DOUBLE PROJECTION ALGORITHMS FOR SOLVING THE SPLIT FEASIBILITY PROBLEMS

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Abstract. We propose two new double projection algorithms for solving the split feasibility problem (SFP). Different from the extragradient projection algorithms, the proposed algorithms do not require fixed stepsize and do not employ the same projection region at different projection steps. We adopt flexible rules for selecting the stepsize and the projection region. The proposed algorithms are shown to be convergent under certain assumptions. Numerical experiments show that the proposed methods appear to be more efficient than the relaxed- CQ algorithm.

1. **Introduction.** The convex feasibility problem(CFP), as an optimization problem [4], is to find a point x^* such that

$$x^* \in \bigcap_{i=1}^{m} C_i,$$

where $m \geq 1$ is an integer, and $C_i, i = 1, 2, \cdots, m$ are nonempty closed convex sets of $\Re^N$. Split feasibility problem (SFP) is the special case of CFP, it is to find a point x^* satisfying

$$x^* \in C, \ Ax^* \in Q, \tag{1}$$

where C and Q are nonempty convex sets of $\Re^N$ and $\Re^M$, respectively, and A is an M by N real matrix. SFP has broad applications in many fields such as approximation theory [9], image reconstruction [5, 13], and so on, which was firstly introduced in Censor and Elfving [6]. The general algorithm is the projection method. Let P_C denote the orthogonal projection onto C; that is, $P_C(x) = \arg \min_{y \in C} \|x - y\|$, over

all $x \in C$. Byrne [3] introduced the so-called CQ algorithm that takes an initial point x^0 arbitrarily, and defines the iterative step as

$$x^{k+1} = P_C(I - \gamma A^T(I - P_Q)A)x^k, \tag{2}$$

where $0 < \gamma < 2/\rho(A^T A)$ and $\rho(A^T A)$ is the spectral radius of $A^T A$. Many projection methods have been developed for solving the SFP, see [1, 2, 3, 10, 19, 20].

Most of these algorithms use invariable stepsize restricted by a Lipschitz constant, which is inflexible and leads to slow convergence. To this case, He *et al* [12] developed a self-adaptive method for solving a variational problem. The numerical results in [12] have shown that the self-adaptive strategy is valid and robust for solving variational inequality problems. Subsequently, a number of self-adaptive projection methods were presented to solve SFP [21, 22, 23, 24, 25], preliminary numerical results show that they are generally promising. The implementation of these algorithms, however, involves the computation of the projections P_C and P_Q and therefore causes additional difficulty in the case where P_C and P_Q do not have closed-form expressions.

Another class of algorithms for SFP that influenced our development for the new algorithms is the extragradient method, which was first introduced by Kinderlehrar [14] to find a solution of variational inequality problem. Later, Nadezhkina and Takahashi introduced an extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the solution set of a variational inequality problem [15]. Furthermore, Ceng *et al* in [8] introduced and analyzed an extragradient method for solving SFP.

In this paper, motivated by self-adaptive method and the extragradient strategy, we propose two double projection algorithms for SFP, which use different variable stepsize at different projection steps, instead of the same stepsize as in [8, 14, 15]. In the same time, the next iteration x^{k+1} generated by our algorithms is a projection either on the current projection region or on the intersection of the set C with a halfspace. The algorithms are shown to be globally convergent to a solution under certain mild assumptions. Numerical experiments show that the proposed methods are more efficient than the existing projection methods.

The main features of the proposed algorithms are

1. The new algorithms employ different variable stepsize at different projection steps, instead of using the fixed stepsize;

2. The Armijo linear search rule at the first projection step is different from the rules in [23, 24, 25]. The purpose of our Armijo linear search is to construct a hyperplane which strictly separates the current point x^k from the solution set;

3. The next iteration x^{k+1} generated by the new algorithms is the projection either on the current projection region or on the intersection of the set C with a halfspace instead of only on the current projection region as the previous algorithms. This will improve the efficiency of convergence without paying essential additional cost in computation.

2. **Preliminaries.** Let I denote the identity operator, $\text{Fix}(T)$ denote the set of the fixed points of an operator T *i.e.*, $\text{Fix}(T) := \{x \mid x = Tx\}$. Select Γ denote the solution set of SFP, that is,

$$\Gamma = \{y \in C \mid Ay \in Q\}. \tag{3}$$

The following definitions and results will be used later on.

