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ORIGINAL RESEARCH

Optimal control of nonlinear Markov jump systems by control parametrisation technique

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Abstract
This paper considers an optimal control problem of nonlinear Markov jump systems with continuous state inequality constraints. Due to the presence of continuous-time Markov chain, no existing computation method is available to solve such an optimal control problem. In this paper, a derandomisation technique is introduced to transform the nonlinear Markov jump system into a deterministic system, which simultaneously gives rise to an equivalent deterministic dynamic optimisation problem. The control parametrisation technique is then used to partition the time horizon into a sequence of subintervals such that the control function is approximated by a piecewise constant function consistent with the partition. The heights of the piecewise constant function on the corresponding subintervals are taken as decision variables to be optimised. In this way, the approximate dynamic optimisation problem is an optimal parameter selection problem, which can be viewed as a finite dimensional optimisation problem. To solve it using a gradient-based optimisation method, the gradient formulas of the cost function and the constraint functions are derived. Finally, a real-world practical problem involving a bioreactor tank model is solved using the method proposed.

1 | INTRODUCTION

A continuous-time Markov jump system, which contains a number of modes, is described by a series of differential equations, each being defined on a specific mode. The switch between the dynamic equations is random, which is determined by a transition rate matrix. The continuous-time Markov jump systems have found many practical applications, such as economic systems [1], DC motor systems [2], robot systems [3], power systems [4], and networked systems [5]. For these systems, their system parameters and structures may change randomly caused by factors, such as data loss in communication, abrupt changes in environment, or failure of components. Due to the important nature of such systems, there is a growing interest among research community towards the study of Markov jump systems. See, for instance, stability analysis [6–8], controller design [9–11], filtering [12–14], and fault detection [15–17].

Optimal control is an important field in the control community. It can deal with many practically important issues, such as reducing energy consumption and improving production efficiency. Due to its practical importance, it has received continuous attention among scholars over the past several decades [18–24]. For an optimal control problem, the purpose is to find a control strategy such that a specific performance measure is minimised subject to specific dynamic systems and various constraints on the state and control variables. Nowadays, the real-world problems are becoming more and more complex and hence they can only be solved by numerical means. Subsequently, there are many effective numerical methods available in the literature for solving various optimal control problems. Examples include iterative dynamic programming [25], control parametrisation [26–31], collocation methods [32, 33], and full parametrisation [34–38].

For the control parametrisation method, the time horizon is partitioned into several subintervals, and the control functions are approximated by piecewise constant functions with the heights on the respective subintervals considered as decision variables to be optimised. Then, the original optimisation problem is approximated by an optimal parameter selection problem, which can be solved numerically based on gradient-based optimisation methods and interior point methods. Many important results related to the control parametrisation methods have been reported in ref. [26–31]. However, there exist many challenging issues yet to be resolved. For example, no numerical methods exist in the literature for solving optimal control problems involving nonlinear Markov jump systems, particularly in the presence of state inequality constraints. This is the motivation behind the study being carried out in this paper. More specifically, we consider an optimal control problem governed by nonlinear Markov jump systems with state inequality constraints. A derandomisation technique is introduced to obtain a representative deterministic optimal problem. Then, the control parametrisation method is used to develop a gradient-based computational method to solve this representative deterministic optimal control problem.

The contributions of this paper are summarised as follows:

(1) A derandomisation method is proposed to transform Markov jump system into a deterministic system with the information of the transition rates. Then, the corresponding representative deterministic optimal control problem is obtained.
(2) The control parametrisation technique is used to approximate the transformed deterministic optimal control problem by an optimal parameter selection problem, which can be regarded as a nonlinear programming problem.
(3) The continuous state inequality constraints are incorporated in the optimisation problem, which are handled by the constraint transcription technique.
(4) The gradient formulas of the cost function and the constraint functions with respect to the control parameters are derived.

The remaining parts of the paper are organised as follows. In Section 2, problem statement is given and some preliminary results are presented. In Section 3, gradient formulas are derived. In Section 4, a real-world practical example involving a bioreactor tank model is considered and solved. Section 5 concludes the paper.

Throughout this paper, $\mathfrak{R}^n$ and $\mathfrak{R}^{n\times m}$ denote the set of real n-dimensional vectors and the set of real $n\times m$ matrices, respectively. I_n and Λ^T denote an $n\times n$ identity matrix and the transpose of the matrix Λ, respectively. diag$\{\cdots\}$ stands for a block-diagonal matrix. $q^i(\cdot)\in\mathfrak{R}^N$ denotes the ith value of the vector $q(\cdot)$. $E\{\cdot\}$ and $\otimes$ represent the expectation operator and Kronecker product, respectively.

2 | PROBLEM STATEMENT AND PRELIMINARIES

Consider the following class of nonlinear Markov jump systems with N modes, defined on a fixed time interval $[0, t_\mathrm{f}]$ in a given probability space (Ξ, Ω, P).

$$\begin{cases}\dot{\boldsymbol{x}}(t) = f_{r(t)}(\boldsymbol{x}(t)) + g_{r(t)}(\boldsymbol{x}(t))\boldsymbol{u}(t),\\ f_{r(t)}(\boldsymbol{0}) = \boldsymbol{0},\ g_{r(t)}(\boldsymbol{0}) = \boldsymbol{0},\\ \boldsymbol{x}(0) = \boldsymbol{x}_0,\ r(0) = r_0,\end{cases} \tag{1}$$

where $\boldsymbol{x}(t)\in\mathfrak{R}^n$ and $\boldsymbol{u}(t)\in\mathfrak{R}^q$ are the system state and control variables, respectively; $f_{r(t)}(\boldsymbol{x}(t))$ and $g_{r(t)}(\boldsymbol{x}(t))$ are the functions with respect to $\boldsymbol{x}(t)$, respectively. $\boldsymbol{x}_0$ and r_0 denote the system initial state and mode, respectively. For convenience, when $r(t)=i$, $f_{r(t)}(\boldsymbol{x}(t))$ and $g_{r(t)}(\boldsymbol{x}(t))$ will be written as $f_i(\boldsymbol{x}(t))$ and $g_i(\boldsymbol{x}(t))$, respectively. Here, we suppose that the stochastic jump process $\{r(t),\ t\geq 0\}$ is a continuous-time Markov chain which takes values in a finite state set $K=\{1,2,\ldots,N\}$ and involves a transition rate matrix $\Lambda=[\lambda_{ij}]_{N\times N}$. The transition probability of the continuous-time Markov jump systems is defined as follows:

