GEOMETRIC CONTROL AND DISTURBANCE DECOUPLING FOR FRACTIONAL SYSTEMS *

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Abstract. We develop a geometric approach for fractional linear time-invariant systems with Caputo-type derivatives. In particular, we generalize the fundamental notions of invariance and controlled invariance to the fractional setting. We then exploit this new geometric framework to address the disturbance decoupling problem via static pseudostate feedback, with and without stability. Our main contribution is a set of necessary and sufficient conditions for the disturbance decoupling problem that are related to the input-output properties of the closed-loop system, and hence they are not just applicable to Caputo-type derivatives but, more broadly, to any type of fractional system. These results show that, while the conditions for guaranteeing the existence of a decoupling pseudostate feedback remain essentially unchanged, the underlying theoretical framework is substantially different, because the fractional derivative is a non-local operator and this property plays a major role in the characterization of the evolution of the pseudostate trajectory. In particular, we show that, unlike the integer case, the infinite-dimensional nature of fractional systems means that feedback control is insufficient to maintain the pseudostate trajectory on a controlled invariant subspace, unless the entire past history of the pseudostate has evolved on that subspace. However, feedforward control can achieve this task under certain necessary and sufficient geometric conditions.

Key words. Controlled invariance, Disturbance decoupling, Fractional systems.

AMS subject classifications. 15A15, 15A09, 15A23

1. Introduction.

Controlled and conditioned invariance are cornerstones of geometric control theory, which was developed independently in [5] and [49]. These concepts were originally introduced for linear time-invariant systems to solve disturbance decoupling and unknown-input observation problems, respectively. However, in the past fifty years, the reach of geometric control has gone well beyond its original target. Indeed, geometry has proved to be a natural setting for solving a plethora of other control and estimation problems that are important in practice, including fault detection, non-interaction, model matching, optimal control and filtering, unknown-input observation, squaring down, $\mathscr{H}_2$-optimal decoupling and filtering and, more recently, monotonic tracking. There is a vast academic literature on these topics, and we refer the readers to the monographs [48, 6, 44, 13] and the numerous references cited therein.

Over the past four decades, the fundamental ideas and results in geometric control, originally developed for linear time-invariant systems, have been adapted/extended to other types of systems, including nonlinear systems [17], infinite dimensional systems [10], singular systems [7, 21] and multidimensional systems [27]. The purpose of this paper is to develop a geometric apparatus for fractional systems, and to show how this geometric framework can address the disturbance decoupling problem for such systems. Indeed, fractional systems are an emerging area of control engineering for which a geometric framework has yet to be developed. Fractional systems involve derivative orders that range over the real set, and this provides additional flexibility to capture complex dynamics [18]. Fractional models have been successfully employed in a rich variety of applications [34], including modeling the dynamics of electronic devices [9], signal processing [29] and robust control [39, 31, 47, 33, 26], just to mention a few. In general, fractional models arise naturally in the mathematical description of any system characterized by memory behavior such as heat diffusion, and fractal structures such as repeated electric components and patterned systems [4, 16].

The focus of this paper is on the development and characterization of the fundamental building blocks of the geometric control approach, namely controlled invariants, output-nulling and stabilizability subspaces. These objects are then employed to address the disturbance decoupling problem by pseudostate feedback. We will show that, while the subspace inclusions used to define controlled invariance and output nulling subspaces do not formally change with respect to the integer case (with the exception of stabilizability subspaces, since the domain of stability changes in the fractional setting), their system-theoretic interpretation in terms of the corresponding trajectories of the underlying dynamical system is profoundly different, and needs to be addressed carefully. The differences

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are due to the non-locality of the fractional derivative operator: unlike the standard integer-order derivative, which depends on the behavior of the function in an arbitrarily small neighborhood of the point at which we compute the derivative, the fractional derivative depends on a finite interval that increases over time. As a result, the evolution of a fractional system from a given time onward is dictated by the entire past behavior of the pseudostate. In other words, the value of the pseudostate variable at just a single point in time does not capture the entire past evolution of the system, and therefore the term "pseudostate" is used in place of the term "state" [41].

After establishing the fundamental building blocks, we then focus our attention on the notion of controlled invariance. To this end, we first show that, as in the integer case, the set of reachable pseudostates from zero initial conditions is a subspace—not just a set—characterized by being the supremal invariant subspace containing the image of the input matrix. This will require going beyond the existing results on reachability/controllability for fractional systems, see e.g. [23, 8, 2, 3].

The next fundamental step is the characterization of controlled invariance for fractional systems. In this context, the non-locality of the fractional operator plays a major role. Indeed, it is well-known that in the classical context, if the state is in a controlled invariant subspace at a certain point in time, then there exists a state feedback that maintains the entire future trajectory on that subspace, irrespective of how the state variable evolved before that time instant. By contrast, we prove that in the fractional case this is only possible if the entire trajectory before the time instant in question evolved on the controlled invariant subspace. On the one hand, this weakens the relationship between the system-theoretic interpretation of controlled invariance and its geometric counterpart. On the other hand, it shows that any other control approach would require knowledge of the entire past trajectory of the system to maintain the evolution of the pseudostate on the controlled invariant subspace. In other words, there are no alternative notions of controlled invariance that can lead to alternative feedback-type solutions to the problem of maintaining the pseudostate trajectory on a given subspace. In fact, if the past trajectory has moved away from a controlled invariant subspace but returns at a specific time, a control function that maintains the future trajectory on that controlled invariant must necessarily include a feedforward component, which takes into account the past evolution of the system. We provide necessary and sufficient conditions for the existence of such a control signal. These properties have no counterpart in the integer setting.

The last part of the paper is devoted to the solution of the disturbance decoupling problem by pseudostate feedback, with and without stability. To this end, we exploit our notion of controlled invariance and output-nulling subspaces to find necessary and sufficient solvability conditions. Interestingly, the input-output nature of the disturbance decoupling problem guarantees that our necessary and sufficient conditions are not constrained by our choice to use Caputo derivative and can be applied to systems with different fractional derivatives [32] and different initialization approaches [14, 30, 40, 35]. In particular, we show that the set of quadruples that solve the disturbance decoupling problem by pseudostate feedback with stability depends on the fractional order of differentiation. This follows from the stability domain and, as a consequence, the dimension of the associated stabilizability subspaces, being a function of the fractional order of differentiation.

Notation. Given a vector space $\mathscr{X}$, we denote by $0_{\mathscr{X}}$ the origin of $\mathscr{X}$. For notational convenience, we do not distinguish between a linear mapping from one finite-dimensional space to another and the corresponding matrix representation with respect to a particular basis. The image and kernel of matrix A are denoted by $\operatorname{im} A$ and $\ker A$, respectively. When A is square, we denote by $\sigma(A)$ the spectrum of A. If $A : \mathscr{X} \longrightarrow \mathscr{Y}$ is a linear map and $\mathscr{J} \subseteq \mathscr{X}$, then the restriction of the map A to $\mathscr{J}$ is denoted by $A\,|\,\mathscr{J}$. If $\mathscr{X} = \mathscr{Y}$ and $\mathscr{J}$ is a subspace of $\mathscr{X}$, we denote by $A\,\mathscr{J}$ the set $\{Ax\,|\,x \in \mathscr{J}\}$, and $\mathscr{J}$ is said to be A-invariant if $A\,\mathscr{J} \subseteq \mathscr{J}$. If $\mathscr{J}$ is A-invariant, then the eigenstructure of A restricted to $\mathscr{J}$ is denoted by $\sigma(A\,|\,\mathscr{J})$. If $\mathscr{J}_1$ and $\mathscr{J}_2$ are A-invariant subspaces and $\mathscr{J}_1 \subseteq \mathscr{J}_2$, then the mapping induced by A on the quotient space $\mathscr{J}_2/\mathscr{J}_1$ is denoted by $A\,|\,\mathscr{J}_2/\mathscr{J}_1$, and its spectrum is denoted by $\sigma(A\,|\,\mathscr{J}_2/\mathscr{J}_1)$. Given a map $A : \mathscr{X} \longrightarrow \mathscr{X}$ and a subspace $\mathscr{S}$ of $\mathscr{X}$, we let $\langle A\,|\,\mathscr{S}\rangle$ denote the smallest A-invariant subspace of $\mathscr{X}$ containing $\mathscr{S}$ and let $\langle \mathscr{S}\,|\,A\rangle$ denote the largest A-invariant subspace contained in $\mathscr{S}$. Given $z \in \mathbb{C}$, we denote by $\operatorname{Arg} z$ the principal argument of z, so that $\operatorname{Arg} z \in (-\pi, \pi]$. Finally, we denote by $\mathbb{R}_+$ the set of non-negative real numbers.

2. Preliminaries. Let $\alpha \in \mathbb{R}_+$ and $t_0 \in \mathbb{R}$. In this paper we adopt the following standard definition of fractional integral operator:

$$\mathscr{I}^\alpha f = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau,$$

see e.g. [32]. Different types of derivatives are used in the development of fractional calculus. The most common are the Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives [32]. In this paper we focus on the Caputo derivative

$$\mathscr{D}^\alpha f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_{t_0}^{t} \frac{f^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\, d\tau. \tag{2.1}$$

Unlike integer-order derivatives, the fractional derivative, irrespective of whether it is a Riemann-Liouville, Caputo, or Grünwald-Letnikov derivative, is a non-local operator because its value at a point t depends on past information on the function f between t_0 and t.

We now use the notion of fractional derivative introduced above to define fractional (commensurate) linear time-invariant systems. Let $\alpha \in (0,2)\setminus\{1\}$ be the order of differentiation. Consider three vector spaces $\mathscr{X} = \mathbb{R}^n$, $\mathscr{U} = \mathbb{R}^m$ and $\mathscr{Y} = \mathbb{R}^p$, which will be referred to as, respectively, the pseudostate space, the input space and the output space. Let $[t_0, +\infty)$ be the time horizon. We study in this paper pseudostate space multivariable commensurate order continuous-time systems in the form[1]

$$\begin{cases} \mathscr{D}^\alpha x(t) = Ax(t) + Bu(t) \\ \quad\; y(t) = Cx(t) + Du(t), \end{cases} \tag{2.2}$$

with the initial condition $x(t_0) = x_0^0$ if $\alpha \in (0,1)$, and $x(t_0) = x_0^0$ and $\dot{x}(t)|_{t=t_0^+} = x_0^1$ if $\alpha \in (1,2)$.[2] In what follows, we refer to α as the *commensurate order* of the system and we denote the initial condition in a compact way as

$$x^{(i)}(t_0) = x_0^i, \quad i \in \{0, \lfloor\alpha\rfloor\}. \tag{2.3}$$

In (2.2), $x : \mathbb{R} \longrightarrow \mathscr{X}$ is the pseudostate, $u : \mathbb{R} \longrightarrow \mathscr{U}$ is the input, and $y : \mathbb{R} \longrightarrow \mathscr{Y}$ is the output of the system. In the following, we restrict our attention to the class of piecewise continuous inputs. Note that this class of functions encompasses virtually every control signal used in practice and guarantees that, if a solution exists, the pseudostate trajectory is $\lceil\alpha\rceil$-times continuously differentiable, and satisfies the differential equation almost everywhere. The reason why we use the terminology pseudostate instead of state is that, by definition, the state should contain enough information to compute the future evolution of the system. For reasons that will become clearer later on, this is not the case when dealing with fractional systems; we refer the reader to [41, 37] for a detailed explanation.

The Caputo derivative has the important property of guaranteeing solution existence and uniqueness from a set of initial conditions that only involve integer derivatives as in (2.3) [24]. At first, this may appear to suggest that the solution can be defined by a finite set of initial conditions, contradicting the "state versus pseudostate" discussion above. However, this is only true when the initial conditions are expressed at the lower end-point of the integral in the fractional derivative operator (i.e., at $t = t_0$), or when the system has been at rest for its entire past history (i.e., for all $t < t_0$). In general, fractional systems are infinite dimensional and their future behavior depends on the entire past history.

