Dynamical Networks of Social Influence: Modern Trends and Perspectives

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Abstract: Dynamics and control of processes over social networks, such as the evolution of opinions, social influence and interpersonal appraisals, diffusion of information and misinformation, emergence and dissociation of communities, are now attracting significant attention from the broad research community that works on systems, control, identification and learning. To provide an introduction to this rapidly developing area, a Tutorial Session was included into the program of IFAC World Congress 2020. This paper provides a brief summary of the three tutorial lectures, covering the most "mature" directions in analysis of social networks and dynamics over them: 1) formation of opinions under social influence; 2) identification and learning for analysis of a network's structure; 3) dynamics of interpersonal appraisals.

1. INTRODUCTION

Systems, control and learning theories have found numerous applications in natural sciences and engineering. However, their potentials in social sciences remain almost undisclosed. Now, the situation is rapidly changing due to the recent remarkable progress in multi-agent systems and complex networks that has revealed common behaviors and principles of functioning in large-scale systems arising in nature and society (Mesbahi and Egerstedt, 2010; Easley and Kleinberg, 2010; Strogatz, 2003).

Whereas the classical studies on social networks (Wasserman and Faust, 1994) primarily focus on "static" structural properties of social networks and ties between individuals, the advanced theory of multi-agent networks has enabled mathematically rigorous analysis of *dynamical processes* unfolding over them: formation of opinions and beliefs, diffusion of information and misinformation, dynamics of interpersonal appraisals and social influence etc. The study on such processes and their interplay with the structural properties is a young and rapidly growing field of research, which has no commonly adopted name yet and lies at the crossroads of systems and control, learning theory, network science and mathematical sociology.

A broad community of researchers working on systems, control and learning is playing an important role in the development of this emerging area. A search with keywords "social dynamics" returns more than 100 papers published in Automatica in 2017-2019 and more than 300 papers published in IEEE journals. A number of surveys on social dynamics have been published recently (Mastroeni et al., 2019; Proskurnikov and Tempo, 2017, 2018). Friedkin (2015) defines the foundational problem of sociology as *coordination and control of social systems*. In spite of this, the novel area of control in social networks is underrepresented on IFAC conferences. In order to fill this gap, a tutorial session is organized during the IFAC World Congress 2020. This paper summarizes the three lectures of this tutorial session and is organized as follows.

Section 2 introduces some preliminary definitions and notation. **Section 3** is devoted to agent-based modeling of information diffusion and evolution of individual opinions, attitudes and beliefs under mechanisms of social influence. **Section 4** is concerned with the problems of *inference* in dynamic social networks, that is, recovering the network's structure from the observed behaviors of individuals.

The "strengths" of social ties are not static and may evolve due to complex socio-psychological processes, changing self-appraisals and interpersonal appraisals of individuals. **Section 5** surveys the recent technical results on modeling the endogenous dynamics of social influence networks under *feedback* mechanisms of reflected appraisals.

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2. PRELIMINARIES AND NOTATION

Henceforth $\mathbf{1}_n = (1, 1, \dots, 1)^\top \in \mathbb{R}^n$ denotes the column of ones and $\boldsymbol{e}_1 = (1, 0, \dots, 0)^\top, \dots, \boldsymbol{e}_n = (0, 0, \dots, 1)^\top$ stand for the canonical coordinate basis of $\mathbb{R}^n$. The *unit simplex* spanned by the basis vectors $\boldsymbol{e}_i$ is denoted by

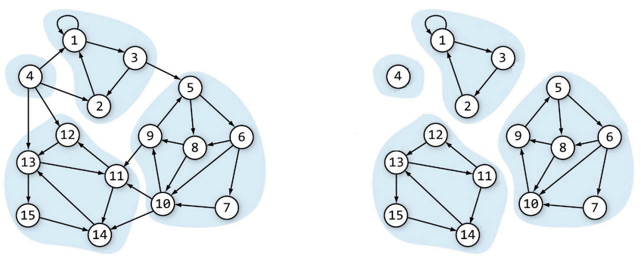
$$\Delta_n = \{\boldsymbol{\xi} \in \mathbb{R}^n : \xi_i \geq 0, \sum\nolimits_{i=1}^{n} \xi_i = 1\}.$$

The symbol $\boldsymbol{I}_n$ denotes the identity $n \times n$ matrix. Let $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ stand for the diagonal $n \times n$ matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. For a given square matrix $\boldsymbol{A}$, we denote $\mathrm{diag}\, \boldsymbol{A} = \mathrm{diag}(a_{11}, \dots, a_{nn})$. A *nonnegative* square matrix $\boldsymbol{A} = (a_{ij})$ is *stochastic* if its rows sum to 1, i.e., $\sum_j a_{ij} = 1\, \forall i$. Due to the Perron-Frobenius theorem, a stochastic matrix has a *left eigenvector* $\boldsymbol{\zeta}$ such that

$$\boldsymbol{\zeta}^\top \boldsymbol{A} = \boldsymbol{\zeta}^\top, \quad \boldsymbol{\zeta} \in \Delta_n. \tag{1}$$

A stochastic matrix is *fully regular* or SIA (stochastic, indecomposable, aperiodic) if the limit $\boldsymbol{A}^\infty = \lim\limits_{k \to \infty} \boldsymbol{A}^k$ exists and has rank 1. In this case, the Perron-Frobenius left eigenvector $\boldsymbol{\zeta} \in \Delta_n$ is *unique* and $\boldsymbol{A}^\infty = \mathbf{1}_n \boldsymbol{\zeta}^\top$.

Any nonnegative square matrix $\boldsymbol{A}$ defines a (directed) *graph* $\mathcal{G}[\boldsymbol{A}] = (\mathcal{V}, \mathcal{E})$, whose sets of nodes and arcs, or edges, are $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} = \{(j, i) : a_{ij} \neq 0\}$, respectively. Note that in multi-agent systems theory, $a_{ij} > 0$ typically measures the influence of agent j onto agent i, which is traditionally depicted as arc $j \to i$. A *walk* of length d in a graph between nodes i_0 and i_d is a sequence of arcs $(i_0, i_1), (i_1, i_2), \dots, (i_{d-1}, i_d) \in \mathcal{E}$. A *path* is a walk whose nodes $i_0, \dots, i_d$ are all distinct. If $i_0, \dots, i_{d-1}$ are all distinct yet $i_d = i_0$, the walk is said to be a cyclic path, or simply a *cycle*. A trivial example of a cycle is *self-loop* $(i, i) \in \mathcal{E}$, corresponding to diagonal entry $a_{ii} > 0$. A graph is called *aperiodic* if the exists **no** integer number $s > 1$ that divides the lengths of all cycles (e.g. if the graph has self-loops). A graph is *strongly connected* (or *strong*) if a (directed) walk exists between any two nodes (in this case, the corresponding matrix $\boldsymbol{A}$ is said to be *irreducible*). A graph that is not strong includes several strongly connected (or strong) *components* (SCC). A SCC is a subgraph, which is strongly connected and *maximal* in the sense that no node and no arc can be added to it without destroying the strong connectivity (Fig. 1). A graph is said to be (out-branching) *rooted* (or to contain an out-branching *spanning tree*, or to be *quasi-strongly connected*) if it has at least one *root* node, which is connected to all other nodes by paths (Fig. 1a). The root nodes, if exist, always constitute a strong component in the graph, which has no incoming arcs.



(a) Graph with root node 4 (b) Graph is not rooted

Fig. 1. Example: strong components of two graphs

3. OPINION FORMATION AND INFORMATION SPREAD UNDER SOCIAL INFLUENCE

Interpersonal *social influence* is a "causal effect of one actor onto another" (Friedkin, 1998), which can manifest in changes of the actors' attitudes and behaviours. Mathematical modeling and quantification of social influence are long-standing problems in sociology and psychology (Sun and Tang, 2011). One of the approaches, to a great deal inspired by Granovetter (1973), relates the ties between individuals to their positions in the network. Another approach, broadly used in analysis of complex networks, measures social ties as *statistical correlations* between the attitudes or other values associated to individuals; a social network is then described by a probabilistic graphical model (Farasat et al., 2015). These two directions of research mainly deal with static social networks and develop in the framework of classical Social Network Analysis.