Definition 2.1. Let $f : \Re^N \to \Re$ be convex. The subdifferential of f at x is defined as
$$\partial f(x) = \{\xi \in \Re^N | \ f(y) \geq f(x) + \langle \xi, y - x \rangle, \quad \forall \ y \in \Re^N\}.$$
An element of $\partial f(x)$ is said to be a subgradient.

Lemma 2.2. [11, 18] *Suppose that $f : \Re^N \to \Re$ is convex. Then its subdifferential are uniformly bounded on any bounded subsets of $\Re^N$.*

Definition 2.3. Given $T : \Re^N \to \Re^N$,
a) T is said to be monotone if
$$\langle T(x) - T(y), x - y \rangle \geq 0, \forall x, y \in \Re^N;$$
b) T is said to be nonexpansive if
$$\|T(x) - T(y)\| \leq \|x - y\|, \forall x, y \in \Re^N;$$
c) T is said to be co-coercive on $\Re^N$ with modulus $\alpha > 0$, if
$$\langle T(x) - T(y), x - y \rangle \geq \alpha \|T(x) - T(y)\|^2, \forall x, y \in \Re^N;$$
d) T is said to be Lipschitz continuous on $\Re^N$ with constant $L > 0$, if
$$\|T(x) - T(y)\| \leq L\|x - y\|, \forall x, y \in \Re^N.$$

Let $P_C(x)$ be the projection of x on C.

Lemma 2.4. [26] *Let C be a nonempty closed convex subset in $\Re^N$. Then for any $x, y \in \Re^N$ and $z \in C$,*
(1) $\langle P_C(x) - x, z - P_C(x) \rangle \geq 0$;
(2) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle$;
(3) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|P_C(x) - x\|^2$;
(4) $\|P_C(x) - P_C(y)\| \leq \|x - y\| - \|P_C(x) - x + y - P_C(y)\|$.

Remark 1. From part (2) of Lemma 2.4, we know that P_C is a monotone, co-coercive with modulus 1 and nonexpansive operator. Moreover, the operator $I - P_C$ is also co-coercive with modulus 1.

Lemma 2.5. [16]. *Let F be a mapping from $\Re^N$ into $\Re^N$. For any $x \in \Re^N$ and $\alpha \geq 0$, define $x(\alpha) = P_C(x - \alpha F(x))$ and $e(x, \alpha) = x - x(\alpha)$. Then, we have*
$$\min\{1, \alpha\}\|e(x, 1)\| \leq \|e(x, \alpha)\| \leq \max\{1, \alpha\}\|e(x, 1)\|.$$

3. **A double projection algorithm and its convergence.** As in [21], the following conditions are supposed to be satisfied:
  (H1) The set C is defined as
$$C = \{x \in \Re^N \mid c(x) \leq 0\},$$
where $c : \Re^N \to \Re$ is convex and C is nonempty.
  The set Q is defined as
$$Q = \{y \in \Re^M \mid q(y) \leq 0\},$$
where $q : \Re^M \to \Re$ is convex and Q is nonempty.
  (H2) For any $x \in \Re^N$ and $y \in \Re^M$, a subgradient $\xi \in \partial c(x)$ and a subgradient $\eta \in \partial q(y)$ can be calculated.
  We define the following halfspaces at point x^k, respectively,
$$C_k = \{x \in \Re^N \mid c(x^k) + \langle \xi^k, x - x^k \rangle \leq 0\},$$

where $\xi^k \in \partial c(x^k)$, and

$$Q_k = \{y \in \Re^M \mid q(Ax^k) + \langle \eta^k, y - Ax^k \rangle \le 0\},$$

where $\eta^k \in \partial q(Ax^k)$.

Obviously, by the definition of subgradient, we know that the orthogonal projections onto C_k and Q_k may be computed directly by reason of the specific forms of C_k and Q_k, see [1].

