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# A CLASS OF TWO-STAGE DISTRIBUTIONALLY ROBUST GAMES

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Abstract. An $N$-person noncooperative game under uncertainty is analyzed, in which each player solves a two-stage distributionally robust optimization problem that depends on a random vector as well as on other players' decisions. Particularly, a special case is considered, where the players' optimization problems are linear at both stages, and it is shown that the Nash equilibrium of this game can be obtained by solving a conic linear variational inequality problem.

## 1. **Introduction.**

The study of game theory can be traced back to Von Neumann's pioneer work in the 1920s and it has been developed extensively since 1950s following Nash's pioneer work [29, 30]. However, most studies assume that the games are played in a deterministic setting, in which the players compete with each other without considering future uncertainties. In such a deterministic noncooperative game, each decision maker behaves selfishly to optimize one's own objective function, which is parameterized by the rivals' decisions. The study aims to find a Nash equilibrium (NE), which is defined as a set of decisions for the players, such that no one can perform better by changing his/her decision if the other players do not change theirs [29, 30]. Generally, there are two ways of computing the NE. The first method is based on the ad hoc study of the games by exploring their special structures, for example, the potential games [28] and supermodular games [43]. The other method is to solve the Nash equilibrium problem by converting it to a variational inequality (VI). This idea was firstly introduced by Bensoussan in [5], and has become a mainstream approach through the study of finite-dimensional VIs and complementarity problems (CPs) [12, 13, 14, 15, 20, 38].

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A more practical game should incorporate random factors in the players' decisions [24]. Since decisions under uncertainty are usually multi-stage in nature, which means that a decision must be made at present to minimize the sum of a current cost and an expected payoff of possible corrective actions in future stages, each player in such game has to solve a multi-stage stochastic optimization problem that is parameterized by a random vector, as well as by the rivals' decisions. Several studies of the multi-stage games have been conducted in the literature [10, 39, 45]. One typical case of such problems has been recently studied by Pang et al [34] and their research indicated that, in contrast to the deterministic games, the stochastic factor may bring essential complications such as nondifferetiability and nonmonotonicity into the VI formulation of the games, hence greatly complicates the solution procedure.

A possible way of getting around the above complications is to consider a "distributionally robust" version of the game under uncertainty, which is the purpose of this paper. In the distributionally robust model, each player solves a parameterized stochastic multi-stage optimization problem, in which the cost of corrective future actions is measured not by the usual expectation, but by the worst-case expectation over a set of possible probability distributions of the random vectors involved. We aim at a basic two-stage model of this paper, where each player solves a two-stage distributionally robust stochastic programming problem parameterized by a random vector and the rivals' decisions. Note that the assumption of each player facing an individual random vector is not restrictive at all since it is allowed in our analysis that some or all the components of the random vectors of different players can be shared. Among the two-stage models, of particular interest to us is the two-stage distributionally robust stochastic linear programming (TSDRSLP) since it could be applicable to areas as production planning [2, 6, 16, 17, 23], finance [26, 27], and other applications [9, 42].

Another advantage of the distributionally robust models is their computational tractability. In the classic multistage stochastic optimization models, for the random vector with a discrete distribution, the probability structure is known as a "scenario tree". Therefore, as the number of random variables increases, the computation overhead is prohibitive, and this is called "curse of dimensionality" in practice. In a distributionally robust stochastic optimization model, however, the distribution of the random vector is assumed not exactly known. It is assumed that the distribution belongs to an "ambiguity set" that is defined via specifying certain constraints on the distribution such as given support and given first and second order moments, etc. Since the second stage objective function in the distributionally robust model is the worst-case expectation over the ambiguity set, the computation of expectation is replaced by a solution to an optimization problem. Consequently, the distributionally robust formulation can avoid the curse of dimensionality and often result in polynomial solvability of the problems. Numerical experiments of this methodology have been very promising [2, 16, 17, 18].

In this paper, we adopt the distributionally robust approach in the study of games with stochastic factors. We introduce a distributionally robust model of the two-stage game under uncertainty, in which each player solves a TSDRSLP problem. We show that this approach results in a monotone conic VI that is computationally tractable under mild assumptions. Therefore, the pitfalls such as nonsmoothness and nonmonotonicity in general stochastic games are removed.

It should be noted that some references [1, 21, 31, 32] have addressed the games with worst-case objectives. However, the treatment therein is restricted to the uncertainty sets rather than the ambiguity sets, therefore they would not able to incorporate more information such as given bound of moments when it is available. Thus, our formulation is more general than the literature above.

The rest of this paper is organized as follows. In Section 2, we study a general model of two-stage distributionally robust game and reduce it to a semi-infinite optimization model. In Section 3, we establish the conic optimization model of the TSDRSLP problem with "WKS-type" of ambiguity set. In Section 4, we develop a conic VI formulation of the two-stage distributionally robust game (TSDRG), which provides a general approach to solving the game problem. Conclusions are presented in Section 5.

**Notations.** We denote a random vector, say $\tilde{z}$, with the tilde sign. Matrices and vectors are represented as upper and lower case letters, respectively. Given a regular (i.e. pointed, proper, and with nonempty interior) cone $\mathcal{K}\mathcal{K}$ such as the positive orthant $\mathbb{R}^n_+$, the second-order cone, or the semidefinite cone, for any two vectors $x, y$, the notation $x \preceq_{\mathcal{K}\mathcal{K}} y$ or $y \succeq_{\mathcal{K}\mathcal{K}} x$ means $y - x \in \mathcal{K}\mathcal{K}$. The dual cone of $\mathcal{K}\mathcal{K}$ is denoted by

$$\mathcal{K}\mathcal{K}^* = \{y : \langle y, x\rangle \geq 0, \forall x \in \mathcal{K}\mathcal{K}\}.$$

The set $\mathcal{P}\mathcal{P}_0(\mathbb{R}^m)$ represents the space of probability distributions of a random vector in $\mathbb{R}^m$.