The most interesting, from a systems and control viewpoint, approach to social influence considers the influence as a *dynamical* mechanism driving the evolution of individual opinions and characteristics related to them. These mechanisms can *co-evolve* along with the individual's opinions and behaviours (due to e.g. reflected appraisal mechanisms described in Section 5), and give rise to models of *coevolutionary* networks (Herrera et al., 2011). The dynamical systems viewpoint opens up the perspective to employ well-developed techniques of multi-agent control for analysis of social systems. This approach gave birth, in particular, to Social Influence Network Theory (Friedkin and Johnsen, 2011), inspired by works of French Jr. (1956).

As discussed in Friedkin (2015), the term "opinion formation" is not very precise, since actually the processes of interpersonal influence alter individual's *cognitive orientations* towards some objects, issues or events. In physical and engineering literature, the term "opinion" is however used in a broader sense, denoting some scalar or multidimensional quantity of interest, associated with a social actor. Opinions understood in this general sense can characterize e.g. sets of cultural traits (Axelrod, 1997) or binary yes/no decisions about participation in a social movements (Granovetter, 1978). Models of "opinion evolution" under social influence can thus describe processes of information diffusion, attitude and belief formation, collective behavior etc. Such models can be divided into two major groups: *macroscopic* models (referred also as statistical, Eulerian, continuum-agent, fluid-based, density-based, kinetic etc.) and *microscopic* (agent-based).

Macroscopic models describe dynamics of opinion distributions, paying no attention to alterations in the opinion of a specific individual. The first models of this kind appeared in 1930s (Rashevsky, 1939) and stemmed from compartmental models used in chemistry and biology (Jacquez, 1985). Such models describe interactions of multiple "compartments", that is, substrates or species (e.g. preys and predators or susceptible, infected and recovered individuals) considered as indecomposable entities. The extension of this approach to social groups (where a compartment may represent a fraction of people supporting some candidate in a presidential election) naturally leads to sociodynamical models in physics (Castellano et al., 2009) and evolutionary game theory (Maynard Smith, 1982). Macroscopic models are important, since they give an efficient computational tool for numerical analysis of *large-scale* social groups, where the number of individuals is huge and the evolution of opinions is nonlinear. Their theoretical analysis is, however, quite complicated, and only a few rigorous results on their stability and convergence exist in the literature, surveyed e.g. in Kolarijani et al. (2021).

Much better studied are *agent-based* models of opinion formation, describing the evolution of opinions of individual actors. In the case where the opinions are discrete (vary in a finite set), the corresponding dynamics are described by a deterministic or stochastic finite automaton, as exemplified e.g. by the voter model (Holley and Liggett, 1975), threshold models (Granovetter, 1978), the model of cultural polarization (Axelrod, 1997) and models of "phase transitions" (Sznajd-Weron and Sznajd, 2000). In spite of recent developments in control of discrete-state systems (such as e.g. Boolean networks or systems over Galois fields), models with discrete opinions primarily remain outside the scope of modern control theory, being studied by methods of probability theory and statistical physics.

Unlike discrete opinion models, agent-based models with *real-valued* opinions are governed by ordinary differential or difference equations, which enables one to apply the rich control-theoretic "armamentarium". Notice that the continuity of opinions does not imply that social actors display or communicate real numbers to each other: the data exchanged by them may be quantized (Frasca et al., 2019) or, more generally, be restricted to a set of discrete values, called "actions"; the relevant class of models is referred as CODA (continuous opinion-discrete action) models (Martins, 2008). The simplest linear models of opinion formation, however, ignore these effects and stipulate the *iterative averaging* (or weighted convex combination) as the basic mechanism that drives opinion evolution.

3.1 The principal linear models of opinion formation

The first principal model describing opinion evolution in a social network is now referred to as the *DeGroot* model (DeGroot, 1974); a special case was previously examined in French Jr. (1956) and Harary (1959). Consider a group with n individuals, where actor i holds an opinion $x_i \in \mathbb{R}$. At each period $k = 0, 1, 2, \ldots$, an actor updates their opinion with a *weighted average* of the opinions displayed by themselves and the others. Mathematically, these dynamics are governed by the equations

$$x_i(k+1) = \sum_{j=1}^{n} a_{ij} x_j(k), \quad \forall i = 1, \ldots, n \tag{2}$$

that can be rewritten in the more compact matrix form

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k), \quad \boldsymbol{x}(k) = (x_1(k), \ldots, x_n(k))^\top. \tag{3}$$

Here $\boldsymbol{A} = (a_{ij})$ is a *stochastic* matrix of *influence weights*.

The most typical behavior of the DeGroot model is the eventual consensus (unanimity) of the opinions, defined as

$$\lim_{k\to\infty} \boldsymbol{x}(k) = c\mathbf{1}_n, \quad c = c(\boldsymbol{x}(0)). \tag{4}$$

Consensus in the DeGroot model is equivalent to the SIA property of the matrix $\boldsymbol{A}$; the relevant criterion is formulated as follows (Proskurnikov and Tempo, 2017).

Theorem 1. For a given stochastic matrix $\boldsymbol{A}$, the following conditions are equivalent: (i) consensus (4) is established for every initial condition; (ii) $\boldsymbol{A}$ is a SIA matrix; (iii) the graph $\mathcal{G}[\boldsymbol{A}]$ is rooted, and the strongly connected component constituted by the roots is aperiodic. If conditions (i)-(iii) hold, then the consensus opinion is $c = \boldsymbol{\zeta}^\top \boldsymbol{x}(0)$, where $\boldsymbol{\zeta}$ is the Perron-Frobenius left eigenvector from (1).

Notice that the Perron-Frobenius eigenvector may be considered as a distribution of *social power* between the individuals. The larger is ζ_i, the more influence actor i has on the consensus opinion of the group. In the extreme case, $\boldsymbol{\zeta} = \boldsymbol{e}_i$, which means that the group's opinion always coincides with the initial opinion of individual i. This holds, in particular, if the group has a unique *stubborn* (or radical) individual i such that $a_{ii} = 1$ (and hence $x_i(k) \equiv x_i(0)$) and i is the single root of the graph $\mathcal{G}[\boldsymbol{A}]$. If $\mathcal{G}[\boldsymbol{A}]$ is strongly connected ($\boldsymbol{A}$ is irreducible), the classical Perron-Frobenius theorem implies that $\zeta_i > 0\,\forall i$, so that every actor contributes to the group's final opinion.

A consensus criterion similar to condition (iii) in Theorem 1 can be obtained for the continuous-time counterpart of the DeGroot model, introduced by Abelson (1964)

$$\dot{x}_i(t) = \sum\nolimits_j a_{ij}(x_j(t) - x_i(t)) \quad \forall i = 1, \ldots, n. \tag{5}$$

Here the matrix $\boldsymbol{A}$ with non-negative entries $a_{ij} \geq 0$ need not be stochastic. Consensus is equivalent to the existence of a root in the graph $\mathcal{G}[\boldsymbol{A}]$ (whereas the aperiodicity condition may be discarded). A detailed analysis of Abelson's model is available in (Proskurnikov and Tempo, 2017).

Since social groups often fail to reach consensus in spite of the connectivity, DeGroot's model is not "rich" enough to explain their behaviors. One of the explanations for persistent disagreement between the actors is their "anchorage" at initial positions. This phenomenon is captured by the seminal *Friedkin-Johnsen* (FJ) model (Friedkin, 2015)

$$\begin{gathered} x_i(k+1) = \lambda_i \sum\nolimits_{j=1}^{n} a_{ij} x_j(k) + (1-\lambda_i) x_i(0), \forall i, \\ \boldsymbol{x}(k) = \boldsymbol{\Lambda}\boldsymbol{A}\boldsymbol{x}(k) + (\boldsymbol{I}_n - \boldsymbol{\Lambda})\boldsymbol{x}(0). \end{gathered} \tag{6}$$

Here $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in [0, 1]$ is actor i's susceptibility to social influence; the maximally susceptible ($\lambda_i = 1$) individuals obey the usual DeGroot's equation (2), whereas totally insusceptible ($\lambda_i = 0$) are stubborn and "stuck" at the initial opinions $x_i(k) \equiv x_i(0)$.