In the following, for every k, we define the function $F_k : \Re^N \to \Re^N$ as

$$F_k(x) := A^T(I - P_{Q_k})Ax$$

and respectively define

$$x(\beta_k) := P_{C_k}(x^k - \beta_k F_k(x^k)) \text{ and } e(x^k, \beta_k) := x^k - x(\beta_k).$$

By Lemma 8.1 in [2], the operator F_k is $1/\rho(A^TA)$-inverse strongly monotone or co-coercive with modulus $1/\rho(A^TA)$ and Lipschitz continuous with $\rho(A^TA)$, where $\rho(A^TA)$ is the largest eigenvalue of the matrix A^TA.

Now, we describe our first double projection algorithm.

Algorithm 3.1
Step 0. Select an point $x^0 \in C$ arbitrarily, parameter $\gamma > 0, l \in (0,1), \lambda > 1, t_k \in \Theta = [t_{\min}, t_{\max}]$ for some fixed $0 < t_{\min} < t_{\max} < 2$. Set $k = 0$.
Step 1. Find $y^k = P_{C_k}(x^k - \beta_k F_k(x^k))$, where $\beta_k = \gamma l^{m_k}$ and m_k is the smallest nonnegative integer such that

$$\langle F_k(x^k), x^k - y^k \rangle \ge \lambda \langle F_k(x^k) - F_k(y^k), x^k - y^k \rangle. \tag{4}$$

Step 2. Compute

$$x^{k+1} = P_{C_k}[x^k - t_k \frac{\langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|^2} F_k(y^k)]. \tag{5}$$

Set $k = k + 1$ and go to Step 1.

In fact, (4) is well defined, we can see that from following lemma.

Lemma 3.1. *There exists a nonnegative number m_k satisfying (4), for $k \ge 0$.*

Proof. By (2) of Lemma 2.4, we have

$$\langle F_k(x^k), x^k - y^k \rangle = \frac{1}{\beta_k} \langle \beta_k F_k(x^k), P_{C_k}(x^k) - P_{C_k}(x^k - \beta_k F_k(x^k)) \rangle$$

$$\ge \frac{1}{\beta_k} \|x^k - y^k\|^2. \tag{6}$$

By the inequality $\langle a, b \rangle \le \frac{\|a\|^2}{2} + \frac{\|b\|^2}{2}$ and the nonexpansiveness of F_k, we get

$$\langle F_k(x^k) - F_k(y^k), x^k - y^k \rangle \le \frac{\|F_k(x^k) - F_k(y^k)\|^2}{2} + \frac{\|x^k - y^k\|^2}{2}$$

$$\le \frac{\rho(A^TA)^2 + 1}{2} \|y^k - x^k\|^2, \tag{7}$$

where $\rho(A^TA)$ is the largest eigenvalue of the matrix A^TA. Obviously, there must exist a constant m such that $\frac{1}{\gamma l^m} \ge \frac{\lambda(\rho(A^TA)^2+1)}{2}$. Hence

$$\langle F_k(x^k), x^k - y^k \rangle \geq \frac{1}{\gamma l^m} \|x^k - y^k\|^2 \geq \frac{\lambda(\rho(A^T A)^2 + 1)}{2} \|x^k - y^k\|^2$$

$$\geq \lambda \langle F_k(x^k) - F_k(y^k), x^k - y^k \rangle,$$

the proof is completed. □

Lemma 3.2. $\frac{l}{\lambda(\rho(A^T A)^2 + 1)} < \beta_k \leq \gamma$ for all $k = 0, 1, \cdots$.

Proof. Obviously, from (4) we know that $\beta_k \leq \gamma$ for all $k = 0, 1, \cdots$,
we only need to show $\frac{l}{\lambda(\rho(A^T A)^2 + 1)} < \beta_k$.

Set $y^{\frac{\beta_k}{l}} = P_{C_k}(x^k - \frac{\beta_k}{l} F_k(x^k))$. From the search rule (4), we know that β_k/l must violate inequality (4), *i.e.*,

$$\langle F_k(x^k), y^{\frac{\beta_k}{l}} - x^k \rangle < \lambda \langle F_k(y^{\frac{\beta_k}{l}}) - F_k(x^k), y^{\frac{\beta_k}{l}} - x^k \rangle.$$

