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Second-order cone reformulation and the price of anarchy of a robust Nash-Cournot game

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Abstract. We study an n-person Nash-Cournot game with incomplete information, in which the opponents' strategies are only known in a perturbed set and the players try to minimize their worst-case costs, which can vary due to data uncertainty. We show that in several interesting cases, this game can be reformulated as second-order cone optimization problems. We also derive a bound of the price of anarchy for this game, which is a bound on the ratio between the cost at the robust Nash-Cournot equilibria and the cost at the system optima.

Key words: Price of anarchy, robust Nash-Cournot equilibria, second order cone optimization, system optimal.

Mathematics Subject Classification: *90B18, 90C33, 65K05, 47N10*

Dedication: This paper is dedicated to Professor Alex Rubinov. Alex has made profound contribution in the development of optimization research, teaching, and application in Asia-Pacific region. He will live in our hearts as a beloved friend and a respected colleague.

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1 Introduction

Suppose that a common commodity is distributed by n Nash players over a network with a single origin-destination pair and m identical, nonintersecting, parallel links $L_1, \cdots, L_m$. Let $\mathcal{M} = \{1, \cdots, m\}$. Assume that the demand d is fixed and $\mathcal{N} = \{1, \cdots, n\}$. Then the feasible set of flows is given by

$$\Omega = \left\{ v \;\middle|\; v = \sum_{k \in \mathcal{N}} v^k,\ e^\top v = d,\ v^k \geq 0, \quad \forall k = 1, ..., n \right\},$$

where e is the m dimensional vector of ones. $v^k = (v_1^k, \cdots, v_m^k) \in R_+^m$ is the vector of flows selected by the kth Nash player, $k \in \mathcal{N}$. Each player treats the other players' route strategies as fixed when routing his own flows. We assume that each player can split flow along the given links. Mathematically, for player $k \in \mathcal{N}$, the optimization problem to be solved is

$$\min_{v \in \Omega} \sum_{a \in \mathcal{M}} t_a(v) v_a^k \equiv \sum_{a \in \mathcal{M}} t_a(v^k + v^{-k}) v_a^k, \tag{1.1}$$

where $t_a(\cdot)$ is a certain cost function on L_a, and

$$v^{-k} = \begin{pmatrix} v_1^{-k} \\ \vdots \\ v_m^{-k} \end{pmatrix} \text{ with } v_a^{-k} = \sum_{i \in \mathcal{N}, i \neq k} v_a^i = v_a - v_a^k \quad \forall a \in \mathcal{M}.$$

Note that we particularly write v as $v^k + v^{-k}$ to show that Problem (1.1) is an optimization problem with respect to v^k, while all other $v^j,\ j \neq k$ are taken as fixed. A point $(v^{1*}, \cdots, v^{n*})$ satisfying

$$v^{k*} \in \underset{v^k + v^{-k*} \in \Omega}{\arg\min} \sum_{a \in \mathcal{M}} t_a(v^k + v^{-k*}) v_a^k,$$

is called a Nash-Cournot equilibrium [2]. Generally speaking, finding a Nash-Cournot equilibrium is very difficult, for example, see [16].

In this paper, we consider the case that each player knows neither his/her opponents' exact strategies nor the probability distribution of their strategies. All the information about his/her opponents is that their strategies are in a given bounded set. We introduce the concept of robust Nash-Cournot equilibrium for an n-player game with quadratic cost functions by using some ideas from robust optimization [4]. This part can be viewed as a direct extension of the work of Hayashi, Yamashita and Fukushima [12] on two-player bimatrix games. We consider the case that both the cost function and the opponents' strategies can be uncertain in the same time (Paper [12] only studied the cases that either the cost function or the opponents' strategies were uncertain). We then show in Section 3 that for some interesting cases, the robust Nash-Cournot equilibrium problem can be reformulated or relaxed as a second-order cone complementarity problem, which can be solved efficiently by modern optimization methods [1, 5, 15].

Define the *system equilibrium problem* as

$$\min_{v \in \Omega} \sum_{a \in \mathcal{M}} t_a(v) v_a.$$

Assume that this problem has an optimal solution $\bar{v}$ with positive optimal value, i.e.

$$Z_s = \sum_{a \in \mathcal{M}} t_a(\bar{v}) \bar{v}_a = \sum_{a \in \mathcal{M}} \sum_{k \in \mathcal{N}} t_a(\bar{v}) \bar{v}_a^k > 0.$$

Let

$$Z_u = \sum_{a\in\mathcal{M}} t_a(v^*)v_a^* = \sum_{a\in\mathcal{M}}\sum_{k\in\mathcal{N}} t_a(v^*)v_a^{k*}$$

be the total cost at a robust Nash-Cournot equilibrium, where v^* is a robust Nash-Cournot equilibrium (also called the user optimum). Then the *price of anarchy* (PA) is defined as the ratio Z_u/Z_s, which was introduced by Koutsoupias and Papadimitriou [14] and has been studied extensively in [6, 7, 10, 14, 17, 18] for nonatomic games and in [19, 20] for atomic games.