In the case of $\boldsymbol{\Lambda} = \boldsymbol{I}_n$, the Friedkin-Johnsen model boils down to (3). For $\boldsymbol{\Lambda} \neq \boldsymbol{I}_n$, the matrix $\boldsymbol{\Lambda}\boldsymbol{A}$ is typically Schur stable (with eigenvalues in the open unit disk) and the opinion vector in (6) converges:

$$\boldsymbol{x}(k) \xrightarrow[k\to\infty]{} \boldsymbol{V}\boldsymbol{x}(0), \quad \boldsymbol{V} = (\boldsymbol{I}_n - \boldsymbol{\Lambda}\boldsymbol{A})^{-1}(\boldsymbol{I}_n - \boldsymbol{\Lambda}). \tag{7}$$

This convergence takes place, for instance, if $\mathcal{G}[\boldsymbol{A}]$ is strongly connected and $\boldsymbol{\Lambda} \neq \boldsymbol{I}_n$. Along with scalar opinions, one may consider *multidimensional* opinions, conveniently represented by *rows* $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{im})$, describing the i-th actor's positions on m issues. Stacking these rows into an $n \times m$ matrix $\boldsymbol{X}$, the FJ model becomes

$$\boldsymbol{X}(k) = \boldsymbol{\Lambda}\boldsymbol{A}\boldsymbol{X}(k) + (\boldsymbol{I}_n - \boldsymbol{\Lambda})\boldsymbol{X}(0). \tag{8}$$

Unlike many other models proposed in the literature, the FJ model has been validated on numerous experiments with small and medium-size groups (Friedkin and Johnsen, 2011; Friedkin et al., 2019; Friedkin and Bullo, 2017). A continuous-time counterpart of the FJ model was proposed earlier by Taylor (1968). The FJ and Taylor models have been recently extended to describe dynamics of *belief systems* that can be considered as multidimensional opinions on several logically related topics (Friedkin et al., 2016a; Parsegov et al., 2017; Ye et al., 2020).

3.2 Further development: nonlinear and stochastic dynamics

Whereas the aforementioned linear models of opinion formation may seem too simplistic, many advanced models studied in the literature essentially stem from them and partially inherit their structures. Due to limited space, we outline here only several classes of advanced models, more details are available in (Mastroeni et al., 2019; Proskurnikov and Tempo, 2018).

Gossip-based interactions. A principal limitation of the DeGroot and the Friedkin-Johnsen models is the assumption of *simultaneous* opinion update, which can apply only to face-to-face interactions in a small group. The standard way to take the spontaneity of social interactions into account is to assume that these interactions are *gossip-based*, that is, at each stage only one pair (or a few pairs) of randomly chosen individuals interact, whereas the opinions of the remaining individuals are unaltered. Randomized gossip-based counterparts of the DeGroot model (2) (with $\boldsymbol{A}(k)$ random) inherit its basic properties, in particular, convergence of the opinions to a consensus value under appropriate connectivity assumptions (Fagnani and Zampieri, 2008). Gossip-based versions of the FJ model (Parsegov et al., 2017) exhibit more sophisticated dynamics: the opinions do not converge but oscillate persistently, whereas the convergence property (7) can be proved only for their expected or time-averaged values.

Dynamic influence weights. Another serious limitation is the assumption of time-invariant distribution of influence between individuals: each actor i assigns to self and other actors j some static weight a_{ij}. In reality, the interrelations between individuals evolve due to numerous endogenous and exogenous processes. One such process, modeled as *reflected appraisal* dynamics, is discussed in Section 5. Many dynamic effects are caused by the phenomena of homophily and social selection: people wish to interact with like-minded individuals and assimilate their opinions more readily than dissimilar ones.

The idea of homophily naturally leads to a plethora of *bounded confidence* models, obtained from the DeGroot (2) and Abelson (5) models by introducing *distance-dependent* influence weights a_{ij}. The idea of using nonlinear coupling functions in models of attitude change in order to explain the opinion disagreement appears in Abelson (1964); however, the first bounded confidence model was proposed in Krause (2000) and is now referred to as the *Hegselmann-Krause* (HK) model (Hegselmann and Krause, 2002). The HK model is a nonlinear modification of (2), where

$$a_{ij}(k) = \begin{cases} \dfrac{1}{|N_i(k)|}, & |x_j(k) - x_i(k)| \le R_i, \\ 0, & |x_j(k) - x_i(k)| > R_i. \end{cases}$$

Here $|N_i(k)|$ denotes the cardinality of the set $N_i(k) = \{j : |x_j(k) - x_i(k)| \le R_i\}$. In other words, each individual assimilates only the opinions falling into his/her *confidence interval* (paying equal attention to them), whereas the opinions beyond this interval are ignored. Even if the initial graph $\mathcal{G}[\boldsymbol{A}(0)]$ is strongly connected, the connectivity can be broken as the opinions evolve. For this reason, unlike the time-invariant DeGroot's model, HK model admits both consensus and *clustering* of the terminal opinions.

The HK model has been thoroughly studied in the *homogeneous case* (where all confidence bounds are equal $R_1 = \ldots = R_n = R$). In particular, it is known the homogeneous model always terminates in at most $O(n^3)$ steps. Even in this situation, however, there many open questions related to the structure of emerging opinions clusters, such as e.g. the so-called "$2R$-conjecture" (Kolarijani et al., 2021). The behavior of opinions in the *heterogeneous* case remains an open problem except for special situations (Chazelle and Wang, 2017). Although the convergence of opinions is observed in extensive numerical simulations, its mathematical proof, to the best of the authors' knowledge, is still elusive. A gossip-based version of the HK model is known as the Deffuant-Weisbuch model (Deffuant et al., 2001). For a survey of recent results on dynamics of bounded confidence models, the reader is referred to (Proskurnikov and Tempo, 2018; Kolarijani et al., 2021).

Negative ties and structural balance. Abelson (1964) suggested that one of the reasons for disagreement can be "boomerang" (reactance, anticonformity) effect: an attempt to convince other people can cause them to adopt an opposing position. Hence, the convex-combination mechanisms (2) and (5) have to be generalized to allow repulsion between the opinions. Although presence of negative ties in opinion formation models has not been secured experimentally and is questioned in some recent works (Takács et al., 2016), "coopetitive" (cooperative and competitive) networks arise in abundance in biology and economics.

Natural extensions of the French-DeGroot and the Abelson models have been proposed by Altafini (2012). For instance, the discrete-time Altafini model (Liu et al., 2017) obeys equation (2), where a_{ij} can be positive and negative, however, the matrix of absolute values $(|a_{ij}|)$ is stochastic. The behavior of such a model is determined by an important property, termed the *structural balance* (Heider, 1946), that is, the possibility of dividing the network into two opposing factions, such that the members within a single faction positively influence each other, whereas the relations between members of different factions is negative. It appears that a structurally balanced and strongly connected graph induces bimodal polarization (or "bipartite consensus") of the opinions: the actors in the two opposing factions converge on the two opposite opinions x_* and $(-x_*)$, where x_* depends on the initial opinion distribution. An imbalanced strongly connected network exhibits a degenerate behavior where all opinions converge to 0. More sophisticated dynamic models (Shi et al., 2019) that are capable to explain clustering of opinions over some imbalanced signed graphs. *Dynamical mechanisms* that render a signed graph structurally balanced constitute a self-standing important area of research, a survey of recent results is available in Cisneros-Velarde et al. (2020). A recent work by Jia et al. (2016) establishes the relation between dynamics of structural balance and reflected appraisal mechanisms (Section 5).

4. INTERPERSONAL INFLUENCE LEARNING

As discussed in the previous sections, the analysis of opinion dynamics in social networks is a rapidly expanding research field, and the theoretical results are reaching a good level of maturity. On the other hand, it should be noted that most literature on the topic adopts the implicit assumption that *the underlying influence graph is known.*

However, while this premise is indeed well justified in several graph-theoretical control problems, such as distributed or cooperative control over networks or distributed decision-making, it becomes less reasonable when dealing with social networks. If we consider for instance the celebrated Friedkin-Johnsen model discussed in Section 3, it is clear that for studying the model's behavior one needs to quantify the social ties between the individuals (that is the numerical value of the entries a_{ij} of the influence matrix). We note that in their experiments with small groups of individuals, Friedkin and Johnsen (2011) were able to quantify these values by means of an ad-hoc "measuring procedure" : in the post-discussion people were asked to distribute "chips" between the individuals they interacted with, and the chip-count was taken as a subjective measure of influence exercised by other group members. Clearly, such a special procedure is hardly replicable in large-scale social networks, such as those arising in online social networks (OSN). The massive data in OSN consist of linked data describing the communications between any two entities, in the form of text, images, audio, and video.Efficient analytic tools and algorithms to reconstruct influence mechanisms are therefore required.