Then, from (6) and (7), we get

$$\frac{l}{\beta_k} \|y^{\frac{\beta_k}{l}} - x^k\|^2 \leq \langle F_k(x^k), y^{\frac{\beta_k}{l}} - x^k \rangle$$

$$< \lambda \langle F_k(y^{\frac{\beta_k}{l}}) - F_k(x^k), y^{\frac{\beta_k}{l}} - x^k \rangle \leq \lambda(\rho(A^T A)^2 + 1) \|y^{\frac{\beta_k}{l}} - x^k\|^2,$$

that is,

$$\frac{l}{\beta_k} < \lambda(\rho(A^T A)^2 + 1).$$

Hence

$$\frac{l}{\lambda(\rho(A^T A)^2 + 1)} < \beta_k.$$

This completes the proof. □

Lemma 3.3. *Suppose* $\Gamma \neq \emptyset$ *and the sequences* $\{x^k\}$ *and* $\{y^k\}$ *are generated by Algorithm 3.1. Then,* $-F_k(y^k)$ *is a descent direction of the function* $\frac{1}{2} \|x - z\|^2$ *at the point* x^k, *where* $z \in \Gamma$.

Proof. From (4) and (6), one has

$$\langle F_k(y^k), x^k - y^k \rangle = \langle F_k(y^k) - F_k(x^k), x^k - y^k \rangle + \langle F_k(x^k), x^k - y^k \rangle$$

$$\geq (1 - \frac{1}{\lambda}) \langle F_k(x^k), x^k - y^k \rangle$$

$$\geq (1 - \frac{1}{\lambda}) \frac{1}{\beta_k} \|x^k - y^k\|^2,$$

that is

$$\langle F_k(y^k), x^k - y^k \rangle \geq (1 - \frac{1}{\lambda}) \frac{1}{\gamma} \|x^k - y^k\|^2 \geq 0. \tag{8}$$

Obviously, for $z \in \Gamma$, $F_k(z) = 0$. Since F_k is monotonic and $z \in \Gamma$, we have

$$\langle F_k(y^k), x^k - z \rangle = \langle F_k(y^k), x^k - y^k \rangle + \langle F_k(y^k), y^k - z \rangle$$

$$\geq \langle F_k(y^k), x^k - y^k \rangle + \langle F_k(z), y^k - z \rangle,$$

that is,

$$\langle F_k(y^k), x^k - z \rangle \geq \langle F_k(y^k), x^k - y^k \rangle. \tag{9}$$

Combining (9) with (8), we obtain the result. □

Remark 2. From the monotonicity of F_k, we know that

$$\langle F_k(y^k), z - y^k \rangle \leq \langle F_k(z), z - y^k \rangle = 0,$$

along with (9), we obtain that the hyperplane

$$H_k := \{x \in \Re^N | \langle F_k(y^k), x - y^k \rangle = 0\}$$

separates the current point from the set Γ.

Remark 3. Lemma 3.3 gives us the reason why we use $-F_k(y^k)$ as the iterative direction at Step 2 to obtain the next iteration. In fact, iterate along this direction makes the iteration to become nearer to the solution point as seen from the proof of the following theorem.

Theorem 3.4. *Let $\{x^k\}$ be a sequence generated by Algorithm 3.1. If $\Gamma \neq \emptyset$, then $\{x^k\}$ converges to a solution of the SFP.*

Proof. Let $\alpha_k = \frac{\langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|^2}$. Pick $z \in \Gamma$, we divide the rest of the proof into three steps.

Step 1. We show that $\{x^k\}$ is bounded.

Since $C \subset C_k, Q \subset Q_k$, then $z = P_C(z) = P_{C_k}(z)$ and $Az = P_Q(Az) = P_{Q_k}(Az)$, hence, $z \in C_k$ and $F_k(z) = 0$ for all $k = 0, 1, 2, \cdots$. Using (3) of Lemma 2.4 and (9), we have

$$
\begin{aligned}
\|x^{k+1} - z\|^2 &= \|P_{C_k}[x^k - t_k \alpha_k F_k(y^k)] - z\|^2 \\
&\leq \|x^k - z - t_k \alpha_k F_k(y^k)\|^2 \\
&= \|x^k - z\|^2 - 2t_k \alpha_k \langle F_k(y^k), x^k - z \rangle + t_k^2 \alpha_k^2 \|F_k(y^k)\|^2 \\
&\leq \|x^k - z\|^2 - 2t_k \alpha_k \langle F_k(y^k), x^k - y^k \rangle + t_k^2 \alpha_k^2 \|F_k(y^k)\|^2.
\end{aligned}
$$

Hence,

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - t_k(2 - t_k) \frac{\langle F_k(y^k), x^k - y^k \rangle^2}{\|F_k(y^k)\|^2}. \tag{10}$$

From (8), we get

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - t_k(2 - t_k)(1 - \frac{1}{\lambda})^2 \frac{1}{\gamma^2} \frac{\|x^k - y^k\|^4}{\|F_k(y^k)\|^2}. \tag{11}$$

By (11), we know that for all k,

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2, \tag{12}$$

which shows that the sequence $\{x^k\}$ is bounded.

