Optimal gradual liquidation of equity from a risky asset*

Nikolai Dokuchaev

Department of Mathematics, Trent University, Ontario, Canada, and

Department of Mathematics & Statistics, Curtin University, GPO Box U1987, Perth,

6845 Western Australia. Email N.Dokuchaev at curtin.edu.au

September 9, 2010

Abstract

We consider a problem of optimal gradual liquidation of equity from a risky asset for continuous time stochastic market model. The owner of the risky asset uses this equity as a source of steady cash flow by borrowing money permanently against this equity. At the terminal time, there is no equity for him in this asset, and the bank gains ownership of this asset. Optimal strategy is obtained explicitly.

JEL classification: C61, D81

Key words and phrases: equity liquidation, optimal strategy, contingent claim replication.

1 Introduction

We consider a problem of optimal gradual liquidation of equity from a risky asset for continuous time stochastic market model. We assume that there is a risky asset with the price $s(t)$, $t \in [0, T]$ (for example, a stock portfolio or a real estate property). We assume that a holder of the risky asset has a saving account and a credit line in a bank. He uses this equity as a collateral for borrowing from the bank. In fact, the owner is going to liquidate his equity in this risky asset by borrowing money against this equity. At the terminal time, there is no equity for him in this asset, and the bank gains ownership of this asset. We assume that the terminal value $s(T)$ of this asset is random and it is unknown at time $t < T$. Therefore, there is a problem of optimal selection of the process of cash flow. We state optimization problem with the goal to

*Applied Economics Letters, 2010 **17**, pp. 1305–1308

describe the most "fair" and most stable and gradual process of liquidation of equity. We don't consider problems of maximization of the payoff to the client or minimization of the payoff to the bank. In our setting, the total amount of money to be paid depends on the market prices of the asset which is beyond the control of the control of both parties. Some alternative settings and references for the problem of optimal liquidation of the equity can be found in Schied and Schneborn (2009).

The problem of gradual liquidation of equity stated in the present paper is reduced to a claim replication problem and optimal stochastic control problem of a new type. Usually, the claim replication problems are solved via the martingale representation method and the related stochastic backward differential equations. We set a new special problem of replication of a random contingent claim such that only the drift coefficient is allowed to be selected. This problem is formulated as an optimal stochastic control problem. We obtained explicit solution of the special claim replication problem and stochastic control problem. These result give the optimal strategy of liquidation of equity as a special case.

2 The problem setting and the main result

Consider a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and standard d-dimensional Wiener process $w(t)$ (with $w(0) = 0$) which generates the filtration $\mathcal{F}_t = \sigma\{w(r) : 0 \le r \le t\}$ augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$. We denote by $L_{22}^{n\times m}$ the class of square integrable random processes adapted to $\mathcal{F}_t$ with values in $\mathbf{R}^{n\times m}$.

We assume that the price of $s(t)$ of a risky asset is a random process. Moreover, we assume that $(s(t), \mathcal{F}_t)$ is a continuous square integrable martingale under $\mathbf{P}$. In particular, this means that

$$s(T) = \mathbf{E}s(T) + \int_0^T k_s(t)dw(t),$$

where $k_s(t) \in L_{22}^{1\times d}$ is an adapted to $\mathcal{F}_t$ square integrable random process taking values in $\mathbf{R}^{1\times d}$.

Let a be the initial debt of the owner of the risky asset (or $-a$ is the initial deposit, if $a < 0$ and the initial balance is positive). We allow $a \ge 0$ as well as $a < 0$. Let $u(t)$ be the process describing the density of the cash flow at time $t \in (0, T)$, such that $u(t)\Delta t$ is the amount of cash borrowed at time interval $(t, t + \Delta t)$, for small $\Delta t > 0$. We assume that the bank interest rate is r, for both loans and savings, where $r \ge 0$ is a constant. This model leads to the following formula for the debt value $x(t)$ at time $t \in [0, T]$

$$x(t) = e^{rt}a + \int_0^t e^{r(t-\tau)}u(\tau)d\tau.$$

We look for optimal choice of $u(t)$ in the class of random square integrable processes $u(t)$ that are adapted to the filtration $\mathcal{F}_t$. In fact, this filtration coincides with the filtration generated by the price process $s(t)$. Therefore, the value $u(t)$ can be calculated using current and historical observations of $s(\tau)$, $\tau \leq t$.