2. **TSDRG: The general case.** Consider an $N$-person game where player $i$'s optimization problem is

$$\min_{x_i \in X_i} \zeta_i(x_i, x_{-i}) \equiv \left\{\theta_i(x_i, x_{-i}) + \sup_{\mathbb{P} \in \mathcal{P}_i} \mathbb{E}_{\mathbb{P}}\left[Q_i(x_i, x_{-i}, \tilde{z}_i)\right]\right\} \tag{2.1}$$

where the set $\mathcal{P}_i$ is the ambiguity set defined as in [44] (details will be given in Section 2.1), $\mathbb{E}_{\mathbb{P}}$ stands for the expectation under probability measure $\mathbb{P}$, $x_i \in \mathbb{R}^{n_i}$ is the vector of player $i$'s first-stage decision variable subject to a feasible region $X_i \subseteq \mathbb{R}^{n_i}$, $\tilde{z}_i$ is the random vector defined on a probability space $(\mathbb{R}^{m_i}, \mathcal{F}, \mathbb{P})$, which is realized after $x_i$ is chosen but before the second-stage decision $y_i \in \mathbb{R}^{k_i}$ is made, where $\mathcal{F}$ is the $\sigma$-algebra generated by subsets of $\mathbb{R}^{m_i}$ and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$, and $Q_i(x_i, x_{-i}, \tilde{z}_i)$ is the corresponding second stage recourse function, i.e.

$$Q_i(x_i, x_{-i}, \tilde{z}_i) = \min_{y_i \in \mathbb{R}^{k_i}} \left\{r_i(y_i) : G_i(x_i, x_{-i}, y_i, \tilde{z}_i) \leq 0\right\}, \tag{2.2}$$

where $G_i : \mathbb{R}^{k_i} \to \mathbb{R}^{m_i}$ for each fixed $x_i$, $x_{-i}$ and $\tilde{z}_i$. Let

$$X = X_1 \times \cdots \times X_N.$$

**The blanket assumption.** We assume that every $r_i(\cdot)$ and every component function of $G_i(\cdot)$ are convex and continuously differentiable and that $Q_i(x_i, x_{-i}, \tilde{z})$ is finite and continuous in $\tilde{z}$ for all $x \in X$.

2.1. **The Weisemann-Kuhn-Sim (WKS)-type of ambiguity set.** Consider the random vector $\tilde{z}_i$ in (2.2). From a practical point of view, it is more flexible to describe the support set $\Omega_i$ by a cone constraint. In addition, it would be technically convenient to allow the random vector $\tilde{z}_i$ to be associated with an artificial random vector $\tilde{u}_i \in \mathbb{R}^{t_i}$ and therefore we introduce a random vector $\tilde{w}_i = (\tilde{z}_i, \tilde{u}_i) \in \mathbb{R}^{m_i} \times \mathbb{R}^{t_i}$, where $\tilde{z}_i$ is an "original" part and $\tilde{u}_i$ is an "auxiliary" part. For instance, the epigraph of a function $\tilde{u}_i = f(\tilde{z}_i)$ is a set of random vectors $(\tilde{z}_i, \tilde{u}_i)$. More advantages of such specification can be seen in [44] , where it is shown that many commonly used statistics can be cast into conic constraints if we employ artificial random variables. Let $\mathcal{PP}_0(\mathbb{R}^{m_i} \times \mathbb{R}^{t_i})$ represent the space of all probability distributions on $\mathbb{R}^{m_i} \times \mathbb{R}^{t_i}$. We define the WKS-type of ambiguity set as

$$\mathcal{PP}_i = \left\{ \ \mathbb{P} \in \mathcal{PP}_0(\mathbb{R}^{m_i} \times \mathbb{R}^{t_i}) : \ \ \mathbb{E}_{\mathbb{P}}[E_i\tilde{z}_i + F_i\tilde{u}_i] = g_i, \mathbb{P}[(\tilde{z}_i, \tilde{u}_i) \in \Omega_i] = 1 \ \ \right\}, \tag{2.3}$$

where $E_i \in \mathbb{R}^{p_i \times m_i}, F_i \in \mathbb{R}^{p_i \times t_i}, g_i \in \mathbb{R}^{p_i}$, and the set $\Omega_i$ is of full dimension, bounded and representable by a cone inequality

$$\Omega_i = \{(z_i, u_i) : G_i z_i + H_i u_i \preceq_{\mathcal{KK}_i} h_i\} \tag{2.4}$$

with $G_i \in \mathbb{R}^{r_i \times m_i}, H_i \in \mathbb{R}^{r_i \times t_i}$, and $h_i \in \mathbb{R}^{r_i}$. When the random vector $\tilde{u}_i$ is absent from (2.3) and (2.4), we may regard $F_i$ and $H_i$ as zero matrices, or null matrices at our convenience. If the cone $\mathcal{KK}_i$ in (2.4) is the positive orthant, the set $\Omega_i$ is obviously a polyhedron. If $\mathcal{KK}_i$ is a second-order cone (a semidefinite cone, respectively), we call $\Omega_i$ a second-order-cone representable set (semidefinite-cone representable set, respectively).