This motivates the line of work that we briefly review in this section, which we refer to as "network influence inference". In this section, we briefly review some recent techniques proposed in the literature to efficiently tackle this challenging problem. We specially focus on modern methodologies that explicitly take into consideration *dynamics* over networks (Wai et al., 2016; Ravazzi et al., 2018), and leverage the technical background and mathematical foundations introduced in Section 3.

We first observe that the problem discussed in this session represents a special instance of the general problem of reconstructing the graph topology from data measured on the nodes (Wainwright and Jordan, 2008; Koller and Friedman, 2009). This problem, which goes under the name of graph learning or network inference, has seen an increasing interest in the past years: we refer the reader to Dong et al. (2019) for an excellent survey of these techniques. Here, we briefly recall two classes of models, which have gained popularity in literature. Both approaches start from the collection of (in general scalar) measurements $x_i(1), ..., x_i(N)$ at each node $i \in \mathcal{V}$.

Probabilistic graphical models (Jordan, 2004; Koller and Friedman, 2009), stem from the assumption that the observed data are realizations of random independent variables whose joint probability distribution depends on the topology of the graph $\mathcal{G}$. Hence, data are interpreted as outcomes of random experiments, and a graphical model is introduced to capture the conditional dependence between random variables. For undirected graphs and continuous variables, the most adopted models are *Markov random fields* (MRF) (Rue and Held, 2005) and *Gaussian graphical models* (Wainwright and Jordan, 2008). Markov random fields require that the random variables at the different nodes satisfy a series of local Markov properties. In particular, these models assume that the values at two nodes should be conditionally independent, given all other variables, whenever these nodes are not connected by an edge (pairwise Markov property). A commonly employed assumption in probabilistic graphical models is that the observations are realizations of a multivariate Gaussian distribution. In this case, the network inference problem translates into the estimation of the covariance matrix from observed data. This is usually done by constructing the Maximum-Likelihood (ML) estimator as in the Graphical Lasso algorithms (Mazumder and Hastie, 2012).

A class of models for describing signals on a graph that is gaining increasing popularity in the signal processing community is represented by *Graph Signal Processing* (GSP)(Shuman et al., 2013). GSP, generalizing classical signal processing concepts and tools, enables the processing and analysis of data that take values on the vertices of a graphs. The "spatial dynamics" of such signals are assumed to be governed by the underlying graph. While the main directions of research in GSP focus on the development of methods for analyzing signals defined over given *known* graphs, re-defining concepts as such as Fourier transform, filtering and frequency response for data residing on graphs, an interest is arising in studying the dual problem of learning the graph topology from measurements of the signals on the graph, under specific assumptions on the characteristics of its graph Fourier transform. In particular, the most common approach for GSP-based graph topology reconstruction is based on the assumption that the underlying graph signal is *smooth* on the graph. That is, the links in the graph should be chosen in such a way that signals on neighboring nodes are close to each other. As a measure of smoothness of the signal on the graph, the so-called Laplacian quadratic form is usually adopted, see Dong et al. (2016) and references therein.

We remark however that the large majority of techniques discussed in this literature consider graphs with non-dynamical (static) variables, and undirected links. The few available extensions to dynamically varying variables and directed topologies usually turn out to be rather complex.

4.1 Sparsity in social networks

One of the key aspects of opinion networks is that usually each individual carries social ties, and interacts, with a small subset of the total set of the considered individuals. This is in accordance with the general observation that many real world networks exhibit power-law degree distributions, that is the fraction of nodes with degree k follows a distribution that decreases as $k^{-\gamma}$, with $\gamma > 1$. This kind of rapid decrease in the node degree distribution has also been confirmed in social networks. For instance, the empirical degree distribution of the focal nodes of the Facebook Ego-Networks, retrieved from the Stanford Network Database (Leskovec and Krevl, 2014) are well approximated by a power-law with $\gamma \in [1.2, 3]$.

4.2 Model-based social network inference

As previously discussed, most available approaches to network inference do not consider signals evolving in a dynamical way. On the other hand, we have now precise models that describe the opinion evolution on a network. This consideration motivates the necessity of designing inference schemes that start from the assumption that the dynamical model of opinion dynamics is available (as described in Section 3), and address the following question: *Given measurements of the evolution of the opinions, and a model of the opinion evolution, how can one estimate the interaction graph and the strength of the connections?*

This research question has generated a recent line of research, which we aim to briefly overview in this tutorial. First, we point out that two different strategies to estimate the interactions in the network can be considered, that we refer to as *persistent measurement* and *sporadic measurement* identification procedures. In the experiments of the first kind, the opinions are observed during T rounds of conversation and the influence matrix is estimated as the matrix best fitting the dynamics for $0, \ldots, T-1$. In such cases, the problem becomes a "classical" system identification problem, and available results on parsimonious systems identification can be used. However, this approach would require knowing the discrete-time indices for the observations made and to store a sufficiently long subsequence of opinions. Moreover, the system could be updated with an unknown interaction rate and the interaction times between agents may be not observable in most practical scenarios (Timme, 2007). These considerations render the persistent measurement approach not applicable to most practical situations, as also discussed in Wai et al. (2016).

To circumvent these issues, a second class of approaches has recently arisen, which only use sporadic data. In particular, we review here two approaches: in the first one, whose general philosophy was introduced in Wai et al. (2016), the agents interact until their opinions stabilize, and the identification problem considers only the initial and the final opinions. In the second one, it is only assumed that one has access to random measurements of the agents' opinions, and statistics of the measurement process are used to estimate the structure of the social network that generated the measurements. We briefly review these two approaches in the next subsections.

4.3 Infinite horizon approach to influence estimation

We consider the problem of reconstructing the inference matrix $\boldsymbol{A}$ starting from observations of the opinion on n individuals, who simultaneously discuss m different (and unrelated) topics $\mathcal{S} = \{0, 1, 2, \ldots, m-1\}$. First, denoting the opinion of individual i on topic s at time k as $x_i(k, s)$, we can introduce the following opinion matrix at time k

$$\begin{aligned}\boldsymbol{X}(k) &= [\boldsymbol{x}(k,0) \ \cdots \ \boldsymbol{x}(k,m-1)] \\ &= \begin{bmatrix} x_1(k,0) & \cdots & x_1(k,m-1) \\ \vdots & & \vdots \\ x_n(k,0) & \cdots & x_n(k,m-1) \end{bmatrix} \in \mathbb{R}^{n \times m}\end{aligned}$$

which collects the opinions of all individuals (rows) on all topics (columns).

We assume that the opinions evolve according to the multidimensional Friedkin-Johnsen (8). The problem we aim at studying is the following: *Given the prejudices* $\boldsymbol{X}(0)$ *and final opinions* $\boldsymbol{X}(\infty) = \lim_{k\to\infty} \boldsymbol{X}(k)$, *estimate the influence matrix* $\boldsymbol{A}$ *from this data.* As will be clarified later, for simplicity of exposition, we also assume the knowledge of the susceptibility matrix $\boldsymbol{\Lambda}$.

First, it should be noted that some assumptions are necessary in order to render the problem well posed. In particular, we assume here that: i) the initial opinions of the individuals are sufficiently different: for all $\ell \in [m]$ there exist $i, j \in \mathcal{V}$ such that $x_i(0,\ell) \neq x_j(0,\ell)$, ii) at least one individual is not completely open-minded: $\boldsymbol{\Lambda} \neq \boldsymbol{I}_n$, iii) there are no totally stubborn individuals: $\lambda_i > 0$ for all $i \in \mathcal{V}$, and iv) dynamics are stable: for any node $v \in \mathcal{V}$ there exists a path from v to a node m such that $\lambda_m < 1$.