$$\begin{aligned}&\Pr\{r(t+\Delta t)=j\,|\,r(t)=i\}\\ &\quad=\begin{cases}\lambda_{ij}\Delta t + o(\Delta t), & \text{if } i\neq j,\\ 1+\lambda_{ii}\Delta t + o(\Delta t), & \text{if } i=j,\end{cases}\end{aligned} \tag{2}$$

where $\Delta t>0$, $\lim_{\Delta t\to 0^+}(\frac{o(\Delta t)}{\Delta t})=0$, λ_{ij} denotes the transition rate from mode i at time t to mode j at time $t+\Delta t$, satisfying $\lambda_{ij}\geq 0$ and $\lambda_{ii}=-\sum_{j=1,i\neq j}^{N}\lambda_{ij}$, $\forall i,j\in K, i\neq j$.

Let U be the control restraint set defined by

$$U=\{\boldsymbol{v}\in\mathfrak{R}^q : \underline{\boldsymbol{u}}\leq\boldsymbol{v}\leq\bar{\boldsymbol{u}}\}$$

where $\underline{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}$ are the given lower and upper bounds of the control $\boldsymbol{u}(t)$, respectively. Any measurable function $\boldsymbol{u}(t)$ defined on $[0,t_\mathrm{f}]$ taking values in U is an admissible control. Let U be the set containing all such measurable functions. Furthermore, let $\boldsymbol{x}(\cdot)$ be the solution of the dynamic system (1) corresponding to $\boldsymbol{u}(t)\in U$.

The optimisation problem under consideration, denoted as (OP), is stated as follows.

Given the dynamistic system (1) with the initial state $\boldsymbol{x}_0$, find an admissible control $\boldsymbol{u}(t)\in U$ to minimise the following objective function:

$$F_0 = E\left\{\Phi_0(\boldsymbol{x}(t_\mathrm{f})) + \int_0^{t_\mathrm{f}} L_0(t,\boldsymbol{x}(t),\boldsymbol{u}(t))\mathrm{d}t\right\}, \tag{3}$$

subject to the state inequality constraints

$$F_v = E\left\{h_v(t,\boldsymbol{x}(t))\right\}\geq 0,\ \ v=1,2,\ldots,v_1, \tag{4}$$

as well as the control constraint

$$\underline{\boldsymbol{u}} \leq \boldsymbol{u}(t) \leq \bar{\boldsymbol{u}}, \tag{5}$$

where Φ_0, L_0, and h_v, $v = 1, 2, \dots, v_1$, are given real-valued functions; $E\{\cdot\}$ denotes expectation; $\underline{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}$ are the given lower and upper bounds of the control $\boldsymbol{u}(t)$, respectively.

Since the system is a Markov jump system defined by Markov chain, problem (OP) is a stochastic optimal control problem. It appears that no computational method is available in the literature for solving such stochastic optimal control problem directly. In this paper, the derandomisation method introduced in ref. [39] is utilised to obtain a representative deterministic optimal control problem involving the information of the transition rates. On this basis, the control parametrisation technique is used to develop a computational method to solve the representative deterministic optimal control problem.

Let $\zeta_{\{i\}}(r(t))$ be the indicator function defined by

$$\zeta_{\{i\}}(r(t)) = \begin{cases} 1, & \text{if } r(t) = i, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

For system (1), the expectations of the states of different jump modes are defined as given below:

$$\boldsymbol{\eta}_i(t) = E\{\boldsymbol{x}(t)\zeta_{\{i\}}(r(t))\}. \tag{7}$$

From Equations (1), (6) and (7), we obtain

$$\begin{aligned} \mathrm{d}\boldsymbol{\eta}_j(t) &= E\{\mathrm{d}\boldsymbol{x}(t)\zeta_{\{j\}}(r(t)) + \boldsymbol{x}(t)\mathrm{d}\zeta_{\{j\}}(r(t))\} \\ &= E\{(f_j(\boldsymbol{x}(t)) + g_j(\boldsymbol{x}(t))\boldsymbol{u}(t))\zeta_{\{j\}}(r(t))\} \\ &\quad + E\{\boldsymbol{x}(t)\}E\{\mathrm{d}\zeta_{\{j\}}(r(t))\}. \end{aligned} \tag{8}$$

Considering the constraints $f_i(\boldsymbol{0}) = \boldsymbol{0}$, $g_i(\boldsymbol{0}) = \boldsymbol{0}$ in Equation (1), we have

$$\mathrm{d}\boldsymbol{\eta}_j(t) = \left(f_j(\boldsymbol{\eta}_j(t)) + g_j(\boldsymbol{\eta}_j(t))\boldsymbol{u}(t) + \sum_{i=1}^{N} \lambda_{ij}\boldsymbol{\eta}_i(t) \right)\mathrm{d}t. \tag{9}$$

Taking all $j = 1, 2, \dots, N$ into account, the system (9) can be written explicitly as:

$$\begin{bmatrix} \mathrm{d}\boldsymbol{\eta}_1(t) \\ \mathrm{d}\boldsymbol{\eta}_2(t) \\ \vdots \\ \mathrm{d}\boldsymbol{\eta}_N(t) \end{bmatrix} = \left(\begin{array}{l} \begin{bmatrix} f_1(\boldsymbol{\eta}_1(t)) \\ f_2(\boldsymbol{\eta}_2(t)) \\ \vdots \\ f_N(\boldsymbol{\eta}_N(t)) \end{bmatrix} + \begin{bmatrix} g_1(\boldsymbol{\eta}_1)\boldsymbol{u}(t) \\ g_2(\boldsymbol{\eta}_2)\boldsymbol{u}(t) \\ \vdots \\ g_N(\boldsymbol{\eta}_N)\boldsymbol{u}(t) \end{bmatrix} \\ + \begin{bmatrix} \lambda_{11} & \lambda_{21} & \cdots & \lambda_{N1} \\ \lambda_{12} & \lambda_{22} & \cdots & \lambda_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1N} & \lambda_{2N} & \cdots & \lambda_{NN} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_1(t) \\ \boldsymbol{\eta}_2(t) \\ \vdots \\ \boldsymbol{\eta}_N(t) \end{bmatrix} \end{array} \right) \mathrm{d}t \quad , \tag{10}$$