The above ambiguity set is less general than the ambiguity set defined in [44] due to the more restrictive format of function $Q_i(x_i, x_{-i}, \tilde{z}_i)$. However, we believe it is general enough to cover possible applications of (2.1).

The term of "distributionally robust" came from a recent paper of Wiesemann, Khun and Sim [44] on distributionally robust convex optimization. In a nutshell, paper [44] considers how to convert a convex constraint

$$\sup_{\mathbb{P} \in \mathcal{PP}} \mathbb{E}_{\mathbb{P}}[v(x, \tilde{z})] \leq \nu$$

to a set of cone constraints. In this paper, we apply the conceptual framework of [44] to (2.1), and discuss how to convert the game (2.1) into a conic VI problem for the linear case. The challenge here is that the $v(x, \tilde{z})$ in [44] is a function with an explicit formula, while in the current paper, $Q_i(x_i, x_{-i}, \tilde{z}_i)$ is an optimal value function of the second-stage problem in a game, which not only depends on player $i$'s decision, but also depends on other players' decisions. Thus, more involved analysis is required.

Historically, the notion of distributionally robust stochastic optimization has been explored to some extent by Scarf [37], Landua [25], Dupacova [11], Kall and Wallace [23], and others, but those studies did not result in efficient algorithms. Stemmed from the recent developments in robust optimization [3] and stochastic VI [36], some papers explored various algorithms for TSDRSLP under different specifications of the set $\mathcal{PP}_i$. See for instances [2, 7, 8, 16, 17, 26, 27, 41]. A major contribution of [44] is to provide a very general format for the set $\mathcal{PP}_i$ that can be tailored into many important applications. We call the format (2.3)-(2.4) the "WKS-type" of ambiguity set to acknowledge it.

2.2. **Reduction of player $i$'s problem to a semi-infinite program.** We next prove that the players' problems can be converted into semi-infinite programming problems.

Note that the worst-case recourse value in (2.1) is the optimal value of the following optimization problem.

$$\begin{array}{ll} \max\limits_{\mathbb{P}} & \mathbb{E}_{\mathbb{P}}\left(Q_i(x_i, x_{-i}, \tilde{z}_i)\right) \\ \text{s. t.} & \mathbb{E}_{\mathbb{P}}(E_i\tilde{z}_i + F_i\tilde{u}_i) = g_i \\ & \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{[(\tilde{z}_i,\tilde{u}_i)\in\Omega_i]}) = 1, \end{array} \tag{2.5}$$

where $\mathbf{1}_{[(\tilde{z}_i,\tilde{u}_i)\in\Omega_i]}$ is the indicator function of $\Omega_i$.

Using the duality theory of linear optimization in probability spaces (see Rockafellar [35]), the dual problem of (2.5) is the following semi-infinite optimization problem.

$$\begin{array}{ll} \min\limits_{\beta_i,\eta_i} & g_i^\top \beta_i + \eta_i \\ \text{s. t.} & (E_i z_i + F_i u_i)^\top \beta_i + \eta_i \geq Q_i(x_i, x_{-i}, z_i), \ \ \forall\, (z_i, u_i) \in \Omega_i, \\ & \beta_i \in \mathbb{R}^{p_i}, \ \eta_i \in \mathbb{R}. \end{array} \tag{2.6}$$

By similar deductions with Lemma 3 in [41], one can obtain the following strong duality proposition for problems (2.5) and its dual (2.6).

**Proposition 2.1.** *Strong duality holds between problems (2.5) and (2.6) in the sense that the primal problem is solvable and* inf *(2.6)* = sup *(2.5).*

A somewhat restrictive condition for the strong duality can be found in [22, Theorem 6.5], which requires that $\Omega_i$ is compact. Since $\Omega_i$ is indeed compact by assumption, together with the generalized Slater condition (called the superconsistent condition in [22]), it also guarantees the strong duality by [22, Theorem 6.5].

In conclusion, Proposition 2.1 indicates that player $i$'s problem (2.1) can be converted to the following semi-infinite program:

$$\begin{array}{ll} \min\limits_{x_i,\beta_i,\eta_i} & \theta_i(x_i, x_{-i}) + g_i^\top \beta_i + \eta_i \\ \text{s. t.} & (E_i z_i + F_i u_i)^\top \beta_i + \eta_i \geq Q_i(x_i, x_{-i}, z_i), \ \ \forall\, (z_i, u_i) \in \Omega_i, \\ & x_i \in X_i. \end{array} \tag{2.7}$$

3. **TSDRG: The linear case.** Consider an important special case of (2.1), in which $\theta_i(x_i, x_{-i}) \equiv c_i^\top x_i$ and

$$Q_i(x_i, x_{-i}, \tilde{z}_i) \equiv \begin{array}{ll} \min\limits_{y_i \geq 0} & d_i^\top y_i \\ \text{s.t.} & A_i(\tilde{z}_i)x_i + A_{-i}(\tilde{z}_i)x_{-i} + D_i(\tilde{z}_i)y_i = b_i(\tilde{z}_i). \end{array} \tag{3.1}$$

In the game, player $i$ has to solve a TSDRSLP

$$\min_{x_i \in X_i} \left\{ c_i^\top x_i + \sup_{\mathbb{P}\in\mathcal{P}_i} \mathbb{E}_{\mathbb{P}}\left[Q_i(x_i, x_{-i}, \tilde{z}_i)\right] \right\}. \tag{3.2}$$