We note that all these assumptions are necessary to guarantee that the identification problem admits a unique solution. In particular, regarding i), it is immediately observed that if the initial opinions are at consensus, then also the final opinions will still be at consensus (with the same value): in this case any stochastic matrix $\boldsymbol{A}$ would be consistent with the data. Assumption iii) and iv) guarantee that all agents' opinions converge to a common value, which is a convex combination of the initial opinions with weights being a function of $\boldsymbol{A}$. Under these assumptions, recovering $\boldsymbol{A}$ amounts at solving the following equations

$$\begin{cases} (\boldsymbol{I}_n - \boldsymbol{\Lambda}\boldsymbol{A})\boldsymbol{X}(\infty) = (\boldsymbol{I}_n - \boldsymbol{\Lambda})\boldsymbol{X}(0), \\ \boldsymbol{A}\boldsymbol{1} = \boldsymbol{1}, \ \boldsymbol{A} \geq 0, \boldsymbol{\Lambda} \geq 0. \end{cases} \tag{9}$$

As discussed in Ravazzi et al. (2018), assumptions i)-iv) are not sufficient to guarantee uniqueness of the solution, due to the ambiguity introduced by the product $\boldsymbol{\Lambda}\boldsymbol{A}$ in (8); see (Ravazzi et al., 2018, Proposition 1). To avoid this ambiguity, for the sake of simplicity, we assume that $\boldsymbol{\Lambda}$ is known. Clearly, whenever $m \geq n$ (and the system in (9) is full rank) the equations (9) admit a unique solution, which can be found by linear programming.

In Ravazzi et al. (2018) it is shown that when the number of topics $m < n$ the sparsity of the network can be exploited to design an algorithm that finds a solution starting from a number of topics that increases only logarithmically with the network size. This is briefly reviewed next.

4.4 Sparse identification

We start from the observation that a social network is typically sparse, in the sense that the interactions among the agents are few when compared to the network dimension. Given $\boldsymbol{\Lambda}$, $\boldsymbol{X}(0)$, and $\boldsymbol{X}(\infty)$, this leads us to estimate the social influence networks by solving a *sparsity problem*. Then, determining the sparsest network that is compatible with the available information can be formulated as the following ℓ_1-minimization problem Candès et al. (2006)

$$\min_{\boldsymbol{A} \in \mathbb{R}^{n\times n}} \|\boldsymbol{A}\|_1, \qquad \text{s.t.} \begin{cases} \boldsymbol{H}\boldsymbol{A}^\top = \boldsymbol{b}^\top, \\ \boldsymbol{A}\boldsymbol{1} = \boldsymbol{1} \end{cases} \tag{10}$$

with $\boldsymbol{H} \doteq \boldsymbol{X}(\infty)^\top = \boldsymbol{X}(0)^\top(\boldsymbol{I}_n - \boldsymbol{\Lambda})(\boldsymbol{I}_n - \boldsymbol{\Lambda}\boldsymbol{A})^{-\top}$ and $\boldsymbol{b} \doteq \boldsymbol{\Lambda}^{-1}[\boldsymbol{X}(\infty) - (\boldsymbol{I}_n - \boldsymbol{\Lambda})\boldsymbol{X}(0)]$. It can be noticed that this problem is separable into n subproblems, and hence each row of $\boldsymbol{A}$ can be learned independently from the others; see again Ravazzi et al. (2018) for details. This type of problem has been extensively studied in the Compressed Sensing (CS) literature (Eldar and Kutyniok, 2012). Several reconstruction algorithms are developed in literature to solve this optimization including iterative algorithms based on convex/nonconvex relaxation with global linear (Daubechies et al., 2010) or local superlinear convergence (Daubechies et al., 2010; Ravazzi and Magli, 2015). Moreover, it is well known that under certain conditions on the matrix $\boldsymbol{H}$, the number of measurements m, and the sparsity of $\boldsymbol{A}$, the exact solution can be obtained (Candès et al., 2006). However, here the sensing matrix $\boldsymbol{H}$ is not fixed, but it depends on the unknowns and is dictated by the evolution of the opinions. This makes it highly nontrivial to apply standard proof techniques from the CS literature [1].

In (Ravazzi et al., 2018, Theorem 1) it is shown that if the initial opinions are independent and have a Gaussian-like distribution and the number of considered topics satisfies the following condition

$$m \geq 4c\frac{(1+\lambda_{\max})^2(1-\lambda_{\min})^2}{(1-\lambda_{\max})^4}d_{\max}\log n \qquad (11)$$

then the solution to (10) provides an exact solution with probability at least $1 - c'\,\mathrm{e}^{-c''m}$, where c, c' and c'' are positive constants, $d_{\max}$ and $\lambda_{\max}$ are the maximum degree of the network and the maximum susceptibility of the agents, respectively.

This result is important because it shows that exact recovery can be generally obtained considering a number of topics that increases only logarithmically with the dimension of the network. Moreover, it provides an explicit dependence of the difficulty of the network reconstruction problem on some characteristic parameters of the network: i) the agents' susceptibility to other opinions cannot be too high: if $\lambda_{\max} \to 1$ then the number of measurements needed for recovery diverges to infinity, ii) it provides insight on the influence of the degree distribution in the social network. For more details and numerical experiments the reader can refer to (Ravazzi et al., 2018).

4.5 Random opinion measurement approach

Recent works (Hojjatinia et al., 2020) propose an approach to the network influence estimation problem based on the availability of measurements of the opinions at random times instants. These techniques assume that the opinions evolve according to gossip-based interactions (see Section 3). In particular, we consider a model proposed in Frasca et al. (2013), in which at each interaction time a subset of nodes is randomly selected from a uniform distribution over $\mathcal{V}$. If the node i is active at time k, agent i interacts with a randomly chosen neighbor j and updates its belief according to a convex combination of its previous belief, the belief of j, and its initial belief.

Given this gossip opinion evolution model, in Hojjatinia et al. (2020) techniques are presented to identify the influence matrix $\boldsymbol{A}$ starting from *partial observations*

$$\boldsymbol{z}(k) = \boldsymbol{P}(k)\boldsymbol{x}(k) \qquad (12)$$

[1] In particular, it can be shown that the sensing matrix $\boldsymbol{H}$ does not satisfy the so-called Restricted Isometry Property (Candes, 2008) yet satisfies the Restricted Eigenvalue Condition (Ravazzi et al., 2018).

where $\boldsymbol{P}(k)$ is a random measurement matrix defined by

$$\boldsymbol{P}(k) = \mathrm{diag}(\boldsymbol{p}(k))$$

and $\boldsymbol{p}(k) \in \{0,1\}^{\mathcal{V}}$ is a random selection vector with known distribution. This setting is rather general, and it captures the following observation schemes:

Intermittent observations. This model captures situations in which the actual rates at which the interactions occur is not perfectly known, and thus sampling time is different from interaction time. This can be obtained setting

$$\boldsymbol{p}(k) = \begin{cases} \mathbf{1} & \text{w.p. } \rho \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

That is, at step $k > 0$ all observations are available with probability ρ.

Independent random sampling This model captures situations in which only a subset of individuals can be contacted (e.g. random interviews). In this case, the selection vector at time k has components $p_i(k)$ distributed according to a Bernoulli distribution with parameters ρ_i for all individuals i, that is the opinions are observed independently with probability $\rho_i \in [0,1]$. When all observations probabilities are equal, i.e. $\rho_i = \rho$ for all i we have *independent and homogeneous sampling*.

The identification problem hence becomes the following: *Given the sequence of observations* $\{\boldsymbol{z}(k)\}_{k=1}^t$ *find an estimate* $\widehat{\boldsymbol{A}}_t$ *of the matrix* $\boldsymbol{A}$. In particular, in Hojjatinia et al. (2020) theoretical conditions are given on the number of samples guaranteeing an error not larger than a fixed tolerance ϵ with high probability.