which can be written in vector form as:

$$\dot{\boldsymbol{\eta}}(t) = f(\boldsymbol{\eta}) + g(\boldsymbol{\eta})\tilde{\boldsymbol{u}}(t) + (\Lambda^T \otimes I_n)\boldsymbol{\eta}(t), \tag{11}$$

where $f(\boldsymbol{\eta}) = [f_1(\boldsymbol{\eta}_1)^T f_2(\boldsymbol{\eta}_2)^T \cdots f_N(\boldsymbol{\eta}_N)^T]^T$, $g(\boldsymbol{\eta}) = \mathrm{diag}\{g_1(\boldsymbol{\eta}_1), g_2(\boldsymbol{\eta}_2), \dots, g_N(\boldsymbol{\eta}_N)\}$, $\Lambda = [\lambda_{ij}]_{N\times N}$,
$\boldsymbol{\eta}(t) = [\boldsymbol{\eta}_1^T(t)\boldsymbol{\eta}_2^T(t) \cdots \boldsymbol{\eta}_N^T(t)]^T$, $\tilde{\boldsymbol{u}}(t) = [\underbrace{\boldsymbol{u}^T(t)\boldsymbol{u}^T(t) \cdots \boldsymbol{u}^T(t)}_{N}]^T$.

Remark 1. It is obvious that system (11) is a nonlinear deterministic system containing the information of the transition rates. There are many methods, such as T–S fuzzy techniques, that have been utilised to study the control of such systems.

Next, we will obtain a representative deterministic optimal control problem for the stochastic optimisation problem (OP) through the use of derandomisation theory.

Let the probability distribution of the initial mode be described by

$$\Pr\{r_0 = i\} = \beta_i \geq 0, \quad i = 1, 2, \dots, N. \tag{12}$$

Define

$$P_0 = [\beta_1 \ \ \beta_2 \ \cdots \ \beta_N]^T,$$

where $\beta_1 + \beta_2 + \cdots + \beta_N = 1$.

By Equation (7), we have the initial conditions

$$\boldsymbol{\eta}_1(0) = \boldsymbol{x}_0\beta_1, \ \boldsymbol{\eta}_2(0) = \boldsymbol{x}_0\beta_2, \ \cdots, \ \boldsymbol{\eta}_N(0) = \boldsymbol{x}_0\beta_N. \tag{13}$$

To obtain the information of probability distribution of mode as a function of time t, we need the following backward Kolmogorov differential equation with the initial condition.

$$\begin{cases} \dfrac{\mathrm{d}\Gamma(t)}{\mathrm{d}t} = \Lambda\Gamma(t), \\ \Gamma(0) = I_N, \end{cases} \tag{14}$$

where $\Gamma(t)$ denotes the transition probability matrix of Markov jump systems (1), and $\Gamma(0)$ is the initial transition probability matrix.

By the Markov chain property and the definition of the expectation, it follows from the use of the derandomisation technique that the cost function and the constraint functions are transformed to be in the form given below:

$$\begin{aligned} F_0 &= \Phi_0\left(\sum_{i=1}^{N} \boldsymbol{\eta}_i(t_\mathrm{f})q^i(t_\mathrm{f}) \right) \\ &\quad + \int_0^{t_\mathrm{f}} L_0\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t)q^i(t), \sum_{i=1}^{N} \tilde{\boldsymbol{u}}_i(t)q^i(t) \right)\mathrm{d}t, \end{aligned} \tag{15}$$

$$F_v = h_v\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t) q^i(t)\right) \geq 0, \ \ v = 1, 2, \ldots, v_1, \tag{16}$$

where $\boldsymbol{\eta}_i(\cdot)$ represents the ith block vector of $\boldsymbol{\eta}(\cdot)$, $q^i(\cdot)$ denotes the ith value of the vector $q(\cdot)$ and $q(\cdot) = \Gamma^T(\cdot)P_0$, and $\tilde{\boldsymbol{u}}_i(t)$ is the ith block vector of $\tilde{\boldsymbol{u}}(t)$ and $\tilde{\boldsymbol{u}}_i(t) = \boldsymbol{u}(t)$, $i = 1, 2, \ldots, N$.

Then, the representative deterministic optimisation problem, described by (NP), is stated as follows:

Given the deterministic system (11) with the initial condition $\boldsymbol{\eta}(0)$, find a control $\tilde{\boldsymbol{u}}(t) \in \tilde{U}$ such that the cost function (15) is minimised subject to the state inequality constraints (16), where $\tilde{U}$ is the set which contains all control functions satisfying the following constraints.

$$\underline{\tilde{\boldsymbol{u}}} \leq \tilde{\boldsymbol{u}}(t) \leq \bar{\tilde{\boldsymbol{u}}} \ \text{ for all } t \in [0, t_f],$$

where $\underline{\tilde{\boldsymbol{u}}} = [\underbrace{\underline{\boldsymbol{u}}^T \ \underline{\boldsymbol{u}}^T \ \cdots \ \underline{\boldsymbol{u}}^T}_{N}]^T$ and $\bar{\tilde{\boldsymbol{u}}} = [\underbrace{\bar{\boldsymbol{u}}^T \ \bar{\boldsymbol{u}}^T \ \cdots \ \bar{\boldsymbol{u}}^T}_{N}]^T$.

By means of the control parametrisation method, the time interval $[0, t_f]$ is subdivided into several uniform or nonuniform subintervals $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, M$, where t_k , $k = 0, \ldots, M$, are such that the following conditions are satisfied.

$$0 = t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq t_M = t_{\mathrm{f}}. \tag{17}$$

Let $\varpi_{[t_{k-1}, t_k)}(t)$ be the indicator function defined by:

$$\varpi_{[t_{k-1}, t_k)}(t) = \begin{cases} 1, & \text{if } t \in [t_{k-1}, t_k), \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Let $u_l(t)$, $l = 1, 2, \ldots, q$, be the lth element of the control vector $\boldsymbol{u}(t) \in \Re^q$, where $\boldsymbol{u}(t) = [u_1(t), \ldots, u_q(t)]^T$, and for $l = 1, 2, \ldots, q$, $u_l(t)$ is in the form given below:

$$u_l(t) = \sum_{k=1}^{M} u_{l,k}(t) \varpi_{[t_{k-1}, t_k)}(t). \tag{19}$$

Here, $u_{l,k}(t)$ denotes the value of the control component $u_l(t)$ in the kth time subinterval. In this paper, $u_{l,k}(t)$ is taken to be a constant parameter σ_l^k. In other words, $u_l(t)$ is approximated as:

$$u_l(t) \approx \sum_{k=1}^{M} \sigma_l^k \varpi_{[t_{k-1}, t_k)}(t). \tag{20}$$