We consider TSDRSLP (3.2) with a fixed recourse, namely, assuming that $D_i(\tilde{z}_i) \equiv D_i$. Besides, assume that the uncertain data $b_i(\tilde{z}_i)$, $A_i(\tilde{z}_i)$, and $A_{-i}(\tilde{z}_i)$, together with the vector $y_i(\tilde{z}_i)$, in (3.1) are affinely dependent on the random vector $\tilde{z}_i$, namely

$$\begin{array}{ll} y_i(\tilde{z}_i) = y_i^0 + \sum\limits_{j=1}^{m_i} \tilde{z}_i^j y_i^j, & b_i(\tilde{z}_i) = b_i^0 + \sum\limits_{j=1}^{m_i} \tilde{z}_i^j b_i^j, \\ A_i(\tilde{z}_i) = A_i^0 + \sum\limits_{j=1}^{m_i} \tilde{z}_i^j A_i^j, & A_{-i}(\tilde{z}_i) = A_{-i}^0 + \sum\limits_{j=1}^{m_i} \tilde{z}_i^j A_{-i}^j \end{array} \tag{3.3}$$

where $b_i^j \in \mathbb{R}^{l_i}$, $A_i^j \in \mathbb{R}^{l_i \times n_i}$, and $A_{-i}^j \in \mathbb{R}^{l_i \times (n-n_i)}$, $j = 0, \ldots, m_i$, $n = \sum_{i=1}^N n_i$, are deterministic and given in advance; and $y_i^0, y_i^1, ..., y_i^{m_i}$ are decision vectors. Since each $y_i^j$ is a $k_i$-dimensional vector, define the $k_i \times (m_i+1)$ matrix $Y_i$ as

$$Y_i = [y_i^0, y_i^1, ..., y_i^{m_i}] = [y_i^0, Y_i^{\bar{0}}] \in \mathbb{R}^{k_i} \times \mathbb{R}^{k_i \times m_i}$$

and denote the $q$th row vector of $Y_i^{\bar{0}}$ by $y_i^q$, i.e.,

$$y_i^q = [(y_i^1)_q, (y_i^2)_q, ..., (y_i^{m_i})_q]^\top \in \mathbb{R}^{m_i}.$$

Note that $y_i^q$ is a column vector.

We shall refer the above fixed-recourse and the affine-dependence assumption as *the linear decision rule*, which is often adopted in dealing with the uncertainties in robust optimization models. See, e.g., Ben-Tal and Nemirovski [4]. It could be thought of as a first-order approximation to a nonlinear relationship between $(A_i, b_i, y_i)$ and $\tilde{z}_i$. Chen et al [8] used it in the context of robust stochastic programming. Also, it is used for dealing with joint chance constraints by Chen et al [7]. It is easy to see that the following equivalence holds.

$$\begin{array}{l} A_i(z_i)x_i + A_{-i}(z_i)x_{-i} + D_i y_i(z_i) = b_i(z_i),\ \forall z_i \in \Omega_i \Longleftrightarrow \\ A_i^j x_i + A_{-i}^j x_{-i} + D_i y_i^j = b_i^j,\ \forall\ j = 0, 1, ..., m_i, \end{array} \tag{3.4}$$

if $\Omega_i \neq \emptyset$. Moreover, we have

**Proposition 3.1.** *Under the linear decision rule with the WKS-type of ambiguity set, it holds that*

$$\begin{array}{rll} \mathbb{E}_{\mathbb{P}}\ [Q_i(x_i, x_{-i}, \tilde{z}_i)] = \mathbb{E}_{\mathbb{P}} \Bigg[ \min\limits_{Y_i, s_i} & d_i^\top y_i^0 + \sum\limits_{j=1}^{m_i} d_i^\top y_i^j \tilde{z}_i^j \Bigg] & \\ \text{s. t.} & A_i^j x_i + A_{-i}^j x_{-i} + D_i y_i^j = b_i^j,\ j = 0, 1, ..., m_i, & \\ & y_i^0(q) + h_i^\top s_i^q \geq 0,\ q = 1, 2, ..., k_i, & (3.5) \\ & G_i^\top s_i^q = y_i^q,\ q = 1, 2, ..., k_i, & \\ & H_i^\top s_i^q = 0,\ q = 1, 2, ..., k_i, & \\ & s_i^q \in \mathcal{KK}_i^*,\ q = 1, 2, ..., k_i. & \end{array}$$

*Proof.* Given (3.4), we only need to show that

$$\left\{ \begin{array}{l} y_i(z_i) \geq 0,\ \forall (z_i, u_i) \in \Omega_i \Longleftrightarrow \\ \min\{y_i^0 + \sum_{j=1}^{m_i} z_i^j y_i^j : G_i z_i + H_i u_i \preceq_{\mathcal{KK}_i} h_i\} \geq 0 \Longleftrightarrow \\ \exists s_i^q \in \mathcal{KK}_i^* \text{ s.t. } \ y_i^0(q) + h_i^\top s_i^q \geq 0,\ G_i^\top s_i^q = y_i^q,\ H_i^\top s_i^q = 0,\ \forall\ q = 1, \ldots, k_i. \end{array} \right.$$