The techniques developed in Hojjatinia et al. (2020) follow an identification approach inspired by the results in Rao et al. (2017) to reconstruct the influence matrix. Under the same conditions i)-iv) outlined above (9), the dynamics converge almost surely and a sort of Yule-Walker relation used for parameter estimation in autoregressive processes holds. More precisely, it can be shown that

$$\mathbb{E}\left[\boldsymbol{x}(k)\boldsymbol{x}(k+1)^\top\right] = \mathbb{E}\left[\boldsymbol{x}(k)\boldsymbol{x}(k)^\top\right] f(\boldsymbol{A},\boldsymbol{\Lambda}) + \mathbb{E}[(\infty)]g(\boldsymbol{\Lambda}). \qquad (13)$$

where $f(\boldsymbol{A},\boldsymbol{\Lambda}), g(\boldsymbol{\Lambda})$ are linear in $\boldsymbol{A}$ and $\boldsymbol{\Lambda}$.

Based on the collection of partial observations $\boldsymbol{z}(k)$, an approximation of matrices $\mathbb{E}\left[\boldsymbol{x}(k)\boldsymbol{x}(k)^\top\right], \mathbb{E}\left[\boldsymbol{x}(k)\boldsymbol{x}(k+1)^\top\right]$ and $\mathbb{E}\left[\boldsymbol{x}(k)\boldsymbol{x}(k)^\top\right]$ leveraging the Birkhoff ergodic theorem and the information on influence matrix is retrieved exploiting relation (13).

Ongoing research focuses on extensions of the inference method to nonlinear opinion dynamics models, working on real data and combining the use of modern machine learning tools and statistical inference models (Longo et al., 2019).

5. DYNAMICS OF REFLECTED SELF–APPRAISAL

The previous two sections explored a number of works stemming from the fundamental DeGroot and FJ models. In this section, we narrow the focus to a specific and much more recent extension of the DeGroot model, termed the DeGroot–Friedkin model (Jia et al., 2015).

It is common to observe that, over the course of discussion on a sequence of topics within a social network, an individual may become increasingly (or decreasingly) confident in their own opinion as a consequence of observing that they are exerting more and more (or less and less) influence on the outcome of the discussion. This is an example of a *feedback mechanism* in social networks, termed *reflected self-appraisal* (Cooley, 1992; Shrauger and Schoeneman, 1979). The DeGroot–Friedkin model aims to capture this phenomenon, and is consequently highly nonlinear; we elect to provide somewhat more details to help elucidate the modeling and analysis, for the benefit of readers who are interested in how system and control theoretic methods can be applied to expand existing models.

Suppose that individuals in a strongly connected influence network are participating in discussions covering different topics *sequentially*, with the topics indexed a topic sequence $\mathcal{S} = \{0, 1, 2, \dots\}$. We stress that in this section, the network sequentially discusses a single topic at a time, unlike Section 4 which assumed the simultaneous discussion of all topics. It is assumed that each topic is discussed through to consensus (which is possible under the assumptions of Theorem 1), and then after each individual undergoes the appraisal process, the network moves on to discuss the next topic in the sequence.

We aim to understand a person's influence on a discussion outcome, and how the resultant updating of their self-confidence (which depends on their influence) affects discussion on the next topic.

The discussion of each single topic occurs according to the DeGroot model (2), and we *define* $a_{ii} \geq 0$ in (2) as individual i's *self-confidence*. Since a_{ii} may depend on the topic, we write it as $a_{ii}(s)$ and for topic $s \in \mathcal{S}$, the opinion $x_i(k, s)$ of individual i evolves for $k = 0, 1, \dots$ as

$$x_i(k+1, s) = a_{ii}(s)x_i(k, s) + (1 - a_{ii}(s)) \sum_{j \neq i}^{n} c_{ij} x_j(k, s), \tag{14}$$

with $a_{ij}(s) \triangleq (1 - a_{ii}(s))c_{ij}$. Here, $c_{ij} \geq 0$ is assumed to be independent of s (though we will relax this assumption in the sequel) and it captures the *relative trust* individual i accords to individual $j \neq i$. With $c_{ii} = 0$, we also require that $\sum_{j=1}^{n} c_{ij} = 1$ for all $i \in \{1, \dots, n\}$. Then it is evident that as $a_{ii}(s)$ evolves along the topic sequence $s = 0, 1, \dots$ (in a manner described in detail below), there continues to hold $\sum_{j=1}^{n} a_{ij}(s) = 1$ for all $i \in \{1, \dots, n\}$ and for all $s \in \mathcal{S}$, as per the DeGroot model in (2). In summary, the opinion discussion updates for each topic $s \in \mathcal{S}$ via

$$\boldsymbol{x}(k+1, s) = \boldsymbol{A}(s)\boldsymbol{x}(k, s), \tag{15}$$

with

$$\boldsymbol{A}(s) = \mathrm{diag}(a_{ii}(s)) + (\boldsymbol{I}_n - \mathrm{diag}(a_{ii}(s)))\boldsymbol{C} \tag{16}$$

row-stochastic, while evidently the matrix $\boldsymbol{C} = (c_{ij})$ is also row-stochastic. The DeGroot–Friedkin model aims to formulate a dynamics for $a_{ii}(s)$ that captures the mechanism of reflected self-appraisal, which we now address.

5.1 Evolution by Reflected Self-Appraisal

Consider a topic $s \in \mathcal{S}$ and let $\mathcal{G}[\boldsymbol{C}]$ be strongly connected. If (case 1) $a_{ii}(s) < 1 \,\forall i$ and $\exists j : a_{jj}(s) > 0$, then $\mathcal{G}[\boldsymbol{A}(s)]$, with $\boldsymbol{A}(s)$ given by (16), is strongly connected and aperiodic (Jia et al., 2015; Ye et al., 2018). If instead (case 2) $\exists k : a_{jj}(s) = 1$ and $a_{ii}(s) < 1 \,\forall i \neq j$, then $\mathcal{G}[\boldsymbol{A}(s)]$ is rooted, with root node v_j (Jia et al., 2015; Ye et al., 2018). For typical, and virtually generic initial $a_{ii}(0)$, these two cases are the only outcomes for all $s \in \mathcal{S}$. We can then assert that

$$\lim_{k \to \infty} \boldsymbol{x}(k, s) = \boldsymbol{\zeta}^\top(s)\boldsymbol{x}(0, s)\mathbf{1}_n = \sum_{i=1}^{n} \zeta_i(s)x_i(0, s)\mathbf{1}_n. \tag{17}$$

Here, $\boldsymbol{\zeta}^\top(s)$ is the left Perron-Frobenius eigenvector of the SIA matrix $\boldsymbol{A}(s)$ from (1); in case 1, $\zeta_i(s) > 0 \,\forall i$, and in case 2, $\boldsymbol{\zeta}^\top(s) = \boldsymbol{e}_j$. Section 3 explained that the i-th entry $\zeta_i(s)$ is the relative contribution, or *social power*, of individual i in forming the consensus value of topic s. The DeGroot–Friedkin model posits on the conclusion of each topic's discussion, the *reflected self-appraisal* mechanism drives each individual i to update their self-confidence for the next topic to be the social power of the current topic:

$$a_{ii}(s+1) = \zeta_i(s). \tag{18}$$

Of course, for the following topic $s+1$, the influence matrix $\boldsymbol{A}(s+1)$ is obtained from (16) but with $s+1$ replacing s. We have so far assumed that $\boldsymbol{C}$ does not depend on s, i.e. that the relative strength of the interactions *between* individuals, taking no account of self-weighting, is constant with s. If individual 1 treats individual 2 twice as reliable as individual 3 for topic 0, that same proportionality will be maintained over all topics. Importantly, what changes is the *overall weight* individual 1 gives to all opinions other than his or her own. This is because in adjusting the self-weight to ensure $a_{ii}(s+1) = \zeta_i(s)$, a countervailing adjustment for the weighting is placed on all others' opinions; setting $a_{ij}(s+1) = (1 - \zeta_i(s))c_{ij}(s)$ ensures that there holds $\sum_{j=1}^{n} a_{ij}(s+1) = 1$ as required in the DeGroot model.

One key task is to determine the behavior of $\zeta_i(s)$ in progression through the sequence of topics $s = 0, 1, 2, \dots$. To do this, let us first impose the following assumption.