Step 2. We claim that

$$\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0. \tag{13}$$

Clearly,

$$\|F_k(y^k)\| = \|F_k(y^k) - z\| + \|z\| \leq \rho(A^T A)\|x^k - \beta_k F_k(x^k) - z\| + \|z\|$$

$$\leq \rho(A^T A)\|x^k - z\| + \gamma\|F_k(x_k)\| + \|z\|. \tag{14}$$

In fact, by the boundedness of $\{x^k\}$ and the continuity of F_k, we know that $\{F_k(x^k)\}$ is also bounded. Thus, from (14), there exists a constant $M > 0$ such that

$\|F_k(y^k)\| \leq M$ for all k. Consequently, we obtain from (11) and the definition of t_k that

$$\lim_{k \to \infty} \|x^k - y^k\| = 0. \tag{15}$$

Moreover, by (3) of Lemma 2.4 and Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
\|x^{k+1} - x^k\| &= \|P_{C_k}[x^k - \alpha_k t_k F_k(y^k)] - x^k\| \\
&\leq \|P_{C_k}[x^k - \alpha_k t_k F_k(y^k)] - P_{C_k}(x^k)\| \\
&\leq \|\alpha_k t_k F_k(y^k)\| \\
&\leq t_k \|x^k - y^k\|.
\end{aligned}
$$

Thus, we get the result.

Step 3. We will show that $x^k \to \bar{x} \in \Gamma$.

Assume that $\bar{x}$ is an accumulation point of $\{x^k\}$ and $x^{k_i} \to \bar{x}$, where $\{x^{k_i}\}_{i=1}^\infty$ is a subsequence of $\{x^k\}$. We are ready to show that $\bar{x}$ is a solution of the SFP.

First, we show that $\bar{x} \in C$. Since $\{x^{k_i+1}\} \in C_{k_i}$, then by the definition of C_{k_i}, we have

$$c(x^{k_i}) + \langle \xi^{k_i}, x^{k_i+1} - x^{k_i} \rangle \leq 0, \forall i = 1, 2, \cdots.$$

Passing onto the limit in this inequality and taking into account (16) and Lemma 2.2, we obtain that

$$c(\bar{x}) \leq 0.$$

Hence, $\bar{x} \in C$.

Next, we need to show $A\bar{x} \in Q$. Define

$$e_k(x, \mu) = x - P_{C_k}(x - \mu F_k(x)), k = 0, 1, 2, \cdots$$

Then from Lemma 2.5, the definition of μ_k and equation (15), we have

$$\lim_{k_i \to \infty} \|e_{k_i}(x^{k_i}, 1)\| \leq \lim_{k_i \to \infty} \frac{\|x^{k_i} - y^{k_i}\|}{\min\{1, \mu_{k_i}\}}$$

$$\leq \lim_{k_i \to \infty} \frac{\|x^{k_i} - y^{k_i}\|}{\min\{1, \underline{\beta}\}} = 0, \tag{16}$$

where $\underline{\beta} = \frac{l}{\lambda(\rho(A^T A)^2 + 1)}$. Using part (1) of Lemma 2.4 and note that $x^* \in C_{k_i}$, we have for all $i = 1, 2, \cdots$,

$$\langle x^{k_i} - F_{k_i}(x^{k_i}) - P_{C_{k_i}}(x^{k_i} - F_{k_i}(x^{k_i})), x^* - P_{C_{k_i}}(x^{k_i} - F_{k_i}(x^{k_i})) \rangle \leq 0,$$

that is,

$$\langle e_{k_i}(x^{k_i}, 1) - F_{k_i}(x^{k_i}), x^{k_i} - x^* - e_{k_i}(x^{k_i}, 1) \rangle \geq 0.$$

From the above inequality and (1) of Lemma 2.4, we know for all $i = 1, 2, \cdots$,

$$
\begin{aligned}
&\langle x^{k_i} - x^*, e_{k_i}(x^{k_i}, 1) \rangle \\
\geq\ & \|e_{k_i}(x^{k_i}, 1)\|^2 - \langle F_{k_i}(x^{k_i}), e_{k_i}(x^{k_i}, 1) \rangle + \langle F_{k_i}(x^{k_i}), x^{k_i} - x^* \rangle \\
=\ & \|e_{k_i}(x^{k_i}, 1)\|^2 - \langle F_{k_i}(x^{k_i}), e_{k_i}(x^{k_i}, 1) \rangle \\
& + \langle F_{k_i}(x^{k_i}) - F_{k_i}(x^*), x^{k_i} - x^* \rangle \\
=\ & \|e_{k_i}(x^{k_i}, 1)\|^2 - \langle F_{k_i}(x^{k_i}), e_{k_i}(x^{k_i}, 1) \rangle \\
& + \langle (I - P_{Q_{k_i}})(Ax^{k_i}) - A^T(I - P_{Q_{k_i}})(Ax^*), Ax^{k_i} - Ax^* \rangle \\
\geq\ & \|e_{k_i}(x^{k_i}, 1)\|^2 - \langle F_{k_i}(x^{k_i}), e_{k_i}(x^{k_i}, 1) \rangle \\
& + \|(I - P_{Q_{k_i}})(Ax^{k_i}) - A^T(I - P_{Q_{k_i}})(Ax^*)\|^2,
\end{aligned}
$$

hence,

$$\langle x^{k_i} - x^*, e_{k_i}(x^{k_i}, 1) \rangle \leq \|e_{k_i}(x^{k_i}, 1)\|^2 - \langle F_{k_i}(x^{k_i}), e_{k_i}(x^{k_i}, 1) \rangle + \|(I - P_{Q_{k_i}})(Ax^{k_i})\|^2. \tag{17}$$

Since

$$\|F_{k_i}(x^{k_i})\| = \|F_{k_i}(x^{k_i}) - F_{k_i}(x^*)\| \leq \rho(A^T A)\|x^{k_i} - x^*\|, \forall i = 1, 2, \cdots,$$

and $\{x^{k_i}\}$ is bounded, the sequence $\{F_{k_i}(x^{k_i})\}$ is also bounded. Therefore, from (16) and (17) we get

$$\lim_{k_i \to \infty} \|(I - P_{Q_{k_i}})(Ax^{k_i})\| = 0,$$

that is,

$$\lim_{k_i \to \infty} P_{Q_{k_i}}(Ax^{k_i}) - Ax^{k_i} = 0. \tag{18}$$

Since $P_{Q_{k_i}}(Ax^{k_i}) \in Q_{k_i}$, we have

$$q(Ax^{k_i}) + \langle \eta^{k_i}, P_{Q_{k_i}}(Ax^{k_i}) - Ax^{k_i} \rangle \leq 0.$$

Letting $k_i \to \infty$, taking into account Lemma 2.2 and (18), we deduce that

$$q(A\bar{x}) \leq 0,$$

that is, $A\bar{x} \in Q$. Therefore, $\bar{x}$ is a solution of the SFP.

Thus, we may use $\bar{x}$ in place of z in (11) and obtain that $\{\|x^k - \bar{x}\|\}$ is convergent. Since there is a subsequence $\{\|x^{k_i} - \bar{x}\|\}$ converging to 0, we have $x^k \to \bar{x}$ as $k \to \infty$. This completes the proof. $\square$

4. **A modified double projection algorithm and its convergence.** In this section, we present a modification of Algorithm 3.1 that is more efficient by computational experience.

Algorithm 4.1

Step 0. Choose an arbitrary point $x^0 \in C$, parameter $\gamma > 0, l \in (0, 1), \lambda > 1, t_k \in \Theta = [t_{min}, t_{max}]$ for some fixed $0 < t_{min} < t_{max} < 2$. Set $k = 0$.

Step 1. Find $y^k = P_{C_k}(x^k - \beta_k F_k(x^k))$, where $\beta_k = \gamma l^{m_k}$ and m_k is the smallest nonnegative integer such that

$$\langle F_k(x^k), x^k - y^k \rangle \geq \lambda \langle F_k(x^k) - F_k(y^k), x^k - y^k \rangle.$$

Construct

$$H_k := \{x \in \Re^N | \langle F_k(y^k), x - y^k \rangle \leq 0\}.$$

Step 2. Compute

$$x^{k+1} = P_{C_k \bigcap H_k}[x^k - t_k \frac{\langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|^2} F_k(y^k)].$$

Set $k = k + 1$ and go to Step 1.

Theorem 4.1. *Suppose that the solution set Γ of the SFP is nonempty. Then any sequence $\{x^k\}$ generated by Algorithm 4.1 converges to a solution of the SFP.*

Proof. The proof of Theorem 4.1 is similar to the proof for Theorem 3.4, so we provide only a sketch. Select $z \in \Gamma$, $z^k = x^k - t_k \frac{\langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|^2} F_k(y^k)$. By Lemma 2.4, we obtain

$$\begin{aligned} 0 &\geq \langle z^k - x^{k+1}, z - x^{k+1} \rangle \\ &= \|x^{k+1} - z^k\|^2 + \langle z^k - x^{k+1}, z - z^k \rangle, \end{aligned}$$

which means
$$\langle z - z^k, x^{k+1} - z^k \rangle \geq \|x^{k+1} - z^k\|^2.$$

Therefore,

$$
\begin{aligned}
\|x^{k+1} - z\|^2 &\leq \|z^k - z\|^2 + \|x^{k+1} - z^k\|^2 + 2\langle z^k - z, x^{k+1} - z^k \rangle \\
&\leq \|z^k - z\|^2 - \|x^{k+1} - z^k\|^2 \\
&= \|x^k - z\|^2 - \|x^{k+1} - z^k\|^2 \\
&\quad + (\frac{\langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|^2})^2 t_k^2 \|F_k(y^k)\|^2 \\
&\quad - 2t_k \frac{\langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|^2} \langle F_k(y^k), x^k - z \rangle \\
&= \|x^k - z\|^2 - \|x^{k+1} - z^k\|^2 - t_k(2 - t_k)(\frac{\langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|})^2 \\
&\quad - \frac{2t_k \langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|^2} \langle F_k(y^k), y^k - z \rangle.
\end{aligned}
$$

From (8) and the monotonicity of F_k, we know that

$$\frac{2t_k \langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|^2} \langle F_k(y^k), y^k - z \rangle > 0.$$

Hence

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \|x^{k+1} - z^k\|^2 - t_k(2 - t_k)(\frac{\langle F_k(y^k), x^k - y^k \rangle}{\|F_k(y^k)\|})^2. \quad (19)$$

The rest of the convergence proof is identical to that of Theorem 3.4. □

Remark 4. The main difference between Algorithm 3.1 and Algorithm 4.1 is the projection region in the second projection step. Algorithm 3.1 selects projection on the current projection region C_k, while Algorithm 4.1 selects projection on the section $C_k \bigcap H_k$(regress projection region) which guarantees that the next iterate is more closer to the solution set. On the other hand, in theory, comparing (19) and (11), we can see that the iterative sequence generated by Algorithm 4.1 is closer to the solution set Γ than the iterative sequence generated by Algorithm 3.1 for the term $\|x^{k+1} - z^k\|^2$ in (19) at each iterate. These are just the aim of our selection projection on the section $C_k \bigcap H_k$ in Algorithm 4.1.

5. **Numerical results.** In this section, we will test two numerical examples (Example 5.1 is selected from [17]) to show our algorithms converge faster than the algorithm in [16] (we denote it by CQ-Algorithm). Throughout the computational experiments, we set $\varepsilon = 10^{-4}$ as the stop criterion. In the algorithms, we take $\lambda = 20, \gamma = 10, l = 0.01$ in Algorithm 3.1 and Algorithm 4.1. All codes are written in MATLAB7.0.

Example 5.1. Let

$$
A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 2 & 0 & 2 \end{bmatrix}
$$

$C = \{x \in \Re^3| \; x_1 + x_2^2 + 2x_3 \leq 0\}$; $Q = \{x \in \Re^3| \; x_1^2 + x_2 - x_3 \leq 0\}$. Find $x \in C$ with $Ax \in Q$.