Let $U = L_{22}^{1\times 1}$.

Let $\theta \in (0, T]$ be given. The problem is stated as the following:

$$\begin{aligned} &\text{Minimize} \quad \mathbf{E}\int_0^T \gamma(t)(u(t)-c)^2 dt \quad \text{over} \quad u(\cdot) \in U, \ c \in \mathbf{R} \\ &\text{subject to} \quad x(T) = s(\theta) \quad \text{a.s.} \end{aligned} \tag{2.1}$$

Here $\gamma(t) > 0$ is a given deterministic discounting coefficient; "a.s." means almost surely, i.e., with probability 1.

This setting attempts to describe the most "fair" and most gradual process of liquidation of equity rather than to maximize the cash flow to the client on minimize it for the bank (after all, the total amount of money to be paid depends mainly on the market price of the asset which is beyond the control). However, the stability is the issue, as well as some estimate of the average flow (the value of c).

Set

$$\rho(t) \triangleq \frac{e^{r(T-t)} - 1}{r}.$$

Theorem 2.1 *Let $\gamma(t) = e^{r(T-t)}$, and let $\theta < T$. Then problem (2.1) has a unique solution $(\widehat{u}(\cdot), \widehat{c})$ in $U \times \mathbf{R}$. For this solution,*

$$\widehat{c} = \rho(0)^{-1}(\mathbf{E}s(T) - e^{rT}a),$$

and $u(t)$ is a path-wise continuous in t process such that

$$\begin{aligned} &u(0) = \widehat{c}, \\ &du(t) = \rho(t)^{-1}ds(t), \quad t \in (0, \theta), \\ &u(t) = u(\theta), \quad t > \theta. \end{aligned}$$

Note that the claim $s(\theta)$ is chosen with $\theta < T$ as an approximation of the terminal value $s(T)$ which cannot be replicated perfectly given our choice of admissible strategies. However, the optimal strategy $u(\cdot)$ obtained above is independent from θ in the following sense. Let $0 < \theta_1 < \theta_2 < T$, then the process $u|_{[0,\theta_1]}$ is the same for $\theta = \theta_1$ and $\theta = \theta_2$.

3 Proofs

Let $U_{s(\theta)}$ be the set of all $u(\cdot) \in U$ such that $x(T) = s(\theta)$.

We have that

$$\int_0^T \gamma(t)(u(t)-c)^2 dt = \int_0^T \gamma(t)u(t)^2 dt - 2c\int_0^T \gamma(t)u(t)dt + c^2\int_0^T \gamma(t)dt.$$

For any $u(\cdot) \in U_{s(\theta)}$,

$$x(T) = e^{rT}a + \int_0^T e^{r(T-\tau)}u(\tau)d\tau = s(\theta). \tag{3.1}$$

Hence

$$c\int_0^T \gamma(\tau)u(\tau)d\tau = c\int_0^T e^{r(T-\tau)}u(\tau)d\tau = c(s(\theta) - e^{rT}a) \quad \forall u(\cdot) \in U_{s(\theta)},$$

and this value does not depend on $u(\cdot)$. Therefore, problem (2.1) can be split into the following two problems:

$$\begin{aligned} &\text{Minimize} \quad \mathbf{E}\int_0^T \gamma(t)u(t)^2 dt \quad \text{over} \quad u(\cdot) \in U, \\ &\text{subject to} \quad x(T) = s(\theta) \quad \text{a.s.} \end{aligned} \tag{3.2}$$

and

$$\text{Minimize} \quad -2c\mathbf{E}\int_0^T e^{r(T-\tau)}u(\tau)d\tau + c^2\mathbf{E}\int_0^T e^{r(T-\tau)}d\tau \quad \text{over} \quad c \in \mathbf{R}. \tag{3.3}$$

By (3.1),

$$-2c\mathbf{E}\int_0^T e^{r(T-\tau)}u(\tau)d\tau + c^2\mathbf{E}\int_0^T e^{r(T-\tau)}d\tau = -2c\mathbf{E}(s(\theta) - e^{rT}a) + c^2\rho(0).$$

Clearly, $\widehat{c} = \rho(0)^{-1}(\mathbf{E}s(\theta) - e^{rT}a)$ is the unique solution of problem (3.3).