Most of the present works on the PA [6, 7, 10, 14, 17, 18, 19, 20] implicitly assume that each player knows the opponents' strategies exactly and can evaluate the cost function exactly. This assumption restricts the applicability of the model to real world networks since in many cases the information is incomplete or is subject to errors. To deal with such situations, recently, Garg and Narahari [8] analyzed the PA by using a Bayesian game under the assumption that each player only knows the probability distribution of the other players' strategies. They prove that the PA is the same as that each player knows the opponents' complete strategies. We treat this problem in a different way. Same as in our reformulation scheme, by specifying data uncertainty sets, we derive some worst-case bounds for the PA, which is the topic of Section 4. Finally, we make some concluding remarks in Section 5.

2 Preliminaries

2.1 Robust Nash-Cournot equilibria

Recall that a Nash-Cournot equilibrium is the equilibrium solution of the n-person game, in which the k-th player's problem is

$$\min_{v\in\Omega} \sum_{a\in\mathcal{M}} t_a(v^k + v^{-k})v_a^k,$$

where $t_a(\cdot)$ is a certain cost function on a; particularly in this paper, t_a is defined so that

$$\sum_{a\in\mathcal{M}} t_a(v^k + v^{-k})v_a^k = (v^k)^\top A(v^k + v^{-k})$$

with a certain square matrix A.

In many applications, a player can not estimate the opponents' strategies accurately and evaluate the cost function exactly. To deal with such situations, Hayashi, Yamashita and Fukushima [12] introduced the concept of robust Nash equilibrium by using the idea of robust optimization [4, 9]. Their definition is for two-player bimatrix game and is assumed that the following statements hold for players 1 and 2:

(i) Player 1 can not estimate Player 2's strategy z exactly, but can estimate that it belongs to a set $Z(z) \subseteq R^m$ containing z. Similarly, Player 2 can not estimate Player 1's strategy y accurately, but can estimate that it belongs to a set $Y(y) \subseteq R^n$ containing y;

(ii) Player 1 can not estimate his/her cost matrix exactly, but can estimate that it belongs to a nonempty set $D_1 \subseteq R^{n\times m}$. Player 2 can not estimate his/her cost matrix exactly, but can estimate that it belongs to a nonempty set $D_2 \subseteq R^{n\times m}$.

(iii) Each player tries to minimize his/her worst cost under (i) and (ii). That is, the cost functions are defined respectively as follows:

$$\tilde{f}_1(y,z) := \max\Big\{y^\top \hat{A}\hat{z} \;\Big|\; \hat{A} \in D_1,\ \hat{z} \in Z(z)\Big\},$$

$$\tilde{f}_2(y,z) := \max\Big\{\hat{y}^\top \hat{B} z \;\Big|\; \hat{B} \in D_2,\ \hat{y} \in Y(y)\Big\}.$$

A point $(\bar{y},\bar{z})$ satisfying $\bar{y} \in \arg\min_{y\in S_1} \tilde{f}_1(y,\bar{z})$ and $\bar{z} \in \arg\min_{z\in S_2} \tilde{f}_2(\bar{y},z)$ is called a robust Nash equilibrium [12], where S_1 and S_2 are strategy sets of Player 1 and Player 2, respectively. Furthermore, they proved that whenever either the opponent's strategies or the cost matrices can be estimate exactly, a robust Nash equilibrium can be obtained as a solution of a second order cone complementarity problem.

We now extend the robust Nash equilibrium to the n-player Nash-Cournot game. Suppose that

(i). Each player k assumes that the opponents' strategies v^{-k} belongs to a set $V_k(v^{-k}) \subseteq R^m_+$ containing v^{-k}.

(ii). Each player k assumes that the cost matrix belongs to a nonempty set D_A containing A.

(iii). Each player tries to minimize his/her worst-case cost under (i) and (ii). That is, the cost function for player k is defined as follows:

$$\tilde{f}_k(v^k,v^{-k}) := \max\Big\{{v^k}^\top \hat{A}\hat{v}^{-k} \;\Big|\; \hat{A} \in D_A,\ \hat{v}^{-k} \in V_k(v^{-k})\Big\}.$$

A point $(v^{1*},\cdots,v^{n*})$ satisfying $v^{k*} \in \arg\min\Big\{\tilde{f}_k(v^k,v^{-k*})\Big| v^k + v^{-k*} \in \Omega\Big\}$ is then defined as a *robust Nash-Cournot equilibrium* of this game.

2.2 The second-order cone optimization problems

Since we will reformulate the game under consideration to a second-order cone optimization problem, we list some useful notations and concepts in this regard.

Let $\mathcal{Q}_n$ denote the second-order cone of dimension n,

$$\mathcal{Q}_n = \{x = (x_1, x^{n-1}) \in R^n \ \mid\ x_1 \geq \|x^{n-1}\|\},$$

where $\|\cdot\|$ denotes the standard Euclidean norm. It is well known that $\mathcal{Q}_n$ induces a partial order on R^n

$$x \succeq_{\mathcal{Q}_n} y \Longleftrightarrow x - y \in \mathcal{Q}_n.$$

If n is evident from the context we drop it from the subscript. A second-order cone programming problem is a convex optimization problem in which a linear function is minimized over the intersection of an affine set and the Cartesian product of second-order cones. It includes linear programs, convex quadratic programs and quadratically constrained convex quadratic programs as special cases. Mathematically, a second-order cone program has the following form

$$\begin{aligned} \min \quad & \sum_{i=1}^{r} c_i^\top x_i \\ \text{s.t.} \quad & \sum_{i=1}^{r} A_i x_i = b, \quad x_i \succeq_{\mathcal{Q}_{n_i}} 0,\ i = 1,\cdots,r, \end{aligned}$$

where r is the number of blocks, $n = \sum_{i=1}^{r} n_i$, $c_i \in R^{n_i}$, $x_i \in R^{n_i}$, $A_i \in R^{m\times n_i}$ and $b \in R^m$.