The first equivalence is obvious, so we only need to prove the second equivalence. Since $\Omega_i$ is of full dimension, the problem

$$\min\{y_i^0(q) + \sum_{j=1}^{m_i} z_i^j y_i^q(j) : G_i z_i + H_i u_i \preceq_{\mathcal{KK}_i} h_i\}$$

satisfies the Slater condition. Therefore the strong duality of cone optimization ([3, Theorem A.2.1]) holds and we have

$$\begin{array}{rl} & \min\{y_i^0(q) + \sum\limits_{j=1}^{m_i} z_i^j y_i^q(j) : G_i z_i + H_i u_i \preceq_{\mathcal{KK}_i} h_i\} \\ = & \max\{y_i^0(q) + h_i^\top s_i^q : G_i^\top s_i^q = y_i^q, H_i^\top s_i^q = 0, s_i^q \in \mathcal{KK}_i^*\} \quad \forall q = 1, \cdots, k_i. \end{array}$$

The proposition follows by noting that

$$\begin{aligned}
&\max\left\{y_i^0(q)+h_i^\top s_i^q : G_i^\top s_i^q = y_i^q, H_i^\top s_i^q = 0, s_i^q \in \mathcal{K}\mathcal{K}_i^*,\ \forall q=1,\dots,k_i\right\} \geq 0\\
&\iff \text{ The system } \left\{\begin{array}{l} y_i^0(q)+h_i^\top s_i^q \geq 0,\\ G_i^\top s_i^q = y_i^q,\\ H_i^\top s_i^q = 0,\\ s_i^q \in \mathcal{K}\mathcal{K}_i^*, \end{array}\right. \quad \text{is feasible for } q=1,2,...,k_i.
\end{aligned}$$

□

In view of Proposition 2.1 and Proposition 3.1, we have the following.

**Proposition 3.2.** *Under the linear decision rule, (3.2) is equivalent to*

$$\begin{aligned}
\min_{x_i,Y_i,S_i,\beta_i,\eta_i}\quad & c_i^\top x_i + g_i^\top \beta_i + \eta_i\\
\text{s.t.}\quad & (E_i z_i + F_i u_i)^\top \beta_i + \eta_i \geq \min_{Y_i,S_i}\left(d_i^\top y_i^0 + \sum_{j=1}^{m_i} d_i^\top y_i^j z_i^j\right), \forall\,(z_i,u_i)\in\Omega_i,\\
& A_i^j x_i + A_{-i}^j x_{-i} + D_i y_i^j = b_i^j,\ j=0,1,...,m_i,\\
& y_i^0(q) + h_i^\top s_i^q \geq 0,\ q=1,2,...,k_i,\\
& G_i^\top s_i^q = y_i^q,\ q=1,2,...,k_i,\\
& H_i^\top s_i^q = 0,\ q=1,2,...,k_i,\\
& x_i\in X_i, \beta_i\in\mathbb{R}^{p_i},\ \eta_i\in\mathbb{R},\ s_i^q\in\mathcal{K}\mathcal{K}_i^*,\ q=1,2,...,k_i.
\end{aligned} \tag{3.6}$$

We may further simplify (3.6) to an easier handling form. To this end, define

$$\mathcal{F}_i := \left\{ (x_i, x_{-i}, Y_i, S_i) : \begin{array}{l} A_i^j x_i + A_{-i}^j x_{-i} + D_i y_i^j = b_i^j,\ j=0,1,...,m_i,\\ x_i\in X_i,\ x_{-i}\in X_{-i},\ s_i^q \in \mathcal{K}\mathcal{K}_i^*,\ y_i^0(q)+h_i^\top s_i^q \geq 0,\\ G_i^\top s_i^q = y_i^q,\ H_i^\top s_i^q = 0,\ \ q=1,2,...,k_i. \end{array} \right\} \tag{3.7}$$

Clearly, $\mathcal{F}_i$ is a closed convex set. Its projections onto the $(Y_i,S_i)$-space are defined as

$$\Pi_{Y_i,S_i} := \{(Y_i,S_i) : \exists\ x_i\in X_i, x_{-i}\in X_{-i} \text{ such that } (x_i,x_{-i},Y_i,S_i)\in\mathcal{F}_i\}.$$

Obviously, we have $\Pi_{Y_i,S_i} \neq \emptyset$, otherwise it is contradicted with the blanket assumption in Section 2.1.

We next prove that the dual problem (3.6) can be written in a simpler form.

**Proposition 3.3.** *Problem (3.6) is equivalent to*

$$\begin{aligned}
\min_{x_i,Y_i,S_i,\beta_i,\eta_i}\quad & c_i^\top x_i + g_i^\top \beta_i + \eta_i\\
\text{s. t.}\quad & (E_i z_i + F_i u_i)^\top \beta_i + \eta_i \geq d_i^\top y_i^0 + \sum_{j=1}^{m_i} d_i^\top y_i^j z_i^j,\ \ \forall\,(z_i,u_i)\in\Omega_i,\\
& A_i^j x_i + A_{-i}^j x_{-i} + D_i y_i^j = b_i^j,\ j=0,1,...,m_i,\\
& y_i^0(q) + h_i^\top s_i^q \geq 0,\ q=1,2,...,k_i,\\
& G_i^\top s_i^q = y_i^q,\ q=1,2,...,k_i,\\
& H_i^\top s_i^q = 0,\ q=1,2,...,k_i,\\
& x_i\in X_i, \beta_i\in\mathbb{R}^{p_i},\ \eta_i\in\mathbb{R},\ s_i^q\in\mathcal{K}\mathcal{K}_i^*,\ q=1,2,...,k_i.
\end{aligned} \tag{3.8}$$