Therefore, it suffices to solve problem (3.2). We obtain below solution of more general problem (3.5) stated for n-dimensional state vector. The solution of problem (3.2) and the proof of Theorem 2.1 follow from Theorem 3.1 below.

A more general stochastic control problem

Let $T > 0$ and $\theta \in [0,T)$ be given. Let f be an n-dimensional $\mathcal{F}_\theta$-measurable random vector, $f \in L_2(\Omega, \mathcal{F}_\theta, \mathbf{P}; \mathbf{R}^n)$.

Let $a \in \mathbf{R}^n$, $\theta < T$, and let $\Gamma(t) > 0$ be a bounded and symmetric matrix process in $\mathbf{R}^{n\times n}$, such that the matrix $\Gamma(t)^{-1}$ is defined for all $t < T$.

Let $U = L_{22}^{n\times 1}$, and let U_f be the set of all $u(\cdot) \in U$ such that

$$\begin{aligned} &\tfrac{dx}{dt}(t) = Ax(t) + bu(t), \quad t \in (0,T), \\ &x(0) = a, \quad x(T) = f \quad \text{a.s..} \end{aligned} \tag{3.4}$$

Consider the problem

$$\text{Minimize} \quad \mathbf{E}\int_0^T u(t)^\top \Gamma(t) u(t)\, dt \quad \text{over} \quad u \in U_f. \tag{3.5}$$

Note that this problem is in fact as a modification of a stochastic control problem with terminal contingent claim from Dokuchaev and Zhou (1999), where the problem was stated for backward stochastic differential equation with non-zero diffusion term. Our setting is different from the one from Dokuchaev and Zhou (1999) since the non-zero diffusion term is not allowed.

By the Martingale Representation Theorem, there exists an unique $k_f \in L_{22}^{n\times d}$ such that

$$f = \mathbf{E}f + \int_0^\theta k_f(t)dw(t).$$

Theorem 3.1 *Let $\theta < T$. Then problem (3.5) has a unique solution $\widehat{u}(\cdot)$. For this solution, $u(t)$ is a path-wise continuous process such that*

$$u(t) = \bar{\mu} + \int_0^t \widehat{\mu}(s)dw(s),$$

where

$$\bar{\mu} = R(0)^{-1}(\mathbf{E}f - q), \quad \widehat{\mu}(t) = R(t)^{-1}k_f(t)\mathbb{I}_{\{t\le\theta\}},$$

and where

$$R(s) \triangleq \int_s^T Q(t)dt, \qquad Q(t) = e^{A(T-t)}b\Gamma(t)^{-1}b^\top e^{A^\top(T-t)}, \qquad q \triangleq e^{AT}a,$$

Proof of Theorem 3.1. Let the function $L(u,\mu) : U \times L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n) \to \mathbf{R}$ be defined as

$$L(u,\mu) \triangleq \frac{1}{2}\mathbf{E}\int_0^T u(t)^\top\Gamma(t)u(t)\,dt + \mathbf{E}\mu(f - x(T)).$$

For a given μ, consider the following problem:

$$\text{Minimize } L(u,\mu) \text{ over } u \in U. \tag{3.6}$$

We solve problem (3.6) using the so-called Stochastic Maximum Principle that gives a necessary condition of optimality for stochastic control problems (see, e.g., Arkin and Saksonov (1979), Bensoussan (1983), Dokuchaev and Zhou (1999), Haussmann (1986), Kushner (1972)). The only $u = u_\mu$ satisfying these necessary conditions of optimality is defined by

$$\begin{aligned}
&\widehat{u}_\mu(t) = \Gamma(t)^{-1} b^\top \psi(t),\\
&\psi(t) = e^{A^\top (T-t)} \mu(t), \quad \mu(t) = \mathbf{E}\{\mu|\mathcal{F}_t\}.
\end{aligned} \tag{3.7}$$

Clearly, the function $L(u, \mu)$ is strictly concave in u, and this minimization problem has an unique solution. Therefore, this u is the solution of (3.6).