REMARK 2.1. There is nothing to gain in choosing $t_0 \neq 0$. In fact, the results for $t_0 = 0$ can be easily adapted to the case $t_0 \neq 0$ by applying an obvious time-shift transformation. Thus, for the sake of readability, from now on we assume $t_0 = 0$.

[1] Note that in the fractional setting the pseudostate space system might not be a realization of higher order fractional differential equations. For this reason, we prefer to avoid the terminology "a state-space representation of a fractional system". A detailed discussion about this issue goes beyond the scope of this paper. We refer the reader to [38, 45] for a detailed explanation; see also [43].

[2] Values of $\alpha \geq 2$ are not considered because it can be shown that (2.2) is not asymptotically stable/stabilizable in this case.

A particular solution of system (2.2), obtained from initial conditions expressed at $t = 0$, can be expressed in terms of the so-called Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which involves a power series that converges for all $z \in \mathbb{C}$ whenever $\alpha, \beta \in \mathbb{R}_+$. For simplicity, we denote $E_{\alpha,1}(z)$ as $E_\alpha(z)$. For the response of the homogeneous system

$$\mathscr{D}^\alpha x(t) = A\,x(t), \tag{2.4}$$

we have the following two cases [32].

- If $\alpha \in (0,1)$, the initial conditions reduce to $x(0) = x_0^0$, and we can write

$$x(t) = E_\alpha(A\,t^\alpha)\,x_0^0. \tag{2.5}$$

- If $\alpha \in (1,2)$, the initial conditions are $x(0) = x_0^0$ and $x^{(1)}(0) = x_0^1$, and we can write the solution as

$$x(t) = E_\alpha(A\,t^\alpha)\,x_0^0 + E_{\alpha,2}(A\,t^\alpha)\,t\,x_0^1. \tag{2.6}$$

Note that, since $\Gamma(k+1) = k!$, function $E_{1,1}(z)$ is the exponential e^z. Consequently, (2.5) is consistent with the solution for a classical integer system when α tends to 1^-. The pseudostate impulse response of (2.2) is given by [24]

$$g(t) = t^{\alpha-1}\,E_{\alpha,\alpha}(A\,t^\alpha)\,B, \tag{2.7}$$

which coincides with the inverse Laplace transform of $(s^\alpha I - A)^{-1}$. By convolution, the pseudostate response of (2.2) from any input u, if it exists, can be written as

$$x(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(A\,(t-\tau)^\alpha\big)\,B\,u(\tau)\,d\tau. \tag{2.8}$$

Since the initial condition (2.3) is expressed at the same point as the lower end-point of the integration interval in the definition of the fractional derivative, this condition is sufficient to compute the future evolution of the pseudostate and output for a given input. However, as mentioned above, because the fractional derivative is non-local, in general the value of the pseudostate at a certain time instant $t^\star > t_0$ does not contain enough information on the past of the system to compute the future evolution from $t^\star$ onward. In other words, a boundary condition expressed as $x^{(i)}(t^\star) = x_0^i$ for $i \in \{0, \lfloor \alpha \rfloor\}$, with $t^\star > t_0$, is insufficient to guarantee solution uniqueness.

Therefore, the classical results in systems theory are not directly applicable to fractional systems defined by the Caputo derivative. In particular, the free evolution of the system cannot be computed by simply replacing the exponential with the Mittag-Leffler function. Despite this limitation, we will show that the Caputo derivative is a convenient starting point for developing intermediate results that eventually lead to solvability conditions for the disturbance decoupling problem; see Section 6. Such conditions are expressed in term of the system's matrices A, B, C and D, and entail input-output properties of the closed-loop transfer function. As such, they are independent from our initial choice of adopting the Caputo derivative, and can be applied to fractional systems defined using different derivatives and different initialization methods, including the initialization function [14] and the frequency distributed approach [45].

In what follows, in line with the above argument, whenever we talk about initial conditions, we assume that such conditions are expressed at the lower end-point $t_0 = 0$. We will deal with the problem of integrating the fractional differential equation from a point in time different from the lower integration end-point in Theorems 5.5 and 5.6. As it will be clear from those results, we will obtain conditions expressed in terms of the entire past trajectory of the pseudostate, in line with the fact that fractional operators are non-local (or infinite-dimensional).

3. Geometric foundations. In the integer case ($\alpha = 1$), a state trajectory of the system (2.2) lies on a subspace $\mathscr{L}$ of the state space $\mathscr{X}$ if and only if the initial state belongs to $\mathscr{L}$ and the first derivative $\dot{x}$ lies almost everywhere on $\mathscr{L}$ [6, Lemma 3.2.1]. This result is the foundation of the geometric approach to control theory, and we now generalize it to the case of fractional derivatives. Let $T > 0$. We denote by $x|_{[0,T]}$ the restriction of a solution of (2.2) to the closed interval $[0,T]$, i.e.,

$$\begin{aligned} x|_{[0,T]} : [0,T] &\longrightarrow \mathscr{X} \\ t &\mapsto x(t). \end{aligned}$$

THEOREM 3.1. *Let $T > 0$. Let $\mathscr{L}$ be a subspace of $\mathscr{X}$. Any pseudostate trajectory $x|_{[0,T]}$ lies in $\mathscr{L}$ if and only if*
(a) $x^{(i)}(0) \in \mathscr{L}, \quad i \in \{0, \lfloor \alpha \rfloor\}$*; and*
(b) $\mathscr{D}^\alpha x(t) \in \mathscr{L}$ *almost everywhere in* $t \in [0,T]$.

Proof: We denote by ν the dimension of $\mathscr{L}$. Let L be a matrix whose columns are a basis for $\mathscr{L}$ if $\nu > 0$, and let L be an $n \times 1$ matrix of zeros if $\nu = 0$.
(Only if). Suppose $x(t) \in \mathscr{L}$ for all $t \in [0,T]$. There exists a function $\xi : [0,T] \longrightarrow \mathbb{R}^\nu$ such that we can write $x(t) = L\xi(t)$ for all $t \in [0,T]$. By the definition of Caputo derivative we find

$$\mathscr{D}^\alpha \left(L\xi(t)\right) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t \frac{L\xi^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\, d\tau = L\left(\frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t \frac{\xi^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\, d\tau\right) \in \mathscr{L}.$$

Moreover, if $x(t) \in \mathscr{L}$ for all $t \in [0,T]$, then $x^{(i)}(0) = x(0) \in \mathscr{L}$. When $\alpha > 1$, the trajectory $x(t)$ is twice piecewise continuously differentiable, so that, in particular, the limit of the difference quotient exists everywhere. Since the difference quotient is in $\mathscr{L}$ and $\mathscr{L}$ is closed (being a subspace), it follows that the limit of the difference quotient is also in $\mathscr{L}$ so that that $x^{(1)}(0) \in \mathscr{L}$. Hence $x^{(i)}(0) \in \mathscr{L}$ for $i = 0$ when $\alpha < 1$ and for $i \in \{0,1\}$ when $\alpha > 1$.

(If). Suppose **(a)**-**(b)** hold. From **(b)**, there exists a function $\zeta : [0,T] \longrightarrow \mathbb{R}^\nu$ such that $\mathscr{D}^\alpha x(t) = L\zeta(t)$ almost everywhere in $[0,T]$. We apply the fractional integral operator $\mathscr{I}^\alpha$ to $\mathscr{D}^\alpha x(t)$, and then exploit a result in [11, p. 54] to obtain

$$\mathscr{I}^\alpha \left(\mathscr{D}^\alpha x(t)\right) = x(t) - \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\mathscr{D}^{(k)} x(t)}{k!}\Big|_{t=0} t^k. \tag{3.1}$$

The left-hand side is in $\mathscr{L}$ for all t since

$$\mathscr{I}^\alpha \left(\mathscr{D}^\alpha x(t)\right) = \mathscr{I}^\alpha \left(L\zeta(t)\right) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} L\zeta(\tau)\, d\tau = L\left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \zeta(\tau)\, d\tau\right) \in \mathscr{L}.$$

Furthermore, from **(a)**

$$\sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\mathscr{D}^{(k)} x(t)}{k!}\Big|_{t=0} t^k = \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{x^{(k)}(0)}{k!} t^k \in \mathscr{L},$$

so that (3.1) gives

$$x(t) = \mathscr{I}^\alpha \left(\mathscr{D}^\alpha x(t)\right) + \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{\mathscr{D}^{(k)} x(t)}{k!}\Big|_{t=0} t^k \in \mathscr{L}.$$

■

Notice that in the proof of the necessity part of Theorem 3.1 we obtain a condition that is stronger than the "almost everywhere" property **(b)** of $\mathscr{D}^\alpha x(t)$. In other words, in one direction the statement of Theorem 3.1 can be strengthened as follows.

COROLLARY 3.2. *Let $T > 0$ and suppose that the pseudostate trajectory $x|_{[0,T]}$ lies in $\mathscr{L}$. Then*

(a) $x^{(i)}(0) \in \mathscr{L}, \quad i \in \{0, \lfloor\alpha\rfloor\}$;
(b) $\mathscr{D}^{\alpha} x(t) \in \mathscr{L}$ *for all* $t \in [0,T]$.

Let us consider the homogeneous system (2.4). In the integer case, it is well-known and easy to prove that the solutions of the homogeneous system lie on A-invariant subspaces. In particular, if the initial pseudostate lies on an A-invariant subspace $\mathscr{J}$, then the entire trajectory evolves in $\mathscr{J}$. This result is generalized to fractional systems as follows.

THEOREM 3.3. *Let x be a solution of (2.4) and let* $\mathscr{J}$ *be a subspace of* $\mathscr{X}$. *The following two statements are equivalent:*
(a) *If* $x^{(i)}(0) \in \mathscr{J}$ *for each* $i \in \{0, \lfloor\alpha\rfloor\}$, *then* $x(t) \in \mathscr{J}$ *for every* $t \in \mathbb{R}_+$;
(b) $A\mathscr{J} \subseteq \mathscr{J}$.

Proof: We first prove that **(b)** implies **(a)**. Assume that $A\mathscr{J} \subseteq \mathscr{J}$, and that for each $i \in \{0, \lfloor\alpha\rfloor\}$ there holds $x^{(i)}(0) \in \mathscr{J}$. Consider first the case $\alpha \in (0,1)$. From (2.5) we find

$$x(t) = \frac{1}{\Gamma(1)}\, x_0^0 + \frac{A\,t^{\alpha}}{\Gamma(\alpha+1)}\, x_0^0 + \frac{A^2\,t^{2\alpha}}{\Gamma(2\alpha+1)}\, x_0^0 + \dots.$$

From **(b)**, we have $A^k x_0^0 \in \mathscr{J}$ for any integer k. Hence, each term in the sum is given by the product of a scalar and a vector in $\mathscr{J}$ for every $t \in \mathbb{R}_+$, and, since $\mathscr{J}$ is a subspace and therefore also a closed set, the sum defining $x(t)$ is also in $\mathscr{J}$.

We now consider $\alpha \in (1,2)$. From (2.6) we find

$$x(t) = \frac{1}{\Gamma(1)}\left(x_0^0 + t\,x_0^1\right) + \frac{A\,t^{\alpha}}{\Gamma(\alpha+1)}\left(x_0^0 + \frac{t}{\alpha+2}\,x_0^1\right) + \frac{A^2\,t^{2\alpha}}{\Gamma(2\alpha+1)}\left(x_0^0 + \frac{t}{2\alpha+2}\,x_0^1\right) + \dots$$

Again, each term in the sum lies in $\mathscr{J}$, and this implies that $x(t) \in \mathscr{J}$.