*Proof.* The first constraint in (3.6) can be written as follows.

$$\begin{array}{l}\forall (z_i,u_i)\in\Omega_i,\ \exists (Y_i,S_i)\in\Pi_{Y_i,S_i}\quad \text{such that}\\ \left(E_iz_i+F_iu_i\right)^\top\beta_i+\eta_i-\left(d_i^\top y_i^0+\sum_{j=1}^{m_i}d_i^\top y_i^j z_i^j\right)\ge 0,\end{array}$$

or equivalently

$$\min_{(z_i,u_i)\in\Omega_i}\ \max_{(Y_i,S_i)\in\Pi_{Y_i,S_i}}\left[\left(E_iz_i+F_iu_i\right)^\top\beta_i+\eta_i-\left(d_i^\top y_i^0+\sum_{j=1}^{m_i}d_i^\top y_i^j z_i^j\right)\right]\ge 0.$$

Clearly, the function in the left hand side of the above inequality is convex in $(z_i,u_i)$ and concave in $(Y_i,S_i)$. Moreover, $\Omega_i$ and $\Pi_{Y_iS_i}$ are closed and convex. By Sion's minimax theorem [40], as $\Omega_i$ is bounded, we have

$$\begin{aligned}&\min_{(z_i,u_i)\in\Omega_i}\ \max_{(Y_i,S_i)\in\Pi_{Y_i,S_i}}\left[\left(E_iz_i+F_iu_i\right)^\top\beta_i+\eta_i-\left(d_i^\top y_i^0+\sum_{j=1}^{m_i}d_i^\top y_i^j z_i^j\right)\right]\\ =&\max_{(Y_i,S_i)\in\Pi_{Y_i,S_i}}\ \min_{(z_i,u_i)\in\Omega_i}\left[\left(E_iz_i+F_iu_i\right)^\top\beta_i+\eta_i-\left(d_i^\top y_i^0+\sum_{j=1}^{m_i}d_i^\top y_i^j z_i^j\right)\right].\end{aligned}$$

The first constraint in (3.6) is therefore equivalent to

$$\begin{array}{l}\exists (Y_i,S_i)\in\Pi_{Y_i,S_i},\ \forall (z_i,u_i)\in\Omega_i\quad \text{such that}\\ \left(E_iz_i+F_iu_i\right)^\top\beta_i+\eta_i-\left(d_i^\top y_i^0+\sum_{j=1}^{m_i}d_i^\top y_i^j z_i^j\right)\ge 0,\end{array}$$

which proves the proposition. □

Now we are ready to show that problem (3.2) can be reformulated as a conic optimization problem – a main result of this section.

**Theorem 3.1.** *Under the linear decision rule with the WKS-type of ambiguity set, the feasible set of problem (3.2) is conic representable. Further, problem (3.2) can be equivalently converted to the following conic optimization problem.*

$$\begin{array}{rl}\min\limits_{x_i,Y_i,S_i,\beta_i,\eta_i,\phi_i} & c_i^\top x_i+g_i^\top\beta_i+\eta_i\\ \text{s. t.} & h_i^\top\phi_i-d_i^\top y_i^0+\eta_i\ge 0,\\ & G_i^\top\phi_i=E_i^\top\beta_i-{Y_i^{\bar{0}}}^\top d_i,\\ & H_i^\top\phi_i=F_i^\top\beta_i,\\ & A_i^jx_i+A_{-i}^jx_{-i}+D_iy_i^j=b_i^j,\ j=0,1,...,m_i,\\ & y_i^0(q)+h_i^\top s_i^q\ge 0,\ q=1,2,...,k_i,\\ & G_i^\top s_i^q=y_i^q,\ q=1,2,...,k_i,\\ & H_i^\top s_i^q=0,\ q=1,2,...,k_i,\\ & x_i\in X_i,\ \phi_i,s_i^q\in\mathcal{KK}_i^*,\ q=1,2,...,k_i.\end{array}\tag{3.9}$$

*Proof.* The first constraint in (3.8) is equivalent to the following

$$\min_{(z_i,u_i)\in\Omega_i}\left[\left(\beta_i^\top E_i-d_i^\top Y_i^{\bar{0}}\right)z_i+\beta_i^\top F_iu_i-d_i^\top y_i^0+\eta_i : G_iz_i+H_iu_i\preceq_{\mathcal{KK}_i}h_i\right]\ge 0.\tag{3.10}$$

Fixing $Y_i,\beta_i,\eta_i$, the left hand side of (3.10) is a convex optimization problem in $(z_i,u_i)$ over a cone. According to our assumption on $\Omega_i$, the feasible set of the convex optimization problem is bounded and is of full dimension. Therefore, it has a strict interior point and the optimal value is finite. By strong duality of finite-dimensional conic optimization problem (cf. [3, Theorem A.2.1]) and looking into the dual problem, it follows that (3.10) is equivalent to the feasibility of system

$$\begin{cases} h_i^\top \phi_i - d_i^\top y_i^0 + \eta_i \geq 0, \\ G_i^\top \phi_i = E_i^\top \beta_i - {Y_i^{\bar{0}}}^\top d_i, \\ H_i^\top \phi_i = F_i^\top \beta_i, \\ \phi_i \in \mathcal{K}\mathcal{K}_i^*. \end{cases} \tag{3.11}$$