Further, we consider the following problem:

$$\text{Maximize } L(\widehat{u}_\mu, \mu) \text{ over } \mu \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n). \tag{3.8}$$

Clearly, the corresponding $x(T)$ is

$$x(T) = \int_0^T e^{A(T-t)} b \widehat{u}_\mu(t) dt + e^{AT} a,$$

and

$$L(\widehat{u}_\mu, \mu) = \frac{1}{2}\mathbf{E}\int_0^T \widehat{u}_\mu(t)^\top \Gamma(t) \widehat{u}_\mu(t)\, dt - \mathbf{E}\mu^\top \int_0^T e^{A(T-t)} b \widehat{u}_\mu(t) dt - \mathbf{E}\mu^\top e^{AT} a + \mathbf{E}\mu^\top f.$$

We have that

$$\begin{aligned}
&\frac{1}{2}\mathbf{E}\int_0^T \widehat{u}_\mu(t)^\top \Gamma(t) \widehat{u}_\mu(t)\, dt - \mathbf{E}\mu^\top \int_0^T e^{A(T-t)} b \widehat{u}_\mu(t) dt\\
&= \frac{1}{2}\mathbf{E}\int_0^T (\Gamma(t)^{-1} b^\top \psi(t))^\top \Gamma(t) \Gamma(t)^{-1} b^\top \psi(t)\, dt - \mathbf{E}\mu^\top \int_0^T e^{A(T-t)} b \Gamma(t)^{-1} b^\top \psi(t) dt\\
&= \frac{1}{2}\mathbf{E}\int_0^T \psi(t)^\top b \Gamma(t)^{-1} \Gamma(t) \Gamma(t)^{-1} b^\top \psi(t)\, dt - \mathbf{E}\mu^\top \int_0^T e^{A(T-t)} b \Gamma(t)^{-1} b^\top \psi(t) dt\\
&= \frac{1}{2}\mathbf{E}\int_0^T \psi(t)^\top b \Gamma(t)^{-1} b^\top \psi(t)\, dt - \mathbf{E}\mu^\top \int_0^T e^{A(T-t)} b \Gamma(t)^{-1} b^\top \psi(t) dt\\
&= \frac{1}{2}\mathbf{E}\int_0^T [e^{A^\top (T-t)} \mu(t)]^\top b \Gamma(t)^{-1} b^\top e^{A^\top (T-t)} \mu(t)\, dt - \mathbf{E}\mu^\top \int_0^T e^{A(T-t)} b \Gamma(t)^{-1} b^\top e^{A^\top (T-t)} \mu(t) dt\\
&= \frac{1}{2}\mathbf{E}\int_0^T \mu(t)^\top e^{A(T-t)} b \Gamma(t)^{-1} b^\top e^{A^\top (T-t)} \mu(t)\, dt - \mathbf{E}\mu^\top \int_0^T e^{A(T-t)} b \Gamma(t)^{-1} b^\top e^{A^\top (T-t)} \mu(t) dt\\
&= \frac{1}{2}\mathbf{E}\int_0^T \mu(t)^\top Q(t) \mu(t)\, dt - \mathbf{E}\mu^\top \int_0^T Q(t) \mu(t) dt = -\frac{1}{2}\mathbf{E}\int_0^T \mu(t)^\top Q(t) \mu(t) dt.
\end{aligned}$$

We have used for the last equality that

$$\mathbf{E}\mu^\top \int_0^T Q(t) \mu(t) dt = \mathbf{E}\int_0^T \mu(t)^\top Q(t) \mu(t) dt.$$

It follows that

$$L(\widehat{u}_\mu, \mu) = \mathbf{E}\mu^\top (f - q) - \frac{1}{2}\mathbf{E}\int_0^T \mu(t)^\top Q(t)\mu(t)\,dt.$$