Therefore, the control $\boldsymbol{u}(t)$ is approximated as a piecewise constant function given by

$$\boldsymbol{u}(t) \approx \sum_{k=1}^{M} \boldsymbol{\sigma}^k \varpi_{[t_{k-1}, t_k)}(t), \tag{21}$$

where $\boldsymbol{\sigma}^k = [\sigma_1^k, \ \sigma_2^k, \ \cdots, \ \sigma_q^k]^T, k = 1, 2, \ldots, M$, and $\boldsymbol{\sigma} = [(\boldsymbol{\sigma}^1)^T, \ (\boldsymbol{\sigma}^2)^T, \ \cdots, \ (\boldsymbol{\sigma}^M)^T]^T$.

Thus, the control $\tilde{\boldsymbol{u}}(t)$ is approximated as

$$\tilde{\boldsymbol{u}}(t) \approx \sum_{k=1}^{M} \tilde{\boldsymbol{\sigma}}^k \varpi_{[t_{k-1}, t_k)}(t), \tag{22}$$

where $\tilde{\boldsymbol{\sigma}}^k = [\underbrace{(\boldsymbol{\sigma}^k)^T, \ (\boldsymbol{\sigma}^k)^T, \ \cdots, \ (\boldsymbol{\sigma}^k)^T}_{N}]^T$, $k = 1, 2, \ldots, M$, and $\tilde{\boldsymbol{\sigma}} = [\underbrace{(\tilde{\boldsymbol{\sigma}}^1)^T, \ (\tilde{\boldsymbol{\sigma}}^2)^T, \ \cdots, (\tilde{\boldsymbol{\sigma}}^M)^T}_{N}]^T$.

With $\tilde{\boldsymbol{u}}(t)$ restricted to take the form of Equations (22), the cost function, the dynamic system and the constraints become, respectively,

$$\tilde{F}_0(\tilde{\boldsymbol{\sigma}}) = \Phi_0\left(\sum_{i=1}^{N} \boldsymbol{\eta}_i(t_{\mathrm{f}}|\tilde{\boldsymbol{\sigma}}) q^i(t_{\mathrm{f}})\right) + \int_0^{t_{\mathrm{f}}} \tilde{L}_0\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t), \tilde{\boldsymbol{\sigma}}\right) \mathrm{d}t, \tag{23}$$

$$\dot{\boldsymbol{\eta}}(t) = f(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) + g(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) \sum_{k=1}^{M} \tilde{\boldsymbol{\sigma}}^k \varpi_{[t_{k-1}, t_k)}(t) + (\Lambda^T \otimes I_n)\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}), \ t \in [0, t_{\mathrm{f}}], \tag{24}$$

$$\boldsymbol{\eta}(0) = [\boldsymbol{\eta}_1^T(0) \quad \boldsymbol{\eta}_2^T(0) \quad \cdots \quad \boldsymbol{\eta}_N^T(0)]^T, \tag{25}$$

$$\tilde{F}_v(\tilde{\boldsymbol{\sigma}}) = h_v\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t)\right) \geq 0, v = 1, 2, \ldots, v_1, \tag{26}$$

$$\underline{\tilde{\boldsymbol{u}}} \leq \tilde{\boldsymbol{\sigma}} \leq \bar{\tilde{\boldsymbol{u}}}, \tag{27}$$

where $\tilde{L}_0(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t), \tilde{\boldsymbol{\sigma}}) = L_0(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t) q^i(t), \sum_{i=1}^{N} \sum_{k=1}^{M} \tilde{\boldsymbol{\sigma}}_i^k q^i(t) \varpi_{[t_{k-1}, t_k)}(t))$, $\tilde{\boldsymbol{\sigma}}_i^k$ denotes the ith block vector of $\tilde{\boldsymbol{\sigma}}^k$ and $\tilde{\boldsymbol{\sigma}}_i^k = \boldsymbol{\sigma}^k$, $i = 1, 2, \ldots, N$, $\boldsymbol{\eta}_i(\cdot|\tilde{\boldsymbol{\sigma}})$ represents the ith block vector of $\boldsymbol{\eta}(\cdot|\tilde{\boldsymbol{\sigma}})$, where $\boldsymbol{\eta}(\cdot|\tilde{\boldsymbol{\sigma}}) = [\boldsymbol{\eta}_1^T(\cdot|\boldsymbol{\sigma}), \boldsymbol{\eta}_2^T(\cdot|\boldsymbol{\sigma}), \ldots, \boldsymbol{\eta}_N^T(\cdot|\boldsymbol{\sigma})]^T$ is the solution of the dynamic system (24) corresponding to $\tilde{\boldsymbol{\sigma}}$, and $q^i(\cdot)$ denotes the ith value of the vector $q(\cdot)$ and $q(\cdot) = \Gamma^T(\cdot)P_0$.

Noted that constraints (26) are continuous state inequality constraints. Clearly, constraints (26) are equivalent to the following equality constraints:

$$\int_0^{t_{\mathrm{f}}} \min\left\{0, h_v\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t)\right)\right\} \mathrm{d}t = 0, \ \ v = 1, 2, \ldots, v_1. \tag{28}$$

Since the function $\min\{0,\gamma\}$ is non-smooth and non-differentiable, it is approximated by a smooth and differential function $L_\varepsilon(\gamma)$, given by

$$L_\varepsilon(\gamma) = \begin{cases} \gamma, & \text{if } \gamma < -\varepsilon, \\ -\dfrac{(\gamma-\varepsilon)^2}{4\varepsilon}, & \text{if } -\varepsilon \le \gamma \le \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \tag{29}$$

where $\varepsilon > 0$ is an adjustable parameter.

This approximation technique is referred to as constraint transcription technique. See ref. [29]. Then, constraints (28) are approximated as

$$\tilde{F}_v^{\varepsilon,\rho}(\tilde{\boldsymbol{\sigma}}) = \rho + \int_0^{t_f} L_\varepsilon\left(h_v\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t)\right)\right)\mathrm{d}t \ge 0, \ v = 1,2,\dots,v_1, \tag{30}$$

where ρ is a positive number.

