STRONG CONVERGENCE OF A KM ITERATIVE ALGORITHM FOR COMPUTING A SPLIT COMMON FIXED-POINT OF QUASI-NONEXPANSIVE OPERATORS

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Abstract. A modified Krasnoselski-Mann iterative algorithm is proposed for solving the split common fixed-point problem for quasi-nonexpansive operators. A parameter sequence is introduced to enhance convergence. It is shown that the proposed iterative algorithm strongly converges to a split common fixed-point in Hilbert spaces. This result extends the applicability of the KM algorithm.

1. Introduction

The convex feasibility problem (CFP) is a classical problem in optimization [1], which is to find a common point in the intersection of finitely many convex sets. It has applications in many areas, including approximation theory [2], image reconstruction from projections [3, 5], control [6], and so on. A popular approach to CFP is the projection algorithm, which incorporates projection technique into the algorithm, see [7]. When there are only two sets in CFP with additional constraints that require the solutions to be in the domain of a linear operator as well as in this operator's range, the problem is called a split feasibility problem (SFP). The SFP in finite-dimensional spaces was probably first introduced by Censor and Elfving [8] for modeling inverse problems, which arise from phase retrievals in medical image reconstruction [9]. Recently, it has been found that the SFP can also be used in various disciplines such as image restoration, computer tomography, and radiation therapy planning [10, 11, 12]. The study on SFP has been extended to infinite-dimensional Hilbert spaces, see, e.g., [9, 11, 13, 14, 15, 16].

This paper is aimed at the split common fixed-point problem (SCFP), which is a generalization of SFP [17]. In particular, we are concerned with the following two-operator SCFP for quasi-nonexpansive mappings in infinite dimensional Hilbert spaces:

$$\text{finding } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.1}$$

2010 *Mathematics Subject Classification.* 37C25, 47H09, 90C25.

Key words and phrases. KM algorithm, strong convergence, fixed point, quasi-nonexpensive operator.

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two quasi-nonexpansive operators with nonempty fixed-point sets $\mathrm{Fix}\,(U) = C$ and $\mathrm{Fix}\,(T) = Q$. This general class of problems includes the class of directed operators considered in [18, 19] as special cases. From the application point of view, it is often more desirable, to cast a problem in the fixed-point framework than in the framework of convex optimization. For instance, in the image recovery problem, it is possible to map the set of images with a certain property to the fixed-point set of a nonlinear quasi-nonexpansive operator. We denote the solution set of the two-operator SCFP (1.1) by

$$\Gamma = \{y \in C \mid Ay \in Q\}. \tag{1.2}$$

The Krasnoselski-Mann (KM for short) algorithm was proposed for solving the fixed-point problem [20]. Byrne [21] first applied KM iteration in his CQ algorithm for solving the SFP. Subsequently, Zhao [16] applied the KM iteration to a perturbed CQ algorithm. Unfortunately, up to our knowledge, only weak convergence results have been proved in the literature on the KM iterative algorithm in Hilbert spaces. To get strong convergence in infinite dimensional Hilbert space, a series of fixed point algorithms [25, 26, 27, 28] are proposed for solving equilibrium problems, fixed point problems and variational inequality problems. Yao [22] introduced a modified KM iterative algorithm which has strong convergence for non-expansive mappings. Cho et al [4] discussed weak and strong convergence for a three-parameter non-expansive mapping. Dang and Gao [15] combined the KM iterative method with the modified CQ algorithm to construct a KM-CQ-Like algorithm which has strong convergence for solving the SFP. We noted that Moudafi [23] gave an extension of the unified framework developed in [17] to quasi-nonexpansive operators for SCFP, which achieves weak convergence.

In this paper we study the strong convergence properties of a modified KM algorithm for solving the SCFP with a quasi-nonexpansive operator T such that $I - T$ is closed at the origin. The algorithm could be thought of as an extension of the algorithms studied in [15, 21, 22]. The differences between the proposed algorithm in this article and the algorithms presented in [15, 21, 22] are:

- Compared with [15], we extend the general split feasibility problem to the split common fixed-point problem by considering the properties of the involving operators instead of considering the properties of the projection operator.
- Compared with [21], we introduce a parameter sequence in the algorithm which can assure strong convergence under suitable conditions, instead of only weak convergence.
- Compared with [22], we use the KM algorithm to solve the SCFP for two quasi-nonexpansive operators instead of only for one non-expansive operator.

