Linear and Multilinear Algebra
Vol. 00, No. 00, January 2009, 1–17

A reduction technique for discrete generalised algebraic and difference Riccati equations

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(*Submitted September 28, 2012*)

This paper proposes a reduction technique for the generalised Riccati difference equation arising in optimal control and optimal filtering. This technique relies on a study on the generalised discrete algebraic Riccati equation. In particular, an analysis on the eigenstructure of the corresponding extended symplectic pencil enables to identify a subspace in which all the solutions of the generalised discrete algebraic Riccati equation are coincident. This subspace is the key to derive a decomposition technique for the generalised Riccati difference equation. This decomposition isolates a "nilpotent" part, which converges to a steady-state solution in a finite number of steps, from another part that can be computed by iterating a reduced-order generalised Riccati difference equation.

Keywords: generalised Riccati difference equation, finite-horizon LQ problem, generalised discrete algebraic Riccati equation, extended symplectic pencil.

AMS Subject Classification: 93C05, 49N05, 15A24.

1. Introduction

Consider the classic finite-horizon Linear Quadratic (LQ) optimal control problem. In particular, consider the discrete linear time-invariant system governed by the difference equation

$$x_{t+1} = A\,x_t + B\,u_t, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and where, for all $t \ge 0$, $x_t \in \mathbb{R}^n$ represents the state and $u_t \in \mathbb{R}^m$ represents the control input. Let the initial state $x_0 \in \mathbb{R}^n$

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[*] Corresponding author. Email: L.Ntogramatzidis@curtin.edu.au. Partially supported by the Italian Ministry for Education and Research (MIUR) under PRIN grant n. 20085FFJ2Z "New Algorithms and Applications of System Identification and Adaptive Control" and by the Australian Research Council under the grant FT120100604. Research carried out while the first author was visiting Curtin University, Perth (WA), Australia.

ISSN: 0308-1087 print/ISSN 1563-5139 online

DOI: 10.1080/03081080xxxxxxxxx
http://www.informaworld.com

be given. The problem is to find a sequence of inputs u_t, with $t = 0, 1, \ldots, T-1$, minimising the cost function

$$J(x_0, u) \stackrel{\text{def}}{=} \sum_{t=0}^{T-1} \begin{bmatrix} x_t^{\mathrm{T}} & u_t^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} + x_T^{\mathrm{T}} P \, x_T. \tag{2}$$

We assume that the weight matrices $Q \in \mathbb{R}^{n \times n}$, $S \in \mathbb{R}^{n \times m}$ and $R \in \mathbb{R}^{m \times m}$ are such that the *Popov matrix* Π is symmetric and positive semidefinite, i.e.,

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} = \Pi^{\mathrm{T}} \geq 0. \tag{3}$$

We also assume that $P = P^{\mathrm{T}} \geq 0$. The set of matrices $\Sigma = (A, B, \Pi)$ is often referred to as *Popov triple*, see e.g. [13]. We recall that, for any time t, the set $\mathcal{U}_t$ of all optimal inputs can be parameterised in terms of an arbitrary m-dimensional signal v_t as $\mathcal{U}_t = \{-K_t \, x_t + G_t \, v_t\}$, where[1]

$$K_t = (R + B^{\mathrm{T}} X_{t+1} B)^{\dagger} (S^{\mathrm{T}} + B^{\mathrm{T}} X_{t+1} A), \tag{4}$$

$$G_t = I_m - (R + B^{\mathrm{T}} X_{t+1} B)^{\dagger} (R + B^{\mathrm{T}} X_{t+1} B), \tag{5}$$

in which X_t is the solution of the Generalised Riccati Difference Equation GRDE(Σ)

$$X_t = A^{\mathrm{T}} X_{t+1} A - (A^{\mathrm{T}} X_{t+1} B + S)(R + B^{\mathrm{T}} X_{t+1} B)^{\dagger} (B^{\mathrm{T}} X_{t+1} A + S^{\mathrm{T}}) + Q \tag{6}$$

iterated backwards from $t = T-1$ to $t = 0$ using the terminal condition

$$X_T = P, \tag{7}$$

see [14]. The equation characterising the set of optimal state trajectories is

$$x_{t+1} = (A - B \, K_t) \, x_t - B \, G_t \, v_t.$$

The optimal cost is $J^* = x_0^{\mathrm{T}} X_0 \, x_0$.

Despite the fact that it has been known for several decades that the generalised discrete Riccati difference equation provides the solution of the classic finite-horizon LQ problem, this equation has not been studied with the same attention and thoroughness that has undergone the study of the standard discrete Riccati difference equation. The purpose of this paper is to attempt to start filling this gap. In particular, we want to show a reduction technique for this equation that

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[1] The symbol $M^{\dagger}$ denotes the Moore-Penrose pseudo-inverse of matrix M.

allows to compute its solution by solving a smaller equation with the same recursive structure, with obvious computational advantages. In order to carry out this task, several ancillary results on the corresponding generalised Riccati equation are established, which constitute an extension of those valid for standard discrete algebraic Riccati equations presented in [12] and [2]. In particular, these results show that the nilpotent part of the closed-loop matrix is independent of the particular solution of the generalised algebraic Riccati equation. Moreover, we provide a necessary and sufficient condition expressed in sole terms of the problem data for the existence of this nilpotent part of the closed-loop matrix. This condition, which appears to be straightforward for the standard algebraic Riccati equation, becomes more involved – and interesting – for the case of the generalised Riccati equation. We then show that every solution of the generalised algebraic Riccati equation coincides along the largest eigenspace associated with the eigenvalue at the origin of the closed-loop, and that this subspace can be employed to decompose the generalised Riccati difference equation into a nilpotent part, whose solution converges to the zero matrix in a finite number of steps (not greater than n) and a part which corresponds to a non-singular closed-loop matrix, and is therefore easy to handle with the standard tools of linear-quadratic optimal control. As a consequence, our analysis permits a generalisation of a long series of results aiming to the closed form representation of the optimal control, see [5, 6, 9, 17] and, for the continuous-time counterpart, [4, 7, 8]. Our analysis of the GRDE is based on the general theory on generalised *algebraic* Riccati equation presented in [15] and on some recent developments derived in [10, 11].

2. The Generalised Discrete Algebraic Riccati Equation

We begin this section by recalling two standard linear algebra results that are used in the derivations throughout the paper.

Lemma 2.1: *Consider* $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = P^T \geq 0$. *Then,*

(1) $\ker P_{12} \supseteq \ker P_{22}$*;*
(2) $P_{12}\, P_{22}^{\dagger}\, P_{22} = P_{12}$*;*
(3) $P_{12}\,(I - P_{22}^{\dagger} P_{22}) = 0$*;*
(4) $P_{11} - P_{12} P_{22}^{\dagger} P_{12}^T \geq 0$.