We now consider the optimisation problem, which is an approximate problem of (NP), as stated below:

Given the dynamic system (24) with the initial condition (25), find a control parameter vector $\tilde{\boldsymbol{\sigma}}$ which satisfies the control constraint (27) such that the cost function (23) is minimised subject to the constraints (30). Let this problem be referred to as Problem (CVP-NP).

Remark 2. It is noted that the optimal cost function value of Problem (CVP-NP) is an approximate optimal cost value of the original Problem (NP). In order to improve the accuracy of the approximate optimal cost function value of Problem (CVP-NP), the value of M should be increased. However, it will give rise to more decision variables, and hence the computational burden will increase.

3 | GRADIENT FORMULAS

To develop a gradient-based optimisation method to solve Problem (CVP-NP), the gradient formulas are required. The gradient formula of the cost function (23) with respect to the control parameter $\tilde{\boldsymbol{\sigma}}$ is given in the following theorem.

Theorem 1. *The gradient formula of $\tilde{F}_0(\tilde{\boldsymbol{\sigma}})$ with respect to the control parameter $\tilde{\boldsymbol{\sigma}}$ is given by*

$$\frac{\partial \tilde{F}_0(\tilde{\boldsymbol{\sigma}})}{\partial \tilde{\boldsymbol{\sigma}}} = \int_0^{t_f} \frac{\partial S_0(t, \boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}), \tilde{\boldsymbol{\sigma}}, \boldsymbol{\mu}_0(t))}{\partial \tilde{\boldsymbol{\sigma}}}\mathrm{d}t = \int_0^{t_f} \left\{ \frac{\partial \tilde{L}_0\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t), \tilde{\boldsymbol{\sigma}}\right)}{\partial \tilde{\boldsymbol{\sigma}}} + \boldsymbol{\mu}_0^T(t) g(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) \frac{\partial \sum_{k=1}^{M} \tilde{\boldsymbol{\sigma}}^k \varpi_{[t_{k-1},t_k)}(t)}{\partial \tilde{\boldsymbol{\sigma}}} \right\} \mathrm{d}t, \ t \in [0, t_f], \tag{31}$$

where $S_0(t, \boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}), \tilde{\boldsymbol{\sigma}}, \boldsymbol{\mu}_0(t)) = \tilde{L}_0(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t), \tilde{\boldsymbol{\sigma}}) + \boldsymbol{\mu}_0^T(t)(f(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) + g(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) \sum_{k=1}^{M} \tilde{\boldsymbol{\sigma}}^k \varpi_{[t_{k-1},t_k)}(t) + (\Lambda^T \otimes I_n)\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}))$, *$\boldsymbol{\eta}_i(\cdot|\tilde{\boldsymbol{\sigma}})$ represents the ith block vector of $\boldsymbol{\eta}(\cdot|\tilde{\boldsymbol{\sigma}})$, where $\boldsymbol{\eta}(\cdot|\tilde{\boldsymbol{\sigma}}) = [\boldsymbol{\eta}_1^T(\cdot|\boldsymbol{\sigma}), \boldsymbol{\eta}_2^T(\cdot|\boldsymbol{\sigma}), \dots, \boldsymbol{\eta}_N^T(\cdot|\boldsymbol{\sigma})]^T$ is the solution of the dynamic system (24) corresponding to $\tilde{\boldsymbol{\sigma}}$, and $q^i(\cdot)$ denotes the ith value of the vector $q(\cdot)$ and $q(\cdot) = \Gamma^T(\cdot)P_0$, and $\boldsymbol{\mu}_0(\cdot)$ is the solution of the following costate system:*

$$\dot{\boldsymbol{\mu}}_0(t) = -\left(\frac{\partial S_0(t, \boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}), \tilde{\boldsymbol{\sigma}}, \boldsymbol{\mu}_0(t))}{\partial \boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})}\right)^T = -\left(\frac{\partial\left(f(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) + g(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) \sum_{k=1}^{M} \tilde{\boldsymbol{\sigma}}^k \varpi_{[t_{k-1},t_k)}(t)\right)}{\partial \boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})} + \Lambda^T \otimes I_n\right)^T \boldsymbol{\mu}_0(t) - \left(\frac{\partial \tilde{L}_0\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t), \tilde{\boldsymbol{\sigma}}\right)}{\partial \boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})}\right)^T, \tag{32}$$

with the terminal condition:

$$\boldsymbol{\mu}_0(t_f) = \left(\frac{\partial \Phi_0\left(\sum_{i=1}^{N} \boldsymbol{\eta}_i(t_f|\tilde{\boldsymbol{\sigma}}) q^i(t_f)\right)}{\partial \boldsymbol{\eta}(t_f|\tilde{\boldsymbol{\sigma}})}\right)^T. \tag{33}$$

Proof. Define

$$S_0(t, \boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}), \tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}_0(t)) = \tilde{L}_0\left(t, \sum_{i=1}^{N} \boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}}) q^i(t), \tilde{\boldsymbol{\sigma}}\right) + \boldsymbol{\tau}_0^T(t) \dot{\boldsymbol{\eta}}(t|\tilde{\boldsymbol{\sigma}})$$

where $\boldsymbol{\tau}_0(t)$ is an absolutely continuous function yet to be determined. Then, $\tilde{F}_0(\tilde{\boldsymbol{\sigma}})$ can be written as:

$$\tilde{F}_0(\tilde{\boldsymbol{\sigma}}) = \Phi_0\left(\sum_{i=1}^{N} \boldsymbol{\eta}_i(t_f|\tilde{\boldsymbol{\sigma}}) q^i(t_f)\right) + \int_0^{t_f} S_0(t, \boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}), \tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}_0(t))\mathrm{d}t - \int_0^{t_f} \boldsymbol{\tau}_0^T(t) \dot{\boldsymbol{\eta}}(t|\tilde{\boldsymbol{\sigma}})\mathrm{d}t. \tag{34}$$