The KKT system of the above second-order cone program is a second-order cone complementarity problem (SOCCP), which has the following format:

$$\text{Find an } z \in \mathcal{Q}_p, \text{ such that } F(z) \in \mathcal{Q}_p \text{ and } z^T F(z) = 0, \tag{2.2}$$

where $\mathcal{Q}_p$ is a second-order cone of certain dimension p (usually $p \geq n+m+r$), and $F : R^p \to R^p$ is a given mapping. The SOCCP can be efficiently solved by a smoothing Newton method, see Chen, Sun, Sun [5] for details.

3 The parametric SOCP reformulation

In this section, we will show that for some interesting cases, the robust Nash-Cournot problem can either be reformulated as a parametric second-order cone program or be relaxed to this type of problems. The exact meaning of "parametric second-order cone program" will be made clear in the sequel.

3.1 Uncertainty in the opponents strategies

We first clarify the meaning of the uncertainty in the opponents strategies.

Assumption 3.1

(a) $V_k(v^{-k}) := \left\{ v^{-k} + \delta v^{-k} \;\middle|\; \|\delta v^{-k}\| \leq \rho_k,\ e^\top \delta v^{-k} = 0 \right\}$, $\forall k \in \mathcal{N}$ *where* $\rho_1, ..., \rho_n$ *are given nonnegative constants;*

(b) $D_A = \{A\}$.

Here, δv^{-k} represents a perturbation vector. The conditions $e^\top \delta v^{-k} = 0$ is for guaranteeing $e^\top(v + \delta v^{-k}) = d$.

Under Assumption 3.1, Player k solves the following problem to determine his/her strategy:

$$\begin{array}{ll} \min_{v^k} & \max\limits_{\delta v^{-k}} \left\{ {v^k}^\top A(v^k + v^{-k} + \delta v^{-k}) \;\middle|\; \|\delta v^{-k}\| \leq \rho_k, e^\top \delta v^{-k} = 0 \right\} \\ \text{s.t.} & v \in \Omega. \end{array} \tag{3.3}$$

Thus, the robust cost function for Player k is

$$\begin{aligned} \tilde{f}_k(v^k, v^{-k}) &= \max \left\{ {v^k}^\top A(v^k + v^{-k} + \delta v^{-k}) \;\middle|\; \|\delta v^{-k}\| \leq \rho_k, e^\top \delta v^{-k} = 0 \right\} \\ &= {v^k}^\top Av + \max \left\{ {v^k}^\top A\delta v^{-k} \;\middle|\; \|\delta v^{-k}\| \leq \rho_k, e^\top \delta v^{-k} = 0 \right\} \\ &= {v^k}^\top Av + \rho_k \left\| \tilde{A}^\top v^k \right\|, \end{aligned}$$

where $\tilde{A} = A(I_m - \frac{1}{m} ee^\top)$ and the last equality follows from the fact that projection of $A^\top v^k$ onto the hyperplane $\{v \mid e^\top v = 0\}$ can be represented as $(I_m - \frac{1}{m} ee^\top)A^\top v^k$, I_m is the m dimensional unit matrix. By introducing an auxiliary variable $v_0 \in R_+$, Problem (3.3) can be reduced to the following optimization problem

$$\begin{array}{ll} \min & {v^k}^\top A v + \rho_k v_0 \\ \text{s.t.} & \|\tilde{A}^\top v^k\| \le v_0, \ v \in \Omega. \end{array} \tag{3.4}$$

Proposition 3.2 *Suppose that the matrix A in (3.4) is positive definite. Then, under Assumption 3.1, the user equilibrium is equivalent to a parametric second-order cone program with parameter v^{-k}.*

Proof. Since A is positive definite, $B := \frac{A+A^\top}{2}$ is symmetric and positive definite. Let

$$u = B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} A v^{-k}.$$

Then (3.4) can be reformulated as

$$\begin{array}{ll} \min & u_0 \\ \text{s.t.} & B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} A v^{-k} = u, \\ & \tilde{A}^\top v^k = w, \\ & (v_0, w) \succeq_{\mathcal{Q}} 0, \ (u_0 - \rho_k v_0, u) \succeq_{\mathcal{Q}} 0, \\ & v \in \Omega, \end{array}$$

which is a second-order cone program parametrically depending on v^{-k}. □

3.2 Entry-wise uncertainty in the cost matrix

In this subsection, we consider the case that the uncertainty in the cost matrix occurs independently from entry to entry; that is,

Assumption 3.3

(a) $V_k(v^{-k}) := \{v^{-k}\}$.

