Revised Manuscript (Clean Version)

Stabilization of a rigid body in a viscous incompressible fluid

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Abstract

This paper addresses the problem of global asymptotic and local exponential stabilization of a rigid body inside a viscous incompressible fluid described by Navier-Stokes equations within a bounded domain in three dimensional space provided that there is no collision between the rigid body and the boundary of the fluid domain. Due to consideration of less regular initial values of the fluid velocity, the forces and moments induced by the fluid on the rigid body are not able to bound. Therefore, the paper handles "fluid work and fluid power" on the rigid body in stability and convergence analysis of the closed-loop system. The control design ensures global asymptotic and local exponential stability of the rigid body while the initial fluid velocity is not required to be small and regular but only under no collision between the rigid body and the boundary of the fluid domain.

Key words: Rigid body; Stabilization; Navier-Stokes equations; Existence; Weak solution.

1 Introduction

Stabilization of a rigid body (e.g., an ocean vehicle), in a viscous fluid has many practical applications in offshore engineering. The fluid loads on a rigid body are usually considered by an approximation approach, see [12, 14, 22, 24] and references therein. In this approach, the fluid loads on a rigid body are approximated and decoupled into two parts. The first part (related to added mass) depends on the acceleration and velocity of the rigid body. The second part depends on the fluid velocity and is considered to be bounded in the Euclidean (pointwise) norm. These approximations are overlooked from the fundamental viewpoint of the fluid-structure interaction, which can be elaborated as follows. The requirement of the first part is oversimplified because it actually depends on the fluid acceleration as shown in Section 5. The requirement of the second part to be bounded in the Euclidean norm requires a strong solution of the Navier-Stokes equations (NSEs) for a viscous incompressible fluid. Currently, existence of this strong solution is local in either time or small initial values [25] under sufficient regularity of initial values.

In comparison with the aforementioned approximation approach, a fundamental NSE approach has been recently considered. In this approach, fully coupled dynamics of both a rigid body and a fluid, where motion of the rigid body is described by nonlinear ordinary differential equations (ODEs) and motion of the fluid is described by NSEs in three dimensional space, is addressed. This approach results in much more complexities but actualities in fluid-structure interactions because we have to deal with: i) existence of an appropriate solution of the fluid and rigid body system; ii) time-varying domain of the fluid; iii) bound of the forces and moments induced by the fluid on the rigid body. There are several works related to the fundamental NSE approach. Existence of a weak solution for a system of a fluid and multiple rigid bodies was proved in [7–9, 13], where dynamics of the fluid and rigid bodies is written as a global fluid and the initial values of the fluid are assumed in $H_0^1(\Omega)$, which is the usual Sobolev space of order 1 with compact support in the domain Ω, because an estimate of the fluid acceleration in a weak form is needed for compactness argument in passing to the limit due to the nonlinear convection term. A similar result was obtained in [6, 18, 19, 27, 28] but using a coordinate transformation to handle difficulty caused by the fluid time-varying domain. In [29], a proportional and derivative control law was designed to stabilize a rigid ball in a fluid with the initial values of the fluid are also assumed in $H_0^1(\Omega)$. This allows to estimate the fluid acceleration in a weak form so that the fluid force acting on the rigid ball is bounded. Feedback stabilization of a rigid body in 1D, 2D, and 3D under similar regularity of the initial data was considered in [2–4], see also [15] for the case of stabilizing a flexible body in a fluid. Stabilization of a rigid ball in compressible fluid was considered in [26], where the global-in-time existence of strong solutions for the corresponding system under a smallness condition on the initial velocities and on the distance between the initial position of the center of the ball was proved, see also [8, 23].

In this paper, we consider the initial values of the fluid

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velocity in $H(\Omega)$, see (7) for definition of this functional space, which is less regular than $H_0^1(\Omega)$. This less regularity of the initial values of the fluid velocity will result in a global solution but will cause a major difficulty: no information on bound of the fluid loads on the rigid body because we do not have an estimate of the fluid acceleration in a weak form for both compactness argument and the bound of the fluid loads on the rigid body. To handle this difficulty, we consider the effect of the "fluid work and fluid power" (instead of the fluid forces and moments) on the rigid body, see discussion just below (39) for detail of the "fluid work and fluid power". Although uniqueness of a weak solution is neither proved nor disproved (this is also a problem for standard NSEs [30]), we show that its boundedness in appropriate norms is sufficient for ensuring global asymptotic and local exponential stability of the closed-loop system provided that there is no collision between the rigid body and the boundary of the fluid domain. Hence, in comparison with the existing works on the approximation approach [12, 14, 22, 24] our control design does not suffer from oversimplifications used in this approach, see the first paragraph of this section. In comparison with the fundamental NSE approach [6–9, 13, 18, 19, 27, 28], our work does not require existence of a strong solution because we only require the initial values of the fluid velocity in $H(\Omega)$. This results in a global solution as long as there is no collision between the rigid body and the boundary of the fluid domain.