Taking integration by parts of the last term gives

$$\tilde{F}_0(\tilde{\boldsymbol{\sigma}}) = \Phi_0\left(\sum_{i=1}^{N}\boldsymbol{\eta}_i(t_f|\tilde{\boldsymbol{\sigma}})q^i(t_f)\right) + \int_0^{t_f} S_0(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\tau}_0(t))\mathrm{d}t$$

$$-\boldsymbol{\tau}_0^T(t_f)\boldsymbol{\eta}(t_f|\tilde{\boldsymbol{\sigma}}) + \boldsymbol{\tau}_0^T(0)\boldsymbol{\eta}(0) + \int_0^{t_f}\dot{\boldsymbol{\tau}}_0^T(t)\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})\mathrm{d}t. \tag{35}$$

Thus,

$$\frac{\partial\tilde{F}_0(\tilde{\boldsymbol{\sigma}})}{\partial\tilde{\boldsymbol{\sigma}}} = \int_0^{t_f}\frac{\partial S_0(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\tau}_0(t))}{\partial\tilde{\boldsymbol{\sigma}}}\mathrm{d}t + \left(\frac{\partial\Phi_0\left(\sum_{i=1}^{N}\boldsymbol{\eta}_i(t_f|\tilde{\boldsymbol{\sigma}})q^i(t_f)\right)}{\partial\boldsymbol{\eta}(t_f|\tilde{\boldsymbol{\sigma}})} - \boldsymbol{\tau}_0^T(t_f)\right)\frac{\partial\boldsymbol{\eta}(t_f|\tilde{\boldsymbol{\sigma}})}{\partial\tilde{\boldsymbol{\sigma}}} + \int_0^{t_f}\left(\frac{\partial S_0(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\tau}_0(t))}{\partial\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})} + \dot{\boldsymbol{\tau}}_0^T(t)\right)\frac{\partial\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})}{\partial\tilde{\boldsymbol{\sigma}}}\mathrm{d}t. \tag{36}$$

Selecting $\boldsymbol{\tau}_0(t) = \boldsymbol{\mu}_0(t)$, we obtain

$$\frac{\partial\tilde{F}_0(\tilde{\boldsymbol{\sigma}})}{\partial\tilde{\boldsymbol{\sigma}}} = \int_0^{t_f}\frac{\partial S_0(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\mu}_0(t))}{\partial\tilde{\boldsymbol{\sigma}}}\mathrm{d}t + \left(\frac{\partial\Phi_0\left(\sum_{i=1}^{N}\boldsymbol{\eta}_i(t_f|\tilde{\boldsymbol{\sigma}})q^i(t_f)\right)}{\partial\boldsymbol{\eta}(t_f|\tilde{\boldsymbol{\sigma}})} - \boldsymbol{\mu}_0^T(t_f)\right)\frac{\partial\boldsymbol{\eta}(t_f|\tilde{\boldsymbol{\sigma}})}{\partial\tilde{\boldsymbol{\sigma}}} + \int_0^{t_f}\left(\frac{\partial S_0(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\lambda}_0(t))}{\partial\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})} + \dot{\boldsymbol{\mu}}_0^T(t)\right)\frac{\partial\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})}{\partial\tilde{\boldsymbol{\sigma}}}\mathrm{d}t. \tag{37}$$

From the costate system (32) with the terminal condition (33), the gradient formula of the function $\tilde{F}_0(\tilde{\boldsymbol{\sigma}})$ with respect to control parameter $\tilde{\boldsymbol{\sigma}}$ is obtained readily as given below:

$$\frac{\partial\tilde{F}_0(\tilde{\boldsymbol{\sigma}})}{\partial\tilde{\boldsymbol{\sigma}}} = \int_0^{t_f}\frac{\partial S_0(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\mu}_0(t))}{\partial\tilde{\boldsymbol{\sigma}}}\mathrm{d}t \tag{38}$$

This completes the proof. □

Next, we give the gradient formulas of the constraint functions (30) with respect to the control parameter $\tilde{\boldsymbol{\sigma}}$.

Theorem 2. *The gradient formulas of the constraints $\tilde{F}_v^{\varepsilon,\rho}(\tilde{\boldsymbol{\sigma}})$ with respect to the control parameter $\tilde{\boldsymbol{\sigma}}$, $v = 1,2,\dots,v_1$, are given by*

$$\frac{\partial\tilde{F}_v^{\varepsilon,\rho}(\tilde{\boldsymbol{\sigma}})}{\partial\tilde{\boldsymbol{\sigma}}} = \int_0^{t_f}\frac{\partial S_v(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\mu}_v(t))}{\partial\tilde{\boldsymbol{\sigma}}}\mathrm{d}t = \int_0^{t_f}\boldsymbol{\mu}_v^T(t)g(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}))\frac{\partial\sum_{k=1}^{M}\tilde{\boldsymbol{\sigma}}^k\varpi_{[t_{k-1},t_k)}(t)}{\partial\tilde{\boldsymbol{\sigma}}}\mathrm{d}t,$$

$$t\in[0,t_f],\ v = 1,2,\dots,v_1, \tag{39}$$

where $S_v(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\mu}_v(t)) = L_\varepsilon(h_v(t,\sum_{i=1}^{N}\boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}})q^i(t))) + \boldsymbol{\mu}_v^T(t)(f(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) + g(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}))\sum_{k=1}^{M}\tilde{\boldsymbol{\sigma}}^k\varpi_{[t_{k-1},t_k)}(t) + (\Lambda^T\otimes I_n)\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}))$, *$\boldsymbol{\eta}_i(\cdot|\tilde{\boldsymbol{\sigma}})$ represents the ith block vector of $\boldsymbol{\eta}(\cdot|\tilde{\boldsymbol{\sigma}})$, where $\boldsymbol{\eta}(\cdot|\tilde{\boldsymbol{\sigma}}) = [\boldsymbol{\eta}_1^T(\cdot|\boldsymbol{\sigma}),\boldsymbol{\eta}_2^T(\cdot|\boldsymbol{\sigma}),\dots,\boldsymbol{\eta}_N^T(\cdot|\boldsymbol{\sigma})]^T$ is the solution of the dynamic system (24) corresponding to $\tilde{\boldsymbol{\sigma}}$, and $q^i(\cdot)$ denotes the ith value of the vector $q(\cdot)$ and $q(\cdot) = \Gamma^T(\cdot)P_0$, and $\boldsymbol{\mu}_v(\cdot)$ is the solution of the following costate system:*