(b) $D_A := \left\{ A + \delta A \in R^{m \times m} \;\middle|\; |\delta A_{ij}| \le \Gamma_{ij} \ (i, j = 1, \cdots, m) \right\}$, *where A and Γ are given matrices with $\Gamma_{ij} \ge 0, \ i, j = 1, \cdots, m$.*

Under Assumption 3.3 and the condition that $v \in \Omega$, the cost function $\tilde{f}_k$ can be written as

$$\begin{aligned} \tilde{f}_k(v^k, v^{-k}) \quad &= \max \left\{ {v^k}^\top \hat{A} v \;|\; \hat{A} \in D_A \right\} \\ &= {v^k}^\top A v + \max \left\{ {v^k}^\top \delta A v \;|\; A + \delta A \in D_A \right\} \\ &= {v^k}^\top A v + \max \left\{ {v^k}^\top \delta A v \;|\; |\delta A_{ij}| \le \Gamma_{ij} \right\} \\ &= {v^k}^\top A v + {v^k}^\top \Gamma v, \end{aligned}$$

where the last equality follows from the fact that $v^k \geq 0$ and $v \geq 0$. Thus, the robust Nash-Cournot problem solved by Player k is

$$\begin{array}{ll} \min & \tilde{f}_k(v^k, v^{-k}) := {v^k}^\top A v + {v^k}^\top \Gamma v \\ \text{s.t.} & v \in \Omega, \end{array} \tag{3.5}$$

where v^{-k} is taken as fixed parameter. Therefore we have

Proposition 3.4 *Suppose that the matrix A and Γ are such that $A + \Gamma$ is positive definite. Then, under Assumption 3.3, the user equilibrium is equivalent to the following parametric second-order cone program with parameter v^{-k}*

$$\begin{array}{ll} \min & u_0 \\ \text{s.t.} & B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} (A + \Gamma) v^{-k} = u, \\ & (u_0, u) \succeq_{\mathcal{Q}} 0, \\ & v \in \Omega, \end{array} \tag{3.6}$$

where $B = \frac{A + A^\top + \Gamma + \Gamma^\top}{2}$.

Proof. Since $A + \Gamma$ is positive definite, $B := \frac{A + A^\top + \Gamma + \Gamma^\top}{2}$ is symmetric and positive definite. It follows from (3.5)

$$\begin{aligned} \tilde{f}_k(v^k, v^{-k}) &= {v^k}^\top A v + {v^k}^\top \Gamma v \\ &= (B^{\frac{1}{2}} v^k)^\top B^{\frac{1}{2}} v^k + {v^k}^\top (A + \Gamma) v^{-k} \\ &= \left\| B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} (A + \Gamma) v^{-k} \right\|^2 - \frac{1}{4} \left\| B^{-\frac{1}{2}} (A + \Gamma) v^{-k} \right\|^2. \end{aligned}$$

Let $u := B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} (A + \Gamma) v^{-k}$, then problem (3.5) and problem (3.6) have same optimal solutions and their optimal objective values will differ by $\frac{1}{4} \| B^{-\frac{1}{2}} (A + \Gamma) v^{-k} \|^2$. □

3.3 Columnwise uncertainty in the cost matrices

We now consider the case where the uncertainty in matrix A occurs columnwise independently. That is,

Assumption 3.5

(a) $V_k(v^{-k}) := \{v^{-k}\}$.

(b) $D_A := \left\{ A + \delta A \in R^{m \times m} \;\middle|\; \|\delta A_j^c\| \leq \gamma_j,\ j = 1, \cdots, m \right\}$, *where A is a given matrix and $\gamma \geq 0$ is a given vector, and δA_j^c denotes the jth column of matrix δA.*

Proposition 3.6 *Suppose that the matrix A in (3.8) is positive definite. Then, under Assumption 3.5, a relaxation of the user equilibrium problem is the following parametric second-order cone program*

$$\begin{aligned}
\min \quad & u_0 \\
\text{s.t.} \quad & B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} (A v^{-k} + \gamma t) = u, \\
& ((u_0 - (\gamma^\top v^{-k}) t + s), u) \succeq_{\mathcal{Q}} 0, \\
& (t, v^k) \succeq_{\mathcal{Q}} 0, \\
& (s, B^{-\frac{1}{2}} (A v^{-k} + \gamma t)) \succeq_{\mathcal{Q}} 0, \\
& v \in \Omega,
\end{aligned} \tag{3.7}$$

where $B = \frac{A + A^\top}{2}$.

Proof. Under Assumption 3.5, the cost function $\tilde{f}_k$ can be written as

$$\begin{aligned}
\tilde{f}_k(v^k, v^{-k}) \quad & = \max \left\{ {v^k}^\top \hat{A} v \;\middle|\; \hat{A} \in D_A \right\} \\
& = {v^k}^\top A v + \max \left\{ {v^k}^\top \delta A v \;\middle|\; \|\delta A_j^c\| \le \gamma_j \right\} \\
& = {v^k}^\top A v + \max_{\|\delta A_j^c\| \le \gamma_j} \sum_{j=1}^m v_j {v^k}^\top \delta A_j^c \\
& = {v^k}^\top A v + \sum_{j=1}^m v_j \left\| v^k \right\| \gamma_j \\
& = {v^k}^\top A v + \gamma^\top v \left\| v^k \right\| .
\end{aligned}$$

Thus, Player k solves the following problem

$$\begin{aligned}
\min \quad & \tilde{f}_k(v^k, v^{-k}) := {v^k}^\top A v + v^\top \gamma \|v^k\| \\
\text{s.t.} \quad & v \in \Omega.
\end{aligned} \tag{3.8}$$

Introducing variables u_0 and t, we can rewrite the optimization problem (3.8) as

$$\begin{aligned}
\min \quad & u_0 \\
\text{s.t.} \quad & {v^k}^\top A v + v^\top \gamma t \le u_0, \\
& \|v^k\| = t, \\
& v \in \Omega.
\end{aligned} \tag{3.9}$$