We now prove that **(a)** implies **(b)**. Let $x^{(i)}(0) \in \mathscr{J}$ for each $i \in \{0, \lfloor\alpha\rfloor\}$, so that $x(t) \in \mathscr{J}$ for every $t \in \mathbb{R}_+$. It follows that $x(t) \in \mathscr{J}$ for every $t \in [0,T]$ and $T > 0$. Thus, from Theorem 3.1, we obtain $\mathscr{D}^{\alpha} x(t) \in \mathscr{J}$ almost everywhere, which, evaluated at $t = 0$, yields $A\,x_0^0 \in \mathscr{J}$. In view of the arbitrariness of x_0^0, we obtain $A\mathscr{J} \subseteq \mathscr{J}$. ■

Although Theorem 3.3 appears to be a natural generalization of a well-known result in the integer case [6, Lemma 3.2.1], here a unique solution cannot be guaranteed from an initial condition specified at non-zero initial times $t^{\star} > 0$, and this weakens the result of the theorem. Indeed, the property of A-invariant subspaces being locii of trajectories of the homogeneous system is maintained in the fractional case only when we consider the initial conditions defined at $t = 0$, i.e., at the lower terminal of the fractional operator. In view of Theorem 3.3 one would expect that if the pseudostate trajectory has evolved from $t = 0$ to a given time $t^{\star}$ on an A-invariant subspace, then it will continue to evolve on the same A-invariant subspace from $t^{\star}$ onward. This is indeed the case as we prove in Section 5 in the more general context of controlled invariance. In fact, the converse implication is also true, and this issue will also be examined in Section 5.

We conclude this section with a result on A-invariance that will be used later. The proof is straightforward and is omitted.

LEMMA 3.4. *Let* $\mathscr{J}$ *be an A-invariant subspace. Consider the system (2.2) with* $x^{(i)}(0) \in \mathscr{J}$ *for* $i \in \{0, \lfloor\alpha\rfloor\}$. *If* $u(t) \in B^{-1}\mathscr{J}$ *for all* $t \geq 0$, *then* $x(t) \in \mathscr{J}$ *for all* $t \geq 0$.[3]

4. Reachable subspace. Reachability and observability have been extensively studied in the fractional setting, [24, 23, 8, 2, 3, 37]. In this section we re-examine reachability from a geometric perspective, with the aim of characterizing reachable pseudostates in terms of subspaces. In particular, we focus on geometrically characterizing the set of pseudostates reachable in finite time from zero initial conditions, under the assumption that the system is at rest for the entire past history. The results developed in this section are therefore related to the input-to-pseudostate properties of the system, and as such are independent of the fractional derivative definition. Note

[3]Note that $B^{-1}\mathscr{V}$ denotes the inverse image of $\mathscr{V}$ through B (interpreted as a function by multiplication), which is always defined.

that the mathematical notion of reachability developed hereafter will be instrumental to the controlled invariance theory for fractional systems, but it might not reflect physical properties of a real system modeled using fractional differential equations.

Most of the approaches for dealing with reachability/controllability issues in the fractional case hinge on obtaining an expression for the controllability Gramian by replacing e^{At} with $t^{\alpha-1}E_{\alpha,\alpha}(A\,t^{\alpha})$, and, mimicking the integer case, defining

$$W_t = \int_0^t (t-\tau)^{2(\alpha-1)} E_{\alpha,\alpha}(A\,(t-\tau)^{\alpha})\,BB^\top E_{\alpha,\alpha}(A^\top(t-\tau)^{\alpha})\,d\tau. \tag{4.1}$$

It was first noted in [23], and several times after (see e.g. [8]), that this definition of Gramian presents convergence issues at $t=\tau$, which justified the introduction of other Gramian definitions that include the factor $(t-\tau)^{2(1-\alpha)}$, called the *neutralizer*, in the integrand function to guarantee convergence; see [23].

In this paper, our focus is not on obtaining a test for complete reachability in the fractional case, but rather on giving a geometric characterization of the subspace of pseudostates that are reachable from zero initial conditions, because this will underpin the definition of inner and outer stabilizable controlled invariant subspaces via the introduction of reachability subspaces. For this purpose, we take a step back, and compute the controllability Gramian with respect to a different inner product. First, we denote by $\mathscr{L}^2_\alpha[0,t]$ the set of functions $\omega:[0,t]\longrightarrow\mathbb{R}^m$ such that the Lebesgue integral

$$\int_0^t (t-\tau)^{\alpha-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau$$

is finite. The set $\mathscr{L}^2_\alpha[0,t]$ is a Hilbert space with inner product

$$\langle u,v\rangle_{\mathscr{L}^2_\alpha[0,t]} = \int_0^t (t-\tau)^{\alpha-1}\,u^\top(\tau)\,v(\tau)\,d\tau.$$

It is easy to see that the inner product properties hold: it is bilinear (from the linearity of the integral), symmetric, and positive semi-definite:

$$\langle v,v\rangle_{\mathscr{L}^2_\alpha[0,t]} = \int_0^t (t-\tau)^{\alpha-1}\,v^\top(\tau)\,v(\tau)\,d\tau \geq 0$$

because $(t-\tau)^{\alpha-1}>0$ almost everywhere in $[0,t]$. Moreover $\langle v,v\rangle_{\mathscr{L}^2_\alpha[0,t]}=0$ if and only if $v=0$ almost everywhere. The next lemma provides a characterization of the space $\mathscr{L}^2_\alpha[0,t]$.

LEMMA 4.1. *For $0<\alpha<\beta<2$, we have*

$$\mathscr{L}^2_\alpha[0,t]\subseteq\mathscr{L}^2_\beta[0,t]. \tag{4.2}$$

Proof: Let $\omega\in\mathscr{L}^2_\alpha[0,t]$, so that $\langle\omega,\omega\rangle_{\mathscr{L}^2_\alpha[0,t]}=\int_0^t(t-\tau)^{\alpha-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau$ exists. We prove that $\langle\omega,\omega\rangle_{\mathscr{L}^2_\beta[0,t]}$ exists for every $\beta>\alpha$. We consider first the case $t\geq 1$, where we have

$$\langle\omega,\omega\rangle_{\mathscr{L}^2_\alpha[0,t]} = \int_0^{t-1}(t-\tau)^{\alpha-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau + \int_{t-1}^t (t-\tau)^{\alpha-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau \tag{4.3}$$

and

$$\langle\omega,\omega\rangle_{\mathscr{L}^2_\beta[0,t]} = \int_0^{t-1}(t-\tau)^{\beta-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau + \int_{t-1}^t (t-\tau)^{\beta-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau. \tag{4.4}$$

Since the first integral on the right-hand side of (4.3) converges, and the function $(t-\tau)^{\alpha-1}$ is bounded and greater than or equal to a positive number (1, if $\alpha<1$, and t^{α}, if $\alpha\geq 1$), for every $\tau\in[0,t-1]$, the integral $\int_0^{t-1} g(\tau)\,\omega^\top(\tau)\,\omega(\tau)\,d\tau$ converges for any bounded function $g(\tau)$. The convergence of the first integral on the right-hand side of (4.4) follows by choosing $g(\tau) = (t-\tau)^{\beta-1}$. The convergence of the second integral on the right-hand side of (4.4) follows by noting that $(t-\tau)^{\alpha-1} > (t-\tau)^{\beta-1}$ for all $\tau \in (t-1,t)$, which implies that $(t-\tau)^{\alpha-1}\omega^\top(\tau)\,\omega(\tau) > (t-\tau)^{\beta-1}\omega^\top(\tau)\,\omega(\tau)$, and therefore $\int_{t-1}^t (t-\tau)^{\alpha-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau > \int_{t-1}^t (t-\tau)^{\beta-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau$. The proof for the case $t<1$ is obtained in the same way by noting that, if $t<1$, then $(t-\tau)^{\alpha-1}\omega^\top(\tau)\,\omega(\tau) > (t-\tau)^{\beta-1}\omega^\top(\tau)\,\omega(\tau)$ for all $\tau \in [0,t)$, so that $\langle \omega,\omega\rangle_{\mathscr{L}^2_\alpha[0,t]} = \int_0^t (t-\tau)^{\alpha-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau > \int_0^t (t-\tau)^{\beta-1}\,\omega^\top(\tau)\,\omega(\tau)\,d\tau = \langle \omega,\omega\rangle_{\mathscr{L}^2_\beta[0,t]}$. ■

For our discussion on reachability, we will consider control functions $u \in \mathscr{L}^2_\alpha[0,t]$. Note that, when $\alpha>1$, this choice is not restrictive since it includes all of the classical input functions in $\mathscr{L}^2[0,t]$. However, when $\alpha<1$, it follows from Lemma 4.1 with $\beta=1$ that $\mathscr{L}^2_\alpha[0,t] \subseteq \mathscr{L}^2[0,t]$, and hence $\mathscr{L}^2_\alpha[0,t]$ may not contain all of the classical control functions. Without the restriction to $\mathscr{L}^2_\alpha[0,t]$, however, the system may not admit a solution because the convolution $g*u$ might not converge. Consider, for instance, an input signal of the form $u(t) = (T-t)^{-0.4}$, where $T>0$ is a parameter, and consider the fractional system $\mathscr{D}^{0.3}x(t) = u(t)$. Then, $\alpha = 0.3$ and $u \in \mathscr{L}^2[0,t] \setminus \mathscr{L}^2_\alpha[0,t]$. From (2.8), the convolution $g*u$ in this case is easily seen to be

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,u(\tau)\,d\tau.$$

Applying this equation to find the solution of the system at time $t=T$, gives

$$x(T) = \frac{1}{\Gamma(\alpha)}\int_0^T (T-\tau)^{-1.1}\,d\tau,$$

which does not converge even though u is clearly in $\mathscr{L}^2[0,t]$.

The following result shows that this situation does not occur if we select the input from $\mathscr{L}^2_\alpha[0,t]$.

LEMMA 4.2. *For $\alpha \in (0,2)$, let $u \in \mathscr{L}^2_\alpha[0,t]$ be any input to the system* (2.2)*. Then, the convolution integral in* (2.8) *is finite.*

Proof: To show that (2.2) admits a solution on $[0,t]$ it suffices to prove that the integral in (2.8) converges. This immediately follows by noting that the function $v(t) = E_{\alpha,\alpha}\big(A\,(t-\tau)^\alpha\big)\,B$ is in $\mathscr{L}^2_\alpha[0,t]$, and that (2.8) can be rewritten as

$$x(t) = \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(A\,(t-\tau)^\alpha\big)Bu(\tau)d\tau = \langle v,u\rangle_{\mathscr{L}^2_\alpha[0,t]}.$$

■

Denoting by $\bar{x}$ the pseudostate reached in the interval $[0,t]$ using the control function $u \in \mathscr{L}^2_\alpha[0,t]$ and the impulse response $g(t)$, we have

$$\bar{x}(t) = \int_0^t \underbrace{(t-\tau)^{\alpha-1}\,E_{\alpha,\alpha}\big(A\,(t-\tau)^\alpha\big)\,B}_{M(\tau)}\,u(\tau)\,d\tau.$$

We define the operator

$$\rho : \mathscr{L}^2_\alpha[0,t] \longrightarrow \mathscr{X}, \quad u \mapsto \int_0^t M(\tau)\,u(\tau)\,d\tau.$$

We recall that, by using $\langle\cdot,\cdot\rangle_{\mathscr{L}^2_\alpha[0,t]}$ and $\langle\cdot,\cdot\rangle_{\mathscr{X}}$ to denote the inner products in $\mathscr{L}^2_\alpha[0,t]$ and $\mathscr{X}$, respectively, the adjoint ρ^* of ρ is the unique operator $\rho^* : \mathscr{X} \longrightarrow \mathscr{L}^2_\alpha[0,t]$ such that

$$\langle \rho\,u,x\rangle_{\mathscr{X}} = \langle u,\rho^* x\rangle_{\mathscr{L}^2_\alpha[0,t]}.$$

With respect to these inner products, the adjoint ρ^* is the following function

$$\begin{aligned}\rho^* : \mathscr{X} &\longrightarrow \mathscr{L}^2_\alpha[0,t],\\ x &\mapsto B^\top E_{\alpha,\alpha}\big(A^\top(t-\tau)^\alpha\big)\,x.\end{aligned}$$

Indeed,

$$\langle \rho\, u, x\rangle_{\mathscr{X}} = \Big(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-\tau)^{\alpha}\big) B u(\tau) d\tau\Big)^{\top} x = \int_0^t (t-\tau)^{\alpha-1} u^{\top}(\tau) B^{\top} E_{\alpha,\alpha}\big(A^{\top}(t-\tau)^{\alpha}\big) x d\tau = \langle u, \rho^* x\rangle_{\mathscr{L}^2_{\alpha}[0,t]}.$$

We obtain the Gramian as

$$\rho\,\rho^* = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(A\,(t-\tau)^{\alpha}\big)\, B B^{\top} E_{\alpha,\alpha}\big(A^{\top}(t-\tau)^{\alpha}\big)\, d\tau.$$

This operator is well defined for all $\alpha > 0$ because it is bounded for all $\alpha > 0$. Since $E_{\alpha,\alpha}\big(A\,(t-\tau)^{\alpha}\big)\,B$ is a function in $\mathscr{L}^2_{\alpha}[0,t]$, the operator ρ^* is indeed an adjoint, and therefore it satisfies the fundamental property of adjoint operators $\mathrm{im}(\rho) = \mathrm{im}(\rho\,\rho^*)$ [42, Lemma 3.5.2], so that the image of our Gramian is exactly the subspace of reachable pseudostates from zero initial conditions.