Assumption 1. The graph $\mathcal{G}[\boldsymbol{C}]$ of the relative interaction matrix $\boldsymbol{C}$ is strongly connected, with $n \geq 3$ nodes. The initial conditions [2] satisfy (i) $\exists j : a_{jj}(0) > 0$ and $a_{ii}(0) < 1, \forall i$ or (ii) $\exists i : a_{ii}(0) = 1$ and $a_{jj}(0) < 1, \forall j \neq i$,

The work (Jia et al., 2015) showed that under Assumption 1, one can derive that

$$\boldsymbol{\zeta}(s+1) = \boldsymbol{F}(\boldsymbol{\zeta}(s)) \tag{19}$$

where

$$\boldsymbol{F}(\boldsymbol{\zeta}) = \begin{cases} \mathbf{e}_i & \text{if } \zeta_i = 1 \text{ for any } i \\ \dfrac{1}{\sum_{i=1}^{n} \frac{\gamma_i}{1-\zeta_i}} \begin{bmatrix} \dfrac{\gamma_1}{1-\zeta_1} \\ \vdots \\ \dfrac{\gamma_n}{1-\zeta_n} \end{bmatrix} & \text{otherwise} \end{cases} \tag{20}$$

with $\gamma_i > 0$ being the i^{th} entry of the dominant left eigenvector $\boldsymbol{\gamma}^\top$ of $\boldsymbol{C}$. Ye et al. (2018) show that on Δ_n the map $\boldsymbol{F} : \Delta_n \mapsto \Delta_n$ is smooth, with $\boldsymbol{\zeta}(s) \in \Delta_n$ for all $s > 0$ under Assumption 1. Much of the subsequent work

[2] The work (Jia et al., 2015) first derived $\boldsymbol{F}$ for restricted initial conditions satisfying $\sum_{i=1}^{n} a_{ii}(0) = 1$. The paper Ye et al. (2018) showed $\boldsymbol{F}$ can be defined for the more general case stated here.

on the DeGroot–Friedkin model has focused on analysing (19). We next cover some of the fundamental results.

5.2 Analysis of the DeGroot–Friedkin Model

We summarise a number of results from the papers (Jia et al., 2015; Ye et al., 2018), but first define a topology class giving rise to distinctive convergence outcomes in the DeGroot–Friedkin model.

Definition 1. (Star Graph). A strongly connected graph $\mathcal{G}[\boldsymbol{C}]$ is termed a star graph if and only if there exists a unique node v_i, the 'centre' node, such that every edge in $\mathcal{G}[\boldsymbol{C}]$ is either outgoing or incoming with respect to v_i.

Theorem 2. Consider the system in (19), with $n \geq 3$, under Assumption 1. If $\exists j : a_{jj}(0) = 1$ and $a_{ii}(0) < 1, \forall i \neq j$, then $\boldsymbol{\zeta}(s) = \mathbf{e}_j, \forall s > 0$. Let $\exists i : a_{ii}(0) > 0$ and $a_{jj}(0) < 1, \forall j$. The following convergence results hold:

(1) If $\mathcal{G}[\boldsymbol{C}]$ is a star graph with centre node designated as v_1 (without loss of generality), then $\lim_{s\to\infty} \boldsymbol{\zeta}(s) = \mathbf{e}_1$. Convergence is asymptotic but not exponentially fast. The remaining fixed points of $\boldsymbol{F}$ are given by $\mathbf{e}_j, j \neq 1$, and are unstable.
(2) If $\mathcal{G}[\boldsymbol{C}]$ is not a star graph, then $\lim_{s\to\infty} \boldsymbol{\zeta}(s) = \boldsymbol{\zeta}^*$ exponentially fast, where $\boldsymbol{\zeta}^* \in \text{int}(\Delta_n)$ is the unique fixed point of the map $\boldsymbol{F}$ in $\Delta_n \setminus \{\mathbf{e}_j, j = 1, \ldots, n\}$. All other fixed points $\boldsymbol{F}$ are given by $\mathbf{e}_j, j = 1, \ldots, n$, and are unstable.

The convergence proof in (Jia et al., 2015) used LaSalle's Invariance Principle to demonstrate asymptotic convergence for both star and non-star $\mathcal{G}[\boldsymbol{C}]$, after a separate proof of uniqueness of $\boldsymbol{\zeta}^*$. Exponential convergence for the non-star case was first established in (Ye et al., 2018) using nonlinear contraction analysis relying on the specific functional form of $\boldsymbol{F}$ in (20), and the method simultaneously proved uniqueness of $\boldsymbol{\zeta}^*$. Further, exponential convergence for star graphs was disproved. An alternative proof for (local) exponential convergence was provided in (Anderson and Ye, 2018), relying on a generalization of the Lefschetz–Hopf Theorem of differential topology that at the same time established uniqueness of $\boldsymbol{\zeta}^*$. The approach did not use the specific functional form of (19), but only drew on certain topological properties of the updating map in (20).

Analysis of Final Social Power: One aspect of particular interest is evaluating the limiting social power vector $\boldsymbol{\zeta}^*$ for non-star $\mathcal{G}[\boldsymbol{C}]$, which also describes for each individual the limiting self-confidence (social power). Since (20) is a highly nonlinear mapping, it is generally speaking not possible to obtain an explicit expression for the fixed point $\boldsymbol{\zeta}^*$. Nonetheless, several conclusions can be obtained.

First, for any $i, j \in \{1, \ldots, n\}$, there holds $\zeta_j^* > \zeta_i^*$ if and only if $\gamma_j > \gamma_i$, and $\zeta_j^* = \zeta_i^*$ if and only if $\gamma_j = \gamma_i$ (Jia et al., 2015). Unsurprisingly, the ordering of the γ_i (often termed eigenvector centrality in other network science literature), also reflects the ordering of the final social powers.

Second, Ye et al. (2018) showed that there holds

$$\zeta_i^* \leq \frac{\gamma_i}{1-\gamma_i}, \quad \forall i \in \{1, \ldots, n\}. \tag{21}$$

Thus, there exists an upper bound on the final social power of all individuals in the network. For $\mathcal{G}[\boldsymbol{C}]$ with $\gamma_i \leq 1/3$ for all i, a bound on the rate of convergence rate of (19) to $\boldsymbol{\zeta}^*$ can also be obtained (Ye et al., 2018, Lemma 3).

Dynamic Relative Interaction Topology: A major extension arises from two intuitive observations that in many social networks, (i) individuals may form new friendships, eliminate old ones, or adjust the level of interactions, and (ii) one individual will have different expertise for different topics, which will affect the respect accorded to him or her by other members of the network. This drives us to consider $\mathcal{G}[\boldsymbol{C}(s)] = \mathcal{G}[\boldsymbol{C}_{\sigma(s)}]$, where $\sigma(s)$ is a switching signal capturing the topic-specific nature of the relative interaction matrix $\boldsymbol{C}(s)$. It is assumed that $\sigma(s)$ is independent of $\boldsymbol{\zeta}(s), s \geq 0$. Coping with such time-variation, including importantly periodic time-variation (such as occurs when a committee meets regularly to cycle through at each meeting a standard set of topics), is difficult, but not impossible. The system becomes

$$\boldsymbol{\zeta}(s+1) = \boldsymbol{F}_{\sigma(s)}(\boldsymbol{\zeta}(s)) \tag{22}$$

with $\boldsymbol{F}_{\sigma(s)}$ defined similarly to that in (20) though $\gamma_i(s) = \gamma_{i,\sigma(s)}$ replaces γ_i, for all $i \in \{1, \ldots, n\}$. We now have a nonlinear switched discrete-time system in (22); while the LaSalle-based approach of (Jia et al., 2015) no longer applies, the nonlinear contraction analysis advanced by Ye et al. (2018) for the original dynamics (19) does. For the system (22) and taking $\mathcal{G}[\boldsymbol{C}(s)]$ to be a strongly connected non-star graph for all s and initial conditions satisfying $\exists j : a_{jj}(0) > 0$ and $a_{ii}(0) < 1, \forall i$, there holds

$$\lim_{s\to\infty} \boldsymbol{\zeta}(s) = \boldsymbol{\zeta}^*(s) \tag{23}$$

exponentially fast. One can term $\boldsymbol{\zeta}^*(s), s \geq 0$ as the "unique limiting trajectory" of (22) determined uniquely by the sequence of switching $\boldsymbol{C}_{\sigma(s)}$. Periodic switching is a special case, and in such instances, $\boldsymbol{\zeta}^*(s)$ is a periodic trajectory. The ordering result detailed above has yet to be established for dynamic topology systems. However, and perhaps surprisingly, the upper bound in (21) and convergence rate result detailed below (21), can be established for networks with dynamic topology, with obvious adjustments because convergence occurs to the unique limiting trajectory $\boldsymbol{\zeta}^*(s)$ as opposed to a fixed point $\boldsymbol{\zeta}^*$.