The paper is organized as follows. In Section 2, we recall some preliminaries. In Section 3, we present a modified KM algorithm and show its strong convergence. In Section 4, we make some concluding remarks.

2. Preliminaries

Throughout the rest of the paper, I denotes the identity operator and $\mathrm{Fix}(T)$ denotes the set of the fixed-points of an operator T i.e., $\mathrm{Fix}(T) := \{x \mid x = T(x)\}$.

Recall that a mapping T is quasi-nonexpansive if

$$\|T(x) - z\| \leq \|x - z\|, \forall x, z \in H \times \mathrm{Fix}(T).$$

A mapping T is called nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|, \forall x, y \in H \times H.$$

Usually, the convergence of fixed-point algorithms requires some additional smoothness properties of the mapping T such as demiclosedness defined as follows.

Definition 2.1. A mapping T is said to be demiclosed if, for any sequence $\{x^k\}$ which weakly converges to y, the sequence $\{T(x^k)\}$ strongly converges to z, then $T(y) = z$.

Lemma 2.1. Let T be a quasi-nonexpansive mappings, for and set $T_\alpha := (1-\alpha)I + \alpha T$. Then, for all $(x, q) \in H \times \mathrm{Fix}\,(T)$ it holds

(1)$\langle x - T(x), x - q\rangle \geq \frac{1}{2}\|x - T(x)\|^2$ and $\langle x - T(x), q - T(x)\rangle \leq \frac{1}{2}\|x - T(x)\|^2$;

(2) $\|T_\alpha(x) - q\|^2 \leq \|x - q\|^2 - \alpha(1-\alpha)\|x - T(x)\|$;

(3) $\langle x - T_\alpha(x), x - q\rangle \geq \frac{\alpha}{2}\|x - T(x)\|^2$.

Proof. By the general equality

$$\langle x, y\rangle = -\frac{1}{2}\|x - y\|^2 + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2,$$

Since T is quasi-nonexpansive, we have

$$\begin{aligned}
\langle x - T(x), x - q\rangle &= -\frac{1}{2}\|T(x) - q\|^2 + \frac{1}{2}\|T(x) - x\|^2 + \frac{1}{2}\|x - q\|^2 \\
&\geq -\frac{1}{2}\|x - q\|^2 + \frac{1}{2}\|T(x) - x\|^2 + \frac{1}{2}\|x - q\|^2 \\
&= \frac{1}{2}\|T(x) - x\|^2
\end{aligned}$$

and

$$\begin{aligned}
\langle x - T(x), q - T(x)\rangle &= -\frac{1}{2}\|x - q\|^2 + \frac{1}{2}\|T(x) - x\|^2 + \frac{1}{2}\|T(x) - q\|^2 \\
&\leq -\frac{1}{2}\|x - q\|^2 + \frac{1}{2}\|T(x) - x\|^2 + \frac{1}{2}\|x - q\|^2 \\
&= \frac{1}{2}\|T(x) - x\|^2.
\end{aligned}$$

Hence, we proved (1).

We obtain (2) from (1) and the fact that

$$\|T_\alpha(x) - q\|^2 = \|x - q\|^2 - 2\alpha\langle x - q, x - T(x)\rangle + \alpha^2\|T(x) - x\|^2.$$

Finally, we obtain (3) by $I - T_\alpha = \alpha(I - T)$ and (1). □

Lemma 2.2 ([21]). Assume $\{a_k\}$ is a sequence of nonnegative real numbers such that

$$a_{k+1} \le (1 - \gamma_k)a_k + \gamma_k\delta_k, k \ge 0,$$

where $\{\gamma_k\}$ is a sequence in $(0, 1)$ and $\{\delta_k\}$ is a sequence in R such that
$(a)\sum_{k=0}^{\infty}\gamma_k = \infty$;
$(b)\limsup_{k\to\infty}\delta_k \le 0$ or $\sum_{k=0}^{\infty}|\delta_k\gamma_k| < \infty$.
Then, $\lim_{k\to\infty} a_k = 0$.