REMARK 4.1. One may wonder where the need to consider the linear operator ρ using $\mathscr{L}^2_{\alpha}[0,t]$ comes from. In fact, we could potentially define $\rho : \mathscr{L}^2[0,t] \longrightarrow \mathscr{X}$. However, the adjoint operator is ill-defined with respect to the standard inner product in $\mathscr{L}^2[0,t]$. In fact, to satisfy the identity $\langle \rho\, u, x\rangle_{\mathscr{X}} = \langle u, \rho^* x\rangle_{\mathscr{L}^2[0,t]}$, the operator ρ^* would be defined by

$$x \mapsto (t-\tau)^{\alpha-1} B^{\top} E_{\alpha,\alpha}\big(A^{\top}\,(t-\tau)^{\alpha}\big)x,$$

which is a map from $\mathscr{X}$ to $\mathscr{L}^2_{\alpha}[0,t]$, and not $\mathscr{L}^2[0,t]$ as required to be a well-defined adjoint for ρ. As a consequence, the fundamental property $\mathrm{im}\,\rho = \mathrm{im}\,\rho\rho^*$ would not hold in general, and in particular when $\alpha \le 0.5$, so that the resulting Gramian operator $W_t = \rho\rho^*$ would not be bounded for all $\alpha \le 0.5$ because of the term $(t-\tau)^{2(\alpha-1)}$ in (4.1).

We recall that $\ker \rho^* = (\mathrm{im}\,\rho)^{\perp}$ [42, Lemma 3.5.2], which generalizes the well-known subspace identity for finite dimensional vector spaces. We now exploit this property to show that the set of pseudostates reachable from zero initial conditions in the interval $[0,t]$ is the image of the matrix $[\; B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B \;]$. In other words, we will prove that

$$\mathrm{im}\,\rho = \mathrm{im}[\; B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B \;],$$

or, equivalently,

$$(\mathrm{im}\,\rho)^{\perp} = \big(\mathrm{im}[\; B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B \;]\big)^{\perp}.$$

This equation is equivalent to

$$x \in \ker \rho^* \quad \Leftrightarrow \quad x \in \ker \begin{bmatrix} B^{\top} \\ B^{\top}A^{\top} \\ \vdots \\ B^{\top}(A^{\top})^{n-1} \end{bmatrix}.$$

We proceed as follows: by definition of ρ^* we have

$$\begin{aligned} x \in \ker \rho^* \quad &\Leftrightarrow \quad B^{\top} E_{\alpha,\alpha}(A^{\top}\,(t-\tau)^{\alpha})\,x = 0, \quad \forall\, \tau \in [0,t] \\ &\Leftrightarrow \quad \sum_{k=0}^{\infty} B^{\top} \frac{(A^{\top})^k\,(t-\tau)^{\alpha k}}{\Gamma(\alpha k+\alpha)}\,x = 0, \quad \forall\, \tau \in [0,t] \\ &\Leftrightarrow \quad \sum_{k=0}^{\infty} B^{\top}\,(A^{\top})^k\,x\,\frac{(t-\tau)^{\alpha k}}{\Gamma(\alpha k+\alpha)} = 0, \quad \forall\, \tau \in [0,t]. \end{aligned}$$

Applying the principle of identity for fractional formal power series, $B^{\top}\,(A^{\top})^k\,x = 0$ for all $k \in \mathbb{N}$. In view of the Cayley-Hamilton theorem this is equivalent to the condition $B^{\top}\,(A^{\top})^k\,x = 0$ for all $k \in \{0,\dots,n-1\}$. Thus,

$x \in \ker \begin{bmatrix} B^\top \\ B^\top A^\top \\ \vdots \\ B^\top (A^\top)^{n-1} \end{bmatrix}$ as required, proving that the set of reachable pseudostates from zero initial conditions in the interval $[0,t]$ is the range of the matrix $[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{n-1}B \end{array}]$.

We stress that this result goes beyond obtaining a reachability test that hinges on the rank of the matrix $[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{n-1}B \end{array}]$. In fact, showing that the set of reachable points is indeed the image of this matrix implies that such set is a *subspace* of $\mathscr{X}$, and it is independent of t (provided $t \neq 0$). Thus, as in the integer case, we can characterize such a subspace geometrically in terms of the system matrices as the smallest A-invariant subspace containing the range of B, see e.g. [44, Corollary 3.3], in symbols $\mathscr{R} = \langle A \,|\, \operatorname{im} B\rangle$. We can therefore denote the subspace of reachable pseudostates from zero initial conditions as $\mathscr{R}$ (i.e., without specifying the end-points of the time interval), and in line with what is well known for the integer case, the system is completely reachable if and only if the rank of $[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{n-1}B \end{array}]$ is equal to n, or, geometrically, if and only if $\mathscr{R} = \mathscr{X}$.

5. Controlled invariance. We now introduce the key concept of controlled invariance.

DEFINITION 5.1. *Consider (2.2). The subspace $\mathscr{V}$ of $\mathscr{X}$ is (A,B)-controlled invariant if, for all $x^{(i)}(0) \in \mathscr{V}$ with $i \in \{0, \lfloor\alpha\rfloor\}$, there exists a control function u such that the corresponding solution $x(t)$ satisfies $x(t) \in \mathscr{V}$ for all $t \geq 0$.*

As with A-invariant subspaces, this notion of controlled invariance is inherently weaker for fractional systems than for the integer systems. Indeed, in an integer system the initial conditions are usually expressed in terms of the initial time $t^\star = 0$, but nothing changes if we express the evolution of the system starting from initial conditions at a generic time $t^\star \neq 0$. By contrast, for fractional systems, the initial condition must be defined at the initial time that corresponds to the lower integration bound $t^\star = t_0$ of the convolution integral defining the Caputo derivative in (2.1).[4] We will show in this section that if we want to characterize the evolution of the fractional system from a certain time $t^\star > 0$ onward, then we will need to provide a condition on the complete past evolution of the system between $t_0 = 0$ and $t^\star$.

The following theorem extends well-known fundamental results of geometric control to fractional systems: it gives two equivalent characterizations of controlled invariance, one in terms of a geometric inclusion and the other in terms of pseudostate feedback.

THEOREM 5.2. *Let $\mathscr{V}$ be a subspace of $\mathscr{X}$. The following statements are equivalent:*
(a) *$\mathscr{V}$ is (A,B)-controlled invariant;*
(b) *$A\mathscr{V} \subseteq \mathscr{V} + \operatorname{im} B$;*
(c) *There exists $F \in \mathbb{R}^{m\times n}$ such that $(A+BF)\mathscr{V} \subseteq \mathscr{V}$.*

Proof: We prove **(a)** $\Rightarrow$ **(b)**. Let $\mathscr{V}$ be (A,B)-controlled invariant. Consider a control input u such that $x(t) \in \mathscr{V}$ for all $t \geq 0$. From Theorem 3.1, we have $\mathscr{D}^\alpha x(t) \in \mathscr{V}$ almost everywhere and $x^{(i)}(0) \in \mathscr{V}$ for $i \in \{0, \lfloor\alpha\rfloor\}$ for all compact intervals $[0,T]$ (with $T > 0$). Thus,

$$\underbrace{A\,x(t)|_{t=0}}_{A\,x(0)} = \underbrace{\mathscr{D}^\alpha x(t)|_{t=0}}_{\in\mathscr{V}} - \underbrace{B\,u(t)|_{t=0}}_{\in\operatorname{im}B}.$$

It follows that $A\,x(0) \in \mathscr{V} + \operatorname{im} B$. In view of the arbitrariness of $x(0)$, **(b)** follows.
The proof **(b)** $\Rightarrow$ **(c)** can be carried out exactly as in the proof of the corresponding point of [44, Theorem 4.2].
We prove **(c)** $\Rightarrow$ **(a)**. In view of Theorem 3.3 applied to the system $\mathscr{D}^\alpha x(t) = (A+BF)\,x(t)$, if $x^{(i)}(0) \in \mathscr{V}$ for each $i \in \{0, \lfloor\alpha\rfloor\}$, then $x(t) \in \mathscr{V}$ for all $t \in \mathbb{R}_+$. Thus, $u(t) = F\,x(t)$ with these initial conditions is such that $x(t) \in \mathscr{V}$ for every $t \in \mathbb{R}_+$. We conclude that $\mathscr{V}$ is (A,B)-controlled invariant. ■

Given a controlled invariant subspace $\mathscr{V}$ of $\mathscr{X}$, any matrix F satisfying $(A+BF)\mathscr{V} \subseteq \mathscr{V}$ is referred to, as in the integer case, as a *friend* of $\mathscr{V}$. A consequence of Theorem 5.2 is that the control function u that maintains the trajectory on $\mathscr{V}$ can always be expressed as a static pseudostate feedback $u = F\,x$ (where F is a friend of $\mathscr{V}$),

[4] Recall that in this paper we have assumed without loss of generality $t_0 = 0$.

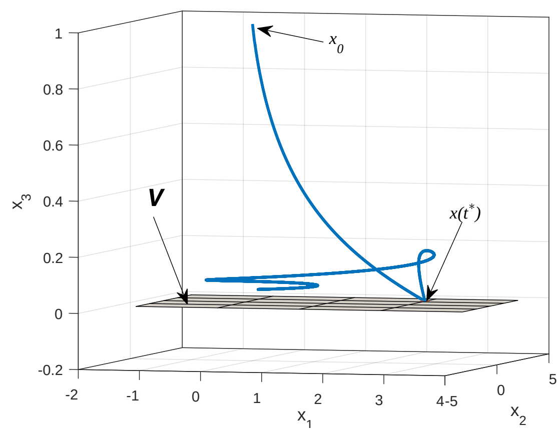

FIG. 5.1. *Pseudostate trajectory for Example 5.1.*

provided that the initial conditions are in $\mathscr{V}$. However, in contrast with the integer case, the initial condition must be defined at $t^\star = t_0 = 0$. In fact, even if $x(t^\star) \in \mathscr{V}$ for $t^\star > 0$ (and $\dot{x}(t)|_{t=t^\star} \in \mathscr{V}$ if $\alpha > 1$), it still may not be possible to maintain the pseudostate trajectory on $\mathscr{V}$ from $t^\star$ onward by using a pseudostate feedback, as the example below shows.