$$\dot{\boldsymbol{\mu}}_v(t) = -\left(\frac{\partial S_v(t,\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}),\tilde{\boldsymbol{\sigma}},\boldsymbol{\mu}_v(t))}{\partial\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})}\right)^T = -\left(\frac{\partial\left(f(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})) + g(\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}}))\sum_{k=1}^{M}\tilde{\boldsymbol{\sigma}}^k\varpi_{[t_{k-1},t_k)}(t)\right)}{\partial\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})} + \Lambda^T\otimes I_n\right)^T\boldsymbol{\mu}_v(t) - \left(\frac{\partial L_\varepsilon\left(h_v\left(t,\sum_{i=1}^{N}\boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}})q^i(t)\right)\right)}{\partial h_v\left(t,\sum_{i=1}^{N}\boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}})q^i(t)\right)}\frac{\partial h_v\left(t,\sum_{i=1}^{N}\boldsymbol{\eta}_i(t|\tilde{\boldsymbol{\sigma}})q^i(t)\right)}{\partial\boldsymbol{\eta}(t|\tilde{\boldsymbol{\sigma}})}\right)^T, \tag{40}$$

with the terminal condition:

$$\boldsymbol{\mu}_v(t_f) = [\underbrace{0,\ 0,\ \cdots,0}_{n*N}]^T. \tag{41}$$

Proof. The proof is similar to that given for Theorem 3.1. Thus, we omit it. □

4 | NUMERICAL EXAMPLE

In this section, we consider a practical application of a bioreactor tank model with three Markov jump parameters reported in ref. [40], which is described by

$$\dot{x}_1(t) = -u_1(t)x_1(t) + x_1(t)(1-x_2(t))\exp\left(\frac{x_2(t)}{\alpha}\right),$$

$$\dot{x}_2(t) = -u_1(t)x_2(t) + \frac{(1+\beta_i)x_1(t)(1-x_2(t))\exp\left(\frac{x_2(t)}{\alpha}\right)}{1+\beta_i-x_2(t)}, \tag{42}$$

where $x_1(t)$ and $x_2(t)$ represent the number of cells and the nutrient concentration at time t, respectively. The input signal $u(t)$ represents the flow rate through the tank. The system parameters α and β_i stand for the nutrient inhibition constant and the growth rate, respectively. In this system, the state variables $x_1(t)$ and $x_2(t)$ are assumed to evolve on the intervals [0.0001,0.9999], while the control variable $u(t)$ is restricted to the interval [0,2]. The Markov jump parameters are set as $\beta_1 = 0.02$, $\beta_2 = 0.03$, $\beta_3 = 0.04$, and system parameter is given as $\alpha = 0.48$. The terminal time $t_f = 0.5$.

The transition rate matrix of the bioreactor tank model is given by

$$\Lambda = \begin{bmatrix} -1 & 0.5 & 0.5 \\ 1.5 & -2 & 0.5 \\ 1 & 1 & -2 \end{bmatrix}.$$

Then, we can obtain the transition probability matrix $\Gamma(t)$ by solving the backward Kolmogorov differential equation (14) with the initial condition $\Gamma(0) = I_3$. The detailed expressions of the matrix are given as below.

$$\Gamma(t) = \begin{bmatrix} \pi_{11}(t) & \pi_{12}(t) & \pi_{13}(t) \\ \pi_{21}(t) & \pi_{22}(t) & \pi_{23}(t) \\ \pi_{31}(t) & \pi_{32}(t) & \pi_{33}(t) \end{bmatrix},$$

where $\pi_{11}(t) = \frac{14+11\exp(-5t/2)}{25} + \frac{t\exp(-5t/2)}{10}$, $\pi_{12}(t) = \frac{6-6\exp(-5t/2)}{25} - \frac{t\exp(-5t/2)}{10}$, $\pi_{13}(t) = \frac{1-\exp(-5t/2)}{5}$, $\pi_{21}(t) = \frac{14-14\exp(-5t/2)}{25} + \frac{t\exp(-5t/2)}{10}$, $\pi_{22}(t) = \frac{6+19\exp(-5t/2)}{25} - \frac{t\exp(-5t/2)}{10}$, $\pi_{23}(t) = \frac{1-\exp(-5t/2)}{5}$, $\pi_{31}(t) = \frac{14-14\exp(-5t/2)}{25} - \frac{2t\exp(-5t/2)}{5}$, $\pi_{32}(t) = \frac{6-6\exp(-5t/2)}{25} + \frac{2t\exp(-5t/2)}{5}$, $\pi_{33}(t) = \frac{1+4\exp(-5t/2)}{5}$.

The initial mode probability distribution is

$$P_0 = [1\ 0\ 0]^T.$$

Our objective is to maximise the nutrient concentration $x_2(t_f)$ (corresponding to minimising $-x_2(t_f)$) at the terminal time $t_f = 0.5$.

Thus, consider the optimisation problem described as follows:

Given system (42) with initial condition $[x_1(0), x_2(0)]^T = [0.8, 0.5]^T$, find a control $u(t)$ satisfying the constraints

$$0 \leq u(t) \leq 2 \quad \text{for} \quad t \in [0, 0.5]$$

such that the cost function

$$F_0 = E\{x_2(0.5)\}$$

is maximised subject to the state constraints

$$0.0001 \leq E\{x_1(t)\} \leq 0.9999,$$

$$0.0001 \leq E\{x_2(t)\} \leq 0.9999.$$

By applying derandomisation method followed by the control parametrisation technique, we obtain the corresponding version of Problem (CVP-NP). It is then solved using the optimisation algorithm where the initial guess of the control is $u(t) = 0.1$.