Lemma 2.2: *Consider* $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ *where* P_{11} *and* P_{22} *are square and* P_{22} *is non-singular. Then,*

$$\det\, P = \det\, P_{22} \cdot \det(P_{11} - P_{12} P_{22}^{-1} P_{21}). \tag{8}$$

We now introduce the so-called Generalised Discrete Algebraic Riccati Equation GDARE(Σ), defined as

$$X = A^{\mathrm{T}} X A - (A^{\mathrm{T}} X B + S)(R + B^{\mathrm{T}} X B)^{\dagger}(B^{\mathrm{T}} X A + S^{\mathrm{T}}) + Q. \tag{9}$$

The algebraic equation (9) subject to the constraint

$$\ker(R + B^{\mathrm{T}} X B) \subseteq \ker(A^{\mathrm{T}} X B + S) \tag{10}$$

is usually referred to as Constrained Generalised Discrete Algebraic Riccati Equation CGDARE(Σ):

$$\begin{cases} X = A^{\mathrm{T}} X A - (A^{\mathrm{T}} X B + S)(R + B^{\mathrm{T}} X B)^{\dagger}(B^{\mathrm{T}} X A + S^{\mathrm{T}}) + Q \\ \ker(R + B^{\mathrm{T}} X B) \subseteq \ker(A^{\mathrm{T}} X B + S) \end{cases} \tag{11}$$

It is obvious that CGDARE(Σ) constitutes a generalisation of the classic Discrete Riccati Algebraic Equation DARE(Σ)

$$X = A^{\mathrm{T}} X A - (A^{\mathrm{T}} X B + S)(R + B^{\mathrm{T}} X B)^{-1}(B^{\mathrm{T}} X A + S^{\mathrm{T}}) + Q, \tag{12}$$

in the sense that any solution of DARE(Σ) is also a solution of CGDARE(Σ) but the *vice-versa* is not true in general. Importantly, however, the inertia of $R + B^{\mathrm{T}} X B$ is independent of the particular solution of the CGDARE(Σ), [15, Theorem 2.4]. This implies that a given CGDARE(Σ) cannot have one solution $X = X^{\mathrm{T}}$ such that $R + B^{\mathrm{T}} X B$ is non-singular and another solution $Y = Y^{\mathrm{T}}$ for which $R + B^{\mathrm{T}} Y B$ is singular. As such, **i)** if DARE(Σ) has a solution, then all solutions of CGDARE(Σ) are solutions of DARE(Σ) and, **ii)** if X is a solution of CGDARE(Σ) such that $R + B^{\mathrm{T}} X B$ is singular, then DARE(Σ) does not admit solutions.

To simplify the notation, for any $X = X^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ we define

$$\begin{aligned} R_X &\stackrel{\text{def}}{=} R + B^{\mathrm{T}} X B \\ S_X &\stackrel{\text{def}}{=} A^{\mathrm{T}} X B + S \\ K_X &\stackrel{\text{def}}{=} (R + B^{\mathrm{T}} X B)^{\dagger} \, (B^{\mathrm{T}} X A + S^{\mathrm{T}}) = R_X^{\dagger} S_X^{\mathrm{T}} \\ A_X &\stackrel{\text{def}}{=} A - B \, K_X \end{aligned}$$

so that (10) can be written as $\ker R_X \subseteq \ker S_X$.

3. GDARE and the extended symplectic pencil

In this section we adapt the analysis carried out in [12] for standard discrete algebraic Riccati equations to the case of CGDARE(Σ). Consider the so-called extended symplectic pencil $N - z\,M$, where

$$M \overset{\text{def}}{=} \begin{bmatrix} I_n & O & O \\ O & -A^{\mathrm{T}} & O \\ O & -B^{\mathrm{T}} & O \end{bmatrix} \qquad \text{and} \qquad N \overset{\text{def}}{=} \begin{bmatrix} A & O & B \\ Q & -I_n & S \\ S^{\mathrm{T}} & O & R \end{bmatrix}.$$

This is an extension that may be reduced to the symplectic structure (see [3, 16]) when the matrix R is invertible. We begin by giving a necessary and sufficient condition for N to be singular. We will also show that, unlike the case in which the pencil $N - z\,M$ is regular, the singularity of N is not equivalent to the fact that the matrix pencil $N - z\,M$ has a generalised eigenvalue at zero.

Lemma 3.1: *Matrix N is singular if and only if at least one of the two matrices R and $A - B\,R^{\dagger}\,S^{T}$ is singular.*

Proof: First note that N is singular if and only if such is $\left[\begin{smallmatrix} A & B \\ S^{\mathrm{T}} & R \end{smallmatrix}\right]$. To see this fact, consider the left null-spaces. Clearly, $\left[\, {v_1}^{\mathrm{T}} \; {v_2}^{\mathrm{T}} \; {v_3}^{\mathrm{T}} \,\right] N = 0$, if and only if $v_2 = 0$ and $\left[\, {v_1}^{\mathrm{T}} \; {v_3}^{\mathrm{T}} \,\right] \left[\begin{smallmatrix} A & B \\ S^{\mathrm{T}} & R \end{smallmatrix}\right] = 0$.
Now, if R is singular, a non-zero vector v_3 exists such ${v_3}^{\mathrm{T}} R = 0$. Since from *(1)* in Lemma 2.1 applied to the Popov matrix $\left[\begin{smallmatrix} Q & S \\ S^{\mathrm{T}} & R \end{smallmatrix}\right]$ the subspace inclusion $\ker R \subseteq \ker S$ holds, we have also $\left[\, 0 \;\; {v_3}^{\mathrm{T}} \,\right] \left[\begin{smallmatrix} A & B \\ S^{\mathrm{T}} & R \end{smallmatrix}\right] = 0$. If R is invertible but $A - B\,R^{\dagger}\,S^{\mathrm{T}} = A - B\,R^{-1}\,S^{\mathrm{T}}$ is singular, from (8) in Lemma 2.2 matrix $\left[\begin{smallmatrix} A & B \\ S^{\mathrm{T}} & R \end{smallmatrix}\right]$ is singular, and therefore so is N. *Vice-versa*, if both R and $A - B\,R^{-1}\,S^{\mathrm{T}}$ are non-singular, $\left[\begin{smallmatrix} A & B \\ S^{\mathrm{T}} & R \end{smallmatrix}\right]$ is non-singular in view of (8) in Lemma 2.2. Thus, N is invertible. ∎

The following theorem (see [11] for a proof) presents a useful decomposition of the extended symplectic pencil that parallels the classic one – see e.g. [12] – which is valid in the case in which the pencil $N - z\,M$ is regular.

Theorem 3.2: *Let X be a symmetric solution of CGDARE(Σ). Let also K_X be the associated gain and A_X be the associated closed-loop matrix. Two invertible matrices U_X and V_X of suitable sizes exist such that*

$$U_X\,(N - z\,M)\,V_X = \begin{bmatrix} A_X - z\,I_n & O & B \\ O & I_n - z\,A_X^T & O \\ O & -z\,B^T & R_X \end{bmatrix}. \tag{13}$$

From Theorem 3.2 we find that if X is a solution of CGDARE(Σ), in view of the triangular structure obtained above we have

$$\det(N - z\,M) = \frac{\det(A_X - z\,I_n)\cdot\det(I_n - z\,A_X^{\mathrm{T}})\cdot\det R_X}{\det U_X\cdot\det V_X}. \tag{14}$$