EXAMPLE 5.1. Consider the completely reachable system described by the matrices

$$A = \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}, \; B = \begin{bmatrix} 1/2 & 1/2 \\ 1 & -1 \\ -1/2 & 1/2 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

with $\alpha = 0.6$. Using **(b)** in Theorem 5.2, it is easy to see that $\mathscr{V} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a controlled invariant subspace, and that a corresponding friend is $F = \begin{bmatrix} 0 & 2 & 4/3 \\ 1 & 1 & -11/3 \end{bmatrix}$. Consider the initial pseudostate $x_0^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We consider the control function $u(t) = F\,x(t) + \begin{bmatrix} v_1(t) \\ 0 \end{bmatrix}$, where $v_1(t)$ is given by

$$v_1(t) = \begin{cases} 2 & \text{if } t \leq t^\star \\ 0 & \text{if } t > t^\star, \end{cases}$$

where $t^\star$ is such that $x(t^\star) \in \mathscr{V}$. Note that $u(t)$ is a pure pseudostate feedback for $t > t^\star$. Also note that $t^\star$ is well defined because $v_1(t) = 2$ brings the third component of the pseudostate variable to zero (i.e., it brings the pseudostate to $\mathscr{V}$) in finite time (which we define as $t^\star$) for any initial pseudostate with a positive third component. For an integer system, the pseudostate is guaranteed to evolve on $\mathscr{V}$ from $t^\star$ onward. By contrast, in the fractional setting the evolution of the pseudostate is determined by the past trajectory from 0 to $t^\star$, and therefore the pseudostate leaves $\mathscr{V}$ even if $x(t^\star) \in \mathscr{V}$ and $u(t) = F\,x(t)$ for $t \geq t^\star$ where F is a friend of $\mathscr{V}$; see Figure 5.1. Later in Theorem 5.5 we show that for any friend of $\mathscr{V}$, not just the particular F used here, the closed-loop system will exhibit this behavior. ■

The behavior exhibited by the pseudostate trajectory in Example 5.1 is a direct consequence of the inherent limitation that we will encounter when trying to compute the future evolution of a fractional system from finite information expressed at a single point in time [36]. In the rest of this Section, we will show how controlled invariance can be extended to take into account the past evolution of the pseudostate trajectory, and we will also outline how to extend this theory to systems not necessarily defined using the Caputo derivative.

The following resultis a simple generalization of [44, Theorem 4.3].

LEMMA 5.3. *Let $\mathscr{V}$ be a controlled invariant subspace. Let F be a friend of $\mathscr{V}$. Let u be a control function. The pseudostate trajectory obtained from $x^{(i)}(0) \in \mathscr{V}$ with $i \in \{0, \lfloor\alpha\rfloor\}$ and $x(t)$ remains in $\mathscr{V}$ for all $t \in [0,T]$ if and only if u takes the form*

$$u(t) = F\,x(t) + v(t), \tag{5.1}$$

where $v(t) \in B^{-1}\mathscr{V}$ for almost all $t \in [0,T]$.

Proof: **(Only if).** Suppose that the pseudostate trajectory arising from the control function u lies in $\mathscr{V}$. Let $v = u - F\,x$. Thus, u can be written as in (5.1). We now show that $v(t) \in B^{-1}\mathscr{V}$ for almost all $t \in [0,T]$. Since $x(t) \in \mathscr{V}$ for all $t \in [0,T]$, then, from Theorem 3.1, we have that $\mathscr{D}^\alpha x(t) \in \mathscr{V}$ almost everywhere in $[0,T]$. Applying u to (2.2) we obtain the closed-loop equation

$$\mathscr{D}^\alpha x(t) = (A + BF)x(t) + B\,v(t).$$

Since $\mathscr{V}$ is a controlled invariant subspace, F is a friend of $\mathscr{V}$ and $x(t) \in \mathscr{V}$ for all $t \in [0,T]$, then $(A+BF)x(t) \in \mathscr{V}$ for all $t \in [0,T]$. Hence, $v(t) \in B^{-1}\mathscr{V}$ for almost all $t \in [0,T]$. **(If).** The opposite implication is obvious and follows along the same lines. ■

We will show that, from the definition of controlled invariance, $x(t^\star) \in \mathscr{V}$ for $t^\star > 0$ implies that the pseudostate can only be maintained in $\mathscr{V}$ if the input and initial conditions are such that $x(t) \in \mathscr{V}$ for all $t \in [0, t^\star]$.[5] This behavior has no counterpart in the integer case, where we can always evolve on a controlled invariant from a time $t^\star > 0$ onward, irrespective of the past trajectory of the pseudostate up to $t^\star$.

The following results formalize the previous point. We first prove a preliminary result.

LEMMA 5.4. *Let $p \in \mathbb{N}$. Let $\xi : [t_1,t_2] \longrightarrow \mathbb{R}^n$ be differentiable p times such that $\xi^{(p)}$ is piecewise continuous. Let $\mathscr{L}$ be a subspace of $\mathbb{R}^n$. The following statements are equivalent:*

(a) *$\xi(t) \in \mathscr{L}$ for all $t \in [t_1,t_2]$;*

(b) *$\xi^{(p)}(t) \in \mathscr{L}$ for $t \in [t_1,t_2]$ except for a finite number of points, and there exists $\bar{t} \in [t_1,t_2]$ such that $\xi^{(i)}(\bar{t}) \in \mathscr{L}$ for each $i \in \{0,\dots,p-1\}$.*

Proof: The implication **(a)** $\Rightarrow$ **(b)** is trivial. We show **(b)** $\Rightarrow$ **(a)**. The implication is obvious if $\mathscr{L} = \mathbb{R}^n$. Thus, we consider the case $\mathscr{L} \subset \mathbb{R}^n$. Let Y denote a basis matrix for $\mathscr{L}^\perp$. From $\xi^{(p)}(t) \in \mathscr{L}$ almost everywhere in $[t_1,t_2]$, we have $Y^\top \xi^{(p)}(t) = 0$ almost everywhere in $[t_1,t_2]$, which is equivalent to

$$Y^\top \int_{t'}^{t} \xi^{(p)}(\tau)\,d\tau = 0 \tag{5.2}$$

for all $t,t' \in [t_1,t_2]$, and (5.2) holds in particular for $t' = \bar{t}$. It follows that

$$Y^\top \int_{\bar{t}}^{t} \xi^{(p)}(\tau)\,d\tau = Y^\top\big(\xi^{(p-1)}(t) - \xi^{(p-1)}(\bar{t})\big) = 0$$

for all $t \in [t_1,t_2]$. This, combined with the condition $\xi^{(i)}(\bar{t}) \in \mathscr{L}$ for all $i \in \{0,\dots,p-1\}$, implies that $Y^\top \xi^{(p-1)}(t) = 0$, which in turn implies $\xi^{(p-1)}(t) \in \mathscr{L}$ for all $t \in [t_1,t_2]$. We repeat the same argument p times to obtain $\xi(t) \in \mathscr{L}$ for all $t \in [t_1,t_2]$. ■

THEOREM 5.5. *Let $\mathscr{V}$ be a controlled invariant subspace and let $x^{(i)}(t^\star) \in \mathscr{V}$ for $t^\star > 0$ and $i \in \{0, \lfloor\alpha\rfloor\}$. There exists a pseudostate feedback control such that $x(t) \in \mathscr{V}$ for all $t > t^\star$ if and only if $x(t) \in \mathscr{V}$ for almost all $t \le t^\star$.*

Proof: **(If).** Let F be a friend of $\mathscr{V}$. From the linearity of the fractional derivative we obtain

$$\frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_{t^\star}^{t} \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\,d\tau = (A+BF)\,x(t) - \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\,I(t), \tag{5.3}$$

[5]Even when $\alpha > 1$, here we do not require the derivative to be in $\mathscr{V}$ at $t = t^\star$ since this condition is automatically implied by the fact that the trajectory is contained in $\mathscr{V}$ for all $t \le t^\star$. This is the case even if the control function has some points of discontinuity: these generate a pointwise discontinuity in the α-th derivative, which is not reflected in a discontinuity of the first derivative.

for all $t > t^\star$, where

$$I(t) = \int_0^{t^\star} \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\, d\tau.$$

Since $x(t) \in \mathscr{V}$ for all $t \in [0,t^*]$, by using Lemma 5.4, it is immediate to see that $x^{(\lceil\alpha\rceil)}(t) \in \mathscr{V}$ for almost all $t \in [0,t^\star]$, and therefore $I(t) \in \mathscr{V}$ for all $t \geq t^*$. Equation (5.3) is a fractional differential equation where the fractional derivative is defined from $t_0 = t^\star$, with initial condition at t_0 in $\mathscr{V}$, the system matrix is $(A+BF)$ so that $\mathscr{V}$ is an invariant subspace, and the input matrix is the identity map. Since $I(t) \in \mathscr{V}$ for all $t \geq 0$, we can apply the result of Lemma 3.4 (choosing B of Lemma 3.4 to be the identity matrix) to find that the solution is in $\mathscr{V}$ for all $t \geq t^*$.

(Only if). There exists a unique decomposition of the pseudostate as $x(t) = x_{\mathscr{V}}(t) + x_\perp(t)$, where, for all $t \leq t^\star$, $x_{\mathscr{V}}(t) \in \mathscr{V}$ and $x_\perp(t) \in \mathscr{V}^\perp$. From the linearity of the fractional derivative we obtain

$$\mathscr{D}^\alpha x(t) = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int_0^t \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\, d\tau = \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \big(I_1(t)+I_2(t)+I_3(t)\big)$$

for all $t > t^\star$, where

$$I_1(t) = \int_0^{t^\star} \frac{x_{\mathscr{V}}^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\, d\tau, \quad I_2(t) = \int_0^{t^\star} \frac{x_\perp^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\, d\tau, \quad I_3(t) = \int_{t^\star}^{t} \frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\, d\tau.$$

For any friend F of $\mathscr{V}$, the closed-loop differential equation obtained from the first equation in (2.2) with the control law $u(t) = F\,x(t)$ becomes

$$\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \big(I_1(t)+I_2(t)+I_3(t)\big) = (A+BF)\,x(t). \tag{5.4}$$

In view of the assumption made on $x(t)$, we have $(A+BF)\,x(t) \in \mathscr{V}$ for all $t \geq t^\star$. Therefore, to satisfy the differential equation (5.4), $\mathscr{D}^\alpha x(t)$ must also be in $\mathscr{V}$ for all $t \geq t^\star$.

By using Lemma 5.4, it is immediate to see that $x_{\mathscr{V}}^{(\lceil\alpha\rceil)}(t) \in \mathscr{V}$ for almost all $t \in [0,t^*]$, and therefore $I_1(t) \in \mathscr{V}$ for all $t > t^\star$. Similarly, we find $I_3(t) \in \mathscr{V}$ and $I_2(t) \in \mathscr{V}^\perp$ for all $t > t^\star$. It follows that, to satisfy the differential equation (5.4), $I_2(t)$ must be zero for all $t > t^\star$. We show that this is only possible if $x_\perp(t) = 0$ for all $t \in [0,t^*]$. Assume that $I_2(t) = 0$ for all $t > t^*$. We can re-write $I_2(t)$ as

$$I_2(t) = \int_0^{t^*} x_\perp^{(\lceil\alpha\rceil)}(\tau)\,K(t,\tau)\,d\tau = 0, \tag{5.5}$$

where

$$K(t,\tau) = \frac{1}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}.$$

We have that, for every $t > t^*$, $K(t,\tau)$ is strictly positive for all $\tau \in [0,t^*]$, and $K(t_1,\tau) - K(t_2,\tau)$ is positive and strictly monotonic in τ for all $t_2 > t_1 > t^*$. We denote by $\tilde{x}_1$ the first component of the vector $x_\perp^{(\lceil\alpha\rceil)}$. For the sake of simplicity, consider the case where $\tilde{x}_1$, which is nonzero by assumption, has only one sign change in $[0,t^*]$, occurring at time $\hat{t}$. Considering that $I_2(t_1) = I_2(t_2) = 0$ we obtain in particular

$$\int_0^{\hat{t}} \tilde{x}_1(\tau)\,K(t_1,\tau)\,d\tau = \int_{\hat{t}}^{t^*} \big(-\tilde{x}_1(\tau)\big)\,K(t_1,\tau)\,d\tau$$
$$\int_0^{\hat{t}} \tilde{x}_1(\tau)\,K(t_2,\tau)\,d\tau = \int_{\hat{t}}^{t^*} \big(-\tilde{x}_1(\tau)\big)\,K(t_2,\tau)\,d\tau.$$

However, the integrands are nonnegative and strictly positive over a set of nonzero Lebesgue measure, and $K(t_1,\tau)-K(t_2,\tau)$ is strictly monotonic in τ. Hence,

$$\int_0^{\hat t}\tilde x_1(\tau)K(t_1,\tau)\,d\tau-\int_0^{\hat t}\tilde x_1(\tau)K(t_2,\tau)\,d\tau<\int_{\hat t}^{t^*}\big(-\tilde x_1(\tau)\big)K(t_1,\tau)\,d\tau-\int_{\hat t}^{t^*}\big(-\tilde x_1(\tau)\big)K(t_2,\tau)\,d\tau,$$

which leads to a contradiction. The same argument can be repeated component-wise and iterated when the function has more than one change of sign. ■

Theorem 5.5 proved that if the pseudostate trajectory exits $\mathscr{V}$ at any point in the interval $[0,t^\star]$, no feedback controls can maintain the trajectory on $\mathscr{V}$ from $t^\star$ onward even if $x(t^\star)\in\mathscr{V}$. We now show, however, that under specific necessary and sufficient conditions, a control with a feedforward component can accomplish this task.