In Section 3, a control law is designed in an appropriate form such that it can be amended to be inverse optimal [20, 21], and suitable for stability analysis of stability of the closed-loop system in Section 5. In Section 4, existence of at least one weak solution of the closed-loop system is shown via a penalization approach. In Section 5, we prove global asymptotic and local exponential stability of the closed-loop system provided that there is no collision between the rigid body and the boundary of the fluid domain. We derive the affect of the fluid on the rigid body via the "fluid work and fluid power". This enables us to consider a proper Lyapunov function for stability analysis of the closed-loop system.

Notation: Let Ω be a open bounded set in $\mathbb{R}^3$, and $T > 0$. $L^p(\Omega)$, where $1 \le p < \infty$, denotes the standard Lebesgue space of measurable p-integrable functions; $L^\infty(\Omega)$ denotes the space of essentially bounded functions; $H^1(\Omega)$ is the usual Sobolev space of order 1, see [1]; $H_0^1(\Omega)$ denotes $H^1(\Omega)$ with compact support; $L^p(0,T;X)$, where $1 \le p < \infty$ and X is a Banach space with the norm denoted by $\|\cdot\|_X$, denotes a Brochner space with the norm $\|\boldsymbol{u}\|_{L^p(0,T;X)} = (\int_0^T \|\boldsymbol{u}\|_X^p dt)^{1/p}$. We also use $\|\cdot\|_E$ to denote the Euclidean norm, i.e., $\|\boldsymbol{x}\|_E = \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ with $\boldsymbol{x}\cdot\boldsymbol{y} = \sum_i (x_i y_i)$. For a scalar, we use $|\cdot|_E$ to denote the absolute value.

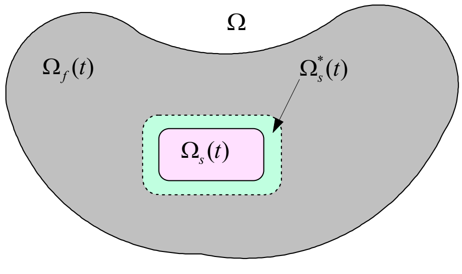

Fig. 1. Domain definition

2 Problem statement

Let $\Omega \subset \mathbb{R}^3$ be a $\mathcal{C}^1$ domain occupied by a viscous incompressible fluid surrounding a rigid body represented by $\Omega_s(t)$, which is a bounded open connected subdomain of Ω, at time t, see Fig. 1, where the domain $\Omega_s^*(t)$ is defined and used in Section 5. We assume that $\Omega_s(0) \Subset \Omega$. The fluid has density $\rho_f > 0$, dynamic viscosity $\mu > 0$, pressure p, velocity $\boldsymbol{u}_f$, and is governed by the NSEs for viscous incompressible fluids [30]:

$$\begin{aligned}&\rho_f(\partial_t \boldsymbol{u}_f + (\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f) - \operatorname{div}(\boldsymbol{\sigma}_f) = 0, \text{ in } \Omega_f(t)\\ &\operatorname{div}(\boldsymbol{u}_f) = 0 \quad \text{in} \quad \Omega_f(t),\end{aligned} \tag{1}$$

where the stress tensor of the fluid $\boldsymbol{\sigma}_f$ is given by

$$\boldsymbol{\sigma}_f = 2\mu \boldsymbol{D}(\boldsymbol{u}_f) - p\boldsymbol{I}_3 \tag{2}$$

with $\boldsymbol{D}(\boldsymbol{u}_f) = \frac{1}{2}(\nabla\boldsymbol{u}_f + (\nabla\boldsymbol{u}_f)^T)$ being the rate tensor of the fluid ($\bullet^T$ denotes the transpose of $\bullet$ and it should not be confused with the time constant T), $\Omega_f(t) \subset \Omega$ being the fluid domain at time t (the boundary (interface between the fluid and the rigid body) of Ω_f depends on t), $\boldsymbol{I}_3$ being the 3×3 identity matrix, and we assume the body force is potential such as gravity, and hence is merged to pressure p.