TABLE 1.   The numerical results of Example 5.1

Initiative point	CQ Algorithm with stepsize $\frac{1}{\rho(A^T A)}$)	Algorithm 3.1	Algorithm 4.1
$x^0 =$ $(-5, -2, -10)^T$	$k = 269; s = 0.063$ $x^* = (0.5071, -1.8186, 01.9072)^T$	$k = 54; s = 0.101$ $x^* = (0.8718; -1.6577; -1.4080)^T$	$k = 28; s = 0.077$ $x^* = (1.1595; -1.0082; 0 - 1.0814)^T$
$x^0 =$ $(-2, -1, -5)^T$	$k = 261; s = 0.063$ $x^* = (0.1098; -1.7655; -1.6134)^T$	$k = 16; s = 0.061$ $x^* = (0.6814; -1.4212; -1.0762)^T$	$k = 1; s = 0.045$ $x^* = (0.4734; -1.7714; -1.3758)^T$
$x^0 =$ $(-6, 0, -1)^T$	$k = 6450; s = 0.525$ $x^* = (-3.9899; -0.6144; 1.8062)^T$	$k = 59; s = 0.096$ $x^* = ((-3.8898; -0.5850; 1.9604)^T$	$k = 1; s = 0.048$ $x^* = (-3.9302, -1.0861, 1.9786)^T$

TABLE 2.   The numerical results of Example 5.2

M, N	CQ Algorithm with stepsize $\frac{1}{\rho(A^T A}$	t_k	Algorithm3.1	Algorithm 4.1
$M = 20, N = 10$	$k = 485, s = 1.040$	0.8	$k = 274, s = 0.312$	$k = 210, s = 0.270$
		1.0	$k = 193, s = 0.100$	$k = 108, s = 0.070$
		1.8	$k = 103, s = 0.067$	$k = 64, s = 0.021$
$M = 100, N = 90$	$k = 3987, s = 3.100$	0.4	$k = 1534, s = 0.690$	$k = 1244, s = 0.530$
		1	$k = 1074, s = 0.500$	$k = 630, s = 0.261$
		1.6	$k = 674, s = 0.201$	$k = 412, s = 0.132$

Example 5.2. Let $A = (a_{ij})_{M \times N}$, $a_{ij} \in (0,1)$ be a random matrix, M, N be two positive integers. $C = \{x \in \Re^N | \sum_{l=1}^N x_l^2 \leq r^2\}$; $Q = \{x \in \Re^M | x \leq b\}$. To ensure the existence of the solution of the problem, the vector b is generated by using the following way: Given a random $N-$dimensional negative vector (each component is negative) $z \in C, r = \|z\|$, taking $b = Az$. Find $x \in C$ with $Ax \in Q$. We take $e_0 = (0, 0, \cdots, 0)$ as the initial point in this example.

The numerical results of Examples 5.1-5.2 can be seen from Tables 1 and 2. In these tables , "k" ," s" and "x^{*}" denote the number of iterations, cpu time in seconds and the solution, respectively.

Table 1 gives the numerical results of Example 5.1 with the CQ-Algorithm, Algorithm 3.1, and Algorithm 4.2, respectively , for the case $t_k = 1$. Table 2 shows the numerical results of Example 5.2 with the same algorithms , respectively, for different t_k.

From Tables 1 and 2, we can see that our algorithms are effective and they converge more quickly than the CQ Algorithm and Algorithm 4.1 converges more quickly than Algorithm 3.1.

6. **Some concluding remarks.** This paper presents two double projection methods with different rules of stepsize selection for solving SFP. The first projection step, different from the self-adaptive projection methods proposed by Zhang *et al* [23] using the co-coercivity and presented by Qu *et al* [16] using the Lipschitz continuity property of the gradient mappings F and F_k, employs a new liner-search rule, which assures the hyperplane H_k separate the current x^k and the solution set Γ. The second projection step of Algorithm 3.1 uses the parameters t_k and α_k to decide the stepsize under current projection region; the second projection step of Algorithm 4.1 modifies the projection region which results in good convergence.

Preliminary numerical results show that our methods are practical and promising for solving SFP.

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