Substituting (3.11) into problem (3.8), the theorem follows. □

4. **The VI formulation of TSDRG: Solution methodology.** As it is well-known, an $N$-person noncooperative game with player $i$ solving a smoothing convex optimization problem

$$\min_{x_i \in X_i(x_{-i})} \theta_i(x_i, x_{-i}),$$

can be equivalently reformulated to a quasi-variational inequality (QVI),

$$-\Theta(x) \in N_X(x), \tag{4.1}$$

where $\Theta(x) := ((\nabla_{x_1}\theta_1)^\top, \cdots, (\nabla_{x_N}\theta_N)^\top)^\top$, $X := \prod_i X_i(x_{-i})$, and $N_X(x)$ is the normal cone of the convex set $X$ at $x$, defined by

$$N_X(x) := \left\{ d \ : \ d^\top (y - x) \leq 0, \ \forall \ y \in X \right\}.$$

Note that player $i$'s constraint set $X_i(x_{-i})$ depends on other players' decisions, which leads that the defining set $X$ depends on the variable $x$. However, when $X_i(x_{-i})$ has some tractable form, (4.1) can reduce to some specific problem, for instance,

- if $X_i(x_{-i}) = \mathbb{R}^{n_i}$ for all $i$, then (4.1) reduces to an equation $\Theta(x) = 0$;
- if $X_i(x_{-i}) = \mathbb{R}_+^{n_i}$ for all $i$, then (4.1) reduces to a complementarity problem (CP)
$$0 \leq \Theta(x) \perp x \geq 0,$$
where $a \perp b$ means $a^\top b = 0$;
- if $X_i(x_{-i}) = K_i$ for all $i$, which is a conic constraint independent of other players' decisions, then (4.1) reduces to a conic complementarity problem
$$0 \preceq_{K^*} \Theta(x) \perp x \succeq_K 0, \text{ where } K = \prod_i K_i;$$
- if $X_i(x_{-i}) = C(x)$ for all $i$, i.e., each player has the shared coupling constraints, then (4.1) reduces to a variational inequality (VI): $-\Theta(x) \in N_C(x)$.

Now, look back into the $N$-person noncooperative TSDRG (3.2). Based on Theorem 3.1, the optimal strategy $x_i^*$ of player $i$ comes from the solution to problem (3.9), thus, $x^* = (x_1^*, \cdots, x_N^*)$ is an equilibrium of the TSDRSG (3.2) if and only if

$$-\ell \in N_{C(x^*)}(v^*), \tag{4.2}$$

where

$$v^* = \begin{pmatrix} v_1^* \\ \vdots \\ v_N^* \end{pmatrix}, \ \ell = \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix}, \ C(x^*) = C_1(x_{-1}^*) \times \cdots \times C_N(x_{-N}^*),$$

with $v_i^* = (x_i^*, (y^*)_i^0, \cdots, (y^*)_i^{m_i}, (s^*)_i^1, \cdots, (s^*)_i^{k_i}, \beta_i^*, \eta_i^*, \phi_i^*)^\top$, $\ell_i = (c_i^\top, 0, 0, g_i^\top, 1, 0)^\top$, and $C_i(x_{-i}^*)$ represents the constraints in (3.9) depending on other players' strategies.

Note that (4.2) is a QVI, but it is monotone since $\ell$ is a constant vector. In literature, QVI has attracted increasing attention [12, 19, 33, 38], since Bensoussan first recognized the relationship between the generalized Nash games and QVIs [5]. In spite of the advantage in modeling complex and realistic problems in the application, the study of QVI is immature because of its analytical difficulties, especially for the computation. However, if the generalized Nash game has a special structure, particularly sharing the coupling constraints for all players, it is possible to reformulate the game as a VI, which has extensive state-of-the-art study. In case of our TSDRG (3.2), for every $j = 0, 1, \cdots, m_i$, if data $A_i^j$ and $b_i^j$ are consistent for all $i = 1, 2, \cdots, N$, which yields the shared coupling constraints, then a VI can be reformulated instead of QVI (4.2) to obtain a Nash equilibrium of game (3.2).

Besides, by introducing proper dual variables, the Karush-Kuhn-Tucker (KKT) system for problem (3.9) parameterized by $x_{-i}$ can be written down:

$$\begin{cases} c_i + \sum_{j=0}^{m_i} (A_i^j)^\top \zeta_i^j = 0, \ g_i - E_i\gamma_i - F_i\delta_i = 0, \ 1 - \alpha_i = 0, \\ -h_i\alpha_i + G_i\gamma_i + H_i\delta_i \in K_i, \\ -\sum_{q=1}^{k_i} \vartheta_i^q h_i + \sum_{q=1}^{k_i} G_i\lambda_i^q + \sum_{q=1}^{k_i} H_i\mu_i^q \in K_i, \\ \alpha_i d_i + D_i^\top \zeta_i^0 - \vartheta_i = 0, \\ \gamma_i(j) d_i + D_i^\top \zeta_i^j + \bar{\lambda}_i^j = 0, \ j = 1, ..., m_i, \\ G_i^\top \phi_i - E_i^\top \beta_i + Y_i^{\bar{0}^\top} d_i = 0, \\ H_i^\top \phi_i - F_i^\top \beta_i = 0, \\ A_i^j x_i + A_{-i}^j x_{-i} + D_i y_i^j - b_i^j = 0, \ j = 0, 1, ..., m_i, \\ G_i^\top s_i^q = y_i^q, \ q = 1, 2, ..., k_i, \\ H_i^\top s_i^q = 0, \ q = 1, 2, ..., k_i, \\ \phi_i \in \mathcal{KK}_i^*, \ s_i^q \in \mathcal{KK}_i^*, \ q = 1, 2, ..., k_i, \\ 0 \le \alpha_i \perp h_i^\top \phi_i - d_i^\top y_i^0 + \eta_i \ge 0, \\ 0 \le \vartheta_i^q \perp y_i^0(q) + h_i^\top s_i^q \ge 0, \ q = 1, 2, ..., k_i, \end{cases} \tag{4.3}$$