Lemma 2.3 ([22]). Let H be a Hilbert space and let $\{x^k\}$ be a sequence in H such that there exists a nonempty set $S \subset H$ satisfying:
(a) For every x^*, $\lim_k \|x^k - x^*\|$ exists.
(b) Any weak cluster point of the sequence $\{x^k\}$ belongs to S.
Then, there exists $z \in S$ such that $\{x^k\}$ weakly converges to z.

3. The algorithm and its asymptotic convergence

We now describe the method and then prove its strong convergence.

3.1. **The algorithm. Algorithm 3.1**

Initialization: Let $x^0 \in H_1$ be arbitrary.
Iterative step: For $k \in N$, set $u = I + \gamma A^T(T - I)A$, $u^k = x^k + \gamma A^T(T - I)Ax^k$, and let

$$y^k = (1 - \beta_k)u^k, \tag{3.1a}$$

$$x^{k+1} = (1 - \alpha_k)y^k + \alpha_k U(y^k), \tag{3.1b}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences in $[0, 1]$,$\gamma \in (0, \frac{1}{\lambda})$, λ is the spectral radius of the operator A^TA.

3.2. **Convergence of the algorithm.**

In this subsection, we establish the strong convergence property for Algorithm 3.1.

Theorem 3.1. Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ be quasi-nonexpansive operators with nonempty $\mathrm{Fix}\,(U) = C$ and let $T : H_2 \to H_2$ be α_2-quasi-nonexpansive operators with nonempty $\mathrm{Fix}\,(T) = Q$. Let $\{\alpha_k\}$ and $\{\beta_k\}$ be two real numbers in $(0, 1)$. Assume the following conditions are satisfied:
$(C1)\lim_{k\to\infty}\beta_k = 0$;
$(C2)\sum_{k=0}^{\infty}\beta_k = \infty$;
$(C3)$ $\alpha_k \in [a, b] \subset (0, 1)$,
Assume that $U - I$ and $T - I$ are demiclosed at 0. If $\Gamma \ne \emptyset$, then any sequence $\{x^k\}$ generated by Algorithm 3.1 strongly converges to a split

common fixed-point.

Proof. First, we prove that $\{x^k\}, \{u(x^k)\}$ and $\{y^k\}$ are all bounded. Taking $z \in \Gamma$, and using (2) in Lemma 2.1, we obtain

$$\begin{aligned}\|x^{k+1} - z\|^2 &= \|(1-\alpha_k)y^k + \alpha_k U(y^k) - z\|^2 \\ &\le \|y^k - z\|^2 - \alpha_k(1-\alpha_k)\|y^k - U(y^k)\|^2 \\ &\le \|y^k - z\|^2. \end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned}\|y^k - z\|^2 &= \|(1-\beta_k)u^k - z\|^2 \\ &= \|(1-\beta_k)(u^k - z) - \beta_k z\|^2 \\ &\le (1-\beta_k)\|u^k - z\|^2 + \beta_k\|z\|^2. \end{aligned} \tag{3.3}$$

From (3.1a), we have

$$\begin{aligned}\|u^k - z\|^2 &= \|x^k + \gamma A^T(T-I)(Ax^k) - z\|^2 \\ &= \|x^k - z\|^2 + \gamma^2\|A^T(T-I)(Ax^k)\|^2 \\ &\quad +2\gamma\langle x^k - z, A^T(T-I)(Ax^k)\rangle \\ &\le \|x^k - z\|^2 + \lambda\gamma^2\|(T-I)(Ax^k)\|^2 \\ &\quad +2\gamma\langle Ax^k - Az, (T-I)(Ax^k)\rangle, \end{aligned}$$

that is

$$\|u^k - z\|^2 \le \|x^k - z\|^2 + \lambda\gamma^2\|(T-I)(Ax^k)\|^2 + 2\gamma\langle Ax^k - Az, (T-I)(Ax^k)\rangle. \tag{3.4}$$

By setting $\theta := 2\gamma\langle Ax^k - Az, (T-I)(Ax^k)\rangle$ and using (1) of Lemma 2.1, we obtain

$$\begin{aligned}\theta &= 2\gamma\langle Ax^k - Az, (T-I)(Ax^k)\rangle \\ &= 2\gamma\langle Ax^k - Az + (T-I)(Ax^k) - (T-I)(Ax^k), (T-I)(Ax^k)\rangle \\ &= 2\gamma(\langle Ax^k - Az, (T-I)(Ax^k)\rangle - \|(T-I)(Ax^k)\|^2) \\ &\le 2\gamma\|(T-I)(Ax^k)\|^2 - \|(T-I)(Ax^k)\|^2) \\ &= -\gamma\|(T-I)(Ax^k)\|^2. \end{aligned}$$

The key inequality above, combined with (3.4), yields

$$\|u^k - z\|^2 \le \|x^k - z\|^2 - \gamma(1-\lambda\gamma)\|(T-I)(Ax^k)\|^2. \tag{3.5}$$

Thus, from (3.2), (3.3), and (3.5), we get

$$\begin{aligned}\|x^{k+1} - z\|^2 &\le \|y^k - z\|^2 \\ &\le (1-\beta_k)\|u(x^k) - z\|^2 + \beta_k\|z\|^2 \\ &\le (1-\beta_k)\|x^k - z\|^2 + \beta_k\|z\|^2 \\ &\le \max\{\|x^k - z\|^2, \|z\|^2\}, \end{aligned}$$

by induction, it is easy to see that

$$\|x^k - z\|^2 \le \max\{\|x^0 - z\|^2, \|z\|^2\}.$$

Hence,$\{x^k\}, \{u(x^k)\}$ and $\{y^k\}$ are all bounded.

From (3.1b), we note that

$$y^k - U(y^k) = \frac{1}{\alpha_k}(y^k - x^{k+1}). \tag{3.6}$$

Thus, from (2) in Lemma 2.1 and (3.6), we have

$$\begin{aligned}\|x^{k+1} - z\|^2 &\leq \|y^k - z\|^2 - \alpha_k(1-\alpha_k)\|Uy^k - y^k\|^2 \\ &= \|y^k - z\|^2 - \frac{(1-\alpha_k)}{\alpha_k}\|y^k - x^{k+1}\|^2. \end{aligned} \tag{3.7}$$

Since $0 < a < \alpha_k < b < 1, \frac{1-\alpha_k}{\alpha_k} \geq \frac{1-b}{b} =: l$. Note that $l > 0$ since $b \in (0,1)$. Therefore, by (3.1a) and (3.7) we obtain

$$\begin{aligned}\|x^{k+1} - z\|^2 &\leq \|y^k - z\|^2 - l\|y^k - x^{k+1}\|^2 \\ &= \|u^k - z - \beta_k u^k\|^2 - l\|u^k - x^{k+1} - \beta_k u^k\|^2 \\ &= \|u^k - z\|^2 - 2\beta_k\langle u^k - z, u^k\rangle + \beta_k^2\|u^k\|^2 \\ &\quad -l\|u^k - x^{k+1}\|^2 + 2l\beta_k\langle u^k, u^k - x^{k+1}\rangle - l\beta_k^2\|u^k\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\|x^{k+1} - z\|^2 \leq{}& \|u^k - z\|^2 - l\|u^k - x^{k+1}\|^2 + \beta_k\{-2\langle u^k, u^k - z\rangle \\ &+2l\langle u^k, u^k - z\rangle + 2l\langle u^k, z - x^{k+1}\rangle + (1-l)\beta_k\|u^k\|^2\}. \end{aligned} \tag{3.8}$$

Since$\{x^k\}$ and $\{u^k\}$ are bounded, there exists a constant $M \geq 0$ such that

$$-2\langle x^k, x^k - z\rangle + 2l\langle u^k, u^k - z\rangle + 2l\langle u^k, z - x^{k+1}\rangle + (1-l)\beta_k\|u^k\|^2 \leq M.$$

From (3.5), we get

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \gamma(1-\lambda\gamma)\|(T-I)(Ax^k)\|^2 - l\|u^k - x^{k+1}\|^2 + M\beta_k.$$

Since $l > 0, \gamma \in (0, \frac{1}{\lambda})$, we have

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \gamma(1-\lambda\gamma)\|(T-I)(Ax^k)\|^2 + M\beta_k \tag{3.9}$$

and

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - l\|u^k - x^{k+1}\|^2 + M\beta_k. \tag{3.10}$$

Consequently, (3.9) and (3.10) can be rewritten respectively as

$$\|x^{k+1} - z\|^2 - \|x^k - z\|^2 + \gamma(1-\lambda\gamma)\|(T-I)(Ax^k)\|^2 \leq M\beta_k \tag{3.11}$$

and

$$\|x^{k+1} - z\|^2 - \|x^k - z\|^2 + l\|x^{k+1} - u^k\|^2 \leq M\beta_k. \tag{3.12}$$

Next, we consider two cases, in each of which we prove that $\{x^k\}$ strongly converges to z.

Case 1. Assume that the sequence $\{\|x^k - z\|\}$ is a monotonically decreasing sequence. Then $\{\|x^k - z\|\}$ is convergent. Clearly, we have

$$\|x^{k+1} - z\|^2 - \|x^k - z\|^2 \to 0, \tag{3.13}$$

(3.13), (C1), and (3.11) imply that

$$\lim_{k\to+\infty}(T-I)(Ax^k) = 0, \tag{3.14}$$

(3.13), (C1), and (3.12) imply that

$$\|x^{k+1} - u^k\| \to 0. \tag{3.15}$$

On the other hand, we note that

$$\|y^k - u^k\| \leq \beta_k \|u^k\| \to 0. \tag{3.16}$$

It is clear from (3.15) and (3.16) that

$$\|y^k - x^{k+1}\| \leq \|y^k - u^k\| + \|x^{k+1} - u^k\| \to 0.$$

Therefore,

$$\|y^k - U(y^k)\| = \frac{1}{\alpha_k}\|y^k - x^{k+1}\| \to 0. \tag{3.17}$$

Denoting by x^* a weak-cluster point of $\{x^k\}$, let $v = 0, 1, 2\cdots$ be the sequence of indices, such that

$$w - \lim_v x^{k_v} = x^*. \tag{3.18}$$

Since $u^k = x^k + \gamma A^T(T - I)Ax^k$ and $y^k = (1 - \beta_k)u^k$, we have

$$w - \lim_v u^{k_v} = x^* \tag{3.19}$$

and

$$w - \lim_v y^{k_v} = x^*. \tag{3.20}$$

Then, from (3.14) and the demiclosedness of $T - I$at 0, we obtain

$$T(Ax^*) = Ax^*,$$

from which it follows that $Ax^* \in Q$. Then (3.17), combined with the demiclosedness of $U - I$ at 0 and the weak convergence of $\{u^{k_v}\}$ to x^*, yields

$$U(x^*) = x^*.$$

Hence $x^* \in C$ and $x^* \in \Gamma$. Therefore, the z in (3.2)-(3.13) can be replaced by x^*.
Since there is no more than one weak-cluster point, the weak convergence of the whole sequence $\{x^k\}$ follows by applying Lemma 2.3 with $S := \Gamma$. The same is true for the whole sequence $\{y^k\}$.

Next, we prove that $\{x^k\}$ strongly converge to x^*. Indeed, from (3.1a) and (3.2), we get

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &\leq \|y^k - x^*\|^2 \\
&= \|(1 - \beta_k)(u^k - x^*) - \beta_k x^*\|^2 \\
&\leq (1 - \beta_k)^2\|u^k - x^*\|^2 - 2\beta_k\langle y^k - x^*, x^*\rangle \\
&\leq (1 - \beta_k)\|u^k - x^*\|^2 - 2\beta_k\langle y^k - x^*, x^*\rangle \\
&\leq (1 - \beta_k)\|x^k - x^*\|^2 - 2\beta_k\langle y^k - x^*, x^*\rangle.
\end{aligned} \tag{3.21}$$

By the weak convergence of $\{y^k\}$, we have $\lim_{k\to\infty}\langle y^k - x^*, x^*\rangle = 0$. Hence, applying Lemma 2.2 to (3.21), we immediately deduce that x^k converges strongly to x^*. Consequently, $u^k x^*$ and y^k converge strongly to x^*.