By the Martingale Representation Theorem, there exist $k_\mu \in L_{22}^{n\times d}$ such that

$$\mu = \bar{\mu} + \int_0^T k_\mu(t)dw(t), \tag{3.9}$$

where $\bar{\mu} \triangleq \mathbf{E}\mu$. It follows that

$$L(\widehat{u}_\mu, \mu) = \mathbf{E}\mu^\top f - \mathbf{E}\mu^\top q - \frac{1}{2}\mathbf{E}\int_0^T \mu(t)^\top Q(t)\mu(t)\,dt.$$

We have that

$$\begin{aligned} L(\widehat{u}_\mu, \mu) &= \bar{\mu}^\top(\bar{f} - q) - \frac{1}{2}\bar{\mu}^\top R(0)\bar{\mu} - \frac{1}{2}\mathbf{E}\int_0^T dt \int_0^t k_\mu(\tau)^\top Q(t) k_\mu(\tau)\,d\tau + \mathbf{E}\int_0^\theta k_\mu(t)^\top k_f(t)\,dt \\ &= \bar{\mu}^\top(\bar{f} - q) - \frac{1}{2}\bar{\mu}^\top R(0)\bar{\mu} - \frac{1}{2}\mathbf{E}\int_0^T k_\mu(\tau)^\top R(\tau) k_\mu(\tau)\,d\tau + \mathbf{E}\int_0^\theta k_\mu(t)^\top k_f(t)\,dt, \end{aligned}$$

Clearly, the solution of problem (3.8) is uniquely defined by (3.9) with

$$\bar{\mu} = R(0)^{-1}(\bar{f} - q), \quad k_\mu(t) = R(t)^{-1}k_f(t)\mathbb{I}_{\{t\le\theta\}}. \tag{3.10}$$

We found that $\sup_\mu \inf_u L(u, \mu)$ is achieved for $(\widehat{u}_\mu, \mu)$ defined by (3.9), (3.10), (3.7). We have that $L(u, \mu)$ is strictly concave in $u \in U$ and affine in $\mu \in L_2(\Omega, \mathcal{F}.\mathbf{P}, \mathbf{R}^n)$. In addition, $L(u, \mu)$ is continuous in $u \in L_{22}^{n\times 1}$ given $\mu \in L_2(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{R}^n)$, and $L(u, \mu)$ is continuous in $\mu \in L_2(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{R}^n)$ given $u \in U$. By Proposition 2.3 from Ekland and Temam (1999), Chapter VI, p. 175, it follows that

$$\inf_{u\in U} \sup_\mu L(u, \mu) = \sup_\mu \inf_{u\in U} L(u, \mu). \tag{3.11}$$

Therefore, (u_μ, μ) defined by (3.9), (3.10), (3.7) is the unique saddle point for (3.11). Further, it is easy to see that

$$\inf_{u\in U_f} \frac{1}{2}\mathbf{E}\int_0^T u(t)^\top \Gamma(t) u(t)\,dt = \inf_{u\in U} \sup_\mu L(u, \mu),$$

and any solution (u, μ) of (3.11) is such that $u \in U_f$. It follows that $u_\mu \in U_f$ and it is the optimal solution for problem (3.5). Then the proof of Theorem 3.1 follows.

Acknowledgment

This work was supported by NSERC grant of Canada 341796-2008 to the author.

References

[1] Arkin, V., Saksonov, M. Necessary optimality conditions for stochastic differential equations, *Soviet. Math. Dokl.*, 20 (1979), 1-5.

[2] Bensoussan, A. Lectures on stochastic control, part I, *Lecture Notes in Math.*, 972 (1983), 1-39.

[3] Dokuchaev, N.G., and Zhou, X.Y. Stochastic controls with terminal contingent conditions. *Journal of Mathematical Analysis and Applications* **238** (1999), 143-165.

[4] Ekland, I., and Temam R. (1999) *Convex Analysis and Variational Problems.* SIAM, Philadelphia.

[5] Haussmann, U.G. A stochastic maximum principle for optimal control of diffusions, Pitman, Boston/London/Melbourne, 1986.

[6] Kushner, H.J. Necessary conditions for continuous parameter stochastic optimization problems, *SIAM J. Contr.*, 10 (1972), 550-565.

[7] Schied, A., and Schöneborn, T. (2009). Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics* **13**, 181-204.