where $\gamma_i(j)$ is the $j$-th element of the dual variable $\gamma_i$, $\bar{\lambda}_i^j = (\lambda_i^1(j), \cdots, \lambda_i^{k_i}(j))^\top$ with $\lambda_i^q(j)$ being the $j$-th element of the dual variable $\lambda_i^q$, $q = 1, \cdots, k_i$. Even if there are no shared coupling constraints, one can concatenate the $N$ KKT systems (4.3) to a mixed complementarity problem (MiCP), which is a generalization of the CP, then apply the iterative algorithms for the MiCP to obtain an equilibrium. Another possible approach stems from a penalty method proposed by Pang and Fukushima [33]. They presented an algorithm for QVI in the case where $K_i$ is the positive orthant, in which they penalized the nonstandard constraint via a penalty term, similarly to the augmented Lagrangian function, then solved a sequence of penalized VIs. Back to our problem, note that the conic constraints set $C_i(x_{-i})$ can be written as

$$C_i(x_{-i}) := \{v_i : \ h_i(v) = 0, \ g_i(v_i) \ge 0, \ f_i(v_i) \in K_i^*\},$$

where $h_i$, $g_i$, $f_i$ are all affine functions. Motivated by their approach, considering $A_{-i}^j$ has the formulation as $A_{-i}^j = [A_1^{ij} \ \cdots \ A_{i-1}^{ij} \ A_{i+1}^{ij} \ \cdots \ A_N^{ij}]$, one can always write the constraints of player $i$ which involved other players' strategies as linear equalities

$$h(v, v^*) := Uv + Wv^* - \hat{b} = 0,$$

with $U, W, \hat{b}$ having the following forms:

$$U = [U' \ D \ 0], \ W = [W' \ 0 \ 0], \ \hat{b} = \begin{pmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_N \end{pmatrix},$$

where

$$U' = \begin{pmatrix} A_1^0 & & & \\ \vdots & & & \\ A_1^m & & & \\ & A_2^0 & & \\ & \vdots & & \\ & A_2^m & & \\ & & \ddots & \\ & & & A_N^0 \\ & & & \vdots \\ & & & A_N^m \end{pmatrix}, D = \begin{pmatrix} D_1 & & & \\ \vdots & & & \\ D_1 & & & \\ & D_2 & & \\ & \vdots & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_N \\ & & & \vdots \\ & & & D_N \end{pmatrix},$$

$$W' = \begin{pmatrix} A_1^0 & A_2^{10} & \cdots & A_N^{10} \\ \vdots & \vdots & \vdots & \vdots \\ A_1^1 & A_2^{1m} & \cdots & A_N^{1m} \\ A_2^{20} & A_2^2 & \cdots & A_N^{20} \\ \vdots & \vdots & \vdots & \vdots \\ A_2^{2m} & A_2^m & \cdots & A_N^{2m} \\ \vdots & \vdots & \vdots & \vdots \\ A_1^{N0} & A_2^{N0} & \cdots & A_N^0 \\ \vdots & \vdots & \vdots & \vdots \\ A_1^{Nm} & A_2^{Nm} & \cdots & A_N^m \end{pmatrix} - U', \ \hat{b}_i = \begin{pmatrix} b_i^0 \\ b_i^1 \\ \vdots \\ b_i^m \end{pmatrix}.$$

Then, denote the conic constraints without the rivals' strategies as

$$C' = \prod_{i=1}^{N} C_i', \text{ where } C_i' := \{v_i : \ g_i(v_i) \geq 0, \ f_i(v_i) \in K_i^*\}.$$

Putting the coupling constraint $h(v, v^*) = 0$ into a penalty term, we may get a solution of the QVI (4.2) iteratively by solving the following VI in the $k$-th iteration:

$$-\ell - U^\top(u^{(k)} + \rho^{(k)} h(v, v)) \in N_{C'}(v),$$

where $\{u^{(k)}\}$ is a bounded sequence of vectors and $\{\rho^{(k)}\}$ is a sequence of increasing positive scalars tending to $+\infty$.

5. **Conclusion.** In this paper, a noncooperative game of $N$-players with uncertainty is considered, in which each player is supposed to make a deterministic decision depending on his rivals' decisions in the first stage and takes a recourse action depending on a random vector as well as on other players' strategies in the second stage. Motivated by the ideas from distributionally robust optimization, a worst-case approach is proposed to model this game with uncertainty. The model has certain naval features such as the distribution of the random vector involved is not assumed to be given, rather, it is assumed to satisfy certain constraints defined by a WKS-type of ambiguity set. It is shown that under a linear decision rule and some mild assumptions, a Nash equilibrium of the proposed game exists and can be found by solving a conic variational inequality problem. Specifically, if all players' optimization problems are linear in both stages, the Nash equilibrium of the two-stage distributionally robust game can be found by solving a deterministic monotone variational inequality problem.

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