THEOREM 5.6. *Let $\mathscr{V}$ be a controlled invariant subspace and let $x^{(i)}(t^\star)\in\mathscr{V}$ for $t^\star>0$ and $i\in\{0,\lfloor\alpha\rfloor\}$. Let*

$$I_2(t)=\int_0^{t^\star}\frac{x_\perp^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\,d\tau.$$

There exists a control function such that $x(t)\in\mathscr{V}$ for all $t>t^\star$ if and only if $I_2(t)\in\operatorname{im}B+\mathscr{V}$ for all $t>t^\star$.

Proof: **(If)**. We can define the control function from $t^\star$ onward as $u(t)=F\,x(t)+\omega(t)$, where F is a friend of $\mathscr{V}$ and ω satisfies

$$B\,\omega(t)=\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}I_2(t)+v(t) \tag{5.6}$$

for all $t>t^\star$, where $v(t)\in\mathscr{V}$ for all $t>t^\star$ since $I_2(t)\in\operatorname{im}B+\mathscr{V}$ for all $t>t^\star$. We rewrite the differential equation as in the proof of Theorem 5.5:

$$\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\big(I_1(t)+I_2(t)+I_3(t)\big)=(A+BF)\,x(t)+B\,\omega(t),$$

which is equivalent to

$$\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_{t^\star}^{t}\frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\,d\tau=(A+BF)\,x(t)-\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}I_1(t)+v(t). \tag{5.7}$$

Eq. (5.7) is a fractional differential equation where the fractional derivative is defined from $t_0=t^\star$, with initial condition at t_0 in $\mathscr{V}$. Since $I_1(t)\in\mathscr{V}$ and $v(t)\in\mathscr{V}$ for all $t\geq 0$, we can apply the result of Lemma 3.4 (choosing B of Lemma 3.4 to be the identity matrix) to find that the solution is in $\mathscr{V}$. **(Only if)**. Following the same procedure as the *(if)* part, by choosing a friend F of $\mathscr{V}$ we obtain the fractional differential equation

$$\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_{t^\star}^{t}\frac{x^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha-\lceil\alpha\rceil+1}}\,d\tau=(A+BF)\,x(t)+B\,\omega(t)-\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}I_1(t)-\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}I_2(t). \tag{5.8}$$

Since $x(t)\in\mathscr{V}$ for all $t\geq t^\star$, the left hand-side is in $\mathscr{V}$ almost everywhere, in view of Theorem 3.1 (adapted to the fractional derivative starting at $t_0=t^\star$) and since F is a friend of $\mathscr{V}$, the term $(A+BF)\,x(t)\in\mathscr{V}$ for all $t\geq t^\star$. Since, by construction, $I_1(t)\in\mathscr{V}$ for all $t\geq 0$, we find $B\,\omega(t)+\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}I_2(t)\in\mathscr{V}$ for almost all $t\geq t^\star$, which, from the continuity of $I_2(t)$, implies that $I_2(t)\in\operatorname{im}B+\mathscr{V}$. ■

REMARK 5.1. As is clear from Example 5.1, the value of the pseudostate at a given point in time can be used to characterize the future evolution of a fractional system with Caputo derivatives only if we assume that the pseudostate has been in a steady-state throughout the entire past of the system. However, adopting the Caputo definition allows us to introduce the concept of controlled invariance in a simple manner, along the lines of integer systems, and link the controlled invariant subspace with its geometric definition, i.e., $A\mathscr{V}\subseteq\mathscr{V}+\operatorname{im}B$. By contrast, using the Riemann-Liouville fractional derivative in place of the Caputo derivative considered in this paper does not allow for a straightforward extension of the concept of invariance and controlled invariance. In fact, a differential equation defined in terms of Riemann-Liouville derivatives involves initial conditions that are expressed in terms of fractional derivatives/integrals. Such conditions are not compatible with a finite pseudostate at $t = 0$ (under the standard assumption that $x(t) = 0$ for all $t < 0$). In other words, the pseudostate trajectories are either such that $x(0) = 0$, or only defined in the open interval $(0,+\infty)$, see, e.g., the counterexample proposed in [41, Section 2]. This, however, does not imply that the concept of controlled invariance developed in this paper fails when using other approaches. In fact, Theorems 5.5 and 5.6, which are central to our theory, explain how to handle the case where the initial conditions are not defined under the (often implicit) assumption the the pseudostate has been at rest throughout the entire past of the system. Roughly speaking, these results state that, if all past pseudostate trajectory has evolved on a controlled invariant subspace, we can maintain the future pseudostate on that controlled invariant subspace. This is in line with the infinite dimensional nature of the system. The aforementioned results can be easily extended to the main approaches available in the literature to solve the initialization problem for fractional systems.

An elegant way to overcome the apparent limitation deriving from the Riemann-Liouville approach is to exploit the so-called *initialization function* proposed in [19, 14, 20, 1], which allows one to use the pseudostate space representation with Riemann-Liouville fractional derivatives. In this case, it is convenient to start from the geometric interpretation of controlled invariant subspace, which is independent from the adopted derivative, and characterizes such subspace in terms of existence of a matrix F such that $(A+BF)\mathscr{V} \subseteq \mathscr{V}$. If we consider such a feedback matrix F, the initialized closed-loop system is

$$\begin{aligned} \mathscr{D}^{\alpha}_{RL}x(t) &= (A+BF)x(t) - \psi(x,\alpha,a,0,t) + Bv(t), \\ y(t) &= (C+DF)x(t) + Dv(t), \end{aligned} \tag{5.9}$$

where $\mathscr{D}_{RL}$ denotes the well-known Riemann-Liouville fractional derivative operator and the initialization function is defined as

$$\psi(x,\alpha,a,0,t) = \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left(\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\int_a^0 \frac{x(\tau)}{(t-\tau)^{\alpha+1-\lceil\alpha\rceil}}\,d\tau\right),$$

see e.g. [14], where $a < 0$, the interval $[a,0]$ is the initialization interval and $x(t) = 0$ for all $t < a$ (which implies that assumptions on the entire past of the system are required, coherently with the infinite-dimensional nature of fractional systems). The controlled invariance condition $(A+BF)\mathscr{V} \subseteq \mathscr{V}$ straightforwardly implies that the pseudostate can evolve on $\mathscr{V}$ using a pseudostate feedback if and only if $\psi(x,\alpha,a,0,t) \in \mathscr{V}$ for all $t \geq 0$. In turn, it is obvious from the same argument of the proof of Theorem 5.5 that $\psi(x,\alpha,a,0,t) \in \mathscr{V}$ for all $t \geq 0$ if and only if $x(t) \in \mathscr{V}$ for all $t \in [a,0]$. More in general, using the same approach of Theorem 5.6, we find that, even if we accept to use a control which is not necessarily a pseudostate feedback, we can evolve on $\mathscr{V}$ if and only if $\psi(x_\perp,\alpha,a,0,t) \in \operatorname{im} B + \mathscr{V}$ for all $t \geq 0$. The previous argument provides a clear extension of Theorems 5.5 and 5.6 and of the underlying controlled invariance theory to the case presented in [14]. The same argument applies *mutatis mutandis* to different initialization functions, such as, for example, the incomplete Gamma function [15] or, more in general, any function $\phi(\cdot)$ such that there exists $k \in \mathbb{R}_+$ such that $|\phi(t)| < kt^{\alpha-1}$, [38].

As second method to deal with the initialization problem and to recover the semigroup property for fractional systems entails the use of a *frequency distributed* model [45], also referred to as diffusive representation. Interestingly, the frequency-distributed approach can be conveniently exploited to define approximated initialization functions that render the initialization problem numerically tractable [41]. Using the frequency distributed representation, the solution of the closed-loop linear differential equation $\mathscr{D}^{\alpha}x = (A+BF)x + Bv$ can be computed as

$$x(t) = \int_0^{\infty} \frac{\sin(\pi\alpha)}{\pi}\omega^{-\alpha} z(\omega,t)d\omega$$

where the vector $z \in \mathbb{R}^n$ satisfies the linear differential equation

$$\frac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + (A+BF)x(t) + Bv(t)$$

subject to infinite-dimensional initial condition $z(\omega,0)=\psi(\omega)$ for all $\omega\in\mathbb{R}_+$. If $z(\omega,0)\in,\mathscr{V}$ for all $\omega\in\mathbb{R}_+$, then obviously $x(0)\in\mathscr{V}$, and the controlled invariance condition $(A+BF)\mathscr{V}\subseteq\mathscr{V}$ guarantees that $(A+BF)x(0)\in\mathscr{V}$. If we limit ourselves to a pseudostate feedback (i.e., if $v(t)=0$), then it is obvious that $z(\omega,t)\in\mathscr{V}$ for all $t>0$ and for all $\omega\in\mathbb{R}_+$, so that we also obtain that the pseudostate can evolve in $\mathscr{V}$, i.e. $x(t)\in\mathscr{V}$ for all $t>0$, as required.

We now parallel the theory of controlled invariance with the theory of output nullingness. Loosely speaking, an output nulling subspace is a controlled invariant subspace in which the pseudostate trajectory can evolve while maintaining the output at zero. Building on the theory of controlled invariance for fractional systems, the extension to output nullingness relies exclusively on the pseudostate-to-output map C and on the input-to-output map D, and therefore it follows along the same lines of the integer case and does not present major issues. A subspace $\mathscr{V}$ is said to be an output nulling subspace if, for any $x^{(i)}(0)\in\mathscr{V}$ with $i\in\{0,\lfloor\alpha\rfloor\}$, there exists a control function u such that the pseudostate trajectory generated by the system remains in $\mathscr{V}$ and the output remains identically at zero. The following result adapts Theorem 5.2 to the case of output nulling subspaces; its proof requires only minor modifications.

THEOREM 5.7. *Let $\mathscr{V}$ be a subspace of $\mathscr{X}$. The following statements are equivalent:*

(a) *$\mathscr{V}$ is output nulling;*

(b) $\begin{bmatrix} A \\ C \end{bmatrix}\mathscr{V}\subseteq(\mathscr{V}\oplus 0_{\mathscr{Y}})+\operatorname{im}\begin{bmatrix} B \\ D\end{bmatrix}$;

(c) *There exists $F\in\mathbb{R}^{m\times n}$ such that $\begin{bmatrix} A+BF \\ C+DF \end{bmatrix}\mathscr{V}\subseteq\mathscr{V}\oplus 0_{\mathscr{Y}}$.*

In view of the last condition, the control function driving the pseudostate on $\mathscr{V}$ if $x^{(i)}(0)\in\mathscr{V}$ with $i\in\{0,\lfloor\alpha\rfloor\}$ can always be expressed as the static pseudostate feedback $u(t)=Fx(t)$, if F satisfies the third condition of Theorem 5.7. In this case, we say that F is an *output nulling friend* of $\mathscr{V}$. We denote by $\mathfrak{F}(\mathscr{V})$ the set of output nulling friends of $\mathscr{V}$. Obviously, an output nulling friend of $\mathscr{V}$ is also a friend of $\mathscr{V}$, but the converse is only true when $D=0$. We observe that Lemma 5.3 can be adapted to output-nulling subspaces as follows. Let $\mathscr{V}$ be an output-nulling subspace. Let F be an output-nulling friend of $\mathscr{V}$. Let u be a control function. The pseudostate trajectory obtained from $x^{(i)}(0)\in\mathscr{V}$ with $i\in\{0,\lfloor\alpha\rfloor\}$ and u remains in $\mathscr{V}$ for all $t\geq 0$ if and only if u takes the form $u(t)=Fx(t)+v(t)$, where $v(t)\in B^{-1}\mathscr{V}\cap\ker D$ for all $t\geq 0$. Note that this condition reduces to $v(t)\in B^{-1}\mathscr{V}$ for all $t\geq 0$ whenever the system is strictly proper, i.e. when $D=0$, in line with the previous considerations.