In what follows, we briefly describe equations of motion of the rigid body in the incompressible fluid. For details of derivation, the reader is referred to [7, 13, 17].

For the rigid body, we define the mass m_s, the density $\rho_s > 0$, the vector of the center of gravity $\boldsymbol{x}_c(t)$ and its velocity vector $\boldsymbol{u}_c(t)$, the modified Rodrigues parameter vector $\boldsymbol{\eta}(t)$ representing the rigid body orientation, see [32], (this vector is related to the principal axis $\boldsymbol{e}$ and the principal angle γ through $\boldsymbol{\eta} = \boldsymbol{e}\tan(\frac{\gamma}{4})$, which is well-defined for all eigenaxis rotations in the range $[0, 2\pi)$; this range can be further relaxed by using quaternion as in [10], and should not be confused with global stability in this paper), angular velocity vector $\boldsymbol{\omega}$, the inertial matrix $\boldsymbol{J}_s$, the transformation matrix $\boldsymbol{R}$, and the velocity vector filed $\boldsymbol{u}_s$ by

$$\begin{aligned}&\rho_s = \frac{m_s}{|\Omega_s(0)|_E},\ \boldsymbol{x}_c = \frac{1}{|\Omega_s(t)|_E}\int_{\Omega_s(t)} \boldsymbol{x}d\boldsymbol{x},\\ &\boldsymbol{u}_s(t,\boldsymbol{x}) = \boldsymbol{u}_c(t) + \boldsymbol{\omega}(t)\times\boldsymbol{r}(t) \text{ for } \boldsymbol{x}\in\Omega_s(t),\\ &\boldsymbol{a}^T\boldsymbol{J}_s\boldsymbol{b} = \rho_s\int_{\Omega_s(0)} (\boldsymbol{a}\times\boldsymbol{r}(t))^T(\boldsymbol{b}\times\boldsymbol{r}(t))\,\forall,\ \boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3,\\ &\boldsymbol{R} = \tfrac{1}{2}\big(\boldsymbol{I} - \boldsymbol{S}(\boldsymbol{\eta}) + \boldsymbol{\eta}\boldsymbol{\eta}^T - \tfrac{1+\|\boldsymbol{\eta}\|_E^2}{2}\boldsymbol{I}_3\big),\end{aligned} \tag{3}$$

where

$$\boldsymbol{r}(t) = \boldsymbol{x} - \boldsymbol{x}_c(t), \tag{4}$$

$|\Omega_s(t)|_E$ denotes the volume of $\Omega_s(t)$, $\boldsymbol{S}(\boldsymbol{\eta})$ denotes the 3×3 skew-symmetric matrix of $\boldsymbol{\eta}$, and it holds that $\boldsymbol{\eta}\cdot\boldsymbol{R}\boldsymbol{\omega} = \frac{1}{4}(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\eta}\cdot\boldsymbol{\omega}$ for all $\boldsymbol{\eta},\boldsymbol{\omega}\in\mathbb{R}^3$. Then, equations of motion of the rigid body are given by

$$\begin{aligned}
&\tfrac{d\boldsymbol{x}_c}{dt} = \boldsymbol{u}_c,\\
&\tfrac{d\boldsymbol{\eta}}{dt} = \boldsymbol{R}\boldsymbol{\omega},\\
&m_s\tfrac{d\boldsymbol{u}_c}{dt} = \int_{\partial\Omega_s(t)}\boldsymbol{\sigma}_f\boldsymbol{n}d\boldsymbol{\tau}+\textstyle\sum_{k=1}^N\boldsymbol{F}_k,\\
&\boldsymbol{J}_s\tfrac{d\boldsymbol{\omega}}{dt} = -\boldsymbol{\omega}\times(\boldsymbol{J}_s\boldsymbol{\omega})+\int_{\partial\Omega_s(t)}\boldsymbol{r}(t)\times(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}\\
&\qquad+\textstyle\sum_{k=1}^N\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k,
\end{aligned}\tag{5}$$

where $\boldsymbol{n}$ is the normal unit vector pointing outside of the rigid body, $\boldsymbol{F}_k$ is the control force, $\boldsymbol{r}_{F_k}$ denotes the relative position vector of $\boldsymbol{F}_k$ with respect to $\boldsymbol{x}_c(t)$ such that $\sum_{k=1}^N\boldsymbol{F}_k\in\mathbb{R}^3$ and $\sum_{k=1}^N\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k\in\mathbb{R}^3$, i.e., the rigid body is fully actuated.