Case 2. Assume that $\{\|x^k - z\|\}$ is not a monotonically decreasing sequence. Set $L_k = \|x^k - z\|^2$ and let $\tau : N \to N$ be a mapping for all $k \geq k_0$ (for some k_0 large enough) by

$$\tau(k) = \max\{n \in N : n \leq k, L_n \leq L_{n+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(k) \to \infty$ as $k \to \infty$ and $L_{\tau(k)} \leq L_{\tau(k)+1}$ for $k \geq k_0$. From (3.11) , it is easy to see that

$$\|(T - I)(Ax^{\tau(k)})\|^2 \leq \frac{M\beta_{\tau(k)}}{\gamma(1 - \lambda\gamma)} \to 0.$$

From (3.12), it is easy to see that

$$\|x^{\tau(k)+1} - u(x^{\tau(k)})\|^2 \leq \frac{M\alpha_{\tau(k)}}{n} \to 0.$$

Thus

$$\|(T - I)(Ax^{\tau(k)})\| \to 0$$

and

$$\|x^{\tau(k)+1} - u(x^{\tau(k)})\| \to 0.$$

By the same arguments as (3.14)-(3.20) in Case 1, we conclude immediately that $x^{\tau(k)}$,$u^{\tau(k)}$ and $y^{\tau(k)}$ weakly converge to a point $\bar{x}^* \in \Gamma$. as $\tau(k) \to \infty$. At the same time, from (3.21), we note that , for all $k \geq k_0$,

$$0 \leq \|x^{\tau(k)+1} - \bar{x}^*\|^2 - \|x^{\tau(k)} - \bar{x}^*\|^2 \leq \beta_{\tau(k)}[2\langle \bar{x}^* - y^{\tau(k)}, \bar{x}^*\rangle - \|x^{\tau(k)} - \bar{x}^*\|^2],$$

which implies that

$$\|x^{\tau(k)} - \bar{x}^*\|^2 \leq 2\langle \bar{x}^* - y^{\tau(k)}, \bar{x}^*\rangle.$$

Hence, we deduce that

$$\lim_{k\to\infty} \|x^{\tau(k)} - \bar{x}^*\| = 0.$$

Therefore,

$$\lim_{k\to\infty} L_{\tau(k)} = \lim_{k\to\infty} L_{\tau(k)+1} = 0.$$

Furthermore, for $k \geq k_0$, it is observed that $L_k \leq L_{\tau(k)+1}$ if $k \neq \tau(k)$ (i.e., if $\tau(k) < k$), because $L_j > L_{j+1}$ for $\tau(k) + 1 \leq j \leq k$. As a consequence, we obtain that for all l$k \geq k_0$,

$$0 \leq L_k \leq \max\{L_{\tau(k)}, L_{\tau(k)+1}\} = L_{\tau(k)+1}.$$

Hence $\lim_{k\to\infty} L_k = 0$, this is, $\{x^k\}$ converges strongly to $\bar{x}^*$. Consequently, it is straightforward to prove that $\{u^k\}$ and $\{y^k\}$ converge strongly to $\bar{x}^*$. This completes the proof. □

4. Concluding remarks

We proposed an algorithm for solving the SCFP in the wide class of quasi-nonexpansive operators. We proved that the algorithm strongly converges to a solution in general Hilbert spaces. It would be interesting to investigate how this algorithm can be extended to solve the *multiple* split common fixed-point problem in infinite Hilbert spaces.

Acknowledgments

The first author's work was partially supported by Natural Science Foundation of Shanghai (14ZR1429200), The third author's work was partially supported by Australian Research Council under Grant DP160102819.

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