Let $B = \frac{A + A^\top}{2}$. Then (3.9) can be rewritten as

$${v^k}^\top B v^k + {v^k}^\top (A v^{-k} + t\gamma) \le u_0 - t\gamma^\top v^{-k}.$$

Setting

$$u = B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} (A v^{-k} + \gamma t),$$

it follows from the above inequality that

$$\|u\|^2 \le u_0 - t\gamma^\top v^{-k} + s,$$

with

$$s = \frac{1}{4}\left\|B^{-\frac{1}{2}}(Av^{-k}+\gamma t)\right\|^2.$$

Thus, the optimization problem (3.8) is equivalent to the following problem

$$\begin{array}{ll}
\min & u_0 \\
\text{s.t.} & B^{\frac{1}{2}}v^k + \dfrac{1}{2}B^{-\frac{1}{2}}(Av^{-k}+\gamma t) = u, \\
& \|v^k\| = t, \\
& \|u\|^2 \le u_0 - t\gamma^\top v^{-k} + s, \\
& \dfrac{1}{4}\|B^{-\frac{1}{2}}(Av^{-k}+\gamma t)\|^2 = s, \\
& v \in \Omega.
\end{array}$$

Relaxing the second and the fourth constraints in the above optimization problem, we get

$$\begin{array}{ll}
\min & u_0 \\
\text{s.t.} & B^{\frac{1}{2}}v^k + \dfrac{1}{2}B^{-\frac{1}{2}}(Av^{-k}+\gamma t) = u, \\
& \|v^k\| \le t, \\
& \|u\|^2 \le u_0 - t\gamma^\top v^{-k} + s, \\
& \dfrac{1}{4}\|B^{-\frac{1}{2}}(Av^{-k}+\gamma t)\|^2 \le s, \\
& v \in \Omega,
\end{array}$$

which is equivalent to (3.7). □

3.4 Uncertainty in both opponents' strategy and the cost matrix (column-wise)

In this subsection, we consider the case that Player k can estimate neither the cost matrix nor his/her opponents' strategies exactly and the uncertainty in the cost matrices occurs in columns. That is, we make the following assumption:

Assumption 3.7

(a) $V_k(v^{-k}) := \left\{v^{-k}+\delta v^{-k} \;\middle|\; \|\delta v^{-k}\| \le \rho_k,\ e^\top \delta v^{-k} = 0,\ v^{-k}+\delta v^{-k} \ge 0\right\}$,

(b) $D_A := \left\{A+\delta A \in R^{m\times m} \;\middle|\; \|\delta A^c_j\| \le \gamma_j,\ j=1,\cdots,m\right\}$, *where A is a given matrix, $\gamma \ge 0$ is a given vector, and δA^c_j denotes the jth column of matrix δA.*

Under Assumption 3.7, the cost function $\tilde{f}_k$ can be written as

$$\begin{aligned}
&\tilde{f}_k(v^k, v^{-k}) \\
= \;& \max\left\{ {v^k}^\top \hat{A}(v^k+v^{-k}+\delta v^{-k}) \;\middle|\; \hat{A}\in D_A,\ v^{-k}+\delta v^{-k} \in V_k(v^{-k})\right\} \\
= \;& {v^k}^\top Av + \max\left\{ {v^k}^\top A\delta v^{-k} + {v^k}^\top \delta A(v+\delta v^{-k}) \;\middle|\; A+\delta A \in D_A,\ v^{-k}+\delta v^{-k}\in V_k(v^{-k})\right\} \\
= \;& {v^k}^\top Av + \max\left\{ {v^k}^\top A\delta v^{-k} + \sum_{j=1}^m (\delta A^c_j)^\top v^k (v+\delta v^{-k})_j \;\middle|\; \|\delta A^c_j\| \le \gamma_j,\ v^{-k}+\delta v^{-k}\in V_k(v^{-k})\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \quad {v^k}^\top Av + \max\left\{ {v^k}^\top A\delta v^{-k} + \sum_{j=1}^{m} \|v^k\|\gamma_j(v+\delta v^{-k})_j \;\middle|\; v^{-k}+\delta v^{-k} \in V_k(v^{-k}) \right\} \\
&\leq \quad {v^k}^\top Av + \|v^k\| v^\top \gamma + \max\Big\{ {v^k}^\top A\delta v^{-k} + \left\|v^k\right\| {\delta v^{-k}}^\top \gamma \;\Big|\; \|\delta v^{-k}\| \leq \rho_k,\ e^\top \delta v^{-k} = 0 \Big\} \\
&= \quad {v^k}^\top Av + \left\|v^k\right\| v^\top \gamma + \rho_k \left\| (I_m - m^{-1}ee^\top)(A^\top v^k + \|v^k\|\gamma) \right\|.
\end{aligned}
$$

Thus, a relaxation of Player k's problem is

$$
\begin{aligned}
\min \quad & {v^k}^\top Av + \|v^k\| v^\top \gamma + \rho_k \left\| (I_m - m^{-1} e_m e_m^\top)(A^\top v^k + \|v^k\|\gamma) \right\| \\
\text{s.t.} \quad & v \in \Omega.
\end{aligned} \tag{3.10}
$$

By introducing auxiliary variables $t, s \in R$, problem (3.10) can be further relaxed to a second-order cone program with parameter v^{-k} as shown in the following proposition. The proof is omitted since it is very similar to the proof of Proposition 3.6.

Proposition 3.8 *Suppose that the matrix A in (3.10) is positive definite. Then, under Assumption 3.7, the user equilibrium can be relaxed to the following parametric second-order cone program*

$$
\begin{aligned}
\min \quad & u_0 \\
\text{s.t.} \quad & B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} (Av^{-k} + \gamma t) = u, \\
& [I_m - m^{-1}ee^\top](A^\top v^k + t\gamma) = r, \\
& (u_0 - (\gamma^\top v^{-k})t - \rho_k s, u) \succeq_{\mathcal{Q}} 0, \\
& (t, v^k) \succeq_{\mathcal{Q}} 0, \\
& (s, r) \succeq_{\mathcal{Q}} 0, \\
& v \in \Omega,
\end{aligned}
$$

where $B = \frac{A+A^\top}{2}$.

3.5 Uncertainty in both opponents' strategy and the cost matrix (entrywise)

In this subsection, we consider the case that Player k can estimate neither the cost matrix nor his/her opponents' strategies exactly and the uncertainty in the cost matrix occurs entrywise independently; that is,

Assumption 3.9

(a) $V_k(v^{-k}) := \{v^{-k} + \delta v^{-k} \mid \|\delta v^{-k}\| \leq \rho_k,\ e^\top \delta v^{-k} = 0\ v^{-k} + \delta v^{-k} \geq 0\}$,

(b) $D_A := \{A + \delta A \in R^{m\times m} \mid |\delta A_{ij}| \leq \Gamma_{ij},\ j = 1, \cdots, m)\}$ *where Γ is a given matrix with* $\Gamma_{ij} \geq 0$.

Under Assumption 3.9, the cost function $\tilde{f}_k$ can be written as

$$\tilde{f}_k(v^k, v^{-k})$$

$$
\begin{aligned}
&= \max\left\{ {v^k}^\top \hat{A}(v+\delta v^{-k}) \;\middle|\; \hat{A}\in D_A,\ v^{-k}+\delta v^{-k}\in V_k(v^{-k}) \right\} \\
&= {v^k}^\top A v \\
&\quad + \max\left\{ {v^k}^\top A\delta v^{-k} + {v^k}^\top \delta A(v+\delta v^{-k}) \;\middle|\; A+\delta A\in D_A,\ v^{-k}+\delta v^{-k}\in V_k(v^{-k}) \right\} \\
&= {v^k}^\top A v + \max\left\{ {v^k}^\top A\delta v^{-k} + \sum_{j=1}^{m}(\delta A_{ij} v_i^k (v+\delta v^{-k})_j \;\middle|\; |\delta A_{ij}|\le \Gamma_{ij},\ v^{-k}+\delta v^{-k}\in V_k(v^{-k}) \right\} \\
&= {v^k}^\top A v + {v^k}^\top \Gamma v + \max\left\{ {v^k}^\top (A+\Gamma)\delta v^{-k} \;\middle|\; (v^{-k}+\delta v^{-k})\in V_k(v^{-k}) \right\} \\
&\le {v^k}^\top A v + {v^k}^\top \Gamma v + \max\left\{ {v^k}^\top (A+\Gamma)\delta v^{-k} \;\middle|\; \|\delta v^{-k}\|\le \rho_k,\ e_m^\top \delta v^{-k}=0 \right\} \\
&= {v^k}^\top (A+\Gamma)v + \rho_k \left\| \left[I_m - m^{-1} e_m e_m^\top \right] (A+\Gamma)^\top v^k \right\|.
\end{aligned}
$$

Thus, a relaxation of Player k's problem is

$$
\begin{aligned}
\min \quad & {v^k}^\top (A+\Gamma)v + \rho_k \left\| \left[I_m - m^{-1} e_m e_m^\top \right] (A+\Gamma)^\top v^k \right\| \\
\text{s.t.} \quad & v\in\Omega.
\end{aligned}
$$

Based on a similar approach to the proofs of Proposition 3.4 and Proposition 3.8, we can obtain the following result.

Proposition 3.10 *Suppose that the matrix A and Γ are such that $A+\Gamma$ is positive definite. Then, under Assumption 3.9, the user equilibrium can be relaxed to the following parametric second-order cone program*

$$
\begin{aligned}
\min \quad & u_0 \\
\text{s.t.} \quad & B^{\frac{1}{2}} v^k + \frac{1}{2} B^{-\frac{1}{2}} (A+\Gamma v^{-k}) = u, \\
& \left[I_m - m^{-1} e e^\top \right] (A+\Gamma)^\top v^k = r, \\
& (u_0 - \rho_k s, u) \succeq_{\mathcal{Q}} 0, \\
& (s, r) \succeq_{\mathcal{Q}} 0, \\
& v\in\Omega,
\end{aligned}
$$

where $B = \frac{A+\Gamma+A^\top+\Gamma^\top}{2}$.

In concluding this section, we explain how the parametric second-order cone programs of the players can be combined into a (non-parametric) second-order complementarity problem (SOCCP)[1]. Let user $k's$ (robust) optimization problem be denoted by $P_k(v^{-k})$, which depends on v^{-k}. Then, $P_k(v^{-k})$ reduces to an SOCCP for any fixed v^{-k}. Now, notice that the robust Nash-Cournot equilibrium problem is to find $\{(v^*)^k\}_{k=1}^n$ such that $(v^*)^k$ solves $P_k(v^{-k})$ for all $k\in\mathcal{N}$ simultaneously. Since the KKT conditions of $P_k(v^{-k})$ can be rewritten as an SOCCP, say KKT$_k(v^{-k})$, by combining all users SOCCPs KKT$_1(v^{-1})$, ...,KKT$_n(v^{-n})$, we can obtain a large SOCCP with variables $(v^1, ..., v^n)$. For more detailed discussion in this matter, see [12].

[1]This paragraph is cited from a referee's comments.

4 The price of anarchy for the robust Nash-Cournot equilibria

We now consider the price of anarchy (PA) for robust Nash-Cournot equilibria and derive several bounds for it. Recall that the PA is the ratio between the total cost at robust Nash-Cournot equilibrium and the system cost. Let v^* denote a robust Nash-Cournot equilibrium and $\bar{v}$ denote the system optimal, then the PA ϱ is

$$\varrho := \frac{Z_u}{Z_s} = \frac{\sum_{a \in \mathcal{M}} t_a(v^*) v_a^*}{\sum_{a \in \mathcal{M}} t_a(\bar{v}) \bar{v}_a} = \frac{(v^*)^\top A v^*}{\bar{v}^\top A \bar{v}}.$$

To derive the bounds, we need to define the degree of asymmetry of a matrix A.

Definition 4.1 *The degree of asymmetry of a positive definite matrix A is defined as*

$$c^2 = \|S^{-1}A\|_S^2 = \sup_{w \neq 0} \frac{\|S^{-1}Aw\|_S^2}{\|w\|_S^2} = \sup_{w \neq 0} \frac{w^\top A^\top S^{-1} A w}{w^\top S w},$$

where

$$S = \frac{A + A^\top}{2}$$

is the symmetrized part of the matrix A and $\|\cdot\|_S$ denotes the S-norm of a vector, i.e. $\|x\|_S = \sqrt{x^\top S x}$ and $\|S^{-1}A\|_S$ is the operator norm of $S^{-1}A$ induced by this vector norm.

It is obvious that $c^2 = 1$ when A is positive definite and symmetric. The constant c^2 was originally introduced by Hammond [11] and has the following property.

Lemma 4.2 *If A^2 is a positive definite matrix, then $c^2 \leq 2$.*

We now analyze the PA under uncertainties. Since the analysis for Assumption 3.1 and the other assumptions is very similar, we just focus on the case where Assumption 3.1 holds.

Theorem 4.3 *Suppose that the matrix A in (3.4) is a positive definite matrix and Assumption 3.1 holds. Furthermore, suppose that $A_{ij} \geq 0$ for all $i, j = 1, \cdots, m$ and there are two scalers $0 \leq \bar{\alpha}, \underline{\alpha} \leq 1$, such that $\bar{v}^k \leq \bar{\alpha}\bar{v}$ and $v^{k*} \geq \underline{\alpha} v^*$ for all $k \in \mathcal{N}$. Then, the PA ϱ satisfies the following inequality*

$$\varrho \leq \frac{2\bar{\alpha} + (1-\underline{\alpha})^2 c^2 + (1-\underline{\alpha}) c \sqrt{(1-\underline{\alpha})^2 c^2 + 4\bar{\alpha}}}{2} + \frac{1}{1-(1-\underline{\alpha})b_2} \frac{\sum_{k \in \mathcal{N}} \rho_k \left(\|\tilde{A}\bar{v}^k\| - \|\tilde{A}v^{k*}\| \right)}{(\bar{v})^\top A \bar{v}} \tag{4.11}$$

where b_2 is given by (4.15)-(4.16) below.

Proof. Since v^* is a solution of the robust Nash-Cournot equilibrium and $\bar{v} \in \Omega$, it follows from (3.4) that

$$\begin{aligned}
&(v^{k*})^\top A v^* + \rho_k \|\tilde{A} v^{k*}\| \\
&\leq (\bar{v}^k)^\top A(\bar{v}^k + v^{-k*}) + \rho_k \|\tilde{A}\bar{v}^k\| \\
&= (\bar{v}^k)^\top A \bar{v}^k + (\bar{v}^k)^\top A v^* - (\bar{v}^k)^\top A v^{k*} + \rho_k \|\tilde{A}\bar{v}^k\| \\
&\leq \bar{\alpha}(\bar{v}^k)^\top A \bar{v} + (1-\underline{\alpha})(\bar{v}^k)^\top A v^* + \rho_k \|\tilde{A}\bar{v}^k\|,
\end{aligned}$$

where the first inequality follows from the fact that v^{k*} is a solution of (3.4) and the last inequality follows from the $\bar{v}^k \geq 0$, $v^* \geq 0$ and $A_{ij} \geq 0$ for all $i, j = 1, \cdots, m$ and the assumptions that $\bar{v}^k \leq \bar{\alpha}\bar{v}$ and $v^{k*} \geq \underline{\alpha} v^*$. Summing up both sides for all $k \in \mathcal{N}$, we get