We impose a homogeneous Dirichlet boundary condition at the boundary $\partial\Omega\cap\partial\Omega_f(t)$ and a continuous condition of the velocity at the interface between the rigid body and the fluid:

$$\begin{aligned}
\boldsymbol{u}_f &= 0 \text{ on } \partial\Omega\cap\partial\Omega_f(t),\\
\boldsymbol{u}_s &= \boldsymbol{u}_f \quad\text{on}\quad \partial\Omega_s(t).
\end{aligned}\tag{6}$$

In derivation of (5), we have used an interface condition that the stress is continuous in normal direction, i.e, $\boldsymbol{\sigma}_f\boldsymbol{n} = \boldsymbol{\sigma}_s\boldsymbol{n}$ on $\partial\Omega_s$ with $\boldsymbol{\sigma}_s$ being the Cauchy stress tensor, i.e., $-\boldsymbol{\sigma}_s\boldsymbol{n}$ is the force applied by the rigid body on the fluid.

From now onwards, we will drop the argument t of Ω_f, Ω_s, and $\boldsymbol{r}$ when it does not lead to a confusion. For use in the rest of the paper, we denote $Q=(0,T)\times\Omega$, and introduce the following function spaces:

$$\begin{aligned}
&V = \{\boldsymbol{v}\in H_0^1(\Omega), \mathrm{div}(\boldsymbol{v})=0\},\\
&H = \{\boldsymbol{v}\in L^2(\Omega), \mathrm{div}(\boldsymbol{v})=0, \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega}=0\},\\
&\mathcal{K}(t) = \{\boldsymbol{v}\in V, \exists(\boldsymbol{v}_v,\boldsymbol{\omega}_v)\in\mathbb{R}^3\times\mathbb{R}^3, \boldsymbol{v}|_{\Omega_s} = \boldsymbol{v}_v+\boldsymbol{\omega}_v\times\boldsymbol{r}\}.
\end{aligned}\tag{7}$$

Note that the elements of $\mathcal{K}(t)$ are given by the rigid body velocity in Ω_s. One can prove the following lemma on the space $\mathcal{K}(t)$, see [5,31].

Lemma 2.1 *The space $\mathcal{K}(t)$ is equivalent to*

$$\mathcal{K}(t) = \{\boldsymbol{v}\in V,\ \boldsymbol{D}(\boldsymbol{v})=0 \text{ in } \Omega_s\}.\tag{8}$$

In this paper, we address the following control objective.

Control Objective 2.1 *Under the initial data:*

$$\begin{aligned}
&\Omega_s(0)\Subset\Omega,\\
&\boldsymbol{u}_f(0,\boldsymbol{x})\in H,\\
&(\boldsymbol{x}_c(0),\boldsymbol{\eta}(0),\boldsymbol{u}_c(0),\boldsymbol{\omega}(0))\in\underline{\Omega}\times\mathcal{D}_\eta\times\mathbb{R}^3\times\mathbb{R}^3,
\end{aligned}\tag{9}$$

where $\underline{\Omega}\Subset\Omega$ and $\mathcal{D}_\eta=(-\infty,\infty)^3$, design the control forces $\sum_{i=1}^k\boldsymbol{F}_i$ and control moments $\sum_{k=1}^N\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k$ to globally asymptotically and locally exponentially stabilize the rigid body at the origin provided that there is no collision between the rigid body and the boundary of the fluid domain. Moreover, the designed control forces and control moments must ensure existence of the weak solution of the closed-loop system consisting of both the rigid body and the fluid, see Definition 4.1 in Section 4 for definition of the weak solution.

Note that the initial values $\boldsymbol{x}_c(0)\in\underline{\Omega}$ is imposed to be compatible with $\Omega_s(0)\Subset\Omega$, and $\boldsymbol{\eta}(0)\in\mathcal{D}_\eta$ is made due to the domain of the modified Rodrigues parameter vector $\boldsymbol{\eta}$.

3 Control design

For convenience, we denote the control force $\boldsymbol{F}_s$ and the control moment $\boldsymbol{M}_s$ as follows

$$\begin{aligned}
\boldsymbol{F}_s &= \textstyle\sum_{k=1}^N\boldsymbol{F}_k,\\
\boldsymbol{M}_s &= