (2000), who applied the GEV to compute the VaR of single and bivariate portfolio positions.

3.2 Peak-Over-Threshold

One major weakness of the block maxima method is its inefficient use of sample data. Only one data sample is selected for each block. In addition, block size may impact the extreme modelling, such that one block may contain more extremes than another. Therefore, modelling of extremes extends into the threshold approach, in which data are modelled over a selected threshold. Blattberg and Gonedes (1974) and Pickands (1975), amongst others, have identified asymptotic behaviours of a collection of random variables over a certain threshold.

As with the block maxima method, we assume that a collection of random variables $\{X_1, X_2, \cdots, X_n\}$ is independent and identically distributed with marginal distribution F. The threshold model aims to find the distribution of

$X_i - u \mid X_i \geq u$, which is characterised by the equation

$$\begin{aligned} F_u(x) &= P\{X - u > x \mid X \geq u\} \\ &= \frac{F(x+u) - F(u)}{1 - F(u)} \qquad (x \geq 0), \end{aligned} \tag{3.1}$$

where u is a high threshold.

For a large class of distributions F, it is possible to find a positive measurable function $\beta(u)$ such that (Coles, 2001)

$$\lim_{u \to x_0} \sup_{0 \leq x < x_0 - u} \mid F_u(x) - G_{\xi, \beta(u)}(y) \mid = 0,$$

where x_0 is the right endpoint of F. If F satisfies the conditions of the GEV Theorem, then it is easy to prove that, for a sufficiently large u, the distribution function of $X - u$, conditional on $X > u$, is approximately

$$H(x) = 1 - (1 + \xi \frac{x}{\tilde{\sigma}})^{-\frac{1}{\xi}},$$

where $\tilde{\sigma} = \sigma + \xi(u - \mu)$ (Embrechts, Resnick, & Samorodnitsky, 1999). This theorem can be understood as a limiting distribution as u increases. This family of distributions is known as the generalised Pareto (GP) distribution, with ξ being the tail index, $\sigma > 0$ being the scale parameter, and $x \geq 0$ being the support.

GP distributions nests several common distributions with different values of the shape parameters ξ. $\xi < 0$ corresponds to short-tailed distributions with a finite right endpoint, such as uniform or beta distributions. $\xi = 0$ corresponds to distributions whose tails decay exponentially, such as normal or lognormal distributions. $\xi > 0$ corresponds to heavy-tailed distribution that decay like power functions, such as Pareto and Cauchy distributions.

We assume that the number of samples over the threshold u is N. Smith and Weissman (1994) has shown that if the excesses are i.i.d. and correspond exactly to a GP distribution, the maximum likelihood estimate (MLE) $\hat{\xi}$ and $\hat{\sigma}$ of the GP parameters ξ and σ will be consistent and asymptotically normally distributed when $N \to \infty$ given $\xi > -\frac{1}{2}$. Under the weaker assumption that the excesses are i.i.d. from an approximate GP distribution, $\hat{\xi}$ and $\hat{\sigma}$ are asymptotically normal. The procedure is asymptotically unbiased, given that $u \to x_0$ and $N \to \infty$. This suggests that the best GP estimator faces a variance and bias trade-off. Increasing the threshold u reduces estimation bias, but enlarges the variance as N becomes smaller.

The threshold theorem suggests the following framework for extreme value modelling. The raw data consists of a sequence of independent and identically distributed random variables. The distribution function $F_z(\cdot)$ can be approximated by GP.

$$F_z(z) = [1 - F_z(u)]F_u(y) + F_z(u), \qquad y > 0.$$

$F_z(u)$ is determined by $\frac{T-N_y}{T}$. The equation can then be simplified to

$$F_z(z) = 1 - \frac{N_y}{T}(1 + \xi\frac{z-u}{v})^{-\frac{1}{\xi}}.$$

We invert the above formula to calculate VaR as follows:

$$VaR^{\alpha} = F_z^{-1}(q) = u + \frac{y}{\xi}[(\frac{T}{N_y}\alpha)^{-\xi} - 1].$$

3.3 Extended Burr XII

We have introduced the GP distribution as the limiting distribution of the peak-over-threshold method. However, the limiting conditions of sufficiently large samples and sufficiently high thresholds are not easy to satisfy in practice, making VaR estimates from GP distribution sensitive to threshold selection. We seek a robust distribution with the various threshold selections to forecast VaR.

Theoretically, for a given monotonically increasing function $g(\cdot)$ such that

$g(X)$ also satisfies the regular condition, the tail of $g\,(X)$ can be approximated by the GP distribution family. In practice, for a finite sample size, a transformation might be necessary to achieve a better approximation for the tail distribution: for instance, the power transformation $X \to X^{1/c}$ with $c > 0$. The Extended Burr XII (EB) distribution has been proposed as an extension of the GP distribution family (Shao, Chen, & Zhang, 2008; Shao, Ip, & Wong, 2004; Shao, Wong, Xia, & Ip, 2004). The EB distribution is defined by the cumulative distribution function

$$
\begin{aligned}
F_{EB}(x; c, k, \lambda) &= 1 - \{1 - k(\frac{x}{\lambda})^c\}^{\frac{1}{k}} \quad k \neq 0 \\
&= 1 - exp\{-(\frac{x}{\lambda})^c\} \quad k = 0
\end{aligned}
$$

and the density function

$$
\begin{aligned}
f_{EB}(x; c, k, \lambda) &= c\lambda^{-1}(\frac{x}{\lambda})^{c-1}\{1 - k(\frac{x}{\lambda})^c\}^{\frac{1}{k}-1} \quad k \neq 0 \\
&= c\lambda^{-1}(\frac{x}{\lambda})^{c-1} exp\{-(\frac{x}{\lambda})^c\} \quad k = 0,
\end{aligned}
$$

where the support is $x \in (0, +\infty)$ for $k > 0$ and $x \in (0, \lambda/(-k)^{1/c})$ for $k < 0$. The EB distribution reduces to Weibull (WB) distribution when $k = 0$.

EB has a close connection with other long-tail distributions. It embeds the generalised Pareto (GP) and log-logistic distributions (Shao, Wong, et al., 2004). It is also an extension of Burr XII (Burr, 1942) in the form $F_{EB}(x; c, \beta, b) = 1 - [1 + (\frac{x}{b})^c]^{(-\beta)}$ under the reparameterisation of $k = -1/\beta$ and $\lambda = -b/\beta^{1/c}$.

EB enjoys some statistical properties which are beneficial to the peak-over-threshold model. It achieves multiple combinations of skewness and kurtosis, allowing flexibility in the approximation of various tail shapes. The parameters of EB have interpretable statistical meanings where ξ is the shape parameter, λ is the scale parameter and the location parameter is determined by ξ and c.

Percentiles and moments of EB have explicit analytic forms as below

$$\begin{aligned}\mu_r = E(X^r) &= \frac{\lambda^r}{(-k)^{r/c+1}} \frac{\Gamma(1+r/c)\Gamma(-1/k-r/c)}{\Gamma(-1/k)} \quad && k<0 \\ &= \lambda^r \Gamma(1+r/c) && k=0 \\ &= \frac{\lambda^r}{k^{r/c+1}} \frac{\Gamma(1+r/c)\Gamma(1/k)}{\Gamma(1/k+r/c+1)} && k>0.\end{aligned}$$

We employ the maximum likelihood estimation (MLE) to estimate the parameters of EB. The log-likelihood function is

$$L(x;c,k,\lambda) = N \ln \frac{c}{\lambda} + (c-1) \sum_{i=1}^{N} \ln \frac{x_i}{\lambda} + (1+\lambda) \sum_{i=1}^{N} \ln \left[1 - k(\frac{x_i}{\lambda})^c\right]^{-\frac{1}{k}} \tag{3.2}$$

where N is the total number of observations and x_i are the samples. The maximum log-likelihood can be obtained through numerical algorithms. In order to find the local maximum of the above likelihood, a numerical method is required. However, without knowing the sign of k (i.e. whether k is positive, negative or zero), it is difficult for computer programs to optimise likelihood. This is a typical issue when dealing with MLE for some distributions known as the embedded model problem (R. Cheng, Evans, & Iles, 1992). To solve this problem, we employ a delta discriminant method (Shao, Ip, & Wong, 2004) in the equation

$$\Delta(x;c,\lambda) = \sum_{i=1}^{n} [(\frac{x_i}{\lambda})^c - \frac{1}{2}(\frac{x_i}{\lambda})^{2c}]. \tag{3.3}$$

By fitting the Weibull distribution, the estimated parameters $\hat{c}, \hat{\lambda}$ can be assigned to the discriminant $\Delta(x;c,\lambda)$. If $\Delta(x;\hat{c},\hat{\lambda}) > 0$, then $\hat{k} > 0$ while if $\Delta(x;\hat{c},\hat{\lambda}) < 0$, then $\hat{k} < 0$, where $\hat{k}$ is the estimated k.

After obtaining the EB parameters $\hat{c}$ $\hat{k}$ $\hat{\lambda}$ from MLE with the x_i over a high threshold u, the unconditional VaR with confidence level α can be computed from the formula

$$VaR^{\alpha} = u + \frac{1}{\hat{\lambda}}[1 - \left(\frac{\alpha N}{N-n}\right)^{-\hat{\xi}}]^{\frac{1}{\hat{c}}}, \tag{3.4}$$

where n is the number of samples below the threshold u.

3.4 Simulation Study

We conduct a simulation study to assess the performance of EB and other candidate distributions in forecasting VaR. Simulations have two primary advantages over empirical studies in evaluating VaR models. First, in simulations the theoretical VaR is known, making the evaluation of VaR among candidate models straightforward. Second, the true data generating process (DGP) is unknown, leading to model misspecification errors. By fitting generated samples to all models, we will be able to assess the robustness and flexibility of fitting candidate models under different DGPs.

The candidate distributions are the generalised Pareto (GP), generalised extreme value (GEV), and Weibull (WB) distributions. GP, as the limiting distribution of the threshold method, is the most frequently adopted distribution (Nolde & Zhou, 2021). GEV (Bali & Weinbaum, 2007) and WB (Q. Chen & Gerlach, 2013; Gebizlioglu et al., 2011; Wang et al., 2019) have shown better performance in some applications. We consider sample sizes of 25, 125 and 250 in the simulation. Each sample size corresponds to the number of observations obtained in short-term (1 year), medium-term (5 year), and long-term (10- year) daily VaR forecast applications, based on a 10% threshold selection and an average of 250 trading days per year. We simulate data from these four distributions across three selected sample sizes, resulting in a total of 12 DGPs. Each of these 12 DGPs produces 999 simulations, and each simulation is fitted to the EB, GP, GEV and WB distributions separately.

We first evaluate the accuracy of the VaR forecasts across the four candidate distributions with the theoretical VaR value. The theoretical VaR is known in a simulation, as opposed to an empirical study, which makes the comparison straightforward. The parameters of each DGP were selected such that each had a 1% VaR value equivalent to the 90% quantile of the threshold method.

In addition to the accuracy of a VaR forecast, risk managers pay close attention to tail fittings, which can be evaluated through the Cramer-Smirnov-von-Mises (CSVM) test (Anderson, 1962; Cramér, 1928):

$$W_n^2 = \frac{1}{12n} \sum_{i=1}^{n} [F(y_{(i)}) - \frac{i-0.5}{n}]^2,$$

where n denotes the number of observations, and $y_{(i)}$ denotes ordered observations, with F as the distribution function. This equation measures the discrepancy between empirical and fitted cumulative probabilities: that is, the smaller the difference between the two, the better the fitting.

Figure 3.1 summarises the 1% VaR forecasts results. The 12 panels in the figure represent the 12 DGPs, with the underlying sample distribution arranged horizontally and the sample size arranged vertically. In each panel, four coloured boxes show the 1% VaR forecasts of the 999 simulations from each of the four fitting distributions (EB, GEV, GP and WB), as indicated in the legend. The black line across each row highlights the theoretical 1% VaR. The value of VaR is on the logarithm scale, as some VaR estimates are extreme outliers.

The results suggest that when data are generated from EB and GP, the medium estimated VaR value of 999 simulations aligns with the theoretical VaR, whereas GEV and WB tend to overestimate VaR, especially in medium and large sample sizes. For data generated from GEV, both GP and WB overestimate VaR in medium and large sample sizes. When data are generated from WB, there are fewer outlier values for all estimation distributions, but the box of GP is slightly over the theoretical value. To summarise, EB tends to have a robust VaR forecast accuracy performance across various DGPs, while the other three candidate distributions overestimate VaR in some scenarios.

Figure 3.2 depicts the same row and column arrangement as Figure 3.1, and

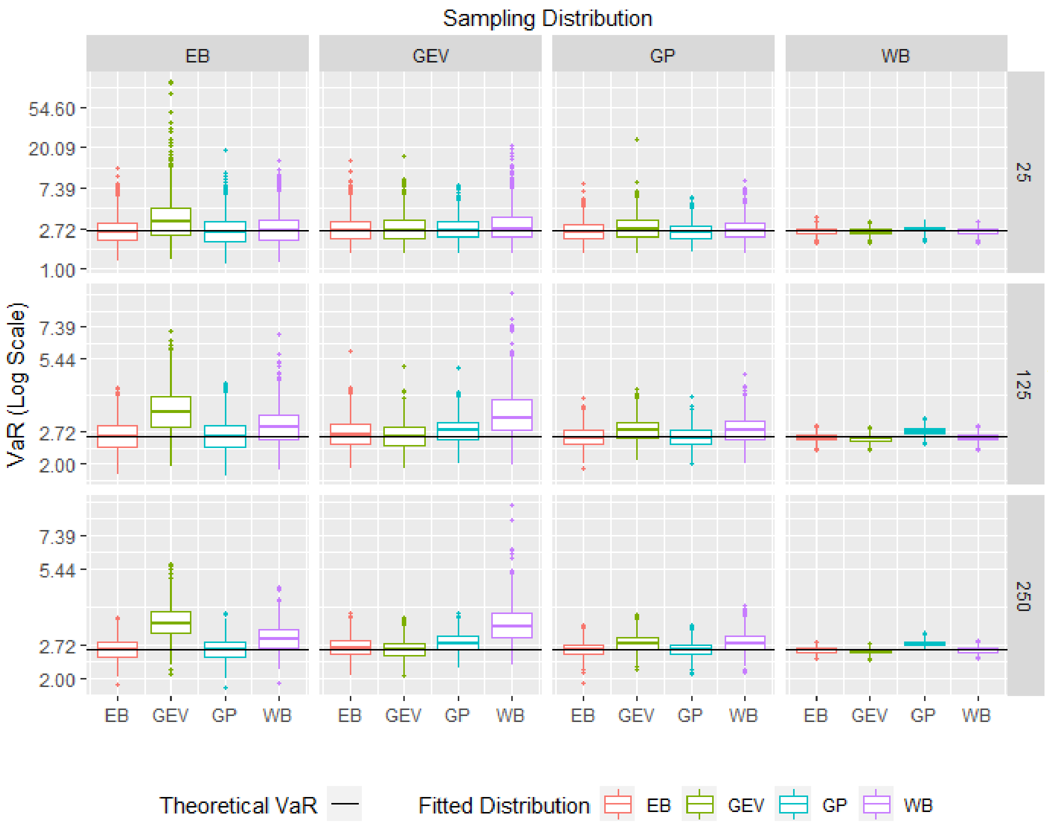

Figure 3.1: This boxplot compares the 1% VaR (y-axis with log-scale) of simulations estimated by Extended Burr XII (EB), generalised extreme value (GEV), generalised Pareto (GP), and Weibull (WB) distributions (x-axis). 999 simulations are generated for each combination of data generating distributions and sample size. The data generating distributions (EB, GEV, GP, and WB) and sample sizes (25, 125, and 250) are shown separately in rows and columns of panels. The theoretical 1% VaR is highlighted in a black line across boxes

assesses the performance of tail fitting by summarising the CSVM statistics within the boxplot. The lower CSVM value indicates a better tail fitting. Data generated from EB has the lowest CSVM medium value across all sample sizes, though there are a few outliers. When data are generated from GEV, both GEV and EB have relatively low CSVM values, but GEV performs slightly better than EB in medium and large samples. When data are generated from GP, both EB and GP have lower CSVM values. When data are generated from WB, EB and WB tend to perform similarly, whereas GEV and GP have higher CSVM values, especially in medium and large samples.

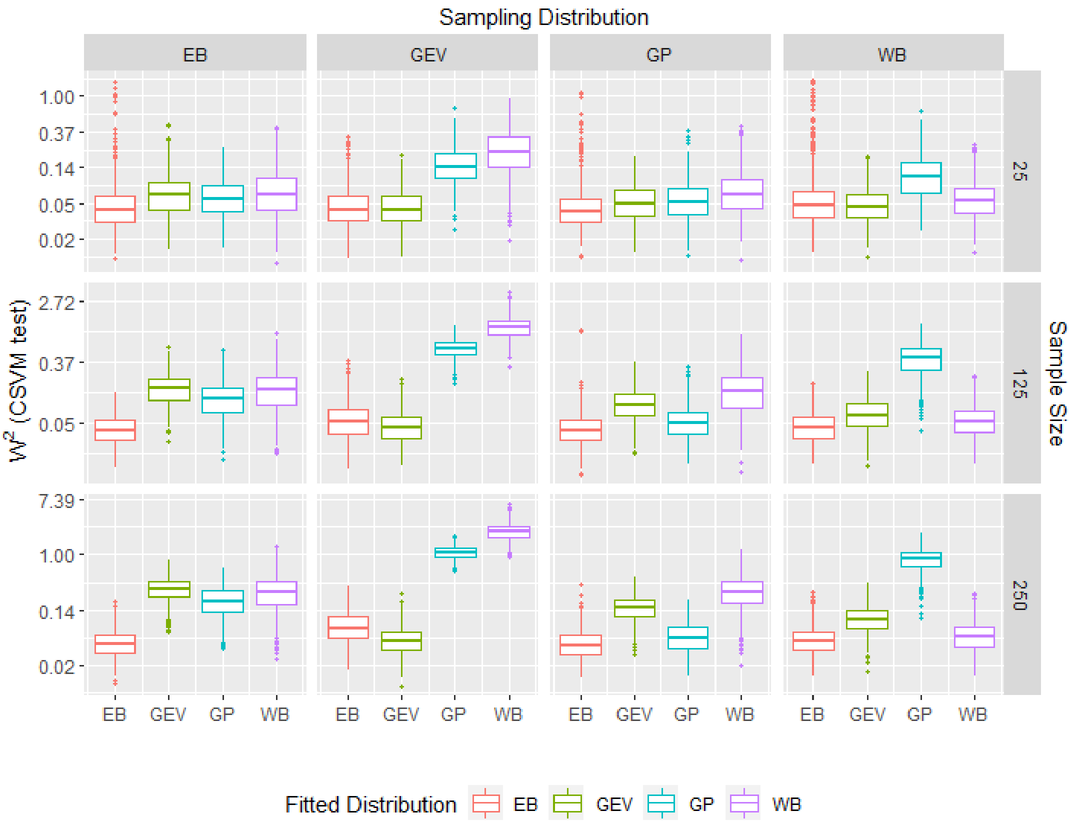

Figure 3.2: This boxplot compares the Cramer-Smirnov-von-Mises (Anderson, 1962; Cramér, 1928) test statistics (y-axis) of simulated samples fitted by Extended Burr XII (EB), generalised extreme value (GEV), generalised Pareto (GP), and Weibull (WB) distributions (x-axis). 999 simulations are generated for each combination of data generating distributions and sample size. The data generating distributions (EB, GEV, GP, and WB) and sample sizes (25, 125, and 250) are shown separately in rows and columns of panels

The simulation study suggests that VaR forecasts from EB have a lower bias to theoretical VaR value, and are robust to threshold and sample size selection. EB has the lowest CSVM value of almost all DGPs, indicating that tail fitting by EB is the best.

3.5 Empirical Study

The simulation study described above suggests that EB may have advantages in unconditional VaR forecasting compared to other candidate distributions. This section evaluates the performance of EB in an empirical study, including exploratory analysis, threshold analysis, and backtesting results.

3.5.1 Data

The performance of the proposed method is evaluated by applying it to the major stock market indices listed in Table 3. The data can be obtained from Yahoo Finance through the Quantmod Package (Ryan, Ulrich, Thielen, Teetor, & Bronder, 2020) in statistical programming software R (R Core Team, 2021).

Table 3.1: Selected Financial Instruments

Financials	Start Date (YMD)	End Date (YMD)	Number of Observation
SP500	1928-01-03	2019-12-30	23108
DJI	1985-01-29	2019-12-30	8802
FTSE	1984-01-03	2019-12-31	9162
HSI	1986-12-31	2019-12-31	8145
N225	1965-01-05	2019-12-30	13530

Table 3.2: Descriptive Statistics of Selected Financial Instruments

Financials	Minimum	Maximum	Mean	Std Dev	Skewness	Kurtosis
SP500	-22.90%	15.36%	0.02%	1.20%	-0.48	19.10
DJI	-25.63%	10.76%	0.03%	1.11%	-0.42	13.37
FTSE	-13.03%	9.38%	0.02%	1.10%	-0.56	10.34
HSI	-24.52%	17.25%	0.03%	1.56%	-0.66	15.97
N225	-16.14%	13.23%	0.02%	1.26%	-0.41	9.49

The descriptive statistics listed in Table 3.2 confirm that the return series of five financial instruments are skewed and have excess kurtosis, indicating that normal distributions and other non-fat-tailed distributions are not appropriate in modelling return series.

3.5.2 Threshold Analysis

The choice of threshold impacts both the distribution fittings and the VaR estimates. We seek a balance between the estimation bias and variance, which can be achieved through two methods (Coles, 2001). The first is the mean residual life plot, an exploratory technique; the second is an estimating procedure that assesses VaR stability at various threshold levels.

In the exploratory technique, some linearity in the mean excess against threshold is expected, since the mean excess is a linear function of higher-level threshold. However, it has two major drawbacks. First, the mean residual life plot can be difficult to interpret because of its sensitivity to the threshold selection, especially at the higher tail. This can be addressed by using the median excess plot, which is relatively robust (Embrechts et al., 1999). Second, the linear relationship is established under GP, and may not exist under other extreme-based distributions.

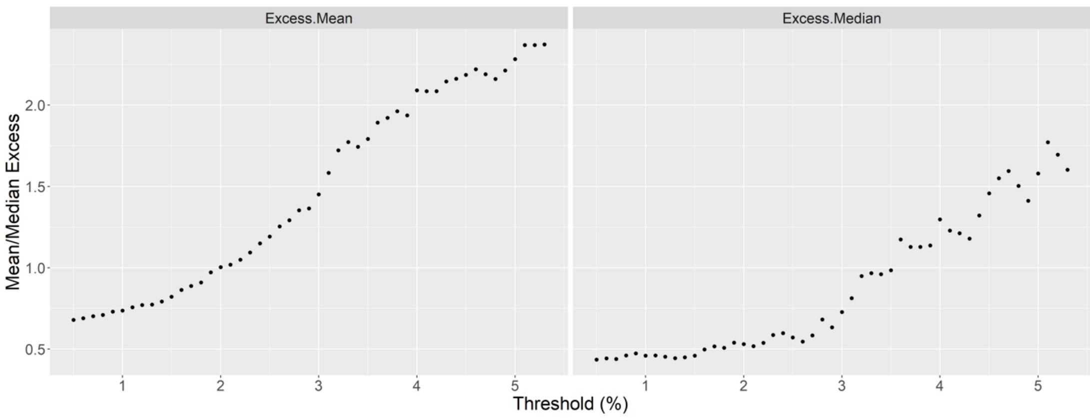

Figure 3.3: Mean/Median Residual Life Plot

Figure 3.3 shows the mean/median residual life plot of SP500[3] with losses on the x-axis and excess on the y-axis. Both the mean and median excess show an upward trend, indicating that the mean and median of the excess increase as the threshold increases. Some non-linear relationships are evident in the mean excess, whereas two linear relationships are evident in the median excess, which

[3]As threshold patterns appear similar across different indexes, we only present the results of SP500.

break around 3. This suggests that, while there is a piecewise linear relationship between the excess and the threshold, it does not aid the decision of threshold selection.

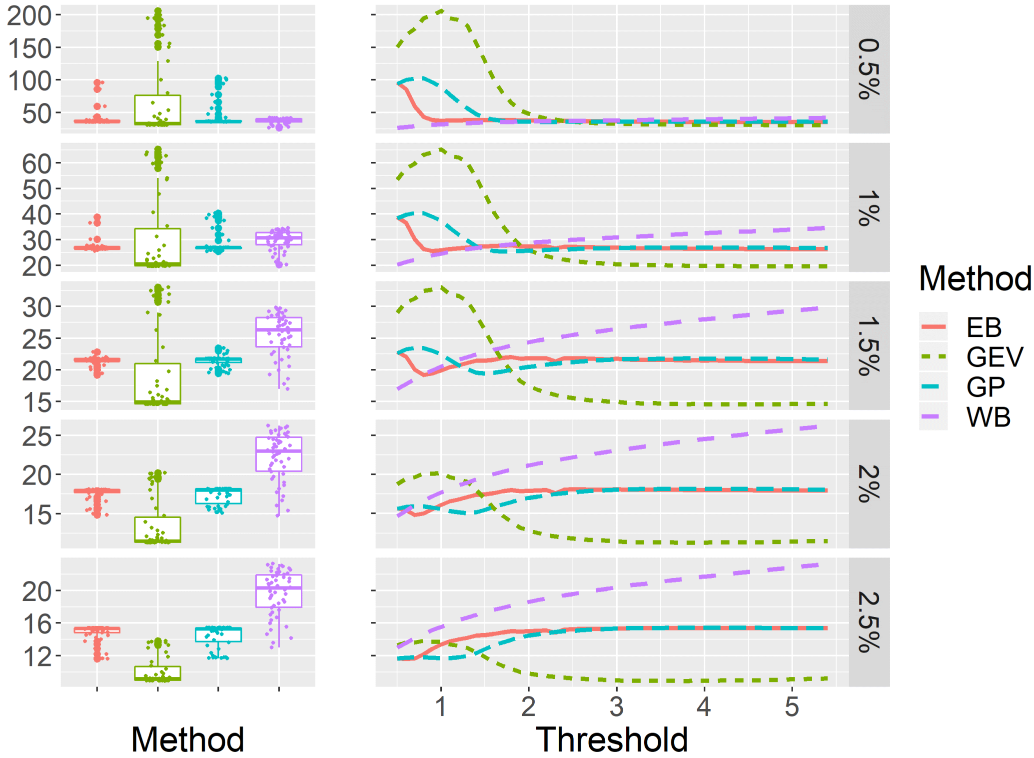

Figure 3.4: Stability of VaR estimates at various threshold levels. The left-hand panel shows the boxplot of VaR estimated across the selected thresholds at 0.5%, 1%, 1.5%, 2%, and 2.5% confidence levels (horizontally), with different methods shown in colour. The right-hand panel shows the line plot with the threshold on the x-axis at five confidence levels, with different methods shown in colour

To estimate the stability of VaR, we compare the VaR estimated at five different confidence levels (shown horizontally in Figure 3.4) with thresholds ranging from 0.5% to 5.4%. The right-hand panel of Figure 4 attests that WB has a steady upward trend with increasing thresholds at all five confidence levels, whereas GEV, GP, and EB are volatile at lower levels of threshold selection, and become stable at higher levels. As the threshold increases, EB becomes stable more quickly, and appears to be less volatile than GEV and GP. This is confirmed by the boxplot in the left-hand panel. Although GP performs similarly to EB, its interquartiles at 2% and 2.5% VaR levels are more widely spread than those of EB. This indicates that EB is more robust than other candidate distributions with various threshold selections at different confidence levels.

3.5.3 Backtesting

UC and MAPE, the backtesting statistics detailed in Section 2.3, are used to evaluate the performance of the proposed methods. We apply the rolling window approach in the 1-year, 5-year and 10-year windows to reflect market changes in the short, medium, and long terms. That is, we use a selected window to build the model, and then forecast VaR for the next day. We also list the empirical VaR and VaR estimates from the historical simulation as a benchmark.

The UC results in Table 4.2 show the hypothetical VaR level α (displayed in the first column from the left), which equals the actual exceedance probability. The p-values of UC for GEV and WB are significant at the 1% confidence level for all possible scenarios, indicating that GEV and WB are not able to accurately model the exceedance probability. The UC tests of the empirical and HS methods are significant in the 1-year and 5-year windows, but not in the 10-year window, indicating that they can accurately capture probability within long-term windows. This may be because both empirical and HS methods rely heavily on a sufficient number of samples to make accurate predictions. The GP and EB perform similarly in the medium and long terms: both methods deliver good predictions of exceedance probability. In the 1-year window, EB continues to capture the exceedance probability well by using an 8% tail threshold. In summary, EB can capture the probability of exceedance under short-, medium-, and long-term windows at different VaR levels, whereas other candidate methods work only in the medium- or long-term windows.

The average MAPE statistics in Table 3.12 show that EB has the lowest percentage error across all windows and VaR levels, except for that of GP's 6% tail in the 10-year window and 8% tail in the 5- and 10-year windows. The percentage error from GEV and WB is approximately five times higher than that of EB, which confirms the results of the likelihood ratio test.

Therefore, we conclude that EB is flexible and robust in estimating VaR under different scenarios compared to other candidate distributions.

Table 3.3: The Unconditional Coverage Test for SP500

		EB			GEV			GP			WB			Empirical	HS
Tail of distribution		6%	8%	10%	6%	8%	10%	6%	8%	10%	6%	8%	10%		
1 Year															
$\alpha = 0.5\%$	LRuc	32.30	2.93	36.05	71.37	60.71	54.89	39.97	46.17	35.09	69.13	44.44	35.41	100.96	69.49
	P-value	0.00	0.09	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha = 1.0\%$	LRuc	13.96	5.24	26.37	172.56	141.01	138.78	26.99	44.56	25.76	212.74	172.56	145.55	74.47	31.51
	P-value	0.00	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha = 1.5\%$	LRuc	9.82	3.30	17.72	295.90	Inf	Inf	17.72	36.14	19.02	360.07	301.82	262.42	47.54	27.74
	P-value	0.00	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha = 2.0\%$	LRuc	5.80	2.83	10.11	Inf	339.77	286.40	10.69	28.70	12.21	518.19	Inf	Inf	50.27	27.31
	P-value	0.02	0.09	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha = 2.5\%$	LRuc	2.11	0.00	10.63	Inf	425.60	359.92	8.85	27.73	12.27	Inf	Inf	531.05	40.56	28.15
	P-value	0.15	0.98	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5 Year															
$\alpha = 0.5\%$	LRuc	7.49	5.25	6.55	83.88	78.70	78.70	6.10	5.25	4.85	92.19	66.83	50.68	13.04	10.65
	P-value	0.01	0.02	0.01	0.00	0.00	0.00	0.01	0.02	0.03	0.00	0.00	0.00	0.00	0.00
$\alpha = 1.0\%$	LRuc	4.11	5.84	3.60	155.44	127.58	123.33	9.35	4.65	7.15	229.60	196.88	148.08	12.27	8.22
	P-value	0.04	0.02	0.06	0.00	0.00	0.00	0.00	0.03	0.01	0.00	0.00	0.00	0.00	0.00
$\alpha = 1.5\%$	LRuc	6.07	3.92	2.75	253.44	Inf	Inf	6.07	3.12	3.92	363.60	305.62	261.57	11.66	7.45
	P-value	0.01	0.05	0.10	0.00	0.00	0.00	0.01	0.08	0.05	0.00	0.00	0.00	0.00	0.01
$\alpha = 2.0\%$	LRuc	3.45	2.33	1.11	Inf	303.21	265.86	3.10	1.92	2.05	480.24	Inf	Inf	7.02	5.19
	P-value	0.06	0.13	0.29	0.00	0.00	0.00	0.08	0.17	0.15	0.00	0.00	0.00	0.01	0.02
$\alpha = 2.5\%$	LRuc	6.47	3.82	3.04	482.16	406.95	355.25	7.12	3.50	3.82	Inf	Inf	494.25	8.77	6.68
	P-value	0.01	0.05	0.08	0.00	0.00	0.00	0.01	0.06	0.05	0.00	0.00	0.00	0.00	0.01
10 Year															
$\alpha = 0.5\%$	LRuc	0.04	0.01	0.00	77.31	69.79	69.79	0.00	0.00	0.09	88.36	67.41	56.43	0.61	1.36
	P-value	0.84	0.92	1.00	0.00	0.00	0.00	1.00	1.00	0.77	0.00	0.00	0.00	0.43	0.24
$\alpha = 1.0\%$	LRuc	0.18	0.02	0.00	189.39	168.73	155.17	0.01	0.32	0.72	226.54	212.29	192.52	1.23	0.59
	P-value	0.67	0.89	0.95	0.00	0.00	0.00	0.94	0.57	0.40	0.00	0.00	0.00	0.27	0.44
$\alpha = 1.5\%$	LRuc	0.94	0.55	0.21	304.29	269.88	230.98	0.83	0.03	0.00	377.91	357.94	317.77	2.34	1.29
	P-value	0.33	0.46	0.65	0.00	0.00	0.00	0.36	0.86	0.96	0.00	0.00	0.00	0.13	0.26
$\alpha = 2.0\%$	LRuc	0.20	0.06	0.01	430.90	Inf	Inf	0.09	0.01	0.01	532.39	494.62	437.94	1.08	0.98
	P-value	0.65	0.80	0.92	0.00	0.00	0.00	0.76	0.92	0.92	0.00	0.00	0.00	0.30	0.32
$\alpha = 2.5\%$	LRuc	0.16	0.01	0.02	Inf	Inf	397.45	0.13	0.05	0.29	662.16	Inf	Inf	0.51	0.64
	P-value	0.69	0.93	0.90	0.00	0.00	0.00	0.72	0.83	0.59	0.00	0.00	0.00	0.48	0.42

Table 3.4: The MAPE of SP500

Tail of distribution		EB			GEV			GP			WB			Empirical	HS
		6%	8%	10%	6%	8%	10%	6%	8%	10%	6%	8%	10%		
1 Year															
$\alpha = 0.5\%$	Exceed (114)	180	133	184	37	42	45	188	194	183	38	51	57	237	214
	MAPE	0.575	0.164	0.610	0.676	0.633	0.606	0.645	0.698	0.601	0.668	0.554	0.501	1.074	0.873
$\alpha = 1.0\%$	Exceed (229)	287	195	310	62	75	76	311	336	309	48	62	73	370	318
	MAPE	0.256	0.149	0.356	0.729	0.672	0.668	0.361	0.470	0.352	0.790	0.729	0.681	0.619	0.391
$\alpha = 1.5\%$	Exceed (343)	402	310	423	80	102	111	423	459	426	60	78	92	477	444
	MAPE	0.173	0.096	0.234	0.767	0.703	0.676	0.234	0.339	0.243	0.825	0.773	0.732	0.391	0.295
$\alpha = 2.0\%$	Exceed (457)	509	422	526	105	127	149	528	575	533	70	92	107	614	572
	MAPE	0.114	0.077	0.151	0.773	0.722	0.674	0.155	0.258	0.166	0.847	0.799	0.766	0.343	0.251
$\alpha = 2.5\%$	Exceed (571)	606	572	650	117	159	186	643	700	656	87	108	122	728	701
	MAPE	0.061	0.001	0.138	0.795	0.722	0.675	0.125	0.225	0.148	0.848	0.811	0.787	0.274	0.227
Average MAPE		0.236	0.097	0.298	0.747	0.690	0.660	0.304	0.398	0.302	0.795	0.733	0.693	0.540	0.407
5 Year															
$\alpha = 0.5\%$	Exceed (109)	139	134	137	29	31	31	136	134	133	26	36	44	149	145
	MAPE	0.272	0.226	0.254	0.735	0.716	0.716	0.245	0.226	0.217	0.762	0.671	0.597	0.364	0.327
$\alpha = 1.0\%$	Exceed (219)	249	255	247	63	75	77	265	251	259	38	48	66	272	262
	MAPE	0.140	0.170	0.130	0.712	0.657	0.648	0.213	0.149	0.185	0.826	0.783	0.698	0.245	0.199
$\alpha = 1.5\%$	Exceed (328)	373	364	358	87	105	113	373	360	364	52	69	84	391	378
	MAPE	0.138	0.111	0.092	0.735	0.680	0.655	0.138	0.098	0.111	0.841	0.790	0.744	0.193	0.153
$\alpha = 2.0\%$	Exceed (437)	476	469	459	108	130	146	474	466	467	71	86	102	493	485
	MAPE	0.089	0.073	0.050	0.753	0.703	0.666	0.085	0.066	0.069	0.838	0.803	0.767	0.128	0.110
$\alpha = 2.5\%$	Exceed (546)	606	592	587	125	152	173	609	590	592	83	102	121	616	607
	MAPE	0.109	0.084	0.075	0.771	0.722	0.683	0.115	0.080	0.084	0.848	0.813	0.779	0.128	0.111
Average MAPE		0.150	0.132	0.120	0.741	0.695	0.674	0.159	0.124	0.133	0.823	0.771	0.717	0.211	0.180
10 Year															
$\alpha = 0.5\%$	Exceed (103)	105	102	103	28	31	31	103	103	100	24	32	37	111	115
	MAPE	0.020	0.010	0.000	0.728	0.699	0.699	0.000	0.000	0.029	0.767	0.689	0.641	0.078	0.117
$\alpha = 1.0\%$	Exceed (206)	212	208	205	44	51	56	207	198	194	33	37	43	222	217
	MAPE	0.029	0.010	0.005	0.786	0.752	0.728	0.005	0.039	0.058	0.840	0.820	0.791	0.078	0.054
$\alpha = 1.5\%$	Exceed (309)	326	322	317	60	71	85	325	312	308	40	45	56	336	329
	MAPE	0.055	0.042	0.026	0.806	0.770	0.725	0.052	0.010	0.003	0.871	0.854	0.819	0.088	0.065
$\alpha = 2.0\%$	Exceed (412)	421	417	414	73	91	114	418	410	410	47	56	71	433	432
	MAPE	0.022	0.012	0.005	0.823	0.779	0.723	0.015	0.005	0.005	0.886	0.864	0.828	0.051	0.049
$\alpha = 2.5\%$	Exceed (515)	524	517	512	88	115	138	523	510	503	60	71	84	531	533
	MAPE	0.018	0.004	0.006	0.829	0.777	0.732	0.016	0.010	0.023	0.884	0.862	0.837	0.031	0.035
Average MAPE		0.029	0.016	0.008	0.794	0.756	0.722	0.018	0.013	0.024	0.849	0.818	0.783	0.065	0.064

Table 3.5: The Unconditional Coverage Test for DJI

		EB			GEV			GP			WB			Empirical	HS
Tail of distribution		6%	8%	10%	6%	8%	10%	6%	8%	10%	6%	8%	10%		
1 Year															
$\alpha = 0.5\%$	LRuc	11.78	2.75	17.73	26.34	11.03	12.32	18.82	22.26	19.94	20.22	9.83	5.83	32.55	22.26
	P-value	0	0.1	0	0	0	0	0	0	0	0	0	0.02	0	0
$\alpha = 1.0\%$	LRuc	8.69	1.64	8.69	59.95	42.49	42.49	9.28	14.63	8.11	62.46	48.52	46.45	22.79	9.89
	P-value	0	0.2	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha = 1.5\%$	LRuc	4.93	4.16	2.94	96.62	70.97	66.92	4.93	6.11	4.22	119.57	101.98	89.02	18.17	6.96
	P-value	0.03	0.04	0.09	0	0	0	0.03	0.01	0.04	0	0	0	0	0.01
$\alpha = 2.0\%$	LRuc	2.08	1.2	3.29	136.46	113.49	95.68	3.29	8.89	2.77	179.93	153.79	139.23	14.2	8.89
	P-value	0.15	0.27	0.07	0	0	0	0.07	0	0.1	0	0	0	0	0
$\alpha = 2.5\%$	LRuc	2.3	0.6	2.51	194.44	145.38	124.84	1.91	11.29	2.95	238.33	197.53	179.61	17.05	8.44
	P-value	0.13	0.44	0.11	0	0	0	0.17	0	0.09	0	0	0	0	0
5 Year															
$\alpha = 0.5\%$	LRuc	9.39	7.73	9.39	16.07	17.86	17.86	11.19	10.28	10.28	16.07	12.87	11.44	8.55	7.73
	P-value	0	0.01	0	0	0	0	0	0	0	0	0	0	0	0.01
$\alpha = 1.0\%$	LRuc	5.2	5.7	5.7	48.29	46.02	41.73	6.22	6.75	6.75	60.94	53.07	43.84	9.75	6.22
	P-value	0.02	0.02	0.02	0	0	0	0.01	0.01	0.01	0	0	0	0	0.01
$\alpha = 1.5\%$	LRuc	4.39	3.33	5.16	82.41	52.15	45.21	6	5.57	6	108.17	98.91	84.99	8.85	8.34
	P-value	0.04	0.07	0.02	0	0	0	0.01	0.02	0.01	0	0	0	0	0
$\alpha = 2.0\%$	LRuc	5.03	5.39	7.33	99.63	78.03	55.17	6.52	8.63	6.92	160.49	146.87	102.02	11.52	13.11
	P-value	0.02	0.02	0.01	0	0	0	0.01	0	0.01	0	0	0	0	0
$\alpha = 2.5\%$	LRuc	9.96	9.11	9.53	136.65	98.41	78.2	9.53	10.4	10.85	196.01	158.21	141.8	10.4	8.7
	P-value	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10 Year															
$\alpha = 0.5\%$	LRuc	0.39	0.39	0.39	20.06	20.06	17.86	0.39	0.2	0.08	22.47	17.86	10.74	0.93	0.93
	P-value	0.53	0.53	0.53	0	0	0	0.53	0.65	0.78	0	0	0	0.33	0.33
$\alpha = 1.0\%$	LRuc	1.55	1.27	1.27	33.79	29.91	29.91	1.55	1.55	1.01	59.22	50.36	33.79	2.98	4.82
	P-value	0.21	0.26	0.26	0	0	0	0.21	0.21	0.32	0	0	0	0.08	0.03
$\alpha = 1.5\%$	LRuc	6.03	5.57	6.03	67.89	58.36	47.9	6.03	4.28	3.89	97.42	78.61	70.45	8.06	6.51
	P-value	0.01	0.02	0.01	0	0	0	0.01	0.04	0.05	0	0	0	0	0.01
$\alpha = 2.0\%$	LRuc	5.61	4.84	3.14	98.7	81.15	62.18	3.45	3.45	3.14	122.48	119.26	104.24	6.44	4.84
	P-value	0.02	0.03	0.08	0	0	0	0.06	0.06	0.08	0	0	0	0.01	0.03
$\alpha = 2.5\%$	LRuc	2.69	2.69	2.2	132.9	99.51	80.45	2.44	1.98	1.76	168.14	154.44	138.77	3.22	4.42
	P-value	0.1	0.1	0.14	0	0	0	0.12	0.16	0.18	0	0	0	0.07	0.04

Table 3.6: The MAPE of DJI

Tail of distribution		EB 6%	EB 8%	EB 10%	GEV 6%	GEV 8%	GEV 10%	GP 6%	GP 8%	GP 10%	WB 6%	WB 8%	WB 10%	Empirical	HS
1 Year															
$\alpha = 0.5\%$	Exceed	67	54	73	14	23	22	74	77	75	17	24	28	85	77
	MAPE	0.567	0.263	0.708	0.673	0.462	0.485	0.731	0.801	0.754	0.602	0.439	0.345	0.988	0.801
$\alpha = 1.0\%$	Exceed	114	74	114	25	33	33	115	123	113	24	30	31	133	116
	MAPE	0.333	0.135	0.333	0.708	0.614	0.614	0.345	0.439	0.322	0.719	0.649	0.637	0.556	0.357
$\alpha = 1.5\%$	Exceed	154	106	148	35	46	48	154	157	152	27	33	38	179	159
	MAPE	0.201	0.173	0.154	0.727	0.641	0.626	0.201	0.224	0.185	0.789	0.743	0.704	0.396	0.24
$\alpha = 2.0\%$	Exceed	190	157	195	44	53	61	195	211	193	30	38	43	222	211
	MAPE	0.111	0.082	0.14	0.743	0.69	0.643	0.14	0.234	0.129	0.825	0.778	0.749	0.298	0.234
$\alpha = 2.5\%$	Exceed	236	225	237	47	65	74	234	264	239	34	46	52	276	257
	MAPE	0.104	0.053	0.109	0.78	0.696	0.654	0.095	0.235	0.118	0.841	0.785	0.757	0.291	0.202
Average MAPE		0.263	0.141	0.289	0.726	0.621	0.604	0.302	0.387	0.302	0.755	0.679	0.638	0.506	0.367
5 Year															
$\alpha = 0.5\%$	Exceed	58	56	58	16	15	15	60	59	59	16	18	19	57	56
	MAPE	0.537	0.484	0.537	0.576	0.602	0.602	0.59	0.564	0.564	0.576	0.523	0.496	0.511	0.484
$\alpha = 1.0\%$	Exceed	96	97	97	24	25	27	98	99	99	19	22	26	104	98
	MAPE	0.272	0.285	0.285	0.682	0.669	0.642	0.299	0.312	0.312	0.748	0.708	0.655	0.378	0.299
$\alpha = 1.5\%$	Exceed	136	133	138	32	46	50	140	139	140	23	26	31	146	145
	MAPE	0.202	0.175	0.219	0.717	0.594	0.558	0.237	0.228	0.237	0.797	0.77	0.726	0.29	0.281
$\alpha = 2.0\%$	Exceed	179	180	185	47	57	70	183	188	184	26	30	46	194	197
	MAPE	0.186	0.193	0.226	0.689	0.622	0.536	0.213	0.246	0.219	0.828	0.801	0.695	0.285	0.305
$\alpha = 2.5\%$	Exceed	233	231	232	54	71	82	232	234	235	34	46	52	234	230
	MAPE	0.235	0.224	0.23	0.714	0.624	0.565	0.23	0.24	0.246	0.82	0.756	0.724	0.24	0.219
Average MAPE		0.286	0.272	0.299	0.676	0.622	0.581	0.314	0.318	0.316	0.754	0.712	0.659	0.341	0.318
10 Year															
$\alpha = 0.5\%$	Exceed	35	35	35	10	10	11	35	34	33	9	11	15	37	37
	MAPE	0.113	0.113	0.113	0.682	0.682	0.65	0.113	0.081	0.049	0.714	0.65	0.523	0.176	0.176
$\alpha = 1.0\%$	Exceed	73	72	72	23	25	25	73	73	71	13	16	23	77	81
	MAPE	0.16	0.144	0.144	0.634	0.603	0.603	0.16	0.16	0.129	0.793	0.746	0.634	0.224	0.288
$\alpha = 1.5\%$	Exceed	119	118	119	27	31	36	119	115	114	17	23	26	123	120
	MAPE	0.261	0.25	0.261	0.714	0.671	0.619	0.261	0.219	0.208	0.82	0.756	0.724	0.303	0.272
$\alpha = 2.0\%$	Exceed	153	151	146	33	40	49	147	147	146	25	26	31	155	151
	MAPE	0.216	0.2	0.16	0.738	0.682	0.611	0.168	0.168	0.16	0.801	0.793	0.754	0.232	0.2
$\alpha = 2.5\%$	Exceed	178	178	176	38	51	60	177	175	174	27	31	36	180	184
	MAPE	0.132	0.132	0.119	0.758	0.676	0.619	0.125	0.113	0.106	0.828	0.803	0.771	0.144	0.17
Average MAPE		0.176	0.168	0.159	0.705	0.663	0.62	0.165	0.148	0.13	0.791	0.75	0.681	0.216	0.221

Table 3.7: The Unconditional Coverage Test for FTSE

		EB			GEV			GP			WB			Empirical	HS
Tail of distribution		6%	8%	10%	6%	8%	10%	6%	8%	10%	6%	8%	10%		
1 Year															
$\alpha = 0.5\%$	LRuc	17.98	9.49	27.37	38.63	38.63	28.16	21.31	26.11	24.87	30.53	20	13.64	47.48	32.66
	P-value	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha = 1.0\%$	LRuc	8.07	0.09	11.06	77.53	61.28	58.86	13.06	21.61	15.95	83.7	66.35	58.86	27.06	19.09
	P-value	0	0.76	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha = 1.5\%$	LRuc	4.79	0.04	9.53	119.72	108.02	97.29	9.53	18.33	11.62	142.98	110.85	108.02	19.01	12.18
	P-value	0.03	0.84	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha = 2.0\%$	LRuc	1.56	0	3.92	175.08	153.16	133.66	4.22	12.12	6.54	203.74	181.85	168.55	16.33	9.35
	P-value	0.21	0.98	0.05	0	0	0	0.04	0	0.01	0	0	0	0	0
$\alpha = 2.5\%$	LRuc	0.26	1.73	4.44	235.09	181.31	162.47	2.93	9.86	2.93	265.38	242.31	221.27	13.79	8.32
	P-value	0.61	0.19	0.04	0	0	0	0.09	0	0.09	0	0	0	0	0
5 Year															
$\alpha = 0.5\%$	LRuc	7.24	4.49	5.79	33.82	33.82	28.35	5.79	7.24	5.79	33.82	25.9	17.63	3.34	2.35
	P-value	0.01	0.03	0.02	0	0	0	0.02	0.01	0.02	0	0	0	0.07	0.13
$\alpha = 1.0\%$	LRuc	7.84	6.19	7.27	67.88	47.4	47.4	7.84	7.27	6.72	80.65	70.89	47.4	5.19	3.85
	P-value	0.01	0.01	0.01	0	0	0	0.01	0.01	0.01	0	0	0	0.02	0.05
$\alpha = 1.5\%$	LRuc	1.37	1.59	1.37	93.62	69.16	60.87	2.07	2.34	2.34	136.05	105.17	83.14	3.22	2.07
	P-value	0.24	0.21	0.24	0	0	0	0.15	0.13	0.13	0	0	0	0.07	0.15
$\alpha = 2.0\%$	LRuc	2.45	2.45	2.69	122.53	107.14	91.13	4.11	3.81	3.23	173.13	145.86	122.53	6.18	3.51
	P-value	0.12	0.12	0.1	0	0	0	0.04	0.05	0.07	0	0	0	0.01	0.06
$\alpha = 2.5\%$	LRuc	0.83	0.83	0.83	171.48	146.4	126.68	0.96	0.96	0.96	218.05	183.75	171.48	2.17	1.59
	P-value	0.36	0.36	0.36	0	0	0	0.33	0.33	0.33	0	0	0	0.14	0.21
10 Year															
$\alpha = 0.5\%$	LRuc	18.2	18.2	17	33.31	36.89	33.31	18.2	19.43	19.43	30.06	21.91	9.28	10.54	12.55
	P-value	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha = 1.0\%$	LRuc	2.6	5.31	5.85	39.39	35.18	26.04	6.41	6.99	6.41	63.51	43.98	35.18	5.31	3.41
	P-value	0.11	0.02	0.02	0	0	0	0.01	0.01	0.01	0	0	0	0.02	0.06
$\alpha = 1.5\%$	LRuc	3.38	4.12	6.27	61.55	45.28	40.05	5.36	5.8	6.27	97.23	79.08	61.55	5.8	3.03
	P-value	0.07	0.04	0.01	0	0	0	0.02	0.02	0.01	0	0	0	0.02	0.08
$\alpha = 2.0\%$	LRuc	2.64	3.52	4.18	88.6	64.97	57.62	3.52	4.53	4.18	127.88	103.98	88.6	3.22	3.52
	P-value	0.1	0.06	0.04	0	0	0	0.06	0.03	0.04	0	0	0	0.07	0.06
$\alpha = 2.5\%$	LRuc	2.42	2.66	2.42	121.14	97.4	81.05	2.66	2.66	1.97	155.55	143.25	121.14	3.74	2.42
	P-value	0.12	0.1	0.12	0	0	0	0.1	0.1	0.16	0	0	0	0.05	0.12

Table 3.8: The MAPE of FTSE

Tail of distribution		EB 6%	EB 8%	EB 10%	GEV 6%	GEV 8%	GEV 10%	GP 6%	GP 8%	GP 10%	WB 6%	WB 8%	WB 10%	Empirical	HS
1 Year															
$\alpha = 0.5\%$	Exceed	75	66	83	10	10	14	78	82	81	13	18	22	97	87
	MAPE	0.701	0.497	0.883	0.773	0.773	0.682	0.769	0.86	0.837	0.705	0.592	0.501	1.2	0.973
$\alpha = 1.0\%$	Exceed	116	91	121	20	26	27	124	135	128	18	24	27	141	132
	MAPE	0.316	0.032	0.372	0.773	0.705	0.694	0.406	0.531	0.452	0.796	0.728	0.694	0.599	0.497
$\alpha = 1.5\%$	Exceed	158	130	169	29	33	37	169	184	173	22	32	33	185	174
	MAPE	0.195	0.017	0.278	0.781	0.75	0.72	0.278	0.391	0.308	0.834	0.758	0.75	0.399	0.316
$\alpha = 2.0\%$	Exceed	193	176	203	34	41	48	204	224	211	26	32	36	232	218
	MAPE	0.094	0.002	0.151	0.807	0.767	0.728	0.157	0.27	0.197	0.853	0.819	0.796	0.316	0.236
$\alpha = 2.5\%$	Exceed	228	240	252	38	55	62	246	268	246	30	36	42	277	264
	MAPE	0.034	0.089	0.143	0.828	0.75	0.719	0.116	0.216	0.116	0.864	0.837	0.809	0.257	0.198
Average MAPE		0.268	0.127	0.365	0.792	0.749	0.709	0.345	0.454	0.382	0.81	0.747	0.71	0.554	0.444
5 Year															
$\alpha = 0.5\%$	Exceed	57	53	55	9	9	11	55	57	55	9	12	16	51	49
	MAPE	0.459	0.357	0.408	0.77	0.77	0.718	0.408	0.459	0.408	0.77	0.693	0.59	0.306	0.254
$\alpha = 1.0\%$	Exceed	104	101	103	18	26	26	104	103	102	14	17	26	99	96
	MAPE	0.331	0.293	0.318	0.77	0.667	0.667	0.331	0.318	0.306	0.821	0.782	0.667	0.267	0.229
$\alpha = 1.5\%$	Exceed	130	131	130	30	40	44	133	134	134	17	26	34	137	133
	MAPE	0.109	0.118	0.109	0.744	0.659	0.625	0.135	0.143	0.143	0.855	0.778	0.71	0.169	0.135
$\alpha = 2.0\%$	Exceed	176	176	177	41	47	54	182	181	179	25	33	41	188	180
	MAPE	0.126	0.126	0.133	0.738	0.699	0.654	0.165	0.158	0.146	0.84	0.789	0.738	0.203	0.152
$\alpha = 2.5\%$	Exceed	208	208	208	45	54	62	209	209	209	31	41	45	216	213
	MAPE	0.065	0.065	0.065	0.77	0.724	0.683	0.07	0.07	0.07	0.841	0.79	0.77	0.106	0.09
Average MAPE		0.218	0.192	0.207	0.758	0.704	0.669	0.222	0.23	0.215	0.825	0.766	0.695	0.21	0.172
10 Year															
$\alpha = 0.5\%$	Exceed	60	60	59	6	5	6	60	61	61	7	10	17	53	55
	MAPE	0.83	0.83	0.799	0.817	0.848	0.817	0.83	0.86	0.86	0.787	0.695	0.482	0.616	0.677
$\alpha = 1.0\%$	Exceed	79	85	86	22	24	29	87	88	87	13	20	24	85	81
	MAPE	0.205	0.296	0.311	0.665	0.634	0.558	0.327	0.342	0.327	0.802	0.695	0.634	0.296	0.235
$\alpha = 1.5\%$	Exceed	117	119	124	32	40	43	122	123	124	19	25	32	123	116
	MAPE	0.189	0.21	0.261	0.675	0.593	0.563	0.24	0.25	0.261	0.807	0.746	0.675	0.25	0.179
$\alpha = 2.0\%$	Exceed	150	153	155	40	51	55	153	156	155	26	34	40	152	153
	MAPE	0.144	0.167	0.182	0.695	0.611	0.581	0.167	0.189	0.182	0.802	0.741	0.695	0.159	0.167
$\alpha = 2.5\%$	Exceed	184	185	184	46	56	64	185	185	182	34	38	46	189	184
	MAPE	0.122	0.128	0.122	0.719	0.658	0.61	0.128	0.128	0.11	0.793	0.768	0.719	0.153	0.122
Average MAPE		0.298	0.326	0.335	0.714	0.669	0.626	0.338	0.354	0.348	0.798	0.729	0.641	0.295	0.276

Table 3.9: The Unconditional Coverage Test for HSI

		EB			GEV			GP			WB			Empirical	HS
Tail of distribution		6%	8%	10%	6%	8%	10%	6%	8%	10%	6%	8%	10%		
1 Year															
$\alpha = 0.5\%$	LRuc	13.09	2.63	10.26	12.25	6.37	8.44	9.39	12.11	10.26	6.37	5.46	2.06	27.13	20.77
	P-value	0	0.1	0	0	0.01	0	0	0	0	0.01	0.02	0.15	0	0
$\alpha = 1.0\%$	LRuc	6.84	2.38	5.78	48.2	35.95	32.41	5.78	12.55	7.97	45.98	39.75	32.41	24.37	13.28
	P-value	0.01	0.12	0.02	0	0	0	0.02	0	0	0	0	0	0	0
$\alpha = 1.5\%$	LRuc	6.13	3.97	5.29	88.55	67.06	62.87	4.9	7.02	3.8	99.71	83.36	80.86	15.47	10.05
	P-value	0.01	0.05	0.02	0	0	0	0.03	0.01	0.05	0	0	0	0	0
$\alpha = 2.0\%$	LRuc	3.59	4.8	1.82	124.83	101.77	94.87	1.07	4.86	1.82	148.84	136.37	130.49	13.83	6.31
	P-value	0.06	0.03	0.18	0	0	0	0.3	0.03	0.18	0	0	0	0	0.01
$\alpha = 2.5\%$	LRuc	1.38	2.36	1.73	170.31	145.11	127.68	1.38	6.63	1.55	202.72	185.87	176.39	8.92	3.8
	P-value	0.24	0.12	0.19	0	0	0	0.24	0.01	0.21	0	0	0	0	0.05
5 Year															
$\alpha = 0.5\%$	LRuc	2.48	2.48	2.48	16.47	16.47	18.45	2.48	2.48	3	18.45	14.64	7.49	2	1.19
	P-value	0.12	0.12	0.12	0	0	0	0.12	0.12	0.08	0	0	0.01	0.16	0.28
$\alpha = 1.0\%$	LRuc	2.76	2.39	1.72	41.34	31.2	27.68	3.15	2.04	1.72	59.55	46.02	37.04	6.03	6.03
	P-value	0.1	0.12	0.19	0	0	0	0.08	0.15	0.19	0	0	0	0.01	0.01
$\alpha = 1.5\%$	LRuc	3.97	3.6	2.02	60.03	49.79	40.85	4.35	2.91	2.6	97.47	82.36	66.87	6.06	6.53
	P-value	0.05	0.06	0.16	0	0	0	0.04	0.09	0.11	0	0	0	0.01	0.01
$\alpha = 2.0\%$	LRuc	3.18	3.18	2.35	90.29	81.08	68.59	3.38	2.35	1.43	132.6	114.01	102.91	4.47	4.47
	P-value	0.07	0.07	0.13	0	0	0	0.07	0.13	0.23	0	0	0	0.03	0.03
$\alpha = 2.5\%$	LRuc	4.38	4.69	3.78	134.53	111.48	83.64	5.02	4.69	2.95	161.4	145.95	134.53	4.38	4.69
	P-value	0.04	0.03	0.05	0	0	0	0.03	0.03	0.09	0	0	0	0.04	0.03
10 Year															
$\alpha = 0.5\%$	LRuc	3.58	3.58	3.58	18.94	18.94	18.94	3.58	3.58	3.58	18.94	16.61	14.49	1.08	1.56
	P-value	0.06	0.06	0.06	0	0	0	0.06	0.06	0.06	0	0	0	0.3	0.21
$\alpha = 1.0\%$	LRuc	0.23	0.12	0.23	40.55	40.55	40.55	0.12	0.23	0.58	55.42	52.11	46.02	0.04	0
	P-value	0.63	0.73	0.63	0	0	0	0.73	0.63	0.45	0	0	0	0.84	0.94
$\alpha = 1.5\%$	LRuc	0.98	0.76	1.79	76.83	59.76	57.25	1.22	1.49	2.12	94.3	83.41	83.41	0.01	0.42
	P-value	0.32	0.38	0.18	0	0	0	0.27	0.22	0.15	0	0	0	0.94	0.52
$\alpha = 2.0\%$	LRuc	0.23	0.34	0.34	101.74	92.69	71.77	0.23	0.34	0.15	130.08	126.14	118.64	0.08	0.23
	P-value	0.63	0.56	0.56	0	0	0	0.63	0.56	0.7	0	0	0	0.77	0.63
$\alpha = 2.5\%$	LRuc	0.52	0.52	0.52	136.56	101.19	82.06	0.52	0.52	0.52	170.3	162.12	143.42	0.4	0.52
	P-value	0.47	0.47	0.47	0	0	0	0.47	0.47	0.47	0	0	0	0.53	0.47

Table 3.10: The MAPE of HSI

		EB			GEV			GP			WB			Empirical	HS
Tail of distribution		6%	8%	10%	6%	8%	10%	6%	8%	10%	6%	8%	10%		
1 Year															
$\alpha = 0.5\%$	Exceed	63	49	60	19	24	22	59	62	60	24	25	30	75	70
	MAPE	0.635	0.272	0.557	0.507	0.377	0.429	0.531	0.609	0.557	0.377	0.351	0.221	0.946	0.817
$\alpha = 1.0\%$	Exceed	101	64	99	25	31	33	99	110	103	26	29	33	124	111
	MAPE	0.31	0.17	0.285	0.676	0.598	0.572	0.285	0.427	0.336	0.663	0.624	0.572	0.609	0.44
$\alpha = 1.5\%$	Exceed	143	95	141	31	40	42	140	145	137	27	33	34	160	150
	MAPE	0.237	0.178	0.22	0.732	0.654	0.637	0.211	0.254	0.185	0.766	0.715	0.706	0.384	0.298
$\alpha = 2.0\%$	Exceed	178	128	171	39	48	51	167	182	171	31	35	37	202	186
	MAPE	0.155	0.17	0.109	0.747	0.689	0.669	0.083	0.181	0.109	0.799	0.773	0.76	0.31	0.207
$\alpha = 2.5\%$	Exceed	209	172	211	44	53	60	209	229	210	34	39	42	235	220
	MAPE	0.085	0.107	0.095	0.772	0.725	0.689	0.085	0.189	0.09	0.824	0.798	0.782	0.22	0.142
Average MAPE		0.284	0.179	0.253	0.687	0.609	0.599	0.239	0.332	0.255	0.686	0.652	0.608	0.494	0.381
5 Year															
$\alpha = 0.5\%$	Exceed	43	43	43	13	13	12	43	43	44	12	14	19	42	40
	MAPE	0.283	0.283	0.283	0.612	0.612	0.642	0.283	0.283	0.313	0.642	0.582	0.433	0.253	0.193
$\alpha = 1.0\%$	Exceed	81	80	78	22	27	29	82	79	78	15	20	24	88	88
	MAPE	0.208	0.193	0.164	0.672	0.597	0.567	0.223	0.179	0.164	0.776	0.702	0.642	0.313	0.313
$\alpha = 1.5\%$	Exceed	121	120	115	34	39	44	122	118	117	20	25	31	126	127
	MAPE	0.203	0.193	0.144	0.662	0.612	0.562	0.213	0.174	0.164	0.801	0.751	0.692	0.253	0.263
$\alpha = 2.0\%$	Exceed	155	155	152	41	45	51	156	152	148	26	32	36	159	159
	MAPE	0.156	0.156	0.134	0.694	0.664	0.62	0.164	0.134	0.104	0.806	0.761	0.731	0.186	0.186
$\alpha = 2.5\%$	Exceed	195	196	193	43	52	65	197	196	190	34	39	43	195	196
	MAPE	0.164	0.17	0.152	0.743	0.69	0.612	0.176	0.17	0.134	0.797	0.767	0.743	0.164	0.17
Average MAPE		0.203	0.199	0.175	0.677	0.635	0.601	0.212	0.188	0.176	0.764	0.713	0.648	0.234	0.225
10 Year															
$\alpha = 0.5\%$	Exceed	18	18	18	8	8	8	18	18	18	8	9	10	22	21
	MAPE	0.339	0.339	0.339	0.706	0.706	0.706	0.339	0.339	0.339	0.706	0.67	0.633	0.192	0.229
$\alpha = 1.0\%$	Exceed	51	52	51	15	15	15	52	51	49	10	11	13	53	55
	MAPE	0.064	0.046	0.064	0.725	0.725	0.725	0.046	0.064	0.101	0.816	0.798	0.761	0.027	0.01
$\alpha = 1.5\%$	Exceed	73	74	70	17	23	24	72	71	69	12	15	15	81	76
	MAPE	0.107	0.094	0.143	0.792	0.719	0.706	0.119	0.131	0.156	0.853	0.816	0.816	0.009	0.07
$\alpha = 2.0\%$	Exceed	104	103	103	23	26	34	104	103	105	15	16	18	106	104
	MAPE	0.046	0.055	0.055	0.789	0.761	0.688	0.046	0.055	0.036	0.862	0.853	0.835	0.027	0.046
$\alpha = 2.5\%$	Exceed	128	128	128	26	38	46	128	128	128	17	19	24	129	128
	MAPE	0.06	0.06	0.06	0.809	0.721	0.662	0.06	0.06	0.06	0.875	0.86	0.824	0.053	0.06
Average MAPE		0.123	0.119	0.132	0.764	0.726	0.697	0.122	0.13	0.138	0.822	0.799	0.774	0.062	0.083

Table 3.11: The Unconditional Coverage Test for N225

		EB			GEV			GP			WB			Empirical	HS
Tail of distribution		6%	8%	10%	6%	8%	10%	6%	8%	10%	6%	8%	10%		
1 Year															
$\alpha = 0.5\%$	LRuc	26.3	5.21	37.93	29.22	22.58	27.45	37.93	40.47	36.69	21.1	15.78	9.49	66.16	44.39
	P-value	0	0.02	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha = 1.0\%$	LRuc	13.86	0.5	20.43	70.26	51.72	55.12	21.88	28.94	21.15	93.2	66.23	53.4	30.63	14.46
	P-value	0	0.48	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha = 1.5\%$	LRuc	6.17	4.98	10.03	112.12	83	81.27	8.78	19.13	10.91	153.58	127.97	109.98	30.1	13.25
	P-value	0.01	0.03	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha = 2.0\%$	LRuc	3.28	3.95	6.74	168.1	135.48	112.87	6.43	18.09	6.43	242.04	190.34	170.47	24.49	14.78
	P-value	0.07	0.05	0.01	0	0	0	0.01	0	0.01	0	0	0	0	0
$\alpha = 2.5\%$	LRuc	1.99	0.62	4.99	222.97	183.99	0	5.24	15.63	4.75	301.21	262.44	232.99	23.5	13.99
	P-value	0.16	0.43	0.03	0	0	0	0.02	0	0.03	0	0	0	0	0
5 Year															
$\alpha = 0.5\%$	LRuc	3.58	5.62	5.07	44.3	41.75	39.31	6.8	6.2	6.8	44.3	30.63	18.82	8.76	4.55
	P-value	0.06	0.02	0.02	0	0	0	0.01	0.01	0.01	0	0	0	0	0.03
$\alpha = 1.0\%$	LRuc	1.43	1.43	1.43	61.49	55.74	53.91	2.4	1.43	1.43	102.86	83.8	67.65	5.88	3.29
	P-value	0.23	0.23	0.23	0	0	0	0.12	0.23	0.23	0	0	0	0.02	0.07
$\alpha = 1.5\%$	LRuc	2.72	2.26	2.72	109.53	88.67	74.15	4.05	2.72	2.26	162.42	144.73	118.81	4.96	2.72
	P-value	0.1	0.13	0.1	0	0	0	0.04	0.1	0.13	0	0	0	0.03	0.1
$\alpha = 2.0\%$	LRuc	2.08	3.1	4.32	154.25	132.1	110.51	6.34	4.86	2.27	228.96	201.28	179.13	4.58	4.58
	P-value	0.15	0.08	0.04	0	0	0	0.01	0.03	0.13	0	0	0	0.03	0.03
$\alpha = 2.5\%$	LRuc	2.9	2.9	2.17	219.48	176.49	0	3.73	2.9	2.17	292.95	264.37	241.05	3.73	2.71
	P-value	0.09	0.09	0.14	0	0	0	0.05	0.09	0.14	0	0	0	0.05	0.1
10 Year															
$\alpha = 0.5\%$	LRuc	1.57	2.3	2.3	35.33	35.33	33	2.3	1.92	2.3	40.37	30.79	19.68	6.4	5.2
	P-value	0.21	0.13	0.13	0	0	0	0.13	0.17	0.13	0	0	0	0.01	0.02
$\alpha = 1.0\%$	LRuc	7.26	7.26	6.78	73.34	53.6	47.97	6.32	7.26	6.32	95.3	86.5	73.34	7.26	4.62
	P-value	0.01	0.01	0.01	0	0	0	0.01	0.01	0.01	0	0	0	0.01	0.03
$\alpha = 1.5\%$	LRuc	4.45	3.83	3.25	97.47	76.86	59.52	3.83	4.13	1.8	161.24	135.95	114.16	3.83	2.98
	P-value	0.03	0.05	0.07	0	0	0	0.05	0.04	0.18	0	0	0	0.05	0.08
$\alpha = 2.0\%$	LRuc	5.13	6.4	5.44	133.47	94.9	81.28	6.07	5.13	3	211.18	188.86	160.55	3	2.55
	P-value	0.02	0.01	0.02	0	0	0	0.01	0.02	0.08	0	0	0	0.08	0.11
$\alpha = 2.5\%$	LRuc	2.5	3.32	3.54	162.92	137.54	111.61	3.1	3.32	2.5	254.96	227.2	196.81	2.69	3.32
	P-value	0.11	0.07	0.06	0	0	0	0.08	0.07	0.11	0	0	0	0.1	0.07

Table 3.12: The MAPE of N225

Tail of distribution		EB 6%	EB 8%	EB 10%	GEV 6%	GEV 8%	GEV 10%	GP 6%	GP 8%	GP 10%	WB 6%	WB 8%	WB 10%	Empirical	HS
1 Year															
$\alpha = 0.5\%$	Exceed	109	83	119	26	30	27	119	121	118	31	35	41	139	124
	MAPE	0.704	0.298	0.861	0.593	0.531	0.578	0.861	0.892	0.845	0.515	0.453	0.359	1.173	0.939
$\alpha = 1.0\%$	Exceed	172	120	182	46	56	54	184	193	183	36	48	55	195	173
	MAPE	0.345	0.062	0.423	0.64	0.562	0.578	0.439	0.509	0.431	0.719	0.625	0.57	0.525	0.353
$\alpha = 1.5\%$	Exceed	227	162	237	66	81	82	234	255	239	49	59	67	272	244
	MAPE	0.183	0.156	0.235	0.656	0.578	0.573	0.22	0.329	0.246	0.745	0.692	0.651	0.418	0.272
$\alpha = 2.0\%$	Exceed	285	225	298	80	95	107	297	326	297	53	71	79	338	319
	MAPE	0.114	0.12	0.165	0.687	0.629	0.582	0.161	0.274	0.161	0.793	0.722	0.691	0.321	0.247
$\alpha = 2.5\%$	Exceed	345	306	360	95	112	128	361	392	359	67	80	91	409	388
	MAPE	0.079	0.043	0.126	0.703	0.65	0.6	0.129	0.226	0.123	0.79	0.75	0.715	0.279	0.213
Average MAPE		0.285	0.136	0.362	0.656	0.59	0.582	0.362	0.446	0.361	0.712	0.648	0.597	0.543	0.405
5 Year															
$\alpha = 0.5\%$	Exceed	74	78	77	16	17	18	80	79	80	16	22	29	83	76
	MAPE	0.256	0.323	0.307	0.729	0.712	0.695	0.357	0.34	0.357	0.729	0.627	0.508	0.408	0.29
$\alpha = 1.0\%$	Exceed	131	131	131	44	47	48	135	131	131	27	34	41	145	138
	MAPE	0.111	0.111	0.111	0.627	0.601	0.593	0.145	0.111	0.111	0.771	0.712	0.652	0.23	0.171
$\alpha = 1.5\%$	Exceed	199	197	199	58	68	76	204	199	197	38	44	54	207	199
	MAPE	0.126	0.114	0.126	0.672	0.615	0.57	0.154	0.126	0.114	0.785	0.751	0.695	0.171	0.126
$\alpha = 2.0\%$	Exceed	258	263	268	74	84	95	275	270	259	47	56	64	269	269
	MAPE	0.094	0.116	0.137	0.686	0.644	0.597	0.167	0.145	0.099	0.801	0.762	0.729	0.141	0.141
$\alpha = 2.5\%$	Exceed	324	324	320	82	100	123	328	324	320	57	66	74	328	323
	MAPE	0.1	0.1	0.086	0.722	0.661	0.583	0.113	0.1	0.086	0.807	0.776	0.749	0.113	0.096
Average MAPE		0.137	0.153	0.153	0.687	0.647	0.608	0.187	0.164	0.153	0.779	0.726	0.667	0.213	0.165
10 Year															
$\alpha = 0.5\%$	Exceed	62	64	64	16	16	17	64	63	64	14	18	24	72	70
	MAPE	0.177	0.215	0.215	0.696	0.696	0.677	0.215	0.196	0.215	0.734	0.658	0.544	0.367	0.329
$\alpha = 1.0\%$	Exceed	134	134	133	31	40	43	132	134	132	23	26	31	134	128
	MAPE	0.272	0.272	0.263	0.706	0.62	0.592	0.253	0.272	0.253	0.782	0.753	0.706	0.272	0.215
$\alpha = 1.5\%$	Exceed	185	183	181	52	62	72	183	184	175	29	37	45	183	180
	MAPE	0.171	0.158	0.146	0.671	0.608	0.544	0.158	0.165	0.108	0.816	0.766	0.715	0.158	0.139
$\alpha = 2.0\%$	Exceed	244	248	245	68	87	95	247	244	236	40	47	57	236	234
	MAPE	0.158	0.177	0.163	0.677	0.587	0.549	0.173	0.158	0.12	0.81	0.777	0.729	0.12	0.111
$\alpha = 2.5\%$	Exceed	289	293	294	87	99	113	292	293	289	53	62	73	290	293
	MAPE	0.098	0.113	0.117	0.67	0.624	0.571	0.109	0.113	0.098	0.799	0.765	0.723	0.101	0.113
Average MAPE		0.175	0.187	0.181	0.684	0.627	0.587	0.182	0.181	0.159	0.788	0.744	0.683	0.204	0.181

3.6 Summary

This chapter introduced the Extended Burr XII distribution (EB) as an extended extreme value approach to estimating VaR. The performance of the proposed method was compared with three competing methods (GP, GEV, and WB) through a simulation study and an empirical study. The simulation study suggested that EB has the best distribution fittings and VaR estimate accuracy of all four candidate distributions. In the empirical study, the VaR estimate was backtested through UC and MAPE in short-, medium-, and long-term windows with different VaR levels. The results indicate that EB outperforms other candidate distributions and is capable of modelling VaR well under a range of different scenarios.

In conclusion, EB provides a robust and flexible approach to estimating VaR in comparison to other extreme-based methods. Researchers and practitioners are encouraged to embed this model in their research and practice.

Chapter 4

Forecast of Dynamic VaR using an Extended GARCH-EVT Approach

In Chapter 3, we used an extended extreme value approach to forecast unconditional VaR. The proposed method provides a useful tool for financial regulation. However, it is not suitable for monitoring risk, since it does not forecast risk on a dynamic basis. In this chapter, we aim to forecast financial market VaR dynamically by considering volatility in our extended extreme value approach.

Unconditional VaR assumes that data are independent and identically distributed (i.i.d.), which is not necessarily the case for financial time series. In fact, financial time series often exhibit serial dependence. As such, one potential approach is to expand EVT to incorporate this type of data. Section 4.1 reviews this approach and identifies its limitations in forecasting dynamic VaR. It then outlines the generalised autoregressive conditional heteroskedasticity (GARCH) model, which captures the volatility characteristics of financial time series. The dynamic VaR forecast is obtained by applying the extended extreme value approach to the residuals of the GARCH model. The last section of the chapter evaluates the proposed method through a large-scale empirical study.

4.1 Extreme Value Analysis for Serially Dependent Data

Unconditional VaR provides a long-term average measure of market risk based on the i.i.d. data assumption. It is best suited to financial risk regulation, since regulation standards are not regularly changed. However, financial time series often exhibit serial dependence, implying that today's risk has an impact upon tomorrow's. Unconditional VaR based on the i.i.d. assumption does not reflect this feature, and therefore is not appropriate for day-to-day risk monitoring. To better model dynamic risk, serial dependence should be considered in the extended extreme value approach.

One attempt has been made by Leadbetter (1983), who expanded the classic EVT to stationary serial dependent data. He argues that extreme value analysis for serially dependent data should focus only on the dependence of extreme events.

Assume that $\{X_1, X_2, \cdots, X_n\}$ is stationary with serial dependence. The maximum of the series, after normalisation, converges to a limit distribution

$$\lim_{n\to\infty} P(\frac{M_n - b_n}{a_n} \leq x) = G^{\theta}(x), \tag{4.1}$$

where G is the limiting distribution under the iid assumption and $0 < \theta < 1$. θ is known as the extremal index[1], which measures the level of dependence of the series. The lower the value of θ, the stronger the dependence in extreme events.

Four different methods are available for estimating the extremal index. The first is the block method, which compares the number of events in selected blocks with the total number of extreme events (Smith & Weissman, 1994). The second is the run estimator, in which the extremal index θ is related to the length of a run (that is, an extreme event followed by a period containing no extreme events) (Smith & Weissman, 1994). The third method is an interval estimator, which

[1] Note that the extreme index should not be confused with the extreme value index. The extreme value index refers to the shape parameters in tail distribution (for instance, ξ for GP). The extreme index measures the serial dependence in extremes for a time series.

captures the length of intervals between extreme events (Ferro & Segers, 2003). The fourth is a pseudo maximum likelihood estimator (Berghaus & Bücher, 2018).

The extremal index performs adequately in estimating the extreme distribution of a conditional process, and allows the level of extreme clustering to be measured. Nevertheless, it is not helpful in monitoring day-to-day risk, since it does not provide information on current market volatility. In addition, the methods listed above require the financial time series to be stationary — a condition that may not be satisfied in a VaR forecast application. Therefore, we turn to another class of statistical models that capture time-varying volatility features: the GARCH model.

4.2 Time-varying Volatility and the GARCH Model

Financial time series typically exhibit a volatility cluster effect, in which large changes in loss tend to be followed by large changes, and large changes in return tend to be followed by large returns. The autoregressive conditional heteroscedasticity (ARCH) model was developed by (Engle, 1982) to model uncertainty in the inflation rate. It aims to capture the time-varying volatility. Bollerslev (1986) have generalised the ARCH model to develop the generalised autroregressive conditional heteroscedasticity (GARCH) model.

The GARCH(p,q) model is defined by

$$X_t = \sigma_t \varepsilon_t \tag{4.2}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{k-j}^2 \tag{4.3}$$

where $\alpha_i \geq 0, \beta_j \geq 0$, and $\omega > 0$. The sequence of innovation ε_t is i.i.d. with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = 1$, which can be interpreted as the volatility of the market at time t. ε_t^2 as the conditional variance of the X_t is determined by the information up to time $t-1$ given in autoregressive structures. It is positively correlated to the past value of itself, and to X_t^2. This model explains the volatility clustering features of the financial market, in which large loss (or return) tends to be followed by large loss (or return), while small loss (or return) tends to be followed by the small loss (or return). The second moments of X_t requires $E(X_t^2) < \infty$.

The GARCH model has the following basic statistical properties:

1. The model admits to a unique and stationary solution if $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$, which is unknown as the properties of uniqueness and stationarity; see Bougerol and Picard (1992) for the proof.

2. Mean zero property, i.e., $E(X_t)$=0.

3. Lack of serial correlation. $E(X_k X_{k+h}) = 0$.

4. Unconditional variance.

5. Heavy tails of X_k.

4.3 Forecasting Conditional VaR by the Extended Extreme Value Approach with the GARCH Model

The GARCH model successfully captures the time-varying volatility characteristics of financial time series, which forms the basis of the dynamic VaR forecast. However, this approach reduces rather than eliminating volatility clustering, as the standardised residuals still have a long tail (McNeil, 1999). Therefore, VaR forecasts from the GARCH model still underestimate VaR.

To better forecast dynamic VaR, we incorporate the GARCH model in the extended extreme value approach following a two-step procedure developed by McNeil and Frey (2000). In the first step, the standardised residuals of the

GARCH model are obtained through QMLE (Bollerslev & Wooldridge, 1992). In the second step, the extended extreme value approach is applied to the standardised residuals. VaR is then forecast.

In a VaR forecast application, GARCH(p,q) is often employed to model the volatility characterised by the equation

$$X_t = \sigma_t Z_t$$
$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \gamma_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{k-j}^2,$$

where σ_t is the volatility at time t; $\omega, \gamma_i, \beta_j$ are the GARCH(p,q) coefficients; and Z_t is a strict white noise process with zero mean and unit variance. To obtain a VaR forecast under GARCH(p,q), the extended extreme value approach is applied to the standardised residuals Z_t after the GARCH coefficients are estimated. VaR through GARCH(p,q) is then

$$VaR_{gr,t}^{\alpha} = \sqrt{\hat{\omega} + \sum_{i=1}^{p} \hat{\gamma}_i X_{t-1}^2 + \sum_{j=1}^{q} \hat{\beta}_j \sigma_{t-1}^2} \times \{u_{gr} + \hat{\lambda}_{gr}[\frac{1}{\hat{k}_{gr}} - \frac{1}{\hat{k}_{gr}}(\frac{N\alpha}{N-n})^{\hat{k}_{gr}}]^{\frac{1}{\hat{c}_{gr}}}\} \tag{4.4}$$

where $\hat{\omega}, \hat{\gamma}_i, \hat{\beta}_j$ are the estimated GARCH(p,q) coefficients, u_{gr} is the selected threshold, $\hat{k}_{gr}, \hat{\lambda}_{gr}, \hat{c}_{gr}$ are the estimated EB parameters, N is the total number of the sample, and α is the VaR confidence level.

The first-order autoregressive model AR(1) is sometimes considered in a VaR forecast task to model dependence among observations. The AR(1) is characterised by the formula

$$X_t = \mu + aX_{t-1} + \varepsilon_t, \tag{4.5}$$

where μ is the long-term autoregressive mean, a is the autoregressive coefficient, and ε_t is the error at time t. To forecast VaR under AR(1), the extended extreme value approach is applied to the error ε_t after fitting AR(1). The VaR with confidence level α at time t is then computed through

$$VaR^{\alpha}_{ar,t} = \hat{\mu} + \hat{a}X_{t-1} + u_{ar} + \hat{\lambda}_{ar}[\frac{1}{\hat{k}_{ar}} - \frac{1}{\hat{k}_{ar}}(\frac{N\alpha}{N-n})^{\hat{k}_{ar}}]^{\frac{1}{\hat{c}_{ar}}}.$$

$\hat{\mu}, \hat{a}$ are estimated long term mean and autoregressive coefficient of AR(1), u_{ar} is a selected threshold from ε_t, and $\hat{\xi}_{ar}, \hat{\lambda}_{ar}, \hat{c}_{ar}$ are estimated EB parameters by the errors ε_t above u_{ar}.

We also consider both autocorrelation and volatility dependence in an AR(1)-GARCH(p,q) model

$$X_t = \mu + aX_{t-1} + \sigma_t Z_t \tag{4.6}$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \gamma_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{k-j}^2, \tag{4.7}$$

Applying the POT method with EB to the standardised residuals, the VaR forecast from AR(1)-GARCH(p,q) is

$$\begin{aligned} VaR^{\alpha}_{ar-gr,t} =& \hat{\mu} + \hat{a}X_{t-1} + \sqrt{\hat{\omega} + \sum_{i=1}^{p} \hat{\gamma}_i X_{t-1}^2 + \sum_{j=1}^{q} \hat{\beta}_j \sigma_{t-1}^2} \times \{u_{ar-gr} + \\ & \hat{\lambda}_{ar-gr}[\frac{1}{\hat{k}_{ar-gr}} - \frac{1}{\hat{k}_{ar-gr}}(\frac{N\alpha}{N-n})^{\hat{k}_{ar-gr}}]^{\frac{1}{\hat{c}_{gr}}}\}, \end{aligned} \tag{4.8}$$

where the subscript of $ar - gr$ represents parameters for AR(1)-GARCH(p,q).

4.4 Large-scale Empirical Study

4.4.1 Data

The dataset contains 29 stock indices, covering both developing and developed countries and 18 currency exchange rates. The stock indices range from 1st January 2000 to 31st December 2019 with 20 years of price history, while the exchange rates range from 1st January 2004 to 31st December 2019 with 16 years of price history. Table A.1 of Appendix A shows the detail of the financial symbols. The dataset is available for download from Yahoo Finance through the Quantmod Package (Ryan et al., 2020) in statistical programming software R (R Core Team, 2021).

The typical procedure for empirical VaR computation splits the dataset into a modelling and validation set. However, as the economic system often evolves with policy change and market conditions, the VaR model backtested by this procedure does not reflect market changes, and performance is affected by the period selected for modelling (Tashman, 2000). Therefore, we adopt a rolling window approach — which has a similar concept to cross-validation in regression practice — to estimate the model and forecast VaR. The rolling window approach adopts new observations and eliminates old ones in the modelling stage, producing VaR forecasts through the updated models. The VaR models backtested under this approach reflect market changes and are more robust.

Loss data are obtained by taking the negative of the log-return defined by $X_t = -100 \times ln\frac{P_t}{P_{t-1}}$, where P_t is the adjusted closing price of a financial instrument. The model selects the threshold at 0.9, 0.91, 0.92, 0.93, and 0.94 quantiles, and the moving window at one to five years with approximately 250 trading days per year. Table 4.1 reports summary statistics of the loss, including mean, standard deviation (Std), skewness (Skew), kurtosis (Kurt), and historical unconditional 1% VaR.

The summary statistics reveal differences in stock indices and exchanges rates. For stock indices, the 50th percentile of the mean is significantly positive, and more than 75% of skewness values are negative, indicating that most stock indices are left-skewed. By contrast, the 50th percentile of the mean for exchange rates is zero, and the percentages of positive and negative skewness are roughly equal, suggesting that exchange rates are somewhat symmetrical. There may be more extreme values in exchange rates than stock indices, as the kurtosis of exchange

Table 4.1: Summary statistics of the loss (negative log-return)

Stats	Mean	Std	Skew	Kurt	VaR (1%)
Panel A: Stock indices (29 series)					
Min	-0.088	0.796	-0.937	3.836	2.221
Q25	-0.022	1.159	0.095	5.459	3.220
Q50	-0.015	1.321	0.301	6.409	3.836
Q75	-0.006	1.482	0.508	8.093	4.371
Max	0.010	6.907	1.983	41.380	16.346
Panel B: Exchanges rates (18 series)					
Min	-0.018	0.079	-1.016	2.527	0.178
Q25	-0.004	0.525	-0.388	6.799	1.321
Q50	0.000	0.628	-0.028	10.514	1.739
Q75	0.002	0.766	0.310	17.544	2.193
Max	0.007	1.091	0.755	181.236	2.440

rates is greater than the kurtosis of market indices.

4.4.2 The Performance of VaR Forecast Accuracy

The empirical study obtains the 1% unconditional and dynamic VaR forecasts for each financial instrument from the peak-over-threshold method, with four distributions based on EVT, five threshold selections, and five rolling windows. We evaluate these forecasts based on their correct coverage probability using the unconditional coverage (UC) test reported in Kupiec (1995), the conditional coverage (CC) test of Christoffersen (1998), and the dynamic quantile (DQ) test of Engle and Manganelli (2004), as listed in Section 2.3. Each test produces a p-value for the null hypothesis that the coverage probability of the proposed model equals the predefined VaR confidence level.

Tables 4.2, 4.3, and 4.4 summarise the percentage of the financial instruments for which the UC, CC, or DQ tests are rejected at a 5% significance level, following Calmon et al. (2021) and Ardia et al. (2018). The percentage of rejection indicates the performance of the candidate methods in forecasting VaR. If a method has a correct coverage probability of 1% for every financial instrument, then the rejection percentage of all instruments will be zero. The shading in the tables highlights the lowest rejection rate of the VaR forecast from candidate methods.

Results from the UC test given in Table 16 show that the unconditional and dynamic VaR forecasts by EB have the lowest percentage rejection rate when the rolling window is 500 trading days or less, and the threshold is 0.91 or higher. When the rolling window is 750 trading days or more, the performance results are mixed, with no single distribution clearly outperforming the others. GEV has the lowest rejection rate in both the market indices and the exchange rate when the threshold is 0.9.

Results from the CC test in Table 4.3 display a similar pattern to those of the UC test, except that CC has a higher average rejection rate, since it makes strong assumptions of serial dependence. The VaR forecast from EB has the lowest rejection rate of all four distributions in all model settings when the rolling window is 500 trading days or less, and the threshold is 0.91 or higher. Unlike UC, no single distribution outperforms the others when the threshold is 0.9.

Results from the DQ test (Table 4.4) have the highest rejection rate among all three probability tests, likely because DQ has the strongest test power. Since DQ makes more realistic assumptions regarding serial dependence, the unconditional VaR forecasts from all distributions with different model settings have been rejected in most cases. However, the dynamic VaR forecast from EB still has the lowest rejection rate of the four candidate distributions when the rolling window is 500 trading days or less, and the threshold is 0.91 or higher.

In summary, for all three probability tests, the unconditional and dynamic VaR forecast from EB has the lowest rejection rate among four candidate distributions in both market indices and exchange rates, when the rolling is 500 trading days or less and the threshold is 0.91 or higher. The VaR forecast from GEV has the lowest rejection rate in the UC test when the threshold is 0.9. For all other model settings, is no single distribution that outperforms the others.

4.4.3 The Performance of VaR Forecasts on Occurrence of Risk Incidents

Table 4.5 and Table 4.6 summarise the occurrence performance of AE and APE statistics for VaR forecasts. Occurrence refers to instances in which the loss is greater than the forecasted VaR value. VaR models with an AE statistic close to one correctly project the occurrence of model failure, while VaR models with an APE statistic close to zero in percentage correctly reflect the occurrence of model failure. VaR models overestimate the occurrence of failure when the AE value is greater than zero, and underestimate the occurrence of failure when AE is less than one.

Table 4.5 shows that VaR forecasts from most scenarios overestimate the occurrence of model failure, with the exception of VaR forecasts from the exchange rate in the 1250 trading days rolling window. However, when the rolling window is 500 trading days or less and the threshold is 0.91 or greater, the AE statistics from EB in the unconditional and dynamic VaR forecast have the closest value to one across four candidate distributions. This indicates that EB is less likely to overestimate VaR. GEV tends to have an AE value close to one when the threshold is 0.9. In all other scenarios, there are no clear patterns.

Table 4.6 shows that VaR forecasts from EB have a lower APE value compared to other candidate distributions when the rolling window is 500 trading days or less, and the threshold is 0.91 or greater. This indicates that the occurrence of failure, in the absolute sense, from EB is smaller. When the rolling window is 750 trading days or more, the results are mixed, and no single distribution outperforms the others. GEV tends to have a better APE performance when the threshold is 0.9.

In summary, the occurrence statistics have a similar pattern to the probability test statistics. Given a shorter rolling window and higher threshold, the unconditional and dynamic VaR is likely to be overestimated by EB. With a longer rolling window, no single distribution is preferred. With a lower threshold, GEV tends to have a better performance.

Table 4.2: Percentage of Assets for which the Unconditional Coverage Test of VaR Predictions is Rejected (%)

Financial Type	Window	Dependence	0.94				0.93				0.92				0.91				0.90			
			EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB
Market indices	250	AR(1)	14	38	76	62	10	66	83	62	21	41	66	41	14	55	55	45	28	48	17	41
		GARCH(1,1)	3	14	55	21	10	31	59	28	10	17	31	14	10	24	31	14	10	17	10	17
		AR(1)-GARCH(1,1)	10	17	72	24	10	38	66	28	7	17	48	21	3	35	35	21	17	28	21	17
	500	AR(1)	28	31	62	35	21	38	55	41	28	35	38	38	31	35	35	31	24	31	10	31
		GARCH(1,1)	0	0	17	0	0	0	3	0	0	3	3	0	0	0	3	0	3	0	0	0
		AR(1)-GARCH(1,1)	0	0	21	0	0	0	10	0	0	3	3	7	0	0	3	0	3	0	0	0
	750	AR(1)	10	10	35	14	10	24	35	17	10	17	28	17	17	21	21	17	17	17	14	17
		GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1000	AR(1)	17	21	38	14	21	24	38	14	21	21	38	14	31	21	21	14	24	21	14	17
		GARCH(1,1)	0	0	10	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	7	0	0	0	7	0	0	0	3	0	0	0	0	0	0	0	0	0
	1250	AR(1)	31	28	41	28	31	28	41	28	28	28	38	28	28	28	31	28	28	31	28	28
		GARCH(1,1)	0	0	17	0	0	0	14	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	17	0	0	0	14	0	0	0	0	0	0	0	0	0	0	0	0	0
Exchange Rate	250	AR(1)	0	6	50	11	0	33	44	22	6	11	17	17	0	33	17	17	0	17	6	11
		GARCH(1,1)	0	0	28	11	0	22	33	11	6	17	17	17	6	22	11	11	6	17	0	17
		AR(1)-GARCH(1,1)	0	0	33	17	0	17	44	17	6	17	11	17	0	22	11	11	6	22	6	17
	500	AR(1)	6	11	44	11	6	17	28	6	6	17	22	6	6	17	0	11	11	17	0	11
		GARCH(1,1)	0	0	11	6	0	6	11	6	0	11	6	6	0	6	6	6	6	6	0	6
		AR(1)-GARCH(1,1)	0	6	11	11	0	6	11	6	0	6	6	6	0	6	6	6	6	6	0	6
	750	AR(1)	6	6	22	0	0	11	22	6	0	0	17	0	0	6	11	0	0	6	0	0
		GARCH(1,1)	0	0	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	11	0	0	0	6	0	0	0	0	0	0	0	0	0	0	0	0	0
	1000	AR(1)	0	0	17	0	0	6	11	0	6	6	11	0	0	6	0	0	6	6	0	6
		GARCH(1,1)	0	0	6	6	0	0	6	6	0	0	0	0	0	6	0	0	0	6	0	0
		AR(1)-GARCH(1,1)	0	0	6	6	0	0	6	0	0	0	6	0	0	6	0	0	0	6	0	0
	1250	AR(1)	6	17	6	22	11	17	6	17	11	17	11	22	22	11	11	22	17	17	17	17
		GARCH(1,1)	0	0	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4.3: Percentage of Assets for which the Conditional Coverage Test of VaR Predictions is Rejected (%)

Financial Type	Window	Dependence	0.94				0.93				0.92				0.91				0.90			
			EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB
Market indices	250	AR(1)	48	76	86	83	59	86	86	90	55	83	72	86	69	86	76	86	66	86	62	48
		GARCH(1,1)	10	21	59	24	7	28	55	21	7	24	35	24	3	35	31	17	7	24	17	10
		AR(1)-GARCH(1,1)	14	17	66	31	14	35	62	38	17	24	45	28	10	35	38	24	10	21	28	14
	500	AR(1)	66	76	83	76	72	79	83	83	72	79	76	83	69	83	76	76	76	79	72	66
		GARCH(1,1)	3	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	3	7	7	3
		AR(1)-GARCH(1,1)	7	10	17	10	10	7	10	10	3	7	10	7	7	7	7	10	7	10	3	7
	750	AR(1)	62	69	76	62	66	69	76	66	66	66	76	66	69	69	72	62	69	69	62	62
		GARCH(1,1)	0	0	3	0	0	0	3	0	0	0	0	0	0	3	0	0	0	3	0	0
		AR(1)-GARCH(1,1)	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1000	AR(1)	66	69	76	69	69	69	69	69	69	69	69	69	69	69	69	69	69	69	55	66
		GARCH(1,1)	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	3	3	0	3	0	3	3	0	3	3	0	0	3	0	0	0	3	0	0
	1250	AR(1)	72	72	79	66	72	72	79	69	72	72	72	69	72	79	69	69	76	79	69	72
		GARCH(1,1)	0	0	3	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	3	10	3	3	0	0	0	3	0	0	3	0	3	0	3	3	3	0	0
Exchange Rate	250	AR(1)	17	28	61	33	17	44	61	28	22	33	33	33	28	50	33	28	22	44	28	17
		GARCH(1,1)	0	0	17	0	0	6	17	6	0	0	6	0	0	17	0	6	0	6	0	0
		AR(1)-GARCH(1,1)	0	0	22	0	0	11	28	0	0	6	0	0	0	11	0	0	0	6	0	0
	500	AR(1)	44	56	67	50	44	50	67	50	39	56	61	44	50	44	50	50	39	50	50	44
		GARCH(1,1)	0	0	6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	11	0	0	0	6	0	0	0	6	0	0	0	0	0	0	0	0	0
	750	AR(1)	56	56	61	56	56	61	61	56	50	56	61	56	50	61	50	56	56	61	44	56
		GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1000	AR(1)	56	50	61	44	50	56	56	44	44	56	50	44	44	56	50	44	50	50	50	56
		GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1250	AR(1)	39	50	39	44	44	44	39	39	39	44	44	44	50	44	33	39	50	44	39	39
		GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
		AR(1)-GARCH(1,1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4.4: Percentage of Assets for which the Dynamic Quantile of VaR Predictions is Rejected (%)

Financial Type	Window	Dependence	0.94				0.93				0.92				0.91				0.90			
			EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB
		AR(1)	100	100	97	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	250	GARCH(1,1)	72	79	93	83	72	86	90	90	72	83	79	79	76	86	83	83	79	86	83	72
		AR(1)-GARCH(1,1)	76	76	86	86	69	90	93	90	72	83	86	86	69	90	86	86	79	90	83	76
		AR(1)	97	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	97
	500	GARCH(1,1)	38	41	62	45	45	41	52	45	45	45	48	48	45	45	41	48	38	45	35	38
		AR(1)-GARCH(1,1)	48	41	62	48	48	52	59	52	55	45	52	55	41	41	52	41	41	45	38	48
		AR(1)	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
Market indices	750	GARCH(1,1)	28	41	52	45	35	48	48	48	41	41	41	45	45	45	48	52	41	41	41	28
		AR(1)-GARCH(1,1)	24	35	55	41	31	45	52	38	38	41	45	41	31	41	45	41	38	41	35	24
		AR(1)	97	100	100	100	100	100	100	100	100	100	97	100	100	100	97	100	100	100	97	97
	1000	GARCH(1,1)	31	35	48	31	41	35	45	31	35	41	41	35	35	38	41	35	41	35	31	31
		AR(1)-GARCH(1,1)	31	38	52	38	38	35	45	35	41	41	45	35	35	38	45	35	38	38	38	31
		AR(1)	100	100	100	97	100	100	100	100	100	100	97	100	100	100	97	100	100	100	97	100
	1250	GARCH(1,1)	28	31	41	24	24	28	41	24	31	31	38	28	21	31	28	24	28	28	28	28
		AR(1)-GARCH(1,1)	38	35	48	35	35	38	45	31	38	38	35	31	35	35	41	38	38	28	31	38
		AR(1)	94	94	94	94	94	94	100	94	94	94	100	94	94	94	94	94	94	94	94	94
	250	GARCH(1,1)	28	50	50	50	28	61	61	50	33	44	44	50	39	61	50	50	39	44	50	28
		AR(1)-GARCH(1,1)	33	33	50	44	22	39	61	44	22	33	50	33	22	39	33	33	33	28	28	33
		AR(1)	89	94	94	94	94	94	94	94	94	94	94	94	94	94	94	94	94	94	94	89
	500	GARCH(1,1)	6	17	28	22	11	17	11	22	17	22	22	22	28	22	6	22	17	17	17	6
		AR(1)-GARCH(1,1)	0	11	22	11	11	11	22	11	0	22	11	6	11	6	22	11	11	6	17	0
		AR(1)	94	94	94	94	89	94	94	94	89	94	94	94	94	94	94	94	89	94	94	94
Exchange Rate	750	GARCH(1,1)	17	11	17	11	17	17	11	11	17	11	11	11	17	17	17	11	11	17	17	17
		AR(1)-GARCH(1,1)	17	17	22	11	17	11	22	11	17	11	17	11	17	11	11	11	11	11	11	17
		AR(1)	89	83	89	83	89	89	89	89	89	89	89	89	83	89	89	89	78	89	89	89
	1000	GARCH(1,1)	22	11	28	28	17	17	22	28	22	22	22	28	17	22	22	28	22	22	17	22
		AR(1)-GARCH(1,1)	17	11	28	28	17	22	22	28	22	22	28	28	17	22	22	22	22	22	17	17
		AR(1)	78	83	83	83	83	83	83	83	78	83	83	83	83	83	83	83	83	83	78	78
	1250	GARCH(1,1)	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17
		AR(1)-GARCH(1,1)	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17

Table 4.5: The Average Value of Actual over Expected Ratio

Financial Type	Window	Dependence	0.94				0.93				0.92				0.91				0.90			
			EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB
		AR(1)	1.27	1.36	1.47	1.39	1.27	1.42	1.47	1.41	1.31	1.37	1.41	1.38	1.30	1.43	1.39	1.37	1.34	1.39	1.31	1.37
	250	GARCH(1,1)	1.25	1.30	1.43	1.33	1.23	1.35	1.43	1.34	1.27	1.30	1.36	1.32	1.24	1.35	1.34	1.31	1.29	1.31	1.29	1.31
		AR(1)-GARCH(1,1)	1.27	1.31	1.45	1.35	1.26	1.38	1.45	1.37	1.29	1.31	1.38	1.34	1.27	1.38	1.36	1.33	1.31	1.34	1.30	1.32
		AR(1)	1.27	1.30	1.44	1.31	1.29	1.31	1.39	1.31	1.30	1.31	1.35	1.31	1.30	1.31	1.32	1.30	1.30	1.31	1.26	1.29
	500	GARCH(1,1)	1.13	1.13	1.27	1.15	1.12	1.13	1.24	1.14	1.13	1.13	1.21	1.14	1.13	1.12	1.18	1.14	1.13	1.12	1.13	1.13
		AR(1)-GARCH(1,1)	1.14	1.13	1.29	1.15	1.14	1.13	1.25	1.15	1.14	1.13	1.22	1.14	1.14	1.11	1.18	1.14	1.13	1.12	1.14	1.12
		AR(1)	1.22	1.24	1.35	1.22	1.21	1.26	1.33	1.22	1.23	1.23	1.28	1.22	1.23	1.25	1.26	1.22	1.23	1.23	1.20	1.22
Market indices	750	GARCH(1,1)	1.11	1.10	1.24	1.12	1.09	1.13	1.21	1.12	1.12	1.10	1.17	1.12	1.09	1.11	1.15	1.10	1.11	1.09	1.11	1.11
		AR(1)-GARCH(1,1)	1.11	1.10	1.24	1.12	1.10	1.13	1.22	1.12	1.11	1.11	1.18	1.11	1.09	1.10	1.16	1.11	1.11	1.08	1.11	1.10
		AR(1)	1.21	1.22	1.33	1.19	1.21	1.22	1.29	1.19	1.22	1.22	1.25	1.20	1.23	1.21	1.22	1.20	1.22	1.21	1.18	1.20
	1000	GARCH(1,1)	1.11	1.11	1.24	1.10	1.10	1.10	1.21	1.09	1.10	1.09	1.17	1.09	1.11	1.09	1.14	1.09	1.11	1.08	1.10	1.09
		AR(1)-GARCH(1,1)	1.11	1.09	1.24	1.11	1.11	1.09	1.21	1.10	1.10	1.10	1.18	1.10	1.10	1.09	1.15	1.09	1.11	1.08	1.11	1.10
		AR(1)	1.23	1.24	1.34	1.21	1.22	1.24	1.32	1.21	1.23	1.24	1.27	1.20	1.23	1.25	1.24	1.21	1.23	1.24	1.19	1.21
	1250	GARCH(1,1)	1.11	1.10	1.24	1.10	1.10	1.10	1.22	1.10	1.10	1.08	1.17	1.09	1.09	1.09	1.14	1.08	1.09	1.07	1.11	1.09
		AR(1)-GARCH(1,1)	1.13	1.10	1.25	1.10	1.10	1.11	1.23	1.10	1.10	1.08	1.18	1.09	1.09	1.08	1.15	1.08	1.09	1.07	1.10	1.08
		AR(1)	1.21	1.33	1.42	1.36	1.17	1.38	1.43	1.36	1.25	1.34	1.36	1.34	1.17	1.39	1.34	1.33	1.24	1.36	1.29	1.33
	250	GARCH(1,1)	1.16	1.25	1.34	1.28	1.17	1.32	1.34	1.29	1.21	1.27	1.29	1.27	1.20	1.31	1.27	1.27	1.23	1.28	1.22	1.26
		AR(1)-GARCH(1,1)	1.16	1.22	1.36	1.29	1.16	1.30	1.36	1.30	1.21	1.27	1.29	1.28	1.20	1.31	1.26	1.27	1.24	1.28	1.22	1.25
		AR(1)	1.17	1.29	1.40	1.26	1.20	1.27	1.36	1.26	1.20	1.28	1.31	1.26	1.23	1.28	1.26	1.26	1.23	1.28	1.23	1.26
	500	GARCH(1,1)	1.10	1.12	1.24	1.15	1.11	1.14	1.21	1.14	1.11	1.15	1.18	1.14	1.15	1.15	1.12	1.15	1.14	1.15	1.11	1.14
		AR(1)-GARCH(1,1)	1.12	1.14	1.27	1.15	1.14	1.14	1.23	1.15	1.13	1.14	1.19	1.14	1.14	1.15	1.16	1.14	1.14	1.15	1.12	1.14
		AR(1)	1.21	1.26	1.36	1.24	1.19	1.29	1.33	1.24	1.24	1.26	1.30	1.24	1.21	1.27	1.26	1.23	1.23	1.27	1.21	1.23
Exchange Rate	750	GARCH(1,1)	1.10	1.12	1.23	1.10	1.09	1.14	1.20	1.10	1.12	1.12	1.16	1.09	1.09	1.14	1.12	1.08	1.12	1.12	1.07	1.09
		AR(1)-GARCH(1,1)	1.11	1.13	1.25	1.10	1.11	1.14	1.23	1.11	1.14	1.12	1.17	1.11	1.11	1.13	1.13	1.11	1.13	1.12	1.09	1.10
		AR(1)	1.13	1.16	1.29	1.13	1.15	1.17	1.25	1.14	1.16	1.16	1.21	1.14	1.17	1.15	1.15	1.14	1.14	1.16	1.11	1.14
	1000	GARCH(1,1)	1.05	1.07	1.19	1.06	1.07	1.06	1.16	1.05	1.07	1.08	1.12	1.05	1.07	1.07	1.07	1.06	1.09	1.05	1.05	1.06
		AR(1)-GARCH(1,1)	1.04	1.06	1.19	1.04	1.05	1.06	1.15	1.04	1.06	1.07	1.12	1.04	1.06	1.06	1.06	1.05	1.08	1.04	1.03	1.05
		AR(1)	0.88	0.90	0.97	0.86	0.88	0.91	0.96	0.86	0.89	0.88	0.92	0.86	0.89	0.90	0.90	0.86	0.88	0.89	0.85	0.87
	1250	GARCH(1,1)	0.99	0.99	1.08	0.97	0.98	1.00	1.05	0.97	1.00	0.98	1.02	0.96	0.97	0.98	1.01	0.95	0.99	0.98	0.96	0.96
		AR(1)-GARCH(1,1)	0.97	0.97	1.07	0.95	0.97	0.98	1.04	0.95	0.97	0.96	1.01	0.94	0.96	0.97	0.98	0.95	0.98	0.97	0.94	0.95

Table 4.6: Absolute Percentage Error

Financial Type	Window	Dependence	0.94				0.93				0.92				0.91				0.90			
			EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB	EB	GP	GEV	WB
Market indices	250	AR(1)	0.27	0.36	0.47	0.39	0.27	0.42	0.47	0.41	0.31	0.37	0.41	0.38	0.30	0.43	0.39	0.37	0.34	0.39	0.31	0.37
		GARCH(1,1)	0.25	0.30	0.43	0.33	0.23	0.35	0.43	0.34	0.27	0.30	0.36	0.32	0.24	0.35	0.34	0.31	0.29	0.31	0.29	0.31
		AR(1)-GARCH(1,1)	0.27	0.31	0.45	0.35	0.26	0.38	0.45	0.37	0.29	0.31	0.38	0.34	0.27	0.38	0.36	0.33	0.31	0.34	0.30	0.32
	500	AR(1)	0.28	0.30	0.44	0.32	0.30	0.32	0.39	0.32	0.30	0.32	0.35	0.32	0.31	0.32	0.32	0.31	0.30	0.32	0.27	0.30
		GARCH(1,1)	0.14	0.13	0.27	0.16	0.13	0.14	0.24	0.16	0.13	0.15	0.21	0.16	0.15	0.14	0.18	0.15	0.14	0.14	0.14	0.14
		AR(1)-GARCH(1,1)	0.15	0.15	0.29	0.17	0.15	0.15	0.25	0.16	0.15	0.15	0.22	0.16	0.15	0.14	0.19	0.15	0.15	0.14	0.15	0.14
	750	AR(1)	0.24	0.24	0.35	0.24	0.23	0.26	0.33	0.24	0.24	0.24	0.29	0.24	0.24	0.25	0.27	0.24	0.24	0.24	0.22	0.23
		GARCH(1,1)	0.13	0.12	0.24	0.13	0.11	0.14	0.21	0.14	0.13	0.13	0.17	0.13	0.12	0.13	0.16	0.13	0.13	0.12	0.13	0.13
		AR(1)-GARCH(1,1)	0.13	0.12	0.24	0.13	0.11	0.14	0.22	0.14	0.13	0.12	0.18	0.13	0.11	0.12	0.16	0.13	0.12	0.11	0.13	0.12
	1000	AR(1)	0.26	0.25	0.35	0.24	0.25	0.26	0.33	0.23	0.26	0.26	0.29	0.24	0.27	0.25	0.26	0.24	0.26	0.25	0.24	0.24
		GARCH(1,1)	0.12	0.13	0.24	0.12	0.13	0.12	0.21	0.13	0.12	0.12	0.17	0.13	0.13	0.12	0.15	0.12	0.13	0.12	0.13	0.13
		AR(1)-GARCH(1,1)	0.13	0.13	0.24	0.14	0.14	0.12	0.21	0.14	0.13	0.13	0.18	0.13	0.13	0.13	0.16	0.13	0.13	0.12	0.13	0.14
	1250	AR(1)	0.30	0.30	0.36	0.29	0.29	0.31	0.35	0.29	0.29	0.30	0.32	0.29	0.30	0.30	0.30	0.28	0.30	0.29	0.29	0.28
		GARCH(1,1)	0.15	0.14	0.24	0.15	0.14	0.14	0.22	0.14	0.14	0.13	0.19	0.14	0.13	0.14	0.17	0.14	0.13	0.13	0.14	0.14
		AR(1)-GARCH(1,1)	0.16	0.14	0.26	0.14	0.14	0.15	0.23	0.15	0.14	0.13	0.20	0.13	0.13	0.13	0.17	0.13	0.13	0.12	0.14	0.13
Exchange Rate	250	AR(1)	0.21	0.33	0.42	0.36	0.19	0.38	0.43	0.36	0.25	0.34	0.36	0.34	0.19	0.39	0.34	0.33	0.24	0.36	0.29	0.33
		GARCH(1,1)	0.16	0.25	0.34	0.28	0.17	0.32	0.34	0.29	0.21	0.27	0.29	0.27	0.20	0.31	0.27	0.27	0.23	0.28	0.22	0.26
		AR(1)-GARCH(1,1)	0.17	0.22	0.36	0.29	0.17	0.30	0.36	0.30	0.21	0.27	0.29	0.28	0.20	0.31	0.26	0.27	0.24	0.28	0.22	0.25
	500	AR(1)	0.19	0.29	0.40	0.26	0.21	0.27	0.36	0.26	0.21	0.28	0.31	0.26	0.23	0.28	0.27	0.26	0.23	0.28	0.24	0.26
		GARCH(1,1)	0.14	0.16	0.24	0.18	0.16	0.18	0.22	0.18	0.15	0.19	0.20	0.18	0.18	0.19	0.15	0.19	0.17	0.19	0.15	0.18
		AR(1)-GARCH(1,1)	0.16	0.16	0.27	0.17	0.16	0.16	0.24	0.17	0.16	0.16	0.20	0.16	0.16	0.17	0.18	0.16	0.16	0.17	0.14	0.16
	750	AR(1)	0.22	0.26	0.36	0.24	0.20	0.29	0.33	0.24	0.25	0.26	0.30	0.24	0.21	0.27	0.26	0.23	0.23	0.27	0.22	0.23
		GARCH(1,1)	0.18	0.18	0.25	0.15	0.17	0.20	0.24	0.16	0.17	0.18	0.21	0.15	0.15	0.19	0.18	0.15	0.17	0.17	0.14	0.16
		AR(1)-GARCH(1,1)	0.17	0.17	0.26	0.17	0.17	0.19	0.25	0.17	0.19	0.16	0.19	0.18	0.16	0.18	0.17	0.17	0.17	0.18	0.15	0.17
	1000	AR(1)	0.20	0.21	0.29	0.19	0.22	0.21	0.27	0.19	0.22	0.19	0.24	0.20	0.20	0.20	0.20	0.19	0.19	0.21	0.17	0.19
		GARCH(1,1)	0.15	0.17	0.22	0.18	0.16	0.16	0.20	0.18	0.16	0.16	0.19	0.18	0.17	0.16	0.17	0.18	0.16	0.15	0.17	0.17
		AR(1)-GARCH(1,1)	0.14	0.16	0.20	0.18	0.16	0.15	0.20	0.17	0.15	0.15	0.18	0.17	0.16	0.15	0.18	0.16	0.15	0.16	0.16	0.15
	1250	AR(1)	0.21	0.22	0.22	0.22	0.22	0.23	0.21	0.21	0.21	0.22	0.20	0.22	0.23	0.20	0.21	0.22	0.22	0.21	0.22	0.23
		GARCH(1,1)	0.16	0.16	0.15	0.20	0.16	0.16	0.16	0.20	0.15	0.19	0.17	0.19	0.17	0.19	0.16	0.20	0.17	0.20	0.18	0.20
		AR(1)-GARCH(1,1)	0.16	0.15	0.14	0.18	0.15	0.15	0.13	0.18	0.15	0.15	0.13	0.18	0.17	0.15	0.16	0.17	0.14	0.16	0.17	0.17

4.5 Summary

Conditional VaR is an effective tool for dynamically monitoring risk. This chapter first explained the limitations of extending EVT directly to stationary and non-stationary time series. It then combined the GARCH model, which effectively characterises volatility in financial time series, with EB to forecast dynamic VaR. The performance of the proposed method was examined through a large-scale empirical study that involved almost 50 financial assets. The empirical research suggests that GARCH + EB outperforms existing GARCH + EVT approaches, and is robust and accurate in forecasting VaR through different backtesting procedures.

Note that, for an extreme risk forecast of a simulation generated from a GARCH process, the GARCH-EVT approach applies the EVT to the standardised residuals of the fitted GARCH model and the standardised residuals follow a normal, student's t or generalised error distribution depending on the setup of the GARCH simulation. Therefore, unlike Chapter 3, we do not provide a simulation study in this chapter as there is no essential difference to the simulation study in Chapter 3.

Chapter 5

A Semi-parametric Model for Dynamic VaR

The extended GARCH-EVT approach employed in Chapter 4 is based on the classic GARCH-EVT framework proposed by McNeil and Frey (2000). This framework consists of two steps. First, the QMLE (Bollerslev & Wooldridge, 1992) is employed to obtain the parameters of the GARCH model, in which the density of innovation is assumed to be unknown. Second, EVT is applied to the standardised residuals of the GARCH model to extrapolate high quantile.

A potential weakness of this framework is that the estimation of EVT in the second step is sensitive to the QMLE estimate in the first step (Chavez-Demoulin, Davison, & McNeil, 2005). In particular, QMLE is a pseudo-likelihood estimate derived by maximising a function related to the logarithm of the likelihood function. Chavez-Demoulin et al. (2005) has pointed out that QMLE decreases the estimation efficiency of the GARCH parameters significantly when the innovation is misspecified. Therefore, we may infer that VaR estimates from the GARCH-EVT framework are less efficient.

In literature, there are a few methods proposed to improve the efficiency of the GARCH estimate. Engle and Gonzalez-Rivera (1991) proposes a semi-parametric GARCH model where the innovation is relaxed and approximated by a non-parametric density. T. Cheng, Gao, and Zhang (2019) develops a kernel density with localized bandwidth selection method to estimate GARCH. Lee and Lee (2011) uses a Gaussian-Mixture in ARMA-GARCH to forecast VaR. Nikolaev, Boshnakov, and Zimmer (n.d.) applies student's t mixture for the GARCH to estimate VaR. However, all these methods has limitation in extrapolating extremes which is VaR aiming for.

This chapter develops a semi-parametric GARCH model, which not only has a better extrapolation capability at extremes, but also improves the VaR estimation efficiency within GARCH framework. The innovation of the semi-parametric GARCH model is approximated by a mixture density, in which the bulk part of the innovation is approximated by the non-parametric density and the tail part is approximated by the extreme value density. Non-parametric density offers greater flexibility in modelling a given dataset, and is not affected by specification bias (Botev, Grotowski, & Kroese, 2010). However, it has less extrapolation power for an extreme quantile, as the data information situated in the tail region is thin (S. Chen & Tang, 2005). Extreme value distribution based on a sound EVT models the tail robustly, which in turn provides an accurate and efficient estimate of extreme quantile.

Section 5.1 introduces the semi-parametric GARCH model with mixture density. Section 5.2 provides an algorithm to estimate the parameters and the VaR. Section 5.3 validates the performance of the proposed method through parameter estimation and VaR forecast. In Section 5.4, we demonstrate the performance of the proposed method through an empirical study.

5.1 A Mixture Approximation of Conditional Density

Let us consider X_t as the loss series of a portfolio. A GARCH(p,q) model is:

$$\begin{aligned}
X_t &= E(X_t|\psi_{t-1}) + \varepsilon_t, \\
\varepsilon_t &= \sigma_t Z_t, \\
Z_t &\sim i.i.d.\ f(0,1), \\
\sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-q}^2 \ .
\end{aligned}$$

ψ_{t-1} is the information up to $t-1$. ε_t is the conditional density with time varying volatility σ_t derived by the GARCH coefficients $\omega, \alpha_i, \beta_j$. The white noise process Z_t satisfies $E(Z_t) = 0$ and $E(Z_t^2) = 1$.

We now approximate the conditional density with a mixture distribution function F and a mixture density function f. F is defined as

$$F(z \mid \boldsymbol{\theta}, \boldsymbol{Z}) = \begin{cases} (1-\phi_u)\frac{H(z|\boldsymbol{Z})}{H(u|\boldsymbol{Z})} & z \leq u, \\ (1-\phi_u) + \phi_u G(z \mid \boldsymbol{\theta}) & z > u. \end{cases} \tag{5.1}$$

H is a non-parametric distribution function defined by the random standardised residuals Z. G is a parametric tail distribution function with parameters $\boldsymbol{\theta}$. u is a selected threshold and ϕ_u is the probability of Z above u[1]. It is also convenient to express Z_t in a mixture structure:

$$Z_t = (1-\Delta)W_t + \Delta Y_t,$$

[1] $H(u \mid \boldsymbol{Z})$ is the cumulative probability at threshold u, which equals $1 - \phi_u$.

where $\Delta \in (0, 1)$ and $P(\Delta = 0) = \pi$. W_t is characterised by a non-parametric density f_1 for the bulk part of the Z_t below threshold u and Y_t is characterised by tail density f_2 above threshold u. The density f is then

$$f(z) = (1 - \pi)f_1(z) + \pi f_2(z),$$

where $\pi = \phi_u$ and

$$f_1(z) = \frac{h(z \mid \boldsymbol{Z})}{H(u \mid \boldsymbol{Z})} I_{(-\infty,u)}(z),$$
$$f_2(z) = g(z \mid \boldsymbol{\theta}) I_{(u,\infty)}(z).$$

5.2 Estimation of the Model and Computation of VaR

Log-likelihood for ε_t given the information up to $t - 1$ is

$$l(\boldsymbol{\varphi}) = \ln L(\boldsymbol{\varphi}) = \sum_{t=1}^{T} \ln g(\varepsilon_t | \psi_{t-1}), \tag{5.2}$$

where $\boldsymbol{\varphi}$ describes the unknown parameters for unconditional mixture density g. It is relatively easy to facilitate estimation of the parameters by working with the standardised residuals $Z_t = \varepsilon_t / \sigma_t$. The log-likelihood function can be written as

$$l(\boldsymbol{\varphi}) = -\sum_{i=1}^{T} \ln \sigma_t + \sum_{i=1}^{T} \ln f(z_t),$$

where z_t is the realisations of the standardised residuals of Z_t.

We develop the following algorithm to maximize the log-likelihood with mixture density and forecast VaR [2]:

[2] In usual practice, the EM algorithm works well in maximising the likelihood of the mixture density. However, in this case, it is hard to separate the likelihood, since the common GARCH parameters drive the standardised residuals. This makes separating the mixture likelihood difficult (MacDonald et al., 2011).

1. Obtain an initial consistent estimate of GARCH parameters $(\omega, \alpha_i, \beta_i)$, which can be estimated from QLME (Bera & Higgins, 1993).

2. Compute the standardised residuals z_t by observations and time-varying volatility through ε_t/σ_t. Standardise z_t, if $E(z_t) \neq 0$ or $E(z_t^2) \neq 1$.

3. Approximate the conditional density f by mixture density $\hat{f}$ proposed in Step 1 by z_t from Step 2.

4. Replace f in likelihood equation 5.2 with $\hat{f}$.

5. Keep $\hat{f}$ fixed and optimise l until it converges.

6. Obtain new standardised residuals z_t'.

7. Apply EVT to the standardised residuals.

8. Obtain a VaR forecast with a given confidence level.

In Step 3, we choose kernel density as the non-parametric part of the mixture. The kernel density is defined as

$$\hat{f}(z;\lambda) = \frac{1}{n\lambda}\sum_{i=1}^{n} K(\frac{z - z_i}{\lambda}),$$

where K is a kernel function and λ is a smoothing parameter (also known as bandwidth). K usually satisfies the conditions

$$K(z) \geq 0 \quad and \quad \int K(z)dz = 1.$$

The kernel can be viewed as spreading a "probability mass" of size $1/n$ for each data point regarding its neighbour points. The choice of K is not critical in the proposed mixture innovation as the tail behaviour associated with the chosen kernel vanishes by averaging (MacDonald et al., 2011). Therefore, we choose a normal density as the kernel with a mean of zero. The bandwidth λ is the standard deviation of the normal kernel, which is selected through global bandwidth estimates (T. Cheng et al., 2019).

For the tail part of the mixture density, we have selected EB. In Chapter 3, we showed that EB has a flexible fitting of the tail under different DGPs and a better extrapolation power over the tail. In Chapter 4, we showed that the EB also captures a better VaR in dynamic VaR forecasts.

5.3 Simulation Study

We conduct a simulation study to evaluate the performance of the proposed method and algorithm. In particular, two estimation procedures are compared.

E-GARCH-EVT The first is the extended GARCH-EVT approach proposed in Chapter 4. Essentially, the extended GARCH-EVT model applies extended EVT to the standardised residuals obtained through the GARCH model estimated by QMLE. This method will be referred to as the classic method.

M-GARCH-EVT The second is the mixture GARCH-EVT approach proposed in this chapter. M-GARCH-EVT uses a mixture density to approximate the innovation of the GARCH model. This method will be referred to as the mixture method.

GARCH(1,1) with $\omega = 0.05, \alpha = 0.2, \beta = 0.7$ is used as the data generating process (DGP) in the simulation. For the innovation of the GARCH model, we consider the following densities:

- Standard normal density,
- Generalised error density with the shape parameter,
- Mixture density of standard normal below 90% quantile and generalised Pareto above 90% threshold,

- Mixture density of generalised error below 90% quantile and generalised Pareto above 90% quantile.

The normal distribution is the standard, with a mean of zero and a standard deviation of one. The GED has a location parameter of zero, a scale parameter of one, and a shape parameter of one. GP has a shape parameter of -0.3 and a scale parameter of one. The sample sizes of 250, 500, 750, 1000, and 1250 are considered. 200 simulations are conducted for each combination of the above DGP settings.

Since the theoretical VaR is known in the simulation study, we employ root mean square error (RMSE) to assess the performance of the VaR forecast from two algorithms on different DGP settings. RMSE of VaR is shown in the following equation:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(V\hat{a}R_i - VaR)^2}{n}},$$

where $\widehat{VaR}_i$ is the VaR estimation from each simulation, VaR is the theoretical VaR and n is the number of samples in each simulation. VaR is evaluated at 2.5%, 2.0%, 1.5%, 1.0%, and 0.5% confidence levels. In addition, the threshold selections in each algorithm are evaluated (selected at 90%, 91%, 92%, 93%, 94%, and 95% of the standardised residual quantile).

Tables 5.1, 5.2, 5.3, and 5.4 show the RMSE summary of the VaR forecast from the simulations with the normal, GED, normal+GP, and GED+GP, respectively. Selected findings from the RMSE results are summarised below.

When data are generated from normal and GED, the E-GARCH-EVT approach has a lower RMSE value, suggesting a more accurate and efficient VaR forecast. By contrast, when data are generated from normal+GP and GED+GP, the M-GARCH-EVT approach has the lower RMSE value. The two methods' average performance is similar when data are generated from GED. GED with shape parameter one has a relatively long tail compared to the normal, while normal+GPD and GED+GPD have a longer tail than GED. The RMSE value of the M-GARCH-EVT is significantly lower than E-GARCH-EVT when data are generated from normal+GPD and GED+GPD. This suggests that when the DGP has a longer tail, the M-GARCH-EVT can adapt fitting to capture the tail behaviour.

When sample size is low, the RMSE value is higher, suggesting that the VaR forecast is less accurate and efficient. This is reasonable, since the fitting tends to be less efficient with a small sample size. The RMSE value increases as the VaR confidence level decreases, since the higher quantile of the tail has higher uncertainty. RMSE shows similar results across different thresholds. However, the VaR forecast performances of the classic and mixture methods are similar with respect to different sample sizes, VaR confidence levels, and thresholds.

In summary, the E-GARCH-EVT model performs better than M-GARCH-EVT when when the DGP is specified non-standard distribution. However, when the DGP is a mixture with a long tail, M-GARCH-EVT can adjust its fittings to better forecast VaR. Overall, we favour M-GARCH-EVT, since the DGP is typically unknown in practice.

Table 5.1: RMSE of VaR Forecast for GARCH Simulation with Normal Density

Sample Size	Threshold	2.50%		2.00%		1.50%		1.00%		0.50%		VaR Average	
		Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic
250	0.90	0.241	0.175	0.245	0.181	0.245	0.184	0.240	0.190	0.270	0.237	0.248	0.193
	0.91	0.269	0.180	0.278	0.184	0.281	0.183	0.273	0.181	0.261	0.209	0.272	0.187
	0.92	0.236	0.156	0.248	0.167	0.256	0.177	0.266	0.189	0.455	0.241	0.292	0.186
	0.93	0.265	0.163	0.284	0.178	0.298	0.193	0.300	0.208	0.296	0.245	0.289	0.197
	0.94	0.295	0.195	0.308	0.206	0.311	0.213	0.299	0.213	0.296	0.230	0.302	0.211
	0.95	0.301	0.171	0.329	0.188	0.346	0.204	0.344	0.218	0.311	0.245	0.326	0.205
500	0.90	0.159	0.089	0.166	0.094	0.171	0.099	0.174	0.108	0.182	0.137	0.170	0.105
	0.91	0.137	0.089	0.144	0.094	0.152	0.100	0.159	0.111	0.177	0.145	0.154	0.108
	0.92	0.146	0.091	0.154	0.098	0.163	0.107	0.172	0.119	0.190	0.152	0.165	0.113
	0.93	0.156	0.109	0.163	0.117	0.168	0.123	0.167	0.129	0.171	0.158	0.165	0.127
	0.94	0.156	0.100	0.170	0.111	0.183	0.123	0.190	0.133	0.194	0.151	0.179	0.124
	0.95	0.157	0.118	0.176	0.134	0.194	0.150	0.206	0.162	0.233	0.175	0.193	0.148
750	0.90	0.117	0.077	0.125	0.083	0.133	0.092	0.143	0.104	0.164	0.133	0.136	0.098
	0.91	0.121	0.075	0.130	0.081	0.139	0.089	0.149	0.101	0.169	0.133	0.142	0.096
	0.92	0.123	0.074	0.130	0.080	0.137	0.088	0.144	0.098	0.159	0.126	0.139	0.093
	0.93	0.125	0.079	0.137	0.087	0.148	0.095	0.156	0.104	0.163	0.127	0.146	0.098
	0.94	0.131	0.080	0.142	0.089	0.153	0.099	0.162	0.111	0.171	0.134	0.152	0.103
	0.95	0.124	0.072	0.145	0.083	0.165	0.097	0.180	0.112	0.183	0.130	0.159	0.099
1000	0.90	0.106	0.065	0.113	0.071	0.120	0.077	0.127	0.085	0.139	0.106	0.121	0.081
	0.91	0.101	0.062	0.107	0.067	0.114	0.073	0.123	0.084	0.142	0.111	0.117	0.079
	0.92	0.105	0.064	0.113	0.071	0.123	0.080	0.134	0.092	0.154	0.120	0.126	0.085
	0.93	0.111	0.068	0.119	0.074	0.127	0.081	0.135	0.090	0.145	0.109	0.127	0.084
	0.94	0.100	0.062	0.106	0.067	0.114	0.076	0.123	0.089	0.143	0.117	0.117	0.082
	0.95	0.114	0.072	0.126	0.081	0.139	0.092	0.153	0.105	0.167	0.126	0.140	0.095
1250	0.90	0.089	0.060	0.095	0.066	0.103	0.073	0.113	0.084	0.131	0.107	0.106	0.078
	0.91	0.091	0.057	0.099	0.062	0.109	0.070	0.123	0.081	0.145	0.107	0.113	0.075
	0.92	0.095	0.060	0.101	0.065	0.108	0.071	0.116	0.080	0.130	0.101	0.110	0.075
	0.93	0.094	0.053	0.103	0.059	0.112	0.067	0.123	0.077	0.137	0.098	0.114	0.071
	0.94	0.093	0.058	0.101	0.064	0.109	0.073	0.118	0.085	0.130	0.106	0.110	0.077
	0.95	0.092	0.053	0.104	0.060	0.118	0.070	0.133	0.083	0.149	0.103	0.119	0.074
Average across sample and threshold		0.148	0.094	0.159	0.102	0.168	0.111	0.175	0.121	0.192	0.147	0.168	0.115

Table 5.2: RMSE of VaR Forecast for GARCH Simulation with Generalised Error Density

Sample Size	Threshold	2.50%		2.00%		1.50%		1.00%		0.50%		VaR Average	
		Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic
250	0.90	0.303	0.247	0.306	0.256	0.304	0.265	0.307	0.285	0.431	0.414	0.330	0.293
	0.91	0.332	0.257	0.344	0.273	0.346	0.285	0.337	0.300	0.397	0.426	0.351	0.308
	0.92	0.333	0.271	0.350	0.291	0.356	0.303	0.351	0.313	0.431	0.503	0.364	0.336
	0.93	0.365	0.318	0.390	0.338	0.402	0.351	0.394	0.356	0.395	0.410	0.389	0.355
	0.94	0.369	0.295	0.389	0.315	0.396	0.329	0.385	0.340	0.445	0.450	0.397	0.346
	0.95	0.338	0.334	0.370	0.364	0.394	0.380	0.401	0.378	0.409	0.423	0.382	0.376
500	0.90	0.193	0.159	0.202	0.169	0.208	0.179	0.215	0.197	0.265	0.270	0.217	0.195
	0.91	0.180	0.157	0.193	0.170	0.207	0.186	0.226	0.211	0.285	0.286	0.218	0.202
	0.92	0.194	0.155	0.207	0.169	0.217	0.184	0.228	0.208	0.291	0.301	0.227	0.203
	0.93	0.190	0.152	0.207	0.169	0.223	0.189	0.239	0.214	0.289	0.284	0.230	0.202
	0.94	0.205	0.163	0.226	0.184	0.246	0.206	0.261	0.231	0.300	0.292	0.248	0.215
	0.95	0.215	0.183	0.247	0.208	0.278	0.236	0.300	0.263	0.330	0.308	0.274	0.240
750	0.90	0.149	0.126	0.156	0.134	0.162	0.143	0.169	0.158	0.207	0.215	0.169	0.155
	0.91	0.151	0.124	0.164	0.136	0.178	0.151	0.198	0.175	0.254	0.243	0.189	0.166
	0.92	0.155	0.129	0.166	0.140	0.177	0.153	0.188	0.170	0.231	0.230	0.183	0.164
	0.93	0.141	0.124	0.156	0.137	0.170	0.153	0.185	0.174	0.227	0.235	0.176	0.165
	0.94	0.157	0.132	0.181	0.154	0.205	0.180	0.227	0.208	0.257	0.255	0.205	0.186
	0.95	0.145	0.125	0.168	0.141	0.195	0.164	0.221	0.191	0.252	0.242	0.196	0.173
1000	0.90	0.129	0.114	0.139	0.124	0.150	0.137	0.167	0.157	0.214	0.212	0.160	0.149
	0.91	0.119	0.100	0.128	0.110	0.139	0.123	0.153	0.143	0.193	0.197	0.146	0.135
	0.92	0.127	0.110	0.138	0.121	0.149	0.134	0.165	0.154	0.209	0.210	0.158	0.146
	0.93	0.126	0.106	0.139	0.118	0.153	0.132	0.169	0.151	0.206	0.197	0.159	0.141
	0.94	0.121	0.105	0.135	0.117	0.152	0.134	0.172	0.158	0.213	0.210	0.159	0.145
	0.95	0.125	0.111	0.142	0.125	0.161	0.141	0.181	0.162	0.212	0.204	0.164	0.149
1250	0.90	0.103	0.092	0.110	0.099	0.117	0.108	0.129	0.123	0.171	0.173	0.126	0.119
	0.91	0.101	0.091	0.110	0.100	0.122	0.112	0.139	0.132	0.188	0.188	0.132	0.125
	0.92	0.112	0.097	0.122	0.107	0.134	0.120	0.149	0.138	0.181	0.178	0.140	0.128
	0.93	0.107	0.096	0.118	0.106	0.130	0.119	0.145	0.136	0.183	0.185	0.137	0.128
	0.94	0.112	0.096	0.123	0.106	0.135	0.118	0.149	0.134	0.176	0.172	0.139	0.125
	0.95	0.115	0.100	0.133	0.115	0.155	0.136	0.182	0.165	0.222	0.214	0.161	0.146
Average across sample and threshold		0.184	0.156	0.199	0.170	0.212	0.185	0.224	0.204	0.269	0.271	0.218	0.197

Table 5.3: RMSE of VaR Forecast for GARCH Simulation with Mixture Density of Normal and GP

Sample Size	Threshold	2.50%		2.00%		1.50%		1.00%		0.50%		VaR Average	
		Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic
250	0.90	0.231	0.213	0.228	0.230	0.219	0.253	0.213	0.295	0.274	0.419	0.233	0.282
	0.91	0.268	0.245	0.272	0.258	0.267	0.274	0.255	0.303	0.271	0.377	0.267	0.291
	0.92	0.253	0.256	0.251	0.260	0.244	0.269	0.238	0.297	0.287	0.383	0.255	0.293
	0.93	0.295	0.285	0.299	0.291	0.296	0.301	0.286	0.324	0.300	0.396	0.295	0.319
	0.94	0.274	0.267	0.280	0.279	0.277	0.292	0.265	0.310	0.311	0.389	0.281	0.307
	0.95	0.271	0.263	0.287	0.266	0.293	0.273	0.288	0.297	0.296	0.372	0.287	0.294
500	0.90	0.148	0.198	0.153	0.211	0.158	0.228	0.166	0.253	0.198	0.305	0.165	0.239
	0.91	0.141	0.180	0.147	0.192	0.150	0.209	0.156	0.235	0.191	0.295	0.157	0.222
	0.92	0.159	0.184	0.167	0.198	0.170	0.217	0.173	0.245	0.196	0.300	0.173	0.229
	0.93	0.163	0.199	0.174	0.212	0.181	0.228	0.184	0.250	0.199	0.297	0.180	0.237
	0.94	0.136	0.179	0.150	0.192	0.163	0.208	0.173	0.233	0.195	0.285	0.163	0.219
	0.95	0.153	0.202	0.171	0.215	0.188	0.230	0.195	0.247	0.200	0.282	0.181	0.235
750	0.90	0.109	0.166	0.112	0.176	0.117	0.190	0.127	0.211	0.159	0.257	0.125	0.200
	0.91	0.105	0.165	0.108	0.172	0.112	0.182	0.120	0.202	0.150	0.249	0.119	0.194
	0.92	0.114	0.177	0.124	0.190	0.135	0.207	0.151	0.232	0.184	0.279	0.142	0.217
	0.93	0.115	0.176	0.123	0.185	0.131	0.198	0.142	0.220	0.167	0.265	0.136	0.209
	0.94	0.111	0.171	0.119	0.180	0.130	0.194	0.143	0.217	0.171	0.263	0.135	0.205
	0.95	0.110	0.182	0.116	0.185	0.130	0.193	0.147	0.211	0.170	0.250	0.135	0.204
1000	0.90	0.109	0.168	0.114	0.178	0.121	0.191	0.132	0.212	0.161	0.253	0.127	0.200
	0.91	0.105	0.165	0.112	0.176	0.121	0.192	0.135	0.214	0.164	0.256	0.127	0.201
	0.92	0.098	0.164	0.106	0.176	0.115	0.192	0.129	0.215	0.159	0.255	0.121	0.200
	0.93	0.103	0.165	0.112	0.178	0.124	0.195	0.139	0.220	0.170	0.264	0.130	0.204
	0.94	0.102	0.163	0.110	0.173	0.121	0.189	0.137	0.212	0.166	0.256	0.127	0.199
	0.95	0.101	0.159	0.109	0.168	0.121	0.183	0.136	0.206	0.161	0.248	0.126	0.193
1250	0.90	0.093	0.156	0.099	0.165	0.106	0.178	0.119	0.198	0.148	0.236	0.113	0.187
	0.91	0.092	0.154	0.098	0.164	0.107	0.178	0.122	0.200	0.155	0.243	0.115	0.188
	0.92	0.100	0.159	0.107	0.170	0.117	0.184	0.130	0.205	0.155	0.241	0.122	0.192
	0.93	0.104	0.167	0.110	0.177	0.120	0.190	0.133	0.209	0.160	0.245	0.125	0.198
	0.94	0.098	0.160	0.104	0.170	0.111	0.184	0.125	0.205	0.154	0.245	0.118	0.193
	0.95	0.107	0.171	0.110	0.177	0.117	0.186	0.130	0.203	0.155	0.237	0.124	0.195
Average across sample and threshold		0.146	0.189	0.152	0.199	0.159	0.213	0.166	0.236	0.194	0.288	0.163	0.225

Table 5.4: RMSE of VaR Forecast for GARCH Simulation with Mixture Density of GED and GP

Sample Size	Threshold	2.50%		2.00%		1.50%		1.00%		0.50%		VaR Average	
		Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic
250	0.90	0.249	0.251	0.253	0.263	0.254	0.280	0.261	0.310	0.311	0.384	0.266	0.298
	0.91	0.280	0.281	0.285	0.290	0.289	0.303	0.298	0.331	0.349	0.406	0.300	0.322
	0.92	0.274	0.274	0.279	0.282	0.280	0.292	0.283	0.311	0.321	0.390	0.287	0.310
	0.93	0.295	0.296	0.298	0.300	0.298	0.307	0.296	0.321	0.324	0.374	0.302	0.320
	0.94	0.288	0.281	0.291	0.288	0.292	0.299	0.297	0.321	0.353	0.384	0.304	0.315
	0.95	0.344	0.342	0.355	0.350	0.359	0.359	0.361	0.378	0.394	0.433	0.363	0.372
500	0.90	0.179	0.213	0.185	0.221	0.193	0.232	0.205	0.251	0.239	0.295	0.200	0.242
	0.91	0.190	0.231	0.196	0.240	0.204	0.250	0.214	0.267	0.245	0.304	0.210	0.258
	0.92	0.204	0.231	0.210	0.239	0.214	0.251	0.220	0.267	0.295	0.301	0.229	0.258
	0.93	0.190	0.219	0.199	0.226	0.211	0.240	0.226	0.263	0.267	0.315	0.219	0.253
	0.94	0.192	0.224	0.202	0.232	0.215	0.244	0.229	0.264	0.259	0.308	0.219	0.254
	0.95	0.190	0.215	0.201	0.219	0.216	0.230	0.232	0.251	0.259	0.296	0.220	0.242
750	0.90	0.167	0.206	0.172	0.213	0.181	0.224	0.193	0.241	0.223	0.277	0.187	0.232
	0.91	0.178	0.218	0.179	0.219	0.182	0.224	0.189	0.235	0.215	0.268	0.189	0.233
	0.92	0.176	0.214	0.180	0.218	0.184	0.224	0.192	0.235	0.209	0.260	0.188	0.230
	0.93	0.184	0.227	0.185	0.230	0.188	0.234	0.194	0.243	0.216	0.267	0.193	0.240
	0.94	0.169	0.204	0.176	0.209	0.185	0.218	0.198	0.235	0.220	0.265	0.190	0.226
	0.95	0.199	0.234	0.201	0.235	0.208	0.241	0.219	0.252	0.238	0.278	0.213	0.248
1000	0.90	0.160	0.202	0.162	0.206	0.168	0.214	0.180	0.228	0.210	0.261	0.176	0.222
	0.91	0.171	0.213	0.174	0.217	0.180	0.225	0.189	0.237	0.211	0.262	0.185	0.231
	0.92	0.174	0.215	0.179	0.222	0.187	0.231	0.199	0.246	0.222	0.273	0.192	0.237
	0.93	0.165	0.206	0.167	0.211	0.173	0.218	0.182	0.230	0.201	0.254	0.178	0.224
	0.94	0.162	0.201	0.165	0.204	0.172	0.212	0.184	0.224	0.205	0.250	0.178	0.218
	0.95	0.170	0.208	0.173	0.209	0.180	0.216	0.192	0.229	0.213	0.257	0.186	0.224
1250	0.90	0.162	0.201	0.163	0.204	0.166	0.208	0.172	0.217	0.189	0.238	0.170	0.214
	0.91	0.168	0.204	0.168	0.205	0.170	0.209	0.175	0.216	0.191	0.236	0.174	0.214
	0.92	0.168	0.204	0.171	0.209	0.176	0.217	0.185	0.229	0.205	0.254	0.181	0.223
	0.93	0.171	0.210	0.173	0.213	0.177	0.218	0.186	0.228	0.207	0.252	0.183	0.224
	0.94	0.172	0.211	0.172	0.213	0.176	0.218	0.183	0.227	0.200	0.248	0.181	0.223
	0.95	0.176	0.214	0.173	0.212	0.174	0.214	0.181	0.221	0.199	0.242	0.181	0.221
Average across sample and threshold		0.199	0.228	0.203	0.233	0.208	0.242	0.217	0.257	0.246	0.294	0.215	0.251

5.4 Empirical Study

5.4.1 Data

The data used to evaluate the empirical performance of the proposed model are same as the data used in Chapter 4, which contain 29 stock indices covering both developing and developed countries and 18 currency exchange rates. Exchange rates include 16 years of price history from 1st January 2004 to 31st December 2019, while the stock indices covers 20 years of price history from 1st January 2000 to 31st December 2019. The details of these financial instruments are shown in Table A.1 and Appendix A. The dataset is available for download from Yahoo Finance through the Quantmod Package A.1 in statistical programming software R (R Core Team, 2021).

In Section 5.4, we adopt a rolling window approach to dynamically evaluate the VaR forecast, which reflects changes in financial markets. Loss data are obtained by taking the negative of the log-return defined by $X_t = -100 \times ln\frac{P_t}{P_{t-1}}$, where P_t is the adjusted closing price of a financial instrument. The model evaluates threshold selection at the 0.9, 0.91, 0.92, 0.93, and 0.94 quantiles, and the moving window at one to five years with approximately 250 trading days per year.

5.4.2 The Performance of VaR Forecast Accuracy

The empirical study obtains the 1% dynamic VaR forecasts from M-GARCH-EVT and E-GARCH-EVT under the classic GARCH-EVT framework. Both methods are evaluated with five threshold selections and five rolling windows. We evaluate these forecasts through the conditional coverage (CC) test of Christoffersen (1998) and the dynamic quantile (DQ) test of Engle and Manganelli (2004) detailed in Section 2.3. Each test produces a p-value for the null hypothesis that the coverage probability of the proposed model equals the predefined VaR confidence level.

Table 5.5 summarises the percentage of financial instruments for which the

CC or DQ test is rejected at a 5% significance level, following Calmon et al. (2021) and Ardia et al. (2018). The percentage of rejection indicates the performance of the candidate method in forecasting VaR. The lower the rejection rate percentage, the better the performance of the VaR forecast. A method with a correct coverage probability of 1% for every financial instrument should have a 0% rejection of all instruments. The shading in the table highlights the lowest rejection rate of the VaR forecast from candidate methods.

The general pattern of the CC and DQ tests shown in Table 5.5 is as follows. When the sample size is small, the percentage is relatively high. This is because the data contain less information, which makes the fitting difficult. The percentage across different thresholds appears quite similar. In particular, results from the CC test suggest that when the sample size is small (250), the M-GARCH-EVT has a greater prediction accuracy over all financial assets, regardless of the threshold selection. However, the two methods have a similar CC test performance with larger sample sizes. Results from the DQ test have a higher rejection rate, since the DQ test has a stronger test power. The sample size pattern of the DQ test is similar to that of the CC test.

Overall, in terms of the VaR forecast accuracy, we favour the M-GARCH-EVT method. When the sample size is large, the two methods appear similar. However, as the sample size decreases, the M-GARCH-EVT delivers a more accurate performance.

Table 5.5: Evaluation of Accuracy of the VaR Forecast

	Threshold\Sample	250		500		750		1000		1250	
		Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture
Conditional Coverage Test	0.90	0.468	0.447	0.234	0.234	0.234	0.149	0.170	0.191	0.106	0.128
	0.91	0.489	0.383	0.234	0.234	0.213	0.213	0.170	0.191	0.106	0.106
	0.92	0.447	0.362	0.191	0.213	0.213	0.213	0.170	0.213	0.128	0.128
	0.93	0.447	0.404	0.213	0.213	0.191	0.191	0.170	0.213	0.128	0.128
	0.94	0.383	0.362	0.213	0.191	0.149	0.106	0.191	0.128	0.128	0.085
Dynamic Quantile Test	0.90	0.936	0.979	0.702	0.66	0.574	0.532	0.468	0.383	0.426	0.426
	0.91	0.936	0.872	0.766	0.723	0.532	0.532	0.511	0.489	0.447	0.426
	0.92	0.957	0.915	0.723	0.596	0.638	0.426	0.532	0.532	0.468	0.426
	0.93	0.936	0.851	0.745	0.617	0.532	0.574	0.447	0.468	0.447	0.404
	0.94	0.915	0.894	0.574	0.617	0.574	0.489	0.468	0.426	0.447	0.34

5.4.3 The Occurrence Performance of the VaR Forecast

Table 5.6 summarises the occurrence performance of AE and APE for VaR forecasts. Occurrence refers to instances in which the loss is greater than the forecasted VaR value. VaR models with an AE statistic close to one correctly project the occurrence of model failure, whereas VaR models with an APE statistic close to zero correctly reflects the occurrence of model failure. A VaR model overestimates the occurrence of failure when the AE value is greater than zero, and underestimates the occurrence of failure when the AE value is less than one.

M-GARCH-EVT has an AE value closer to one in all combinations of threshold selections and sample sizes. When the sample size is large, the AE value is close to one, suggesting a better VaR forecast. This is consistent with the pattern in the CC and DQ tests. However, in AE statistics, as the threshold increases, both methods have a better VaR forecast. This is because the asymptotic properties of the EVT are more likely to be satisfied when the threshold is larger. The APE statistics display a similar pattern: M-GARCH-EVT has a lower value across all combinations of threshold and sample size. However, APE does not show a strong pattern in sample size; the performances of sample sizes above 500 appear to be similar.

In summary, the results from the occurrence statistics strongly favour the M-GARCH-EVT approach, which outperforms E-GARCH-EVT in both AE and APE for all combinations of results.

Table 5.6: Evaluation of Occurence of the VaR Forecast

	Threshold\Sample	250		500		750		1000		1250	
		Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture	Classic	Mixture
Actual Exceedance	0.90	1.40	1.37	1.20	1.17	1.09	1.09	1.09	1.07	1.09	1.08
	0.91	1.38	1.35	1.19	1.16	1.09	1.07	1.10	1.08	1.09	1.07
	0.92	1.37	1.35	1.19	1.15	1.10	1.07	1.10	1.06	1.09	1.06
	0.93	1.33	1.30	1.18	1.15	1.09	1.07	1.10	1.06	1.09	1.06
	0.94	1.33	1.29	1.14	1.1	1.08	1.05	1.09	1.04	1.08	1.06
Absolute Percentage Error	0.90	0.40	0.37	0.21	0.19	0.15	0.15	0.15	0.13	0.17	0.16
	0.91	0.38	0.35	0.21	0.18	0.16	0.14	0.16	0.14	0.17	0.15
	0.92	0.37	0.35	0.21	0.18	0.15	0.13	0.15	0.15	0.17	0.14
	0.93	0.34	0.3	0.2	0.17	0.15	0.14	0.15	0.13	0.16	0.14
	0.94	0.34	0.3	0.17	0.14	0.14	0.11	0.14	0.12	0.15	0.15
Absolute Deviation Mean	0.90	0.83	0.83	0.84	0.85	0.79	0.79	0.79	0.79	0.81	0.8
	0.91	0.83	0.83	0.83	0.84	0.78	0.81	0.79	0.79	0.80	0.81
	0.92	0.82	0.83	0.83	0.84	0.79	0.78	0.79	0.79	0.81	0.80
	0.93	0.84	0.84	0.82	0.83	0.79	0.79	0.79	0.79	0.80	0.81
	0.94	0.83	0.85	0.83	0.84	0.78	0.78	0.79	0.80	0.81	0.8
Absolute Deviation Max	0.90	5.82	5.74	5.49	5.46	4.79	4.83	4.81	4.82	4.75	4.66
	0.91	5.82	5.68	5.49	5.44	4.77	4.86	4.82	4.78	4.74	4.67
	0.92	5.81	5.72	5.48	5.41	4.79	4.72	4.8	4.73	4.74	4.68
	0.93	5.80	5.62	5.46	5.38	4.78	4.68	4.78	4.71	4.72	4.64
	0.94	5.74	5.63	5.45	5.40	4.78	4.71	4.78	4.65	4.72	4.65

5.5 Summary

This chapter has proposed a model in which the GARCH residuals are approximated by a mixture of non-parametric density for the bulk part, and extreme value density for the tail part. The proposed model aims to improve VaR forecasting efficiency in the classic GARCH-EVT framework. In the classic GARCH-EVT framework, the EVT estimation is sensitive to the GARCH parameter estimation, which may impact the VaR forecast. We provide an algorithm to estimate the GARCH parameters and forecast VaR for the proposed model. The performance of the proposed method is evaluated through a simulation study and an empirical study, and compared to the classic GARCH-EVT framework. Both studies show that a GARCH-EVT approach with mixture density can forecast VaR accurately and efficiently.

As the reader may notice, AR(1) component is included in the dynamic VaR forecast in Chapter 4 but not Chapter 5, which potentially makes the two chapters inconsistent. We should explain the reason for this as follows. In the literature, It has been demonstrated that the AR(1) component does not contribute significantly to the performance of extreme risk forecast by using existing extreme-based methods (Bali, 2007; Nolde & Zhou, 2021). However, since a new extreme-based method is proposed to forecast VaR, we aim to confirm the usefulness of the AR(1) component in the newly proposed method. As the empirical results of chapter 4 show that AR(1) has an insignificant impact on extreme risk forecasting, we reconfirm the conclusion from the literature. Therefore, for simplicity, it is not necessary to include AR(1) component in the mixture model in Chapter Chapter 5 again.

Chapter 6

Conclusion and Discussion

6.1 Summary of VaR Methods

VaR is a frequently used risk measure that assesses the worst loss from a risk event by ascribing a confidence level and a specified time window. In the context of finance, this concerns an investment loss from changing market conditions. While VaR is conceptually simple, its estimation is not an easy task. The primary difficulties arise from selecting the risk distribution and dependence structure among the observations. Therefore, a wide range of methods have been developed to better forecast VaR.

Variance–covariance based methods are the most common. These methods assume that the financial time series is normally and independently distributed. These two assumptions simplify the computation of VaR and are easy to implement. However, in practice, financial time series tend to be fat-tailed, indicating that the normality assumption usually underestimates VaR.

Non-parametric methods, such as the historical simulation method and its extensions, are designed to relax the strong distribution assumption of a financial time series. Non-parametric methods do not make any assumptions about distribution or serial dependence, and their implementation is as easy as that of variance–covariance methods. Since they are based on historical information, more data will increase the methods' robustness. However, in practice, the large-scale data required to simulate historical events may not always be available. Even when large-scale data are available, data at the tail will still be limited. Therefore, the power of non-parametric methods is limited by their extrapolation power.

In the domain of financial risk management, the GARCH model and its extensions are successful in modelling dependence structures among observations and characterising the volatility clustering effect in the financial time series.

The EVT, a sound statistical theory, focuses on modelling tail. As such, it provides a relatively accurate and robust VaR forecast compared to other methods. The GARCH model reduces the leptokurtosis[1] of the financial time series, but does not eliminate it. EVT can therefore be applied to the standardised residuals of the GARCH model to better forecast VaR.

VaR methods can be classified into two groups. Unconditional VaR assumes the serial independence of data, while conditional or dynamic VaR poses a specific dependence structure. Both VaR measures have utility in a financial application. Unconditional VaR measures the long-term average of the VaR risk, and is relevant to financial risk regulation applications. Dynamic VaR considers market volatility, and is best suited for individual companies monitoring market risk.

6.2 Contribution

In this thesis, we promote the use of an extended extreme value approach to forecast VaR. Our chief contributions are twofold, and correspond to the research motivations stated in Chapter 1.

First, the extended extreme value approach is robust and accurate in forecasting unconditional VaR. This approach is applicable to financial regulation, and can be employed to set the long term average capital reserve at the industry level.

[1] The statistical property of having a greater kurtosis than a normal distribution.

Second, the combined extended extreme value approach and GARCH model is robust and accurate in forecasting dynamic VaR. This provides financial institutions with an excellent tool for managing day-to-day market risk. In addition, we have outlined the shortcomings of applying EVT to the standardised residuals of the GARCH model — namely, that the estimation of EVT is determined by the estimation of the GARCH model. This allowed the development of a semi-parametric GARCH model with a mixture residual, consisting of a mixture of Kernel density for bulk part of the residual and EB for the tail part of the residual. We have also developed an algorithm to estimate the parameters of the semi-parametric model. The newly developed model remedies the shortcomings of the GARCH-EB VaR approach and the GARCH-EVT approaches.

Third, by noticing the inefficiency of the classic GARCH-EVT framework in forecast dynamic VaR, we propose an extended GARCH-EVT approach, where the innovation is a mixture density. The bulk part of the mixture density is approximated by a non-parametric density while the tail of the mixture density is approximated by an extreme density. An algorithm is developed to estimate the parameters of the proposed method and forecast VaR. The performance of the proposed method is evaluated through a simulation study and an empirical study, and compared to the classic GARCH-EVT framework. Both studies show that a GARCH-EVT approach with mixture density can forecast VaR accurately and efficiently.

6.3 Limitations and Future Research

There are several properties that a risk measure should satisfy. Unfortunately, VaR is not sub-additive, and is therefore not a coherent risk measure (Artzner et al., 1999). This implies that the above methods cannot be used to obtain the VaR of a portfolio by summing up VaR of individual asset in the portfolio. Moreover, the computed VaR cannot be used to tackle portfolio optimisation problems (Rockafellar & Uryasev, 2002). All VaR methods have these shortcomings.

Therefore, our approach could be extended to another frequently used risk measure, Expected Shortfall (ES), which measures average risk beyond VaR estimates. ES is a coherent risk measure but does not hold elicitability (Gneiting, 2011). There is an extensive body of literature dedicated to developing alternative methods of forecasting ES (see Y. Zhang, Peng, Qu, Shi, and Erdem (2021) and Taylor (2019)). Some key differences between VaR and ES are listed below:

- VaR is more common in the banking sector, while ES is more common in the insurance sector.
- VaR is not a sub-additive risk measure, while ES is.
- ES is not an elicitable risk measure, while VaR is. This means that ES is not properly backtestable.
- ES does not exist in the infinite mean model, making it difficult to forecast.
- ES has a larger confidence interval.
- Both VaR and ES are law-invariant risk measures.
- ES contains "what if" information, meaning that it delivers an expectation of loss if the loss exceeds a certain threshold.

Another limitation of the proposed model is its computational complexity. While methods based on extended EVT are accurate and robust, the estimation procedure is complex compared to the variance–covariance and simulation-based methods. This feature leads to some difficulties in implementation. The final limitation arises from the EB distribution itself. Since EB is a three-parameter distribution, extremely small samples may lead to an unstable estimation. This has been demonstrated in the simulation study in Chapter 3.

Appendices

Appendix A

Tables

Table A.1: Financial Symbols with Descriptions

Exchanges Rates		**Stock Indices**	
Symbols	**Description**	**Symbols**	**Description**
EURUSD=X	EUR/USD	^GSPC	S&P 500
JPY=X	USD/JPY	^DJI	Dow Jones Industrial Average
GBPUSD=X	GBP/USD	^IXIC	NASDAQ Composite
NZDUSD=X	NZD/USD	^NYA	NYSE COMPOSITE (DJ)
EURJPY=X	EUR/JPY	^XAX	NYSE AMEX COMPOSITE INDEX
GBPJPY=X	GBP/JPY	^RUT	Russell 2000
EURGBP=X	EUR/GBP	^VIX	Cboe Volatility Index
EURCAD=X	EUR/CAD	^FTSE	FTSE 100
EURSEK=X	EUR/SEK	^GDAXI	DAX PERFORMANCE-INDEX
EURHUF=X	EUR/HUF	^FCHI	CAC 40
HKD=X	USD/HKD	^N100	EURONEXT 100
SGD=X	USD/SGD	^BFX	BEL 20
INR=X	USD/INR	^N225	Nikkei 225
MXN=X	USD/MXN	^HSI	HANG SENG INDEX
PHP=X	USD/PHP	000001.SS	SSE Composite Index
THB=X	USD/THB	399001.SZ	Shenzhen Component
MYR=X	USD/MYR	^STI	STI Index
ZAR=X	USD/ZAR	^AXJO	S&P/ASX 200
		^AORD	ALL ORDINARIES
		^BSESN	S&P BSE SENSEX
		^JKSE	Jakarta Composite Index
		^KLSE	FTSE Bursa Malaysia KLCI
		^KS11	KOSPI Composite Index
		^TWII	TSEC weighted index
		^GSPTSE	S&P/TSX Composite index
		^BVSP	IBOVESPA
		^MXX	IPC MEXICO
		^MERV	MERVAL
		^TA125.TA	^TA125.TA

Appendix B

Mathematical Proofs

B.1 Proof of Equation 3.4

From the equation 2.1, we know that

$$\begin{aligned} VaR^{\alpha} &= F^{-1}(1-\alpha) \\ &= u + x, \end{aligned} \tag{B.1}$$

where u is a pre-selected constant.

From the equation 3.1, a conditional random variable $X|X > u$ over the threshold u has following cumulative probability.

$$\begin{aligned} F_u(x) &= P\{X - u > x \mid X \geq u\} \\ &= \frac{F(x+u) - F(u)}{1 - F(u)} \\ &= \frac{1 - \alpha - \frac{n}{N}}{1 - \frac{n}{N}}. \end{aligned} \tag{B.2}$$

The conditional random variable $X|X > u$ has a EB distribution leading to the following equation

$$1 - \{1 - k(\frac{x}{\lambda})^c\}^{\frac{1}{k}} = \frac{1 - \alpha - \frac{n}{N}}{1 - \frac{n}{N}}. \tag{B.3}$$

The value of x can be obtained as

$$x = \hat{\lambda}[\frac{1}{\hat{k}} - \frac{1}{\hat{k}}(\frac{N\alpha}{N-n})^{\hat{k}}]^{\frac{1}{\hat{c}}}. \tag{B.4}$$

Therefore, the VaR at confidence level α is approximated as

$$VaR^{\alpha} = u + \hat{\lambda}[\frac{1}{\hat{k}} - \frac{1}{\hat{k}}(\frac{N\alpha}{N-n})^{\hat{k}}]^{\frac{1}{\hat{c}}}. \tag{B.5}$$

B.2 Proof of Equation 4.8

From equation 4.6, we know that

$$VaR^{\alpha}_{ar-gr,t} = \mu + aX_{t-1} + \sigma_t VaR^{\alpha}_{Z_t}. \tag{B.6}$$

From equation 4.7, σ_t can be obtained as

$$\sigma_t = \sqrt{\hat{\omega} + \sum_{i=1}^{p} \hat{\gamma}_i X^2_{t-1} + \sum_{j=1}^{q} \hat{\beta}_j \sigma^2_{t-1}}. \tag{B.7}$$

From equation 3.4, $VaR^{\alpha}_{Z_t}$ is

$$VaR^{\alpha}_{Z_t} = u_{ar-gr} + \hat{\lambda}_{ar-gr}[\frac{1}{\hat{k}_{ar-gr}} - \frac{1}{\hat{k}_{ar-gr}}(\frac{N\alpha}{N-n})^{\hat{k}_{ar-gr}}]^{\frac{1}{\hat{c}_{ar-gr}}}. \tag{B.8}$$

By substituting equation B.7 and B.8 into B.6, the VaR forecast at day t with AR(1)-GARCH(p,q) model is

$$\begin{aligned} VaR^{\alpha}_{ar-gr,t} =& \hat{\mu} + \hat{a}X_{t-1} + \sqrt{\hat{\omega} + \sum_{i=1}^{p} \hat{\gamma}_i X^2_{t-1} + \sum_{j=1}^{q} \hat{\beta}_j \sigma^2_{t-1}} \times \{u_{ar-gr} + \\ & \hat{\lambda}_{ar-gr}[\frac{1}{\hat{k}_{ar-gr}} - \frac{1}{\hat{k}_{ar-gr}}(\frac{N\alpha}{N-n})^{\hat{k}_{ar-gr}}]^{\frac{1}{\hat{c}_{ar-gr}}}\}. \end{aligned} \tag{B.9}$$

Appendix C

Copyright Information

Glossary

ARCH Autoregressive Conditional Heteroskedasticity. 15

DGP Data Generating Process. 74

EB Extended Burr XII distribution. 28, 29

EVT Extreme Value Theory. 15, 21, 23, 51

GARCH Generalised Autoregressive Conditional Heteroskedasticity. 15, 51

GEV Generalised Extreme Value Distribution. 16, 21, 24–26

GP Generalised Pareto Distribution. 16, 21, 26, 28

i.i.d. Independent and identically distributed. 13, 51

QMLE Quasi Maximum Likelihood. 69

RMSE Root Mean Square Error. 75

VaR Value at Risk. 11

WB Weibull Distribution. 16

References

Abad, P., & Benito, S. (2013, August). A detailed comparison of value at risk estimates. *Mathematics and Computers in Simulation*, *94*, 258–276. Retrieved 2017-07-25, from http://linkinghub.elsevier.com/retrieve/pii/S0378475412001218 doi: 10.1016/j.matcom.2012.05.011

Abad, P., Benito, S., & López, C. (2014, January). A comprehensive review of Value at Risk methodologies. *The Spanish Review of Financial Economics*, *12*(1), 15–32. Retrieved 2021-09-16, from https://linkinghub.elsevier.com/retrieve/pii/S217312681300017X doi: 10.1016/j.srfe.2013.06.001

Anderson, T. W. (1962, September). On the Distribution of the Two-Sample Cramer-von Mises Criterion. *The Annals of Mathematical Statistics*, *33*(3), 1148–1159. Retrieved 2021-02-23, from http://projecteuclid.org/euclid.aoms/1177704477 doi: 10.1214/aoms/1177704477

Ardia, D., Bluteau, K., Boudt, K., & Catania, L. (2018). Forecasting risk with Markov-switching GARCH models: A large-scale performance study. *International Journal of Forecasting*, 15.

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999, July). Coherent Measures of Risk. *Mathematical Finance*, *9*(3), 203–228. Retrieved 2018-07-19, from http://doi.wiley.com/10.1111/1467-9965.00068 doi: 10.1111/1467-9965.00068

Bali, T. G. (2003). An extreme value approach to estimating volatility and value at risk. *The Journal of Business*, *76*(1), 83–108. Retrieved 2017-08-09, from http://www.jstor.org/stable/10.1086/344669

Bali, T. G. (2007). A generalized extreme value approach to financial risk measurement. *Journal of Money, Credit and Banking*, *39*(7), 1613–1649. Retrieved 2017-07-19, from http://onlinelibrary.wiley.com/doi/10.1111/j.1538-4616.2007.00081.x/full doi: https://doi.org/10.1111/j.1538-4616.2007.00081.x

Bali, T. G., & Theodossiou, P. (2007, February). A conditional-SGT-VaR approach with alternative GARCH models. *Annals of Operations Research*, *151*(1), 241–267. Retrieved 2022-01-20, from http://link.springer.com/10.1007/s10479-006-0118-4 doi: 10.1007/s10479-006-0118-4

Bali, T. G., & Weinbaum, D. (2007, February). A conditional extreme value volatility estimator based on high-frequency returns. *Journal of Economic Dynamics and Control*, *31*(2), 361–397. Retrieved 2017-08-15, from http://linkinghub.elsevier.com/retrieve/pii/S0165188906000170 doi: 10.1016/j.jedc.2005.10.002

Barone-Adesi, G., Giannopoulos, K., & Vosper, L. (1999). VaR without correlations for portfolios of derivative securities. *The Journal of Futures Markets*, *19*(5), 583–602. doi: https://doi.org/10.1002/(SICI)1096-9934(199908)19:5⟨583::AID-FUT5⟩3.0.CO;2-S

Barone-Adesi, G., Giannopoulos, K., & Vosper, L. (2002, March). Backtesting derivative portfolios with filtered historical simulation (FHS). *European Financial Management*, *8*(1), 31–58. Retrieved 2019-12-10, from http://doi.wiley.com/10.1111/1468-036X.00175 doi: https://doi.org/10.1111/1468-036X.00175

Basel Accord III. (2019). , 1626.

Beder, T. S. (1995, September). VAR: Seductive but Dangerous. *Financial Analysts Journal*, *51*(5), 12–24. Retrieved 2022-01-19, from https://www.tandfonline.com/doi/full/10.2469/faj.v51.n5.1932 doi: 10.2469/faj.v51.n5.1932

Bekiros, S. D., & Georgoutsos, D. A. (2005, July). Estimation of Value-at-Risk by extreme value and conventional methods: a comparative evaluation of their predictive performance. *Journal of International Financial Markets, Institutions and Money*, *15*(3), 209–228. Retrieved 2020-01-23, from https://linkinghub.elsevier.com/retrieve/pii/S1042443104000745 doi: 10.1016/j.intfin.2004.05.002

Bera, A. K., & Higgins, M. L. (1993, December). ARCH MODELS: PROPERTIES, ESTIMATION AND TESTING. *Journal of Economic Surveys*, *7*(4), 305–366. Retrieved 2019-09-05, from http://doi.wiley.com/10.1111/j.1467-6419.1993.tb00170.x doi: 10.1111/j.1467-6419.1993.tb00170.x

Berghaus, B., & Bücher, A. (2018, October). Weak convergence of a pseudo maximum likelihood estimator for the extremal index. *The Annals of Statistics*, *46*(5). Retrieved 2022-01-24, from https://projecteuclid.org/journals/annals-of-statistics/volume-46/issue-5/Weak-convergence-of-a-pseudo-maximum-likelihood-estimator-for-the/10.1214/17-AOS1621.full doi: 10.1214/17-AOS1621

Blattberg, R. C., & Gonedes, N. J. (1974). A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices. *The Journal of Business*, *47*(2), 244–280. Retrieved 2017-09-13, from http://www.jstor.org/stable/2353383 doi: 10.2307/2353383

Bollerslev, T. (1986, April). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *31*(3), 307–327. Retrieved 2022-01-04, from https://linkinghub.elsevier.com/retrieve/pii/0304407686900631 doi: 10.1016/0304-4076(86)90063-1

Bollerslev, T., & Wooldridge, J. M. (1992, January). Quasi-maximum likelihood estimation and inference in dynamic models with time-varying covariances. *Econometric Reviews*, *11*(2), 143–172. Retrieved 2022-02-07, from http://www.tandfonline.com/doi/abs/10.1080/07474939208800229 doi: 10.1080/07474939208800229

Botev, Z. I., Grotowski, J. F., & Kroese, D. P. (2010, October). Kernel density estimation via diffusion. *The Annals of Statistics*, *38*(5). Retrieved 2021-12-03, from https://projecteuclid.org/journals/annals-of-statistics/volume-38/issue-5/Kernel-density-estimation-via-diffusion/10.1214/10-AOS799.full doi: 10.1214/10-AOS799

Bougerol, P., & Picard, N. (1992, October). Strict Stationarity of Generalized Autoregressive Processes. *The Annals of Probability*, *20*(4). Retrieved 2022-01-24, from https://projecteuclid.org/journals/annals-of-probability/volume-20/issue-4/Strict-Stationarity-of-Generalized-Autoregressive-Processes/10.1214/aop/1176989526.full doi: 10.1214/aop/1176989526

Brooks, C., Clare, A., Dalle Molle, J., & Persand, G. (2005, March). A comparison of extreme value theory approaches for determining value at risk. *Journal of Empirical Finance*, *12*(2), 339–352. Retrieved 2021-09-16, from https://linkinghub.elsevier.com/retrieve/pii/S0927539804000854 doi: 10.1016/j.jempfin.2004.01.004

Burr, I. W. (1942, June). Cumulative frequency functions. *The Annals of Mathematical Statistics*, *13*(2), 215–232. Retrieved 2020-02-10, from http://projecteuclid.org/euclid.aoms/1177731607 doi: 10.1214/aoms/1177731607

Byström, H. N. (2004, June). Managing extreme risks in tranquil and volatile markets using conditional extreme value theory. *International Review of Financial Analysis*, *13*(2), 133–152. Retrieved 2020-01-23, from https://linkinghub.elsevier.com/retrieve/pii/S1057521904000110 doi: 10.1016/j.irfa.2004.02.003

Calmon, W., Ferioli, E., Lettieri, D., Soares, J., & Pizzinga, A. (2021, April). An Extensive Comparison of Some Well-Established Value at Risk Methods. *International Statistical Review*, *89*(1), 148–166. Retrieved 2021-08-

18, from https://onlinelibrary.wiley.com/doi/10.1111/insr.12393 doi: 10.1111/insr.12393

Chavez-Demoulin, V., Davison, A. C., & McNeil, A. J. (2005, April). Estimating value-at-risk: a point process approach. *Quantitative Finance*, *5*(2), 227–234. Retrieved 2018-02-16, from http://www.tandfonline.com/doi/abs/10.1080/14697680500039613 doi: 10.1080/14697680500039613

Chen, Q., & Gerlach, R. H. (2013, October). The two-sided weibull distribution and forecasting financial tail risk. *International Journal of Forecasting*, *29*(4), 527–540. Retrieved 2020-01-23, from https://linkinghub.elsevier.com/retrieve/pii/S0169207013000332 doi: 10.1016/j.ijforecast.2013.01.007

Chen, S., & Tang, C. (2005, April). Nonparametric Inference of Value-at-Risk for Dependent Financial Returns. *Journal of Financial Econometrics*, *3*(2), 227–255. Retrieved 2019-07-10, from https://academic.oup.com/jfec/article-lookup/doi/10.1093/jjfinec/nbi012 doi: 10.1093/jjfinec/nbi012

Cheng, R., Evans, B., & Iles, T. (1992). Embedded models in non-linear regression. *Journal of the Royal Statistical Society. Series B (Methodological)*, *54*(3), 877–888. Retrieved 2018-11-09, from https://www.jstor.org/stable/2345866

Cheng, T., Gao, J., & Zhang, X. (2019, August). Nonparametric localized bandwidth selection for Kernel density estimation. *Econometric Reviews*, *38*(7), 733–762. Retrieved 2021-06-24, from https://www.tandfonline.com/doi/full/10.1080/07474938.2017.1397835 doi: 10.1080/07474938.2017.1397835

Cheng, W.-H., & Hung, J.-C. (2011, January). Skewness and leptokurtosis in GARCH-typed VaR estimation of petroleum and metal asset returns. *Journal of Empirical Finance*, *18*(1), 160–173. Retrieved 2020-01-28, from https://linkinghub.elsevier.com/retrieve/

pii/S092753981000040X doi: 10.1016/j.jempfin.2010.05.004

Christoffersen, P. F. (1998, November). Evaluating interval forecasts. *International Economic Review*, *39*(4), 841. Retrieved 2018-07-10, from https://www.jstor.org/stable/2527341?origin=crossref doi: 10.2307/2527341

Cifter, A. (2011, June). Value-at-risk estimation with wavelet-based extreme value theory: Evidence from emerging markets. *Physica A: Statistical Mechanics and its Applications*, *390*(12), 2356–2367. Retrieved 2020-01-23, from https://linkinghub.elsevier.com/retrieve/pii/S0378437111001609 doi: 10.1016/j.physa.2011.02.033

Coles, S. (2001). *An introduction to statistical modeling of extreme values*. London: Springer. Retrieved from https://doi.org/10.1007/978-1-4471-3675-0 (OCLC: 751837244)

Cramér, H. (1928, January). On the composition of elementary errors: First paper: Mathematical deductions. *Scandinavian Actuarial Journal*, *1928*(1), 13–74. Retrieved 2020-02-07, from http://www.tandfonline.com/doi/abs/10.1080/03461238.1928.10416862 doi: 10.1080/03461238.1928.10416862

Danielsson, J., & de Vries, C. G. (1997). Beyond the Sample: Extreme Quantile and Probability Estimation. *Tinbergen Institute Discussion Paper*, 39.

Embrechts, P., Resnick, S. I., & Samorodnitsky, G. (1999, April). Extreme Value Theory as a Risk Management Tool. *North American Actuarial Journal*, *3*(2), 30–41. Retrieved 2019-02-13, from http://www.tandfonline.com/doi/abs/10.1080/10920277.1999.10595797 doi: 10.1080/10920277.1999.10595797

Engle, R. F. (1982, July). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*, *50*(4), 987. Retrieved 2022-01-24, from https://www.jstor.org/stable/1912773?origin=crossref doi: 10.2307/1912773

Engle, R. F., & Gonzalez-Rivera, G. (1991, October). Semiparametric ARCH Models. *Journal of Business & Economic Statistics*, *9*(4), 345–359. Retrieved 2021-06-02, from http://www.tandfonline.com/doi/abs/10.1080/07350015.1991.10509863 doi: 10.1080/07350015.1991.10509863

Engle, R. F., & Manganelli, S. (2004, October). CAViaR: conditional autoregressive value at risk by regression quantiles. *Journal of Business & Economic Statistics*, *22*(4), 367–381. Retrieved 2019-12-10, from http://www.tandfonline.com/doi/abs/10.1198/073500104000000370 doi: 10.1198/073500104000000370

Fan, Y., Zhang, Y.-J., Tsai, H.-T., & Wei, Y.-M. (2008, November). Estimating 'Value at Risk' of crude oil price and its spillover effect using the GED-GARCH approach. *Energy Economics*, *30*(6), 3156–3171. Retrieved 2020-01-28, from https://linkinghub.elsevier.com/retrieve/pii/S014098830800056X doi: 10.1016/j.eneco.2008.04.002

Ferro, C. A. T., & Segers, J. (2003, May). Inference for clusters of extreme values: *Clusters of Extreme Values*. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, *65*(2), 545–556. Retrieved 2022-01-24, from https://onlinelibrary.wiley.com/doi/10.1111/1467-9868.00401 doi: 10.1111/1467-9868.00401

Fisher, R. A., & Tippett, L. H. C. (1928, April). Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Mathematical Proceedings of the Cambridge Philosophical Society*, *24*(02), 180. Retrieved 2017-07-19, from http://www.journals.cambridge.org/abstract_S0305004100015681 doi: 10.1017/S0305004100015681

Fong Chan, K., & Gray, P. (2006, April). Using extreme value theory to measure value-at-risk for daily electricity spot prices. *International Journal of Forecasting*, *22*(2), 283–300. Retrieved 2020-01-23, from https://linkinghub.elsevier.com/retrieve/pii/S0169207005001226 doi: 10.1016/j.ijforecast.2005.10.002

Gebizlioglu, O. L., Şenoğlu, B., & Kantar, Y. M. (2011, April). Comparison of certain value-at-risk estimation methods for the two-parameter Weibull loss distribution. *Journal of Computational and Applied Mathematics*, *235*(11), 3304–3314. Retrieved 2020-01-23, from https://linkinghub.elsevier.com/retrieve/pii/S0377042711000550 doi: 10.1016/j.cam.2011.01.044

Gnedenko, B. (1943, July). Sur La Distribution Limite Du Terme Maximum D'Une Serie Aleatoire. *The Annals of Mathematics*, *44*(3), 423. Retrieved 2021-12-09, from https://www.jstor.org/stable/1968974?origin=crossref doi: 10.2307/1968974

Gneiting, T. (2011, June). Making and Evaluating Point Forecasts. *Journal of the American Statistical Association*, *106*(494), 746–762. Retrieved 2022-02-09, from http://www.tandfonline.com/doi/abs/10.1198/jasa.2011.r10138 doi: 10.1198/jasa.2011.r10138

González-Rivera, G., Lee, T.-H., & Mishra, S. (2004, October). Forecasting volatility: A reality check based on option pricing, utility function, value-at-risk, and predictive likelihood. *International Journal of Forecasting*, *20*(4), 629–645. Retrieved 2019-12-20, from https://linkinghub.elsevier.com/retrieve/pii/S0169207003001420 doi: 10.1016/j.ijforecast.2003.10.003

Hansen, B. E. (1994, August). Autoregressive conditional density estimation. *International Economic Review*, *35*(3), 705. Retrieved 2020-01-28, from https://www.jstor.org/stable/2527081?origin=crossref doi: 10.2307/2527081

Hendricks, D. (1996). Evaluation of Value-at-Risk Models Using Historical Data. *Economic Policy Review*, 32.

Hull, J., & White, A. (1998, September). Incorporating volatility updating into the historical simulation method for value-at-risk. *The Journal of Risk*, *1*(1), 5–19. Retrieved 2022-01-19, from http://www.risk.net/journal-of-risk/technical-paper/2161156/incorporating-volatility-updating-historical-simulation-method-value-risk doi: 10.21314/JOR.1998.001

Jenkinson, A. F. (1955, April). The frequency distribution of the annual maximum (or minimum) values of meteorological elements. *Quarterly Journal of the Royal Meteorological Society*, *81*(348), 14. doi: https://doi.org/10.1002/qj.49708134804

Jorion, P. (2007). *Value at risk: the new benchmark for managing financial risk* (3rd ed ed.). New York: McGraw-Hill. (OCLC: ocm69104260)

Kupiec, P. H. (1995, November). Techniques for verifying the accuracy of risk measurement models. *The Journal of Derivatives*, *3*(2), 73–84. Retrieved 2018-12-13, from http://jod.iijournals.com/lookup/doi/10.3905/jod.1995.407942 doi: 10.3905/jod.1995.407942

Leadbetter, M. R. (1983). Extremes and local dependence in stationary sequences. *Probability Theory and Related Fields*, *65*(2), 291–306. doi: https://doi.org/10.1007/BF00532484

Lee, S., & Lee, T. (2011, September). Value-at-risk forecasting based on Gaussian mixture ARMA–GARCH model. *Journal of Statistical Computation and Simulation*, *81*(9), 1131–1144. Retrieved 2021-08-18, from http://www.tandfonline.com/doi/abs/10.1080/00949651003752320 doi: 10.1080/00949651003752320

Longerstaey, J., & Spencer, M. (1996). RiskMetrics - technical document (4th ed.).

Longin, F. M. (2000, July). From value at risk to stress testing: The extreme value approach. *Journal of Banking & Finance*, *24*(7), 1097–1130. Retrieved 2019-03-06, from http://linkinghub.elsevier.com/retrieve/pii/S0378426699000771 doi: 10.1016/S0378-4266(99)00077-1

MacDonald, A., Scarrott, C., Lee, D., Darlow, B., Reale, M., & Russell, G. (2011, June). A flexible extreme value mixture model. *Computational Statistics*

& Data Analysis, *55*(6), 2137–2157. Retrieved 2021-07-18, from https://linkinghub.elsevier.com/retrieve/pii/S0167947311000077 doi: 10.1016/j.csda.2011.01.005

Malevergne, Y., & Sornette, D. (2006). *Extreme financial risks: from dependence to risk management*. Heidelberg: Springer.

Markowitz, H. (1952). Portfolio Selection. *The Journal of Finance*, *7*(1), 77–91. Retrieved 2017-09-20, from http://www.jstor.org/stable/2975974 doi: 10.2307/2975974

Mcaleer, M., & da Veiga, B. (2008, January). Forecasting value-at-risk with a parsimonious portfolio spillover GARCH (PS-GARCH) model. *Journal of Forecasting*, *27*(1), 1–19. Retrieved 2019-12-20, from http://doi.wiley.com/10.1002/for.1049 doi: 10.1002/for.1049

McNeil, A. J. (1999). Extreme value theory for risk managers. *Departement Mathematik ETH Zentrum*.

McNeil, A. J., & Frey, R. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *Journal of Empirical Finance*, *7*(3-4), 271–300. doi: https://doi.org/10.1016/S0927-5398(00)00012-8

Nikolaev, N. Y., Boshnakov, G. N., & Zimmer, R. (n.d.). Heavy-tailed mixture GARCH volatility modeling and value-at-risk estimation. , *40*(6), 2233–2243. Retrieved 2022-07-11, from https://linkinghub.elsevier.com/retrieve/pii/S0957417412011621 doi: 10.1016/j.eswa.2012.10.038

Nolde, N., & Zhou, C. (2021, March). Extreme Value Analysis for Financial Risk Management. *Annual Review of Statistics and Its Application*, *8*(1), 217–240. Retrieved 2021-08-18, from https://www.annualreviews.org/doi/10.1146/annurev-statistics-042720-015705 doi: 10.1146/annurev-statistics-042720-015705

Pickands, J. (1975). Statistical inference using extreme order statistics. *The Annals of Statistics*, *3*(1), 119–131. Retrieved 2019-02-27, from https://

www.jstor.org/stable/2958083 doi: http://dx.doi.org/10.1214/aos/1176343003

Pritsker, M. (2006). The hidden dangers of historical simulation. *Journal of Banking & Finance*, *30*(2), 561–582. doi: https://doi.org/10.1016/j.jbankfin.2005.04.013

R Core Team. (2021). R: A language and environment for statistical computing [Computer software manual]. Vienna, Austria. Retrieved from https://www.R-project.org/

Richardson, M. P., Boudoukh, J. K., & Whitelaw, R. F. (1998). The Best of Both Worlds: A Hybrid Approach to Calculating Value at Risk. *SSRN Electronic Journal*. Retrieved 2022-01-19, from http://www.ssrn.com/abstract=51420 doi: 10.2139/ssrn.51420

Rockafellar, R. T., & Royset, J. O. (2014, December). Random variables, monotone relations, and convex analysis. *Mathematical Programming*, *148*(1-2), 297–331. Retrieved 2017-08-01, from http://link.springer.com/10.1007/s10107-014-0801-1 doi: 10.1007/s10107-014-0801-1

Rockafellar, R. T., & Uryasev, S. (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, *26*(7), 1443–1471. Retrieved 2017-07-19, from http://www.sciencedirect.com/science/article/pii/S0378426602002716

Rockafellar, R. T., & Uryasev, S. (2013). The fundamental risk quadrangle in risk management, optimization and statistical estimation. *Surveys in Operations Research and Management Science*, *18*(1), 33–53. Retrieved 2017-07-19, from http://www.sciencedirect.com/science/article/pii/S1876735413000032

Rockafellar, R. T., Uryasev, S., & Zabarankin, M. (2006, January). Generalized deviations in risk analysis. *Finance and Stochastics*, *10*(1), 51–74. Retrieved 2019-07-15, from http://link.springer.com/10.1007/s00780-005-0165-8 doi: 10.1007/s00780-005-0165-8

Ryan, J. A., Ulrich, J. M., Thielen, W., Teetor, P., & Bronder, S. (2020). quantmod: Quantitative financial modelling framework [Computer software manual]. (R package version 0.4.18)

Shao, Q., Chen, Y. D., & Zhang, L. (2008, January). An extension of three-parameter Burr III distribution for low-flow frequency analysis. *Computational Statistics & Data Analysis*, *52*(3), 1304–1314. Retrieved 2019-06-07, from https://linkinghub.elsevier.com/retrieve/pii/S0167947307002526 doi: 10.1016/j.csda.2007.06.014

Shao, Q., Ip, W.-C., & Wong, H. (2004, June). Determination of embedded distributions. *Computational Statistics & Data Analysis*, *46*(2), 317–334. Retrieved 2020-09-25, from https://linkinghub.elsevier.com/retrieve/pii/S0167947303001713 doi: 10.1016/S0167-9473(03)00171-3

Shao, Q., Wong, H., Xia, J., & Ip, W.-C. (2004, August). Models for extremes using the extended three-parameter Burr XII system with application to flood frequency analysis / Modèles d'extrêmes utilisant le système Burr XII étendu à trois paramètres et application à l'analyse fréquentielle des crues. *Hydrological Sciences Journal*, *49*(4). Retrieved 2017-09-11, from http://www.tandfonline.com/doi/abs/10.1623/hysj.49.4.685.54425 doi: 10.1623/hysj.49.4.685.54425

Smith, R. L., & Weissman, I. (1994, September). Estimating the Extremal Index. *Journal of the Royal Statistical Society: Series B (Methodological)*, *56*(3), 515–528. Retrieved 2022-01-24, from https://onlinelibrary.wiley.com/doi/10.1111/j.2517-6161.1994.tb01997.x doi: 10.1111/j.2517-6161.1994.tb01997.x

Tashman, L. J. (2000, October). Out-of-sample tests of forecasting accuracy: an analysis and review. *International Journal of Forecasting*, *16*(4), 437–450. Retrieved 2019-02-10, from http://linkinghub.elsevier.com/retrieve/pii/S0169207000000650 doi: 10.1016/S0169-2070(00)00065-0

Taylor, J. W. (2019, January). Forecasting Value at Risk and Expected Shortfall Using a Semiparametric Approach Based on the Asymmetric Laplace Distribution. *Journal of Business & Economic Statistics*, *37*(1), 121–133. Retrieved 2021-08-18, from https://www.tandfonline.com/doi/full/10.1080/07350015.2017.1281815 doi: 10.1080/07350015.2017.1281815

Theodossiou, P. (1998). Financial data and the skewed generalized t distribution. *Management Science*, *44*(12), 1650–1661. doi: https://www.jstor.org/stable/2634700

Wang, C., Chen, Q., & Gerlach, R. (2019, June). Bayesian realized-GARCH models for financial tail risk forecasting incorporating the two-sided Weibull distribution. *Quantitative Finance*, *19*(6), 1017–1042. Retrieved 2020-01-23, from https://www.tandfonline.com/doi/full/10.1080/14697688.2018.1540880 doi: 10.1080/14697688.2018.1540880

Youssef, M., Belkacem, L., & Mokni, K. (2015, September). Value-at-Risk estimation of energy commodities: A long-memory GARCH–EVT approach. *Energy Economics*, *51*, 99–110. Retrieved 2020-01-23, from https://linkinghub.elsevier.com/retrieve/pii/S0140988315001851 doi: 10.1016/j.eneco.2015.06.010

Zhang, L., Shao, Q., & Xu, H. (2022). A mixture innovation with extended extreme value approach for value at risk. *Journal of Applied Statistics*. (Manuscript submitted for publication)

Zhang, L., Shao, Q., Xu, H., & Wang, S. (2021). Improving the forecast skill of value at risk by an extension of extreme value approach. *Communications in Statistics-Simulation and Computation*. (Manuscript submitted for publication)

Zhang, Y., Peng, Y., Qu, X., Shi, J., & Erdem, E. (2021, April). A Finite Mixture GARCH Approach with EM Algorithm for Energy Forecasting Applications. *Energies*, *14*(9), 2352. Retrieved 2021-12-03, from https://www.mdpi.com/1996-1073/14/9/2352 doi: 10.3390/en14092352

Živković, S., & Aktan, B. (2011, October). Decay factor optimisation in time weighted simulation — Evaluating VaR performance. *International Journal of Forecasting*, *27*(4), 1147–1159. Retrieved 2022-01-19, from https://linkinghub.elsevier.com/retrieve/pii/S0169207011000045 doi: 10.1016/j.ijforecast.2010.09.007