$$\begin{aligned}
&(v^*)^\top A v^* + \sum_{k\in\mathcal{N}} \rho_k \|\tilde{A} v^{k*}\| \\
&\leq \bar{\alpha}\bar{v}^\top A\bar{v} + (1-\underline{\alpha})\bar{v}^\top A v^* + \sum_{k\in\mathcal{N}} \rho_k \|\tilde{A}\bar{v}^k\| \\
&= \bar{\alpha}\bar{v}^\top A\bar{v} + (1-\underline{\alpha})\bar{v}^\top A S^{-1} S v^* + \sum_{k\in\mathcal{N}} \rho_k \|\tilde{A}\bar{v}^k\| \\
&\leq \bar{\alpha}\bar{v}^\top A\bar{v} + (1-\underline{\alpha})\|\bar{v}^\top A S^{-1}\|_S \|v^*\|_S + \sum_{k\in\mathcal{N}} \rho_k \|\tilde{A}\bar{v}^k\| \\
&\leq \bar{\alpha}\bar{v}^\top A\bar{v} + c(1-\underline{\alpha})\|\bar{v}\|_S \|v^*\|_S + \sum_{k\in\mathcal{N}} \rho_k \|\tilde{A}\bar{v}^k\|, \qquad (4.12)
\end{aligned}$$

where the second inequality follows from Cauchy-Schwarz inequality and the last one follows from the norm inequality.

For any two vectors x and y in R^n, we have

$$2\sqrt{b_1 b_2}\|x\|_S\|y\|_S \leq b_1\|x\|_S^2 + b_2\|y\|_S^2$$

if $b_1, b_2 \geq 0$. This implies that

$$c\|x\|_S\|y\|_S \leq b_1\|x\|_S^2 + b_2\|y\|_S^2 \qquad (4.13)$$

if $b_1, b_2 \geq 0$ and $b_1 b_2 \geq c^2/4$. It follows from (4.12) and (4.13) that

$$(1-(1-\underline{\alpha})b_2)(v^*)^\top A v^* \leq (\bar{\alpha} + (1-\underline{\alpha})b_1)\bar{v}^\top A\bar{v} + \sum_{k\in\mathcal{N}} \rho_k(\|\tilde{A}\bar{v}^k\| - \|\tilde{A}v^{k*}\|).$$

If $b_2 \geq 1/(1-\underline{\alpha})$, then the above inequality may hold trivially (at least in some cases). Thus, we need to add the constraint that $b_2 < 1/(1-\underline{\alpha})$. We may find the best upper bound by solving

$$\begin{aligned}
\min \quad & \frac{\bar{\alpha} + (1-\underline{\alpha})b_1}{1-(1-\underline{\alpha})b_2} \\
\text{s. t.} \quad & b_1 b_2 \geq c^2/4 \\
& 0 \leq b_2 < 1/(1-\underline{\alpha}).
\end{aligned}$$

Since we just want to find an upper bound of the price of anarchy, we can tighten the second constraint in the above optimization problem

$$\begin{aligned}
\min \quad & \frac{\bar{\alpha} + (1-\underline{\alpha})b_1}{1-(1-\underline{\alpha})b_2} \\
\text{s. t.} \quad & b_1 b_2 \geq c^2/4 \qquad (4.14) \\
& 0 \leq b_2 \leq 1.
\end{aligned}$$

The optimal objective value of (4.14) is

$$t = \frac{2\bar{\alpha} + (1-\underline{\alpha})^2 c^2 + (1-\underline{\alpha})c\sqrt{(1-\underline{\alpha})^2 c^2 + 4\bar{\alpha}}}{2}, \qquad (4.15)$$

which is obtained at

$$b_2 = \frac{t - \bar{\alpha} - \sqrt{(t-\bar{\alpha})^2 - (1-\underline{\alpha})^2 c^2 t}}{2} \qquad \text{and} \qquad b_1 = \frac{c^2}{4b_2}. \tag{4.16}$$

This completes the proof. □

Remark. Unlike the game with complete information, in our bound, there is a term inherited from the uncertainties of the data (the second term in the right hand of (4.11)). If $\rho_k = 0$ for all $k \in \mathcal{N}$, the case reduces to the one with complete information and the bound (4.11) reduces to

$$\varrho \leq \frac{2\bar{\alpha} + (1-\underline{\alpha})^2 c^2 + (1-\underline{\alpha})c\sqrt{(1-\underline{\alpha})^2 c^2 + 4\bar{\alpha}}}{2}, \tag{4.17}$$

which appears to be new in the literature. Furthermore, if $n = 1$, that is, if there is only a monopoly player controls all users in the network, we have $\bar{\alpha} = \underline{\alpha} = 1$ and the bound given in (4.17) is $\varrho = 1$, indicating there is no efficiency loss. For $n \geq 2$, it is possible that $\bar{\alpha} = 1$ and $\underline{\alpha} = 0$. For example, this occurs for the network game with unsplittable flows. For this case, the bound in (4.17) reduces to

$$\varrho \leq \frac{2 + c^2 + c\sqrt{c^2+4}}{2}.$$

Moreover, if the cost matrix is symmetric, i.e., $c = 1$, then $\varrho \leq \frac{3+\sqrt{5}}{2} = 2.618$, which is just the bound derived by Awerbuch et al. [3]. □

5 Conclusions

We considered a traffic game with incomplete information. We proved that in some interesting cases, the robust Nash-Cournot equilibrium problem can be reformulated as a second-order cone program. We also gave some bounds of the PA of the robust Nash-Cournot equilibria, which appears not to have been considered in the literature.

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