Clearly, the set of output-nulling subspaces is a subset of the set of controlled invariant subspaces. As such, a counterpart of Theorem 5.5 holds for output-nulling subspaces. Let $\mathscr{V}$ be an output-nulling subspace and let $x(t^\star)\in\mathscr{V}$ for $t^\star>0$. There exists a pseudostate feedback control such that $x(t)\in\mathscr{V}$ and $y(t)=0$ for almost all $t>t^\star$ if and only if $x(t)\in\mathscr{V}$ for almost all $t\leq t^\star$.

Note that in the previous consideration we did not require the output to be zero in $[0,t^\star]$. Whenever $D=0$, this is automatically implied by the condition $x(t)\in\mathscr{V}$ for $t\in[0,t^\star]$, since $\mathscr{V}\subseteq\ker C$. However, when $D\neq 0$, the pseudostate might evolve on $\mathscr{V}$ with a corresponding output which is not zero in $[0,t^\star]$. This, however, does not prevent the possibility of evolving on $\mathscr{V}$ from $t^\star$ on by also ensuring the output to be zero for $t\geq t^\star$. In fact, if $\mathscr{V}$ is an output-nulling subspace, the condition $(C+DF)\mathscr{V}=0_{\mathscr{Y}}$ (point **(c)** of Theorem 5.7) is algebraic in nature, and can be enforced for any point of $\mathscr{V}$ by using an output-nulling friend of $\mathscr{V}$ irrespectively of how the pseudostate evolves in $\mathscr{V}$.
The adaptation of Theorem 5.6 to the case where we consider output nulling subspaces instead of simple controlled invariant subspaces follows straightforwardly by substituting the condition $I_2(t)\in\operatorname{im}B$ with the inclusion $I_2(t)\in B\ker D$ for all $t>t^\star$.

Since the condition for a subspace to be output nulling is formally the same as the integer case, it follows immediately that the set of output nulling subspaces is closed under addition. This implies that we can define the largest output nulling subspace $\mathscr{V}^\star$, which can be interpreted as the set of all initial conditions for which a control function exists for which the output can be maintained identically at zero. The sequence

$$\left\{\begin{array}{rcl}\mathscr{V}_0 &=& \mathscr{X}\\ \mathscr{V}_{i+1} &=& \begin{bmatrix} A \\ C\end{bmatrix}^{-1}\left((\mathscr{V}_i\oplus 0_{\mathscr{Y}})+\operatorname{im}\begin{bmatrix} B \\ D\end{bmatrix}\right), \qquad i\in\mathbb{N}\end{array}\right. \tag{5.10}$$

is monotonically decreasing and converges to $\mathscr{V}^\star$ in at most $n-1$ steps, i.e., $\mathscr{V}_0 \supset \mathscr{V}_1 \supset \ldots \supset \mathscr{V}_h = \mathscr{V}_{h+1} = \ldots$ implies $\mathscr{V}^\star = \mathscr{V}_h$, with $h \le n-1$, [44, p. 162]. From the results of Section 4, given an output nulling subspace $\mathscr{V}$, we can define the reachability subspace $\mathscr{R}_{\mathscr{V}}$ on $\mathscr{V}$ as the set of points that can be reached from zero initial conditions by means of control functions that maintain the pseudostate on $\mathscr{V}$ and the output at zero. Given an output nulling friend F of $\mathscr{V}$, there holds $\mathscr{R}_{\mathscr{V}} \stackrel{\text{def}}{=} \langle A+BF \,|\, \mathscr{V} \cap B \ker D\rangle$, [44, Thm. 7.16]. Since the properties of assignability of the spectrum of $A+BF$ are only dependent upon the matrices A and B, as in the integer case the eigenvalues of $A+BF$, for $F \in \mathfrak{F}(\mathscr{V})$, can be divided into two multi-sets: the eigenvalues of the mapping $A+BF \,|\, \mathscr{V}$ and the eigenvalues of $A+BF \,|\, \frac{\mathscr{X}}{\mathscr{V}}$. In turn, the eigenvalues of $A+BF \,|\, \mathscr{V}$ can be divided into two multi-sets: the eigenvalues of $A+BF \,|\, \mathscr{R}_{\mathscr{V}}$ are all freely assignable with a suitable choice of $F \in \mathfrak{F}(\mathscr{V})$, whereas those of $A+BF \,|\, \frac{\mathscr{V}}{\mathscr{R}_{\mathscr{V}}}$ are fixed, i.e., they are independent from $F \in \mathfrak{F}(\mathscr{V})$. Likewise, the eigenvalues of $A+BF \,|\, \frac{\mathscr{V}+\langle A \,|\, \operatorname{im} B\rangle}{\mathscr{V}}$ are freely assignable with a suitable choice of $F \in \mathfrak{F}(\mathscr{V})$, whereas those of $A+BF \,|\, \frac{\mathscr{X}}{\mathscr{V}+\langle A \,|\, \operatorname{im} B\rangle}$ are fixed for all $F \in \mathfrak{F}(\mathscr{V})$. We say that $\mathscr{V}$ is

- *internally stabilizable* if there exists $F \in \mathfrak{F}(\mathscr{V})$ such that, for all $\lambda \in \sigma(A+BF \,|\, \mathscr{V})$, we have [22, 12]

$$|\operatorname{Arg}\lambda| > \alpha\,\frac{\pi}{2}, \tag{5.11}$$

  or, equivalently, if $\forall \lambda \in \sigma(A+BF \,|\, \frac{\mathscr{V}}{\mathscr{R}_{\mathscr{V}}})$ (5.11) holds;
- *externally stabilizable* if there exists $F \in \mathfrak{F}(\mathscr{V})$ such that $\forall \lambda \in \sigma(A+BF \,|\, \frac{\mathscr{X}}{\mathscr{V}})$, or, equivalently, if $\forall \lambda \in \sigma\big(A+BF \,|\, \frac{\mathscr{X}}{\mathscr{V}+\langle A \,|\, \operatorname{im} B\rangle}\big)$, (5.11) holds true.[6]

An output nulling subspace that is internally stabilizable is also referred to as a *stabilizability output nulling subspace*. As in the integer case, the set of stabilizability output nulling subspaces is closed under addition, and thus it admits a maximum, that we denote by $\mathscr{V}^\star_{g,\alpha}$: this subspace is the set of all initial pseudostates for which an input exists that maintains the output at zero and the pseudostate trajectory converges to the origin.

An output nulling subspace $\mathscr{R}$ for which an output nulling friend F exists such that the spectrum of $A+BF \,|\, \mathscr{R}$ is arbitrary is called a *reachability output nulling subspace*. The set of reachability output nulling subspaces is closed under addition, and thus it admits a maximum, that we denote by $\mathscr{R}^\star$: there holds $\mathscr{R}^\star \subseteq \mathscr{V}^\star_{g,\alpha} \subseteq \mathscr{V}^\star$. The subspace $\mathscr{R}^\star$ is also the output nulling reachability subspace on $\mathscr{V}^\star$, i.e., $\mathscr{R}^\star = \mathscr{R}_{\mathscr{V}^\star}$. This subspace can be interpreted as the set of all initial pseudostates that are reachable from zero initial conditions by control inputs that maintain the output at zero. The spectrum of $A+BF \,|\, \frac{\mathscr{V}^\star}{\mathscr{R}^\star}$ is the *invariant zero structure* of the system. The invariant zeros can be alternatively characterized by the values of $\lambda \in \mathbb{C}$ such that the matrix pencil

$$P(\lambda) = \begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}$$

loses rank with respect to its normal rank (i.e., its rank as a polynomial matrix), [44, Thm. 7.19]. A *minimum-phase invariant zero z* is an invariant zero which satisfies the condition

$$|\operatorname{Arg} z| > \alpha\,\frac{\pi}{2}.$$

It follows that, while $\mathscr{V}^\star$ is the same for the fractional system (2.2) and its corresponding integer system

$$\begin{cases} \dot{x}(t) = A\,x(t)+B\,u(t) \\ y(t) = C\,x(t)+D\,u(t), \end{cases} \tag{5.12}$$

because such subspace depends solely on the matrices A, B, C and D involved, the same is not true for $\mathscr{V}^\star_{g,\alpha}$. In fact, only the invariant zero directions given by the projection onto the pseudostate space $\mathscr{X}$ of the null-space of $P(\lambda)$ evaluated at the minimum-phase invariant zeros are part of the span of $\mathscr{V}^\star_{g,\alpha}$, [28]. Given a fractional system, if we denote the stabilizability output nulling subspace of the corresponding integer system as $\mathscr{V}^\star_g$, we have the following obvious inclusions: $\mathscr{V}^\star_{g,\alpha} \supseteq \mathscr{V}^\star_g$ if $\alpha < 1$ and $\mathscr{V}^\star_{g,\alpha} \subseteq \mathscr{V}^\star_g$ if $\alpha > 1$.

[6] Recall the equality $\mathscr{R} = \langle A \,|\, \operatorname{im} B\rangle$ from Section 4.

6. Disturbance decoupling problem by pseudostate feedback. We now consider the problem of the decoupling of a disturbance w acting on the system. The disturbance decoupling problem has a central role in control theory, and it is of particular interest in the context of this paper because the geometric approach for integer systems was originally developed to address it. Moreover, the disturbance decoupling problem is the prototype of a large number of richer control problems for linear time-invariant systems, including model matching, LQ-H_2 and non-interacting control problems. Consider the system

$$\begin{cases} \mathscr{D}^\alpha x(t) = A x(t) + B u(t) + H w(t) \\ y(t) = C x(t) + D u(t), \end{cases} \tag{6.1}$$

with $x^{(i)}(0) = x_0^i$ for $i \in \{0, \lfloor \alpha \rfloor\}$, where, for all $t \geq 0$, the vector $w(t) \in \mathscr{W} = \mathbb{R}^q$ represents a disturbance.

We are concerned with the following problems.

PROBLEM 6.1. *Find under which conditions there exists a pseudostate-feedback control $u = F x$ such that the transfer function between the disturbance w and the output y is equal to zero, i.e.,*

$$T_F(s) = (C + DF)\left(s^\alpha I - (A + BF)\right)^{-1} H = 0. \tag{6.2}$$

PROBLEM 6.2. *Find under which conditions there exists $u = F x$ such that (6.2) holds and for all $\lambda \in \sigma(A + BF)$, we have $|\operatorname{Arg}\lambda| > \alpha \frac{\pi}{2}$.*

Notice that with no loss of generality in (6.1) we did not consider a feedthrough term w to y. In fact, any non-zero feedthrough from w to y would render both Problem 6.1 and 6.2 unsolvable. An obvious necessary condition for Problem 6.2 to be solvable is the asymptotic stabilizability of the pair (A, B), which is equivalent to the existence of a feedback matrix F such that for all $\lambda \in \sigma(A + BF)$ there holds $|\operatorname{Arg}\lambda| > \alpha\pi/2$.[7]

The solution of Problem 6.1 is given in the following theorem, where we denote by $\mathscr{V}^\star$ the supremal output-nulling subspace of the quadruple (A, B, C, D).