When R_X is non-singular, the dynamics represented by this matrix pencil are decomposed into a part governed by the generalised eigenstructure of $A_X - z\,I_n$, a part governed by the finite generalised eigenstructure of $I_n - z\,A_X^{\mathrm{T}}$, and a part which corresponds to the dynamics of the eigenvalues at infinity. When X is a solution of DARE(Σ), the generalised eigenvalues[1] of $z\,N - M$ are given by the eigenvalues of A_X, the reciprocal of the non-zero eigenvalues of A_X, and a generalised eigenvalue at infinity whose algebraic multiplicity is equal to m plus the algebraic multiplicity of the eigenvalue of A_X at the origin. The matrix pencil $I_n - z\,A_X^{\mathrm{T}}$ has no generalised eigenvalues at $z = 0$. This means that $z = 0$ is a generalised eigenvalue of the matrix pencil $U_X\,(N - z\,M)\,V_X$ if and only if it is a generalised eigenvalue of the matrix pencil $A_X - z\,I_n$, because certainly $z = 0$ cannot cause the rank of $I_n - z\,A_X^{\mathrm{T}}$ to be smaller than its normal rank and because the normal rank of $N - z\,M$ is $2\,n + m$. This means that the Kronecker eigenstructure of the eigenvalue at the origin of $U_X\,(N - z\,M)\,V_X$ coincides with the Jordan eigenstructure of the eigenvalue at the origin of the closed-loop matrix A_X. Since the generalised eigenvalues of $N - z\,M$ do not depend on the particular solution $X = X^{\mathrm{T}}$ of CGDARE(Σ), the same holds for the generalised eigenvalues and the Kronecker structure of $U_X\,(N - z\,M)\,V_X$ for any non-singular U_X and V_X. Therefore, the nilpotent structure of the closed-loop matrix A_X – which is the Jordan eigenstructure of the generalised eigenvalue at the origin of A_X – if any, is independent of the particular solution $X = X^{\mathrm{T}}$ of CGDARE(Σ). Moreover, since

$$U_X\,N\,V_X = \begin{bmatrix} A_X & O & B \\ O & I_n & O \\ O & O & R_X \end{bmatrix}, \tag{15}$$

we see that, when R_X is invertible, N is singular if and only if A_X is singular. Since from Lemma 3.1 matrix N is singular if and only if at least one of the two matrices R and $A - B\,R^{\dagger}\,S^{\mathrm{T}}$ is singular, we also have the following result.

Lemma 3.3: **(see e.g. [2])** *Let R_X be invertible. Then, A_X is singular if and only if at least one of the two matrices R and $A - B\,R^{\dagger}\,S^{T}$ is singular.*

However, when the matrix R_X is singular, it is no longer true that A_X is singular if and only if R or $A - B\,R^{\dagger}\,S^{\mathrm{T}}$ is singular. Indeed, (15) shows that the algebraic multiplicity of the eigenvalue at the origin of N is equal to the sum of the algebraic

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[1] Recall that a generalised eigenvalue of a matrix pencil $N - z\,M$ is a value of $z \in \mathbb{C}$ for which the rank of the matrix pencil $N - z\,M$ is lower than its normal rank.

multiplicities of the eigenvalue at the origin of A_X and R_X. Therefore, the fact that N is singular does not necessarily imply that A_X is singular. Indeed, Lemma 3.3 can be generalised to the case where R_X is possibly singular as follows.

Proposition 3.4: *The closed-loop matrix* A_X *is singular if and only if* $\operatorname{rank} R < \operatorname{rank} R_X$ *or* $A - B\,R^{\dagger}\,S^{T}$ *is singular.*

Proof: Given a square matrix Z, let us denote by $\mu(Z)$ the algebraic multiplicity of its eigenvalue at the origin. Then, we know from (15) that $\mu(N) = \mu\left(\left[\begin{smallmatrix} A & B \\ S^{\mathrm{T}} & R \end{smallmatrix}\right]\right) = \mu(A_X) + \mu(R_X)$. Consider a basis in the input space that isolates the invertible part of R. In other words, in this basis R is written as $R = \left[\begin{smallmatrix} R_1 & O \\ O & O \end{smallmatrix}\right]$ where R_1 is invertible, while $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ and $S = \begin{bmatrix} S_1 & O \end{bmatrix}$ are partitioned accordingly. It follows that $\mu\left(\left[\begin{smallmatrix} A & B \\ S^{\mathrm{T}} & R \end{smallmatrix}\right]\right) = \mu(R) + \mu\left(\left[\begin{smallmatrix} A & B_1 \\ S_1^{\mathrm{T}} & R_1 \end{smallmatrix}\right]\right)$. As such,

$$\mu(A_X) = \mu\left(\begin{bmatrix} A & B \\ S^{\mathrm{T}} & R \end{bmatrix}\right) - \mu(R_X) = \mu\left(\begin{bmatrix} A & B_1 \\ S_1^{\mathrm{T}} & R_1 \end{bmatrix}\right) + \mu(R) - \mu(R_X). \tag{16}$$

First, we show that if $\operatorname{rank} R < \operatorname{rank} R_X$, then A_X is singular. Since $\operatorname{rank} R < \operatorname{rank} R_X$, then obviously $\mu(R) > \mu(R_X)$, so that (16) gives $\mu(A_X) > 0$.

Let now $A - B\,R^{\dagger}\,S^{\mathrm{T}}$ be singular, and let $\operatorname{rank} R = \operatorname{rank} R_X$. From (16) we find that $\mu(A_X) = \mu\left(\left[\begin{smallmatrix} A & B_1 \\ S_1^{\mathrm{T}} & R_1 \end{smallmatrix}\right]\right)$. However, $A - B\,R^{\dagger}\,S^{\mathrm{T}} = A - B_1\,R_1^{-1}\,S_1^{\mathrm{T}}$. If $A - B\,R^{\dagger}\,S^{\mathrm{T}}$ is singular, there exists a non-zero vector k such that $\begin{bmatrix} k^{\mathrm{T}} & -k^{\mathrm{T}}\,B_1\,R_1^{-1} \end{bmatrix} \left[\begin{smallmatrix} A & B_1 \\ S_1^{\mathrm{T}} & R_1 \end{smallmatrix}\right] = 0$. Hence, $\mu\left(\left[\begin{smallmatrix} A & B_1 \\ S_1^{\mathrm{T}} & R_1 \end{smallmatrix}\right]\right) > 0$, and therefore also $\mu(A_X) > 0$.

To prove that the converse is true, it suffices to show that if $A - B\,R^{\dagger}\,S^{\mathrm{T}}$ is non-singular and $\operatorname{rank} R = \operatorname{rank} R_X$, then A_X is non-singular. To this end, we observe that $\operatorname{rank} R = \operatorname{rank} R_X$ is equivalent to $\mu(R) = \mu(R_X)$ because R and R_X are symmetric. Thus, in view of (16), it suffices to show that if $A - B\,R^{\dagger}\,S^{\mathrm{T}}$ is non-singular, then $\mu\left(\left[\begin{smallmatrix} A & B_1 \\ S_1^{\mathrm{T}} & R_1 \end{smallmatrix}\right]\right) = 0$. Indeed, assume that $A - B\,R^{\dagger}\,S^{\mathrm{T}} = A - B_1\,R_1^{-1}\,S_1^{\mathrm{T}}$ is non-singular, and take a vector $[\, v_1^{\mathrm{T}} \; v_2^{\mathrm{T}} \,]$ such that $[\, v_1^{\mathrm{T}} \; v_2^{\mathrm{T}} \,] \left[\begin{smallmatrix} A & B_1 \\ S_1^{\mathrm{T}} & R_1 \end{smallmatrix}\right] = 0$. Then, since R_1 is invertible we get $v_2^{\mathrm{T}} = -v_1^{\mathrm{T}}\,B_1\,R_1^{-1}$ and $v_1^{\mathrm{T}}\,(A - B_1\,R_1^{-1}\,S_1^{\mathrm{T}}) = 0$. Hence, $v_1 = 0$ since $A - B_1\,R_1^{-1}\,S_1^{\mathrm{T}}$ is non-singular, and therefore also $v_2 = 0$. ■

Remark 1: We recall that $\mu(R_X)$ is invariant for any symmetric solution X of CGDARE(Σ), [15]. Hence, as a direct consequence of (16), we have that $\mu(A_X)$ is the same for any symmetric solution X of CGDARE(Σ). This means, in particular, that the closed-loop matrix corresponding to a given symmetric solution of CGDARE(Σ) is singular if and only if the closed-loop matrix corresponding to any other symmetric solution of CGDARE(Σ) is singular. In the next section we show that a stronger result holds: when present, the zero eigenvalue has the same Jordan structure for any pair A_X and A_Y of closed-loop matrices corresponding to any pair X, Y of symmetric solutions of CGDARE(Σ). Moreover, the generalised eigenspaces corresponding to the zero eigenvalue of A_X and A_Y coincide. The restriction of A_X and A_Y to this generalised eigenspace also coincide. Finally, X and Y coincide along this generalised eigenspace.

4. The subspace where all solutions coincide

Given a solution $X = X^{\mathrm{T}}$ of CGDARE(Σ), we denote by $\mathcal{U}$ the generalised eigenspace corresponding to the eigenvalue at the origin of A_X, i.e., $\mathcal{U} \stackrel{\text{def}}{=} \ker(A_X)^n$, where $(A_X)^n$ denotes the n-th power of A_X. Notice that, in principle, $\mathcal{U}$ could depend on the particular solution X. In this section, and in particular in Theorem 4.4, we want to prove not only that $\mathcal{U}$ does *not* depend on the particular solution X, but also that all solutions of CGDARE(Σ) are coincident along $\mathcal{U}$. In other words, given two solutions $X = X^{\mathrm{T}}$ and $Y = Y^{\mathrm{T}}$ of CGDARE(Σ), we show that $\ker(A_X)^n = \ker(A_Y)^n$ and, given a basis matrix[1] U of the subspace $\mathcal{U} = \ker(A_X)^n = \ker(A_Y)^n$, the change of coordinate matrix $T = [\,U \quad U_c\,]$ yields

$$T^{-1}\, X\, T = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix} \quad \text{and} \quad T^{-1}\, Y\, T = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & Y_{22} \end{bmatrix}. \tag{17}$$

We begin by presenting a first simple result.

Lemma 4.1: *Two symmetric solutions X and Y of CGDARE(Σ) are coincident along the subspace $\mathcal{U}$ if and only if $\mathcal{U} \subseteq \ker(X-Y)$.*

Proof: Suppose X and Y are coincident along the subspace $\mathcal{U}$, and are already written in the basis defined by T in (17). In this basis $\mathcal{U}$ can be written as $\mathcal{U} = \operatorname{im} \left[\begin{smallmatrix} I \\ O \end{smallmatrix}\right]$. If (17) holds, then we can write $X - Y = \left[\begin{smallmatrix} O & O \\ O & \star \end{smallmatrix}\right]$. Then, $(X-Y)\mathcal{U} = \left[\begin{smallmatrix} O & O \\ O & \star \end{smallmatrix}\right]\left[\begin{smallmatrix} I \\ O \end{smallmatrix}\right] = \{0\}$. *Vice-versa*, if $(X-Y)\mathcal{U} = \{0\}$ and we write $X - Y = \left[\begin{smallmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^{\mathrm{T}} & \Delta_{22} \end{smallmatrix}\right]$, we find that $\left[\begin{smallmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^{\mathrm{T}} & \Delta_{22} \end{smallmatrix}\right]\left[\begin{smallmatrix} I \\ O \end{smallmatrix}\right] = \{0\}$ implies $\Delta_{11} = 0$ and $\Delta_{12} = 0$. ■

We now present two results that will be useful to prove Theorem 4.4. Let $X = X^{\mathrm{T}} \in \mathbb{R}^{n \times n}$. Similarly to [12], we define the function

$$\mathcal{D}(X) \stackrel{\text{def}}{=} X - A^{\mathrm{T}}\, X\, A + (A^{\mathrm{T}}\, X\, B + S)(R + B^{\mathrm{T}}\, X\, B)^{\dagger}(B^{\mathrm{T}}\, X\, A + S^{\mathrm{T}}) - Q. \tag{18}$$

If in particular $X = X^{\mathrm{T}}$ is a solution of GDARE(Σ), then $\mathcal{D}(X) = 0$. Recall that we have defined $R_X = R + B^{\mathrm{T}}\, X\, B$, $S_X = A^{\mathrm{T}}\, X\, B + S$ and $R_Y = R + B^{\mathrm{T}}\, Y\, B$, $S_Y \stackrel{\text{def}}{=} A^{\mathrm{T}}\, Y\, B + S$.

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[1] Given a subspace $\mathcal{S}$, a basis matrix S of $\mathcal{S}$ is such that $\operatorname{im} S = \mathcal{S}$ and $\ker S = \{0\}$.

Lemma 4.2: *Let $X=X^T \in \mathbb{R}^{n\times n}$ and $Y=Y^T \in \mathbb{R}^{n\times n}$ be such that (10) holds, i.e.,*

$$\ker R_X \subseteq \ker S_X \tag{19}$$

$$\ker R_Y \subseteq \ker S_Y. \tag{20}$$

Let $A_X = A - B\,K_X$ with $K_X = R_X^\dagger\, S_X^T$ and $A_Y = A - B\,K_Y$ with $K_Y = R_Y^\dagger\, S_Y^T$. Moreover, let us define the difference $\Delta \overset{def}{=} X - Y$. Then,

$$\mathcal{D}(X) - \mathcal{D}(Y) = \Delta - A_Y^T\,\Delta\,A_Y + A_Y^T\,\Delta\,B\,R_X^\dagger\,B^T\,\Delta\,A_Y. \tag{21}$$

The proof can be found in [1, p.382].

The following lemma is the counterpart of Lemma 2.2 in [12] where the standard DARE was considered.

Lemma 4.3: *Let $X=X^T \in \mathbb{R}^{n\times n}$ and $Y=Y^T \in \mathbb{R}^{n\times n}$ be such that (19-20) hold. Let $\Delta = X - Y$. Then,*

$$\mathcal{D}(X) - \mathcal{D}(Y) = \Delta - A_Y^T\,\Delta\,A_X. \tag{22}$$

Proof: First, notice that

$$A_Y^{\mathrm{T}} \Delta\, B = [A^{\mathrm{T}} - (A^{\mathrm{T}} Y\,B + S)\,R_Y^\dagger B^{\mathrm{T}}]\Delta\, B.$$

We now show that $\ker R_X \subseteq \ker(A_Y^{\mathrm{T}}\Delta\, B)$. To this end, let P_X be a basis of the null-space of R_X. Hence, $(R + B^{\mathrm{T}} X B) P_X = 0$. Then,

$$\begin{aligned}
A_Y^{\mathrm{T}}\,\Delta\,B\,P_X &= \Big(A^{\mathrm{T}} - (A^{\mathrm{T}}\,Y\,B + S)\,R_Y^\dagger\,B^{\mathrm{T}}\Big)\,(X - Y)\,B\,P_X \\
&= A^{\mathrm{T}}\,X\,B\,P_X - (A^{\mathrm{T}}\,Y\,B + S)\,R_Y^\dagger\,B^{\mathrm{T}}\,X\,B\,P_X - A^{\mathrm{T}}\,Y\,B\,P_X \\
&\quad + (A^{\mathrm{T}} Y\,B + S)\,R_Y^\dagger\,B^{\mathrm{T}}\,Y\,B\,P_X \\
&\quad + (A^{\mathrm{T}} Y\,B + S)\,R_Y^\dagger\,R\,P_X - (A^{\mathrm{T}} Y\,B + S)\,R_Y^\dagger\,R\,P_X \\
&= A^{\mathrm{T}}\,X\,B\,P_X + (A^{\mathrm{T}} Y\,B + S)\,R_Y^\dagger\,R_Y\,P_X - A^{\mathrm{T}}\,Y\,B\,P_X \\
&= A^{\mathrm{T}}\,X\,B\,P_X + S_Y\,P_X - A^{\mathrm{T}}\,Y\,B\,P_X = (A^{\mathrm{T}}\,X\,B + S)\,P_X,
\end{aligned}$$

which is zero since $\ker R_X \subseteq \ker S_X$ in view of (19) in Lemma 4.2. Now we want to prove that

$$A_Y^{\mathrm{T}} \Delta\,(A_Y - A_X) = A_Y^{\mathrm{T}}\,\Delta\,B\,R_X^\dagger\,B^{\mathrm{T}}\,\Delta\,A_Y. \tag{23}$$

Consider the term

$$A_Y^{\mathrm{T}} \Delta (A_Y - A_X) = A_Y^{\mathrm{T}} \Delta\,B\,(R_X^\dagger S_X - R_Y^\dagger S_Y). \tag{24}$$

Since $R_X^{\dagger} R_X$ is an orthogonal projection that projects onto $\operatorname{im} R_X^{\mathrm{T}} = \operatorname{im} R_X$, we have $\ker R_X = \operatorname{im}(I_m - R_X^{\dagger} R_X)$. Since as we have shown $\ker R_X \subseteq \ker(A_Y^{\mathrm{T}} \Delta\, B)$, from $\ker R_X = \operatorname{im}(I_m - R_X^{\dagger} R_X)$ we also have $A_Y^{\mathrm{T}} \Delta\, B\,(I_m - R_X^{\dagger} R_X) = 0$, which means that $A_Y^{\mathrm{T}} \Delta\, B\, R_X^{\dagger}\, R_X = A_Y^{\mathrm{T}}\, \Delta\, B$. We use this fact on (24) to get

$$\begin{aligned} A_Y^{\mathrm{T}} \Delta (A_Y - A_X) &= A_Y^{\mathrm{T}} \Delta\, B\, R_X^{\dagger} [(B^{\mathrm{T}} X A + S) - R_X\, R_Y^{\dagger} (B^{\mathrm{T}} Y A + S)] \\ &= A_Y^{\mathrm{T}} \Delta\, B\, R_X^{\dagger} [(B^{\mathrm{T}} X A + S - B^{\mathrm{T}}\, Y\, A + B^{\mathrm{T}} Y\, A) - R_X\, R_Y^{\dagger} (B^{\mathrm{T}} Y A + S)] \\ &= A_Y^{\mathrm{T}} \Delta\, B\, R_X^{\dagger} [B^{\mathrm{T}} \Delta\, A + (I_m - R_X\, R_Y^{\dagger})(B^{\mathrm{T}} Y A + S)]. \end{aligned} \tag{25}$$

Since $R_X = R + B^{\mathrm{T}} X\, B - B^{\mathrm{T}} Y\, B + B^{\mathrm{T}} Y\, B = R_Y + B^{\mathrm{T}}\, \Delta\, B$, eq. (25) becomes

$$\begin{aligned} A_Y^{\mathrm{T}} \Delta (A_Y - A_X) &= A_Y^{\mathrm{T}} \Delta\, B\, R_X^{\dagger} [B^{\mathrm{T}} \Delta\, A + (I_m - R_Y\, R_Y^{\dagger} - B^{\mathrm{T}} \Delta\, B\, R_Y^{\dagger})(B^{\mathrm{T}} Y A + S)] \\ &= A_Y^{\mathrm{T}} \Delta B\, R_X^{\dagger} B^{\mathrm{T}} \Delta\, (A - B\, R_Y^{\dagger})(B^{\mathrm{T}} Y A + S) = \Delta B\, R_X^{\dagger} B^{\mathrm{T}} \Delta\, A_Y, \end{aligned}$$

since from Lemma 2.1 $(I_m - R_Y\, R_Y^{\dagger})(B^{\mathrm{T}} Y A + S) = 0$ from $\ker R_Y \subseteq \ker(A^{\mathrm{T}} Y\, B + S)$. Eq. (23) follows by recalling that $A_Y = A - B\, R_Y^{\dagger}\, S_Y$. Plugging (23) into (21) yields

$$\mathcal{D}(X) - \mathcal{D}(Y) = \Delta - A_Y^{\mathrm{T}}\, \Delta A_Y + A_Y^{\mathrm{T}} \Delta (A_Y - A_X) = \Delta - A_Y^{\mathrm{T}}\, \Delta A_X.$$

■

Now we are ready to prove the main result of this section. This result extends the analysis of Proposition 2.1 in [12] to solutions of CGDARE(Σ).

Theorem 4.4: *Let $\mathcal{U} = \ker(A_X)^n$ denote the generalised eigenspace corresponding to the eigenvalue at the origin of A_X. Then*

(1) All solutions of CGDARE(Σ) are coincident along $\mathcal{U}$, i.e., given two solutions X and Y of CGDARE(Σ),

$$(X - Y)\,\mathcal{U} = \{0\};$$

(2) $\mathcal{U}$ does not depend on the solution X of CGDARE(Σ), i.e., given two solutions X and Y of CGDARE(Σ), there holds

$$\ker(A_X)^n = \ker(A_Y)^n.$$

Proof: Let us prove *(1)*. Consider a non-singular $T \in \mathbb{R}^{n \times n}$. Define the new quintuple

$$\tilde{A} \stackrel{\text{def}}{=} T^{-1} A\, T, \qquad \tilde{B} \stackrel{\text{def}}{=} T^{-1} B, \quad \tilde{Q} \stackrel{\text{def}}{=} T^{\mathrm{T}}\, Q\, T, \quad \tilde{S} \stackrel{\text{def}}{=} T^{\mathrm{T}} S, \quad \tilde{R} \stackrel{\text{def}}{=} R.$$

It is straightforward to see that X satisfies GDARE(Σ) with respect to (A, B, Q, R, S) if and only if $\tilde{X} \stackrel{\text{def}}{=} T^{\mathrm{T}} X\, T$ satisfies GDARE(Σ) with respect to

$(\tilde{A},\tilde{B},\tilde{Q},\tilde{R},\tilde{S})$, which for the sake of simplicity is denoted by $\tilde{\mathcal{D}}$, so that $\tilde{\mathcal{D}}(\tilde{X})=0$. The closed-loop matrix in the new basis is related to the closed-loop matrix in the original basis by

$$\tilde{A}_{\tilde{X}}=\tilde{A}-\tilde{B}\,(\tilde{R}+\tilde{B}^{\mathrm{T}}\tilde{X}\,\tilde{B})^{\dagger}(\tilde{B}^{\mathrm{T}}\tilde{X}\,\tilde{A}+\tilde{S}^{\mathrm{T}})=T^{-1}\,A_X\,T.$$

Moreover, if $\tilde{\mathcal{U}}=\ker(\tilde{A}_{\tilde{X}})^n$, then $\tilde{\mathcal{U}}=T^{-1}\mathcal{U}$ since $(\tilde{A}_{\tilde{X}})^n\tilde{\mathcal{U}}=0$ is equivalent to $T^{-1}(A_X)^nT\tilde{\mathcal{U}}=T^{-1}(A_X)^n\,\mathcal{U}=0$. We choose an orthogonal change of coordinate matrix T as $T=[\,U\quad U_c\,]$, where U is a basis matrix of $\mathcal{U}$. In this new basis

$$\begin{aligned}\tilde{A}_{\tilde{X}}&=T^{-1}\,A_X\,T=\begin{bmatrix}U\;U_c\end{bmatrix}^{\mathrm{T}}A_X\begin{bmatrix}U\;U_c\end{bmatrix}\\&=\begin{bmatrix}U^{\mathrm{T}}A_X\,U\;\star\\U_c^{\mathrm{T}}A_X\,U\;\star\end{bmatrix}=\begin{bmatrix}U^{\mathrm{T}}A_X\,U & \star\\ O & U_c^{\mathrm{T}}A_X\,U_c\end{bmatrix},\end{aligned}$$

where the zero in the bottom left corner is due to the fact that the rows of $U_c^{\mathrm{T}}A_X$ are orthogonal to the columns of U. Moreover, the submatrix $N_0\stackrel{\text{def}}{=}U^{\mathrm{T}}A_X\,U$ is nilpotent with the same nilpotency index[1] of A_X. Notice also that $H_X\stackrel{\text{def}}{=}U_c^{\mathrm{T}}A_X\,U_c$ is non-singular. Let $\tilde{X}$ be a solution of CGDARE($\tilde{\Sigma}$) in this new basis, and let it be partitioned as

$$\tilde{X}=\begin{bmatrix}\tilde{X}_{11}\;\tilde{X}_{12}\\\tilde{X}_{12}^{\mathrm{T}}\;\tilde{X}_{22}\end{bmatrix},$$

where $\tilde{X}_{11}$ is $\nu\times\nu$, with $\nu=\dim\mathcal{U}$. Consider another solution $\tilde{Y}$ of CGDARE($\tilde{\Sigma}$), partitioned as $Y=\begin{bmatrix}\tilde{Y}_{11}\;\tilde{Y}_{12}\\\tilde{Y}_{12}^{\mathrm{T}}\;\tilde{Y}_{22}\end{bmatrix}$. Let $\Delta\stackrel{\text{def}}{=}\tilde{X}-\tilde{Y}$ be partitioned in the same way. Since $\tilde{X}$ and $\tilde{Y}$ are both solutions of CGDARE($\tilde{\Sigma}$), we get $\tilde{\mathcal{D}}(\tilde{X})=\tilde{\mathcal{D}}(\tilde{Y})=0$. Thus, in view of Lemma 4.3, there holds

$$\Delta-\tilde{A}_{\tilde{Y}}^{\mathrm{T}}\,\Delta\,\tilde{A}_{\tilde{X}}=0. \tag{26}$$

If Δ is partitioned as $\Delta=[\,\Delta_1\quad\Delta_2\,]$ where Δ_1 has ν columns, eq. (26) becomes

$$\begin{bmatrix}\Delta_1\;\Delta_2\end{bmatrix}-\tilde{A}_{\tilde{Y}}^{\mathrm{T}}\begin{bmatrix}\Delta_1\;\Delta_2\end{bmatrix}\begin{bmatrix}N_0&\star\\O&H_X\end{bmatrix}=\begin{bmatrix}\Delta_1-\tilde{A}_{\tilde{Y}}^{\mathrm{T}}\Delta_1\,N_0\;\star\end{bmatrix}=0,$$

from which we get $\Delta_1=\tilde{A}_{\tilde{Y}}^{\mathrm{T}}\,\Delta_1\,N_0$. Thus,

$$\Delta_1=\tilde{A}_{\tilde{Y}}^{\mathrm{T}}\Delta_1\,N_0=(\tilde{A}_{\tilde{Y}}^{\mathrm{T}})^2\Delta_1\,N_0^2=\ldots=(\tilde{A}_{\tilde{Y}}^{\mathrm{T}})^n\Delta_1\,(N_0)^n,$$

---

[1] With a slight abuse of nomenclature, we use the term *nilpotency index* of a matrix M to refer to the smallest integer ν for which $\ker(M)^\nu=\ker(M)^{\nu+1}$, which is defined also when M is not nilpotent.

which is equal to zero since $(N_0)^n$ is the zero matrix. Hence, $\Delta_1 = 0$. Thus, we have also

$$\Delta\,\mathcal{U} = \begin{bmatrix} O & \star \end{bmatrix} \left(\operatorname{im}\begin{bmatrix} I \\ O \end{bmatrix}\right) = \{0\}.$$

Since Δ is symmetric, we get

$$\tilde{X} - \tilde{Y} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^{\mathrm{T}} & \tilde{X}_{22} \end{bmatrix} - \begin{bmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{12}^{\mathrm{T}} & \tilde{Y}_{22} \end{bmatrix} = \begin{bmatrix} O & O \\ O & \tilde{X}_{22} - \tilde{Y}_{22} \end{bmatrix},$$

which leads to $\tilde{X}_{11} = \tilde{Y}_{11}$ and $\tilde{X}_{12} = \tilde{Y}_{12}$.

Let us prove *(2)*. Since $\ker R_Y$ coincides with $\ker R_X$ by virtue of [10, Theorem 4.3], we find

$$\begin{aligned} A_X - A_Y &= B\,(R_Y^{\dagger} S_Y^{\mathrm{T}} - R_X^{\dagger}\, S_X^{\mathrm{T}}) \\ &= B\, R_Y^{\dagger}(S_Y^{\mathrm{T}} - R_Y\, R_X^{\dagger}\, S_X^{\mathrm{T}}). \end{aligned} \tag{27}$$

Plugging

$$S_Y^{\mathrm{T}} = B^{\mathrm{T}}\, Y\, A + S^{\mathrm{T}} = B^{\mathrm{T}}\, \Delta\, A + S^{\mathrm{T}} + B^{\mathrm{T}}\, X\, A = B^{\mathrm{T}}\, \Delta\, A + S_X^{\mathrm{T}} \tag{28}$$

and

$$R_Y = R + B^{\mathrm{T}}\, Y\, B - B^{\mathrm{T}}\, X\, B + B^{\mathrm{T}}\, X\, B = R_X + B^{\mathrm{T}}\, \Delta\, B \tag{29}$$

into (27) yields

$$\begin{aligned} A_X - A_Y &= B\, R_Y^{\dagger}(B^{\mathrm{T}}\, \Delta\, A - B^{\mathrm{T}}\, \Delta\, B\, R_X^{\dagger}\, S_X^{\mathrm{T}}) \\ &= B\, R_Y^{\dagger} B^{\mathrm{T}}\, \Delta\, A_X. \end{aligned}$$

This means that the identity

$$A_X - A_Y = B\, R_Y^{\dagger} B^{\mathrm{T}}\, \Delta\, A_X$$

holds. By partitioning $\Delta = \left[\begin{smallmatrix} O & \star \\ O & \star \end{smallmatrix}\right]$, we find that also $B\, R_Y^{\dagger} B^{\mathrm{T}}\, \Delta = \left[\begin{smallmatrix} O & \star \\ O & \star \end{smallmatrix}\right]$, so that

$$\begin{aligned} A_Y &= A_X - B\, R_Y^{\dagger} B^{\mathrm{T}}\, \Delta\, A_X \\ &= \begin{bmatrix} N_0 & \star \\ O & H_X \end{bmatrix} - \begin{bmatrix} O & \star \\ O & \star \end{bmatrix} \begin{bmatrix} N_0 & \star \\ O & H_X \end{bmatrix} = \begin{bmatrix} N_0 & \star \\ O & H_Y \end{bmatrix}. \end{aligned}$$

Thus, $\ker(A_Y)^n \supseteq \ker(A_X)^n$. If we interchange the role of X and Y, we obtain the opposite inclusion $\ker(A_Y)^n \subseteq \ker(A_X)^n$. Notice, in passing, that this also implies that H_Y is non-singular. ■

5. The Generalised Riccati Difference Equation

Consider the GRDE(Σ) along with the terminal condition $X_T = P = P^{\mathrm{T}} \geq 0$. Let us define

$$\mathcal{R}(X) \stackrel{\text{def}}{=} A^{\mathrm{T}} X A - (A^{\mathrm{T}} X B + S)(R + B^{\mathrm{T}} X B)^{\dagger}(B^{\mathrm{T}} X A + S^{\mathrm{T}}) + Q.$$

With this definition, GRDE(Σ) can be written as $X_t = \mathcal{R}(X_{t+1})$. Moreover, GDARE($\Sigma$) can be written as

$$\mathcal{D}(X) = X - \mathcal{R}(X) = 0.$$

We have the following important result.

Theorem 5.1: *Let $X_\circ = X_\circ^T$ be a solution of CGDARE(Σ). Let ν be the index of nilpotency of $A_{X_\circ}$. Moreover, let X_t be a solution of (6-7) and define $\Delta_t \stackrel{\text{def}}{=} X_t - X_\circ$. Then, for $\tau \geq \nu$, we have $\Delta_{T-\tau}\mathcal{U} = \{0\}$.*

Proof: Since $X_\circ = X_\circ^{\mathrm{T}}$ is a solution of CGDARE(Σ), we have $\mathcal{D}(X_\circ) = 0$. This is equivalent to saying that $X_\circ = \mathcal{R}(X_\circ)$. From the definition of Δ_t we get in particular $\Delta_T = X_T - X_\circ$. With these definitions in mind, we find

$$\begin{aligned}\Delta_t &= \mathcal{R}(X_{t+1}) - \mathcal{R}(X_\circ) = X_{t+1} - \mathcal{D}(X_{t+1}) - X_\circ \\ &= \Delta_{t+1} - \mathcal{D}(X_{t+1}) = \Delta_{t+1} - \mathcal{D}(X_{t+1}) + \mathcal{D}(X_\circ) \\ &= \Delta_{t+1} - [\mathcal{D}(X_{t+1}) - \mathcal{D}(X_\circ)]. \end{aligned} \tag{30}$$

However, we know from (21) that

$$\begin{aligned}&\mathcal{D}(X_{t+1}) - \mathcal{D}(X_\circ) \\ &\quad = \Delta_{t+1} - A_{X_\circ}^{\mathrm{T}} [\Delta_{t+1} - \Delta_{t+1} B (R + B^{\mathrm{T}} X_{t+1} B)^{\dagger} B^{\mathrm{T}} \Delta_{t+1}] A_{X_\circ}, \end{aligned} \tag{31}$$

which, once plugged into (30), gives

$$\begin{aligned}\Delta_t &= \Delta_{t+1} - \Delta_{t+1} + A_{X_\circ}^{\mathrm{T}} [\Delta_{t+1} + \Delta_{t+1} B (R + B^{\mathrm{T}} X_{t+1} B)^{\dagger} B^{\mathrm{T}} \Delta_{t+1}] A_{X_\circ} \\ &= A_{X_\circ}^{\mathrm{T}} [I_n - \Delta_{t+1} B (R + B^{\mathrm{T}} X_{t+1} B)^{\dagger} B^{\mathrm{T}}] \Delta_{t+1} A_{X_\circ} = F_{t+1} \Delta_{t+1} A_{X_\circ}, \end{aligned} \tag{32}$$

where

$$F_{t+1} \stackrel{\text{def}}{=} A_{X_\circ}^{\mathrm{T}} - A_{X_\circ}^{\mathrm{T}} \Delta_{t+1} B (R + B^{\mathrm{T}} X_{t+1} B)^{\dagger} B^{\mathrm{T}}.$$

It follows that we can write

$$\begin{aligned}\Delta_{T-1} &= F_T\,\Delta_T\,A_{X_\circ},\\ \Delta_{T-2} &= F_{T-1}\,\Delta_{T-1}\,A_{X_\circ} = F_{T-1}\,F_T\,\Delta_T\,(A_{X_\circ})^2,\\ &\;\vdots \qquad (33)\\ \Delta_{T-\tau} &= \left(\prod_{i=T-\tau+1}^{T} F_i\right)\Delta_T\,(A_{X_\circ})^{\tau}. \qquad (34)\end{aligned}$$

This shows that for $\tau \geq \nu$ we have $\ker \Delta_{T-\tau} \supseteq \ker(A_{X_\circ})^n$. ■

Now we show that the result given in Theorem 5.1 can be used to obtain a reduction for the generalised discrete-time Riccati difference equation. Consider the same basis induced by the change of coordinates used in Theorem 4.4, so that the first ν components of this basis span the subspace $\mathcal{U} = \ker(A_X)^n$. The closed-loop matrix in this basis can be written as

$$A_{X_\circ} = \begin{bmatrix} N_0 & \star \\ O & Z \end{bmatrix},$$

where N_0 is nilpotent and Z is non-singular. Hence, $(A_{X_\circ})^\nu = \left[\begin{smallmatrix} O & \star \\ O & Z^\nu \end{smallmatrix}\right]$, where we recall that ν is the nilpotency index of $A_{X_\circ}$. By writing (34) in this basis, for $\tau \geq \nu$ we find

$$\Delta_{T-\tau} = \begin{bmatrix} \star & \star \\ \star & \star \end{bmatrix} \begin{bmatrix} O & \star \\ O & Z^\tau \end{bmatrix} = \begin{bmatrix} O & \star \\ O & \star \end{bmatrix} = \begin{bmatrix} O & O \\ O & \star \end{bmatrix},$$

where the last equality follows from the fact that $\Delta_{T-\tau}$ is symmetric.

Now, let us rewrite the Riccati difference equation (32) as

$$\Delta_t = A_{X_\circ}^{\mathrm{T}}\,\Delta_{t+1} A_{X_\circ} - A_{X_\circ}^{\mathrm{T}}\,\Delta_{t+1}\,B(R + B^{\mathrm{T}} X_{t+1} B)^{\dagger} B^{\mathrm{T}}\,\Delta_{t+1} A_{X_\circ}. \qquad (35)$$

For $t \le T-\nu$, we get $\Delta_t = \begin{bmatrix} O & O \\ O & \Psi_t \end{bmatrix}$, and the previous equation becomes

$$\begin{aligned}
\begin{bmatrix} O & O \\ O & \Psi_t \end{bmatrix} &= \begin{bmatrix} N_0^{\mathrm{T}} & O \\ \star & Z^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} O & O \\ O & \Psi_{t+1} \end{bmatrix} \begin{bmatrix} N_0 & \star \\ O & Z \end{bmatrix} \\
&\quad - \begin{bmatrix} N_0^{\mathrm{T}} & O \\ \star & Z^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} O & O \\ O & \Psi_{t+1} \end{bmatrix} B\,(R + B^{\mathrm{T}} X_{t+1}\, B)^{\dagger} B^{\mathrm{T}} \begin{bmatrix} O & O \\ O & \Psi_{t+1} \end{bmatrix} \begin{bmatrix} N_0 & \star \\ O & Z \end{bmatrix} \\
&= \begin{bmatrix} O & O \\ O & Z^{\mathrm{T}} \Psi_{t+1}\, Z \end{bmatrix} \\
&\quad - \begin{bmatrix} O & O \\ O & Z^{\mathrm{T}}\, \Psi_{t+1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \left(R + \begin{bmatrix} B_1^{\mathrm{T}} & B_2^{\mathrm{T}} \end{bmatrix} (\Delta_{t+1} + X_\circ) \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)^{\dagger} \begin{bmatrix} B_1^{\mathrm{T}} & B_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} O & O \\ O & \Psi_{t+1} Z \end{bmatrix}.
\end{aligned}$$

By partitioning $X_\circ$ as $X_\circ = \begin{bmatrix} X_{\circ,11} & X_{\circ,12} \\ X_{\circ,12}^{\mathrm{T}} & X_{\circ,22} \end{bmatrix}$, we get

$$\begin{aligned}
\begin{bmatrix} O & O \\ O & \Psi_t \end{bmatrix} &= \begin{bmatrix} O & O \\ O & Z^{\mathrm{T}}\, \Psi_{t+1}\, Z \end{bmatrix} - \begin{bmatrix} O & O \\ O & Z^{\mathrm{T}}\, \Psi_{t+1} \end{bmatrix} \begin{bmatrix} \star & \star \\ \star & B_2\,(R_0 + B_2^{\mathrm{T}}\, \Psi_{t+1}\, B_2)^{\dagger}\, B_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} O & O \\ O & \Psi_{t+1}\, Z \end{bmatrix} \\
&= \begin{bmatrix} O & O \\ O & Z^{\mathrm{T}}\, \Psi_{t+1}\, Z \end{bmatrix} - \begin{bmatrix} O & O \\ O & Z^{\mathrm{T}}\, \Psi_{t+1}\, B_2\,(R_0 + B_2^{\mathrm{T}}\, \Psi_{t+1}\, B_2)^{\dagger}\, B_2^{\mathrm{T}}\, \Psi_{t+1}\, Z \end{bmatrix},
\end{aligned}$$

where $R_0 \overset{\text{def}}{=} R + B_2^{\mathrm{T}}\, X_{\circ,22}\, B_2$. Therefore, Ψ_t satisfies the reduced homogeneous Riccati difference equation

$$\Psi_t = Z^{\mathrm{T}}\, \Psi_{t+1}\, Z - Z^{\mathrm{T}}\, \Psi_{t+1}\, B_2\,(R_0 + B_2^{\mathrm{T}}\, \Psi_{t+1}\, B_2)^{\dagger}\, B_2^{\mathrm{T}}\, \Psi_{t+1}\, Z. \tag{36}$$

The associated generalised discrete Riccati algebraic equation is

$$\Psi - Z^{\mathrm{T}}\, \Psi\, Z + Z^{\mathrm{T}}\, \Psi\, B_2\,(R_0 + B_2^{\mathrm{T}}\, \Psi\, B_2)^{\dagger}\, B_2^{\mathrm{T}}\, \Psi\, Z = 0. \tag{37}$$

Being homogeneous, this equation admits the solution $\Psi = 0$. This fact has two important consequences:

- The closed-loop matrix associated with this solution is clearly Z, which is non-singular. On the other hand, we know that the nilpotent part of the closed-loop matrix is independent of the particular solution of CGDARE(Σ) considered. This means that all solutions of (37) have a closed-loop matrix that is non-singular;
- Given a solution Ψ of (37), the null-space of $R_0 + B_2^{\mathrm{T}}\, \Psi\, B_2$ coincides with the null-space of R_0, since the null-space of $R_0 + B_2^{\mathrm{T}}\, \Psi\, B_2$ does not depend on the particular solution of (37) and we know that the zero matrix is a solution of (37).

As a result of this discussion, it turns out that given a reference solution $X_\circ$ of CGDARE(Σ), the solution of GDRE(Σ) with terminal condition $X_T = P$ can be computed backward as follows:

(1) For the first ν steps, i.e., from $t = T$ to $t = T - \nu$, X_t is computed by iterating the GDRE(Σ) starting from the terminal condition $X_T = P$;

(2) In the basis that isolates the nilpotent part of A_X, we have

$$\Delta_{T-\nu} = \begin{bmatrix} O & O \\ O & \Psi_{T-\nu} \end{bmatrix}.$$

From $t = T-\nu-1$ to $t = 0$, the solution of GDRE(Σ) can be found iterating the reduced order GDRE in (36) starting from the terminal condition $\Psi_{T-\nu}$.

Remark 1 : The advantage of using the reduced-order generalised difference Riccati algebraic equation (36) consists in the fact that the closed-loop matrix of any solution of the associated generalised discrete Riccati algebraic equation is non-singular. Hence, when the reduced-order pencil given by the Popov triple $\left(Z, B_2, \left[\begin{smallmatrix} 0 & 0 \\ 0 & R_0 \end{smallmatrix}\right]\right)$ is regular, the solution of the reduced-order generalised difference Riccati algebraic equation (36) can also be computed in closed-form, using the results in [6]. Indeed, consider a solution Ψ of (37) with its non-singular closed-loop matrix A_Ψ and let Y be the corresponding solution of the closed-loop Hermitian Stein equation

$$A_\Psi\, Y\, A_\Psi^{\mathrm{T}} - Y + B_2\,(R_0 + B_2^{\mathrm{T}}\,\Psi\,B_2)^{-1} B_2^{\mathrm{T}} = 0. \tag{38}$$

The set of solutions of the extended symplectic difference equation for the reduced system is parameterised in terms of $K_1, K_2 \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ as

$$\begin{bmatrix} \Xi_t \\ \Lambda_t \\ \Omega_t \end{bmatrix} = \begin{bmatrix} I_{n-\nu} \\ \Psi \\ -K_\Psi \end{bmatrix} (A_\Psi)^t\, K_1 + \begin{bmatrix} Y\, A_\Psi^{\mathrm{T}} \\ (\Psi\, Y - I_{n-\nu}) A_\Psi^{\mathrm{T}} \\ -K_\star \end{bmatrix} (A_\Psi^{\mathrm{T}})^{T-t-1}\, K_2, \quad 0 \le t \le T, \tag{39}$$

where $K_\star \overset{\text{def}}{=} K_\Psi\, Y\, A_\Psi^{\mathrm{T}} - (R_0 + B_2^{\mathrm{T}}\,\Psi\,B_2)^{-1}\, B_2^{\mathrm{T}}$. The values of the parameter matrices K_1 and K_2 can be computed so that the terminal condition satisfies $X_T = I_n$ and $\Lambda_T = \Psi_{T-\nu}$. Such values exist because A_Ψ is non-singular, and are given by

$$K_1 = (A_\Psi)^{-T}\,(I_{n-\nu} - Y\,(\Psi - \Psi_{T-\nu}))$$
$$K_2 = \Psi - \Psi_{T-\nu}.$$

Then, the solution of (36) is given by $\Psi_t = \Lambda_t\, \Xi_t^{-1}$.

6. Concluding remarks

In this paper we have considered the generalised Riccati difference equation with a terminal condition which arises in finite-horizon LQ optimal control. We have shown in particular that it is possible to identify and deflate the singular part of such equation using the corresponding generalised algebraic Riccati equation. The two advantages of this technique are the reduction of the dimension of the Riccati equation at hand as well as the fact that the reduced problem is non-singular, and can therefore be handled with the standard tools of the finite-horizon LQ theory.

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