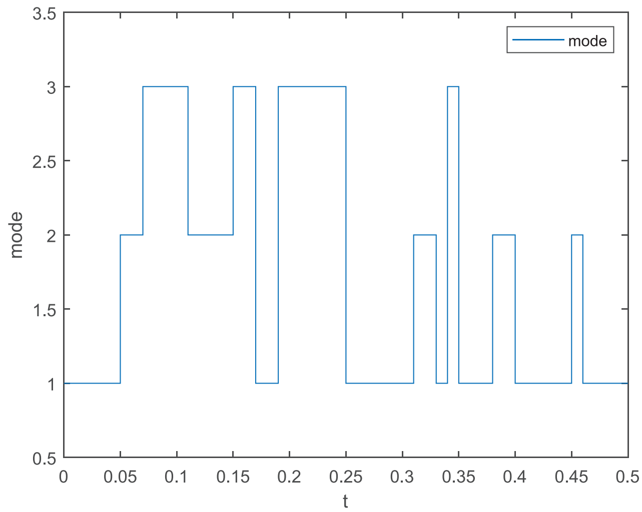

FIGURE 1 One evolution of system mode

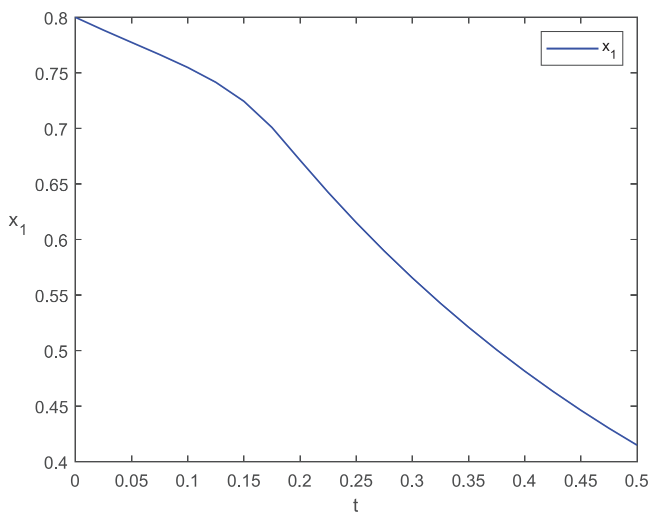

FIGURE 2 Trajectory of the cells number

Here, the time interval is subdivided into $M = 20$. The maximum value of the cost function at the terminal time $t_f = 0.5$ obtained is $F_0 = E\{x_2(0.5)\} = 0.699461553$.

Figures 1–4 show the evolution of system mode, the trajectory of the cells number, the trajectory of the nutrient concentration, and the input trajectory.

To verify the validity of the obtained control, we test run 800 times of simulations under the obtained control. The optimal values of the cost function of the 800 test runs is mostly concentrated around 0.6945, which is relatively close to the obtained optimal value.

Figure 5 shows the histogram of the results of the test runs, and most of the results are distributed near the obtained optimal results of Problem (CVP-NP), which further indicates the effectiveness of the proposed method.

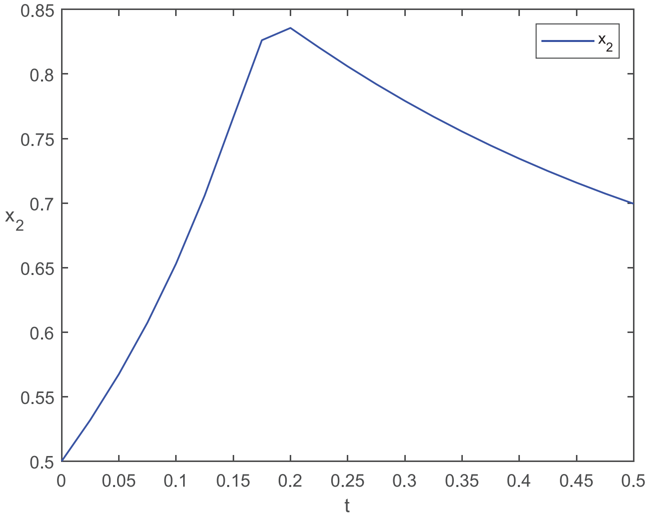

FIGURE 3 Trajectory of the nutrient concentration

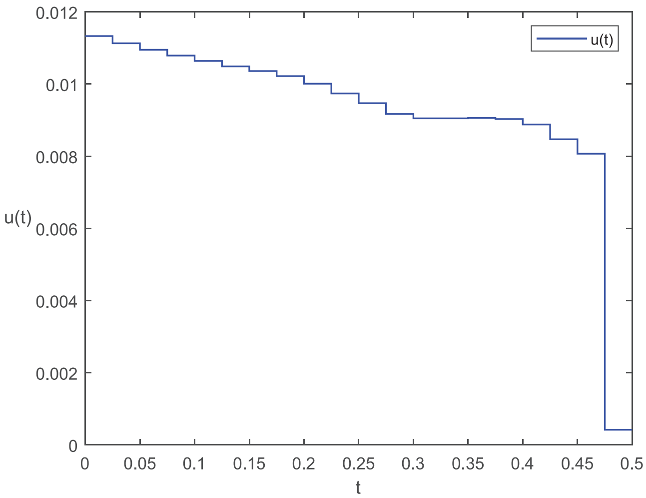

FIGURE 4 Trajectory of the input

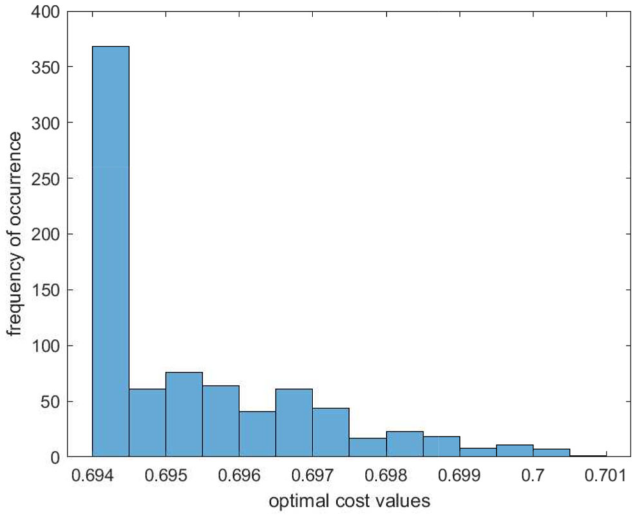

FIGURE 5 Results of the test runs

5 | CONCLUSION

This paper considered the optimal control computation for an optimal control problem described by a nonlinear Markov jump system with continuous state inequality constraints. By the utilisation of a derandomisation technique, a representative deterministic optimisation problem is obtained. On this basis, we obtained an equivalent deterministic optimal control problem. Then, based on the control parametrisation technique, an approximate finite dimensional optimisation problem is obtained, which can be computed numerically using gradient-based optimisation methods. For this, the gradient formulas of the cost function and the constraint functions are derived. Finally, a real-world practical problem involving a bioreactor tank model is solved using the method proposed.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study

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