THEOREM 6.1. *Problem 6.1 is solvable if and only if*

$$\operatorname{im} H \subseteq \mathscr{V}^\star. \tag{6.3}$$

Proof: First, we observe that Problem 6.1 is solvable if and only if there exists a feedback matrix F such that the closed-loop transfer function $T_F(s)$ is zero as in (6.2). We show that there exists F such that $T_F(s) = 0$ if and only if there exists F such that $(C + DF)\langle A + BF \,|\, \operatorname{im} H\rangle = 0$. **(Only if)**. Suppose by contradiction that $(C + DF)\langle A + BF \,|\, \operatorname{im} H\rangle \neq \{0\}$. Let $\bar{x} \in \langle A + BF \,|\, \operatorname{im} H\rangle \setminus \ker(C + DF)$. Since $\langle A + BF \,|\, \operatorname{im} H\rangle$ is the reachable subspace for the pair $(A + BF, H)$, there exists w such that for an arbitrarily small $\bar{t} > 0$ we have $x(\bar{t}) = \bar{x}$. Thus, $y(\bar{t}) = (C + DF)\bar{x} \neq 0$. This implies $T_F(s) \neq 0$. **(If)**. Obvious since $(C + DF)\langle A + BF \,|\, \operatorname{im} H\rangle = 0$ implies that the reachable subspace of $(A + BF, H)$ is unobservable.
We now show that the $\langle A + BF \,|\, \operatorname{im} H\rangle$ is equivalent to (6.3). Since $\langle A + BF \,|\, \operatorname{im} H\rangle$ is the smallest $(A + BF)$-invariant subspace containing $\operatorname{im} H$, the feedback matrix F solves Problem 6.1 if and only if there exists an $(A + BF)$-invariant subspace $\mathscr{V}$ containing $\operatorname{im} H$ and contained in $\ker(C + DF)$, i.e., if and only if there exists an output-nulling subspace for the quadruple (A, B, C, D) containing $\operatorname{im} H$. Since $\mathscr{V}^\star$ is the supremal output-nulling subspace for (A, B, C, D), this is equivalent also to the condition $\operatorname{im} H \subseteq \mathscr{V}^\star$. ■

As in the integer case, the proof of Theorem 6.1 is constructive: if the solvability condition $\operatorname{im} H \subseteq \mathscr{V}^\star$ is satisfied, a decoupling feedback matrix F can be computed as a friend of $\mathscr{V}^\star$. The solution of Problem 6.2 can be carried out in a similar way. The solvability condition involves the largest stabilizability output-nulling subspace $\mathscr{V}^\star_{g,\alpha}$ of the quadruple (A, B, C, D). Let us now consider Problem 6.2.

[7] Another equivalent and easily checkable condition for the asymptotic stabilizability can be stated as for the integer case using the change of coordinate matrix $T = [\,T_1 \quad T_2\,]$, where $\operatorname{im} T_1 = \langle A, \operatorname{im} B\rangle$. Indeed, as in the integer case $T^{-1}(A + BF)T = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}$ and $T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$. Then, (A, B) is asymptotically stabilizable if and only if $\forall \lambda \in \sigma(A_{2,2})$ we have $|\operatorname{Arg}\lambda| > \alpha\pi/2$.

THEOREM 6.2. *Problem 6.2 is solvable if and only if (A,B) is asymptotically stabilizable and*

$$\operatorname{im} H \subseteq \mathscr{V}^{\star}_{g,\alpha}. \tag{6.4}$$

Proof: **(If)**. Since $\mathscr{V}^{\star}_{g,\alpha} \subseteq \mathscr{V}^{\star}$, it is obvious that (6.4) implies (6.3), so that $T_F(s)=0$. We now focus on the stability. Since $\mathscr{V}^{\star}_{g,\alpha}$ is a stabilizability output-nulling subspace, it has a friend F such that $|\operatorname{Arg}\lambda| > \alpha\pi/2$ for all $\lambda \in \sigma(A+BF\,|\,\mathscr{V}^{\star}_{g,\alpha})$. The asymptotic stabilizability of (A,B) implies that for any friend F of $\mathscr{V}^{\star}_{g,\alpha}$ we have $|\operatorname{Arg}\lambda| > \alpha\pi/2$ for all $\lambda \in \sigma\big(A+BF\,|\,\mathscr{X}/(\mathscr{V}^{\star}_{g,\alpha}+\langle A\,|\operatorname{im}B\rangle)\big)$. It follows that there exists a friend F of $\mathscr{V}^{\star}_{g,\alpha}$ such that $\forall\lambda \in \sigma(A+BF)$, $|\operatorname{Arg}\lambda| > \alpha\pi/2$. **(Only if)**. If the closed loop system is stable, the pair (A,B) is obviously stabilizable. Condition (6.4) follows on noting that $\mathscr{V}^{\star}_{g,\alpha}$ is the largest stabilizability output-nulling subspace. ■

REMARK 6.1. As as mentioned in Section 2, given a general fractional differential equation of arbitrary order, the pseudostate-space system (2.2) might not be a valid realization in the sense that, using the Caputo derivative, it might be impossible to define coherent initial conditions such that the free evolution of the original system matches the free evolution of the pseudostate-space realization [38]. However, disturbance decoupling problems are independent from the initial condition, as they only depend on the input-output (disturbance to output) properties of the fractional system. As such, the solvability conditions of Theorem 6.1 and Theorem 6.2 remain valid for the original fractional differential equation, and, even more importantly, are independent from the adopted definition of fractional derivative (provided that existence and uniqueness are satisfied). In fact, input-output properties are independent of the adopted definition of fractional derivative, and from the initialization approach. Using the Laplace transform, we find that the initialization function approach proposed in [14] leads to

$$Y(s) = \big(C(Is^{\alpha}-A)^{-1}B+D\big)U(s)+C(Is^{\alpha}-A)^{-1}B\,\psi(s)$$

from which it is obvious that the input/output transfer function is completely independent from the initialization function ψ. Similarly, if we consider a frequency-distributed model we find [46]

$$Y(s) = C(s^{\alpha}-A)^{-1}s^{\alpha}\int_0^{\infty}\frac{\sin(\pi\alpha)\omega^{-\alpha}z(\omega,0)}{\pi(s+\omega)}d\omega+\big(C(s^{\alpha}-A)^{-1}B+D\big)U(s),$$

where, again, the input output behavior is independent from $z(\omega,0)$, as required.

6.1. An illustrative example. Consider the completely reachable system

$$A=\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\; B=\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix},\; H=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\; C=\begin{bmatrix} -1 & 0 & 1 \end{bmatrix},\; D=\begin{bmatrix} 0 & 0 \end{bmatrix}.$$

with $\alpha = 1/3$. For this system, using (5.10), it is easily found that $\mathscr{V}^{\star} = \mathscr{R}^{\star} = \mathscr{V}^{\star}_{g,\alpha} = \operatorname{span}\left\{\left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right],\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right]\right\}$. By construction $\mathscr{V}^{\star} \subseteq \ker(C+DF)$. Clearly $\operatorname{im}H \subseteq \mathscr{V}^{\star}_{g,\alpha}$, so that the conditions of Theorem 6.2 are satisfied. We

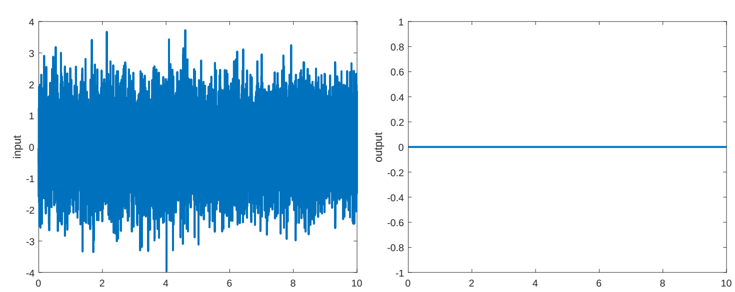

FIG. 6.1. *Disturbance (left) and output (right) with zero initial pseudostate.*

assign the closed-loop eigenvalues restricted to $\mathscr{R}^\star$ to be $\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$, and the remaining one to be -2. This choice ensures that the closed-loop system is asymptotically stable because $\alpha = 1/3$. Using the approach in [25] we obtain the solution $F = \frac{1}{3}\begin{bmatrix} -2 & 6 & 2 \\ 7 & 3 & -4 \end{bmatrix}$. Figure 6.1 shows that F indeed decouples the disturbance (which is modeled by a random process) from the output, starting from zero initial pseudostate. Figure 6.2 left shows that the closed-loop system is indeed stabilized: when the disturbance is equal to zero, given an initial pseudostate $x_0^0 = \begin{bmatrix} 2 \\ 1/2 \\ 1 \end{bmatrix} \notin \mathscr{R}^\star$, the pseudostate trajectory converges asymptotically to zero. In the center plot of Figure 6.2 we plotted the pseudostate evolution in the same setting of the previous case, but with $\alpha = 2/3$ (in which case F places two closed-loop eigenvalues on the stability boundary) and with $\alpha = 1$ (in which the system becomes an integer system, but in this case F is not stabilizing), respectively. Note that in both cases the first and third components of the pseudostate variable converge: this is due to the presence of the closed-loop eigenvalue at -2. Note that in these cases the condition of Theorem 6.2 are still satisfied, but the closed-loop system is not asymptotically stable. In fact, the solvability of the disturbance decoupling problem with stability does not imply that every friend of $\mathscr{V}^\star_{g,\alpha}$ can be used. By contrast, if stability is not required and the condition of Theorem 6.1 is satisfied, then every friend of $\mathscr{V}$ solves Problem 6.1.

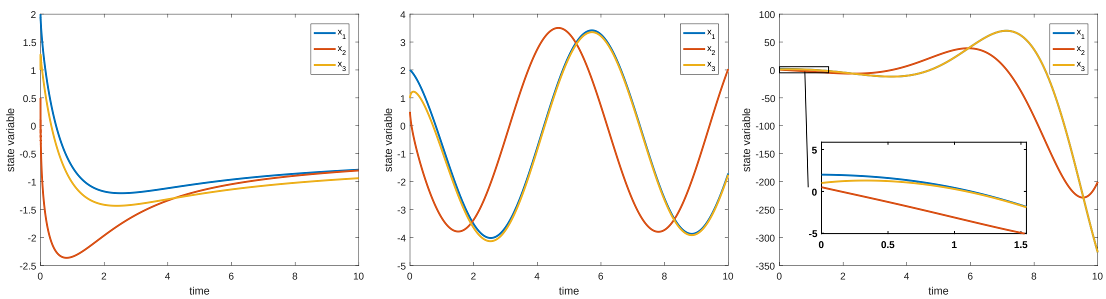

FIG. 6.2. *Pseudostate response with* $\alpha = \frac{1}{3}$ *(left),* $\alpha = \frac{2}{3}$ *(center) and* $\alpha = 1$ *(right).*

Concluding remarks. We have developed a geometric approach to the theory of fractional systems. The crucial step in this development is the definition of controlled invariance, which appears to be a much richer notion than its counterpart for integer systems. Indeed, in the fractional case the property of a controlled invariant subspace as the locus of trajectories of a linear time-invariant system is the same as the integer one if we consider $t = 0$ as the initial pseudostate where the boundary conditions are assigned. By contrast, assigning the pseudostate at an arbitrary time instant on a controlled invariant subspace ensures that the trajectory can be kept on this subspace only if its past evolved in the same subspace. In particular, we have proved that if the pseudostate trajectory, in that interval, leaves the controlled invariant subspace, there are no pseudostate-feedback trajectories that can maintain the future evolution on that subspace. We have also given necessary and sufficient conditions for the solution of the disturbance decoupling problem with and without a stability requirement. The next step will be the investigation of the concept of conditioned invariance for fractional systems, in relation with the existence of unknown-input observation. This will also allow us to extend the family of disturbance decoupling problems that can be addressed for fractional systems to include dynamic output feedbacks.

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