The key conclusion from examining dynamically changing relative interaction topologies is that sequential opinion discussion *removes initial social power/self-confidence exponentially fast*, and self-confidence/social power evolving via reflected self-appraisal eventually depends only on the sequence of topology structures, i.e. the distinct agent-to-agent interactions. We illustrate with a simulation example; Fig. 2 depicts 6 individuals discussing topics over a periodically-varying network. Keeping the network unchanged, we initialize the system with two different sets of initial conditions, $\hat{\boldsymbol{a}}$ and $\tilde{\boldsymbol{a}}$, satisfying Assumption 1, and with $\hat{a}_{ii}(0) \neq \tilde{a}_{ii}(0)$ for every $i \in \{1, \ldots, n\}$. For individual i, the social power trajectory $\zeta_i(s)$ converges exponentially fast (by about the 8th topic) to the unique, periodic trajectory $\zeta_i^*(s), s \geq 0$ independently of the initial self-confidence (dotted line for $\hat{a}_{ii}(0)$, solid line for $\tilde{a}_{ii}(0)$).

Behavior in Self-Appraisal Dynamics: The DeGroot-Friedkin model hypothesizes that reflected self-appraisal leads to the update (18), in which an individual's self-confidence for topic $s+1$, viz. $a_{ii}(s+1)$, is precisely their so-

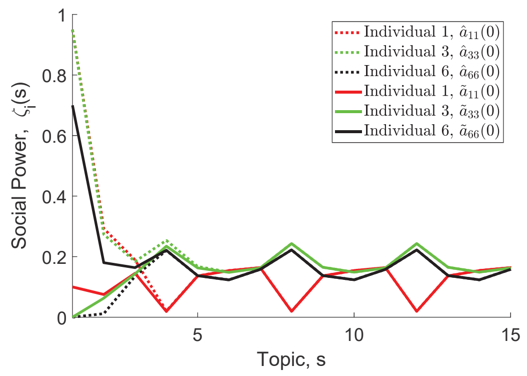


Fig. 2. Evolution of social powers over a sequence of topics for a network of 6 individuals, with periodically-varying network structure. For $i = 1, 3, 6$, we set $\hat{a}_{ii}(0) \neq \tilde{a}_{ii}(0)$. For any individual i, their social power trajectories from different initial conditions (dotted line for $\hat{a}_{ii}(0)$ and solid line for $\tilde{a}_{ii}(0)$) converge to the same unique trajectory $\zeta_i^*(s), s \geq 0$ (Figure from Anderson and Ye (2019)).

cial power in topic s, viz. $\zeta_i(s)$. However, some individuals are under-confident, other individuals are overconfident, and other individuals again may over-react, either positively or negatively depending on the discussion outcome. It is interesting to examine the inclusion of such behaviors using a modification of the basic model, as was done by Ye and Anderson (2019), and as we now describe.

The behavioral aspects are incorporated using a function $\phi_i(\zeta)$ with the following properties.

Assumption 2. For every $i \in \{1, \ldots, n\}$, $\phi_i(x) : [0, 1] \to [0, 1]$ is a smooth, surjective, and monotonically increasing function satisfying $\phi_i = 0 \Leftrightarrow x = 0$ and $\phi_i = 1 \Leftrightarrow x = 1$.

For each individual $i \in \{1, \ldots, n\}$, the self-appraisal dynamics are proposed to be

$$a_{ii}(s + 1) = \phi_i(\zeta_i(s)). \tag{24}$$

An individual i who is *humble* has $\phi_i(\zeta_i) < \zeta_i$ on $(0, 1)$; an *arrogant* individual has $\phi_i(\zeta_i) > \zeta$ on $(0, 1)$; an *emotional* individual has $\phi_i(\zeta_i) < \zeta_i$ on $(0, a)$ for some $a \in (0, 1)$, $\phi_i(a) = a$ and $\phi_i(\zeta_i) > \zeta_i$ on $(a, 1)$; an *unreactive* individual has $\phi_i(\zeta_i) > \zeta_i$ on $(0, a)$ for some $a \in (0, 1)$, $\phi_i(a) = a$ and $\phi_i(\zeta_i) < \zeta_i$ on $(a, 1)$; finally, a *well-adjusted* individual might have $\phi_i(\zeta_i) = \zeta_i$ (i.e. the original model in (18)). Examples of each type of individual are depicted in Fig. 3. This change requires an adjustment of the mapping $\boldsymbol{F}$ in (19), which is replaced with $\bar{\boldsymbol{F}} = \boldsymbol{F} \circ \boldsymbol{\Phi}$:

$$\bar{\boldsymbol{F}} : \boldsymbol{\zeta}(s) \to \boldsymbol{\zeta}(s+1) = \frac{1}{\sum_{j=1}^{n} \frac{\gamma_j}{1-\phi_j(\zeta_j(s))}} \begin{bmatrix} \frac{\gamma_1}{1 - \phi_1(\zeta_1(s))} \\ \vdots \\ \frac{\gamma_n}{1 - \phi_n(\zeta_n(s))} \end{bmatrix}$$

Now it is the properties of this mapping $\bar{\boldsymbol{F}}$ that need to be analysed. Some preliminary results for networks with humble, unreactive or well-adjusted individuals are reported in Ye and Anderson (2019). However, comprehensive convergence results are missing except for networks of just just humble or well-adjusted individuals; then, Theorem 2 applies with some obvious adjustments. Interestingly, simulations reveal that multiple attractive equilibria can exist, and that it is also possible for a single individual (either arrogant or emotional) to end up with all the limiting social power in a network, even when $\mathcal{G}[\boldsymbol{C}]$ is not a star graph.

5.3 Future possible developments

Apart from further study of the dynamics with behavior included, we now briefly touch upon two other directions, as candidates for future work. The first direction is to relax the constraint that self-confidence is updated only when each topic has been fully discussed. Such a model was first alluded to in (Jia et al., 2015), and the work (Jia et al., 2019) recently presented a more complete analysis. The second is to replace the DeGroot update in (14) with an equivalent of the FJ model, via adjustment of (6) by inclusion of the topic index s. For the case of constant C, limited preliminary results can be found in Mirtabatabaei et al. (2014); extensive simulations and empirical validation are reported in Friedkin et al. (2016b); Friedkin and Bullo (2017).

It is worth noting that most works on the DeGroot–Friedkin model focus on *modelling*. In terms of *control*, strategies for increasing or limiting individual social power are of interest, but are yet to be explored. If an individual could optimise the order in which topics in a periodic agenda were addressed, they may be able to acquire more social power. If two or more individuals formed a coalition to mutually support each other's opinion, they may individually acquire more social power. A committee chair could call out arrogant behavior of an individual, to limit their acquisition of social power.

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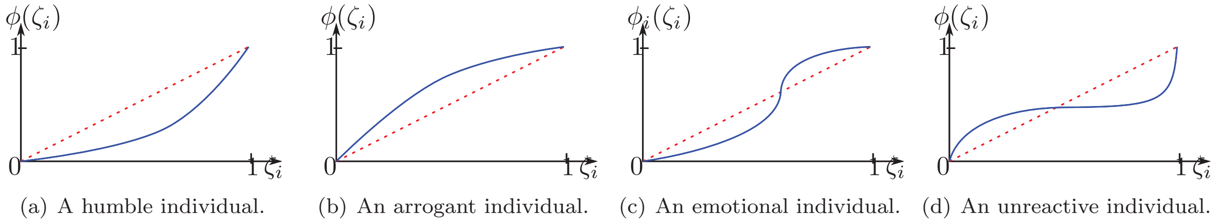

Fig. 3. Illustrative examples of $\phi_i(\zeta_i)$ (blue lines) which distort individual i's self-confidence a_{ii} from true social power ζ_i. The dotted red line is a "well-adjusted" individual from the original DeGroot–Friedkin model, $\phi_i(\zeta_i) = \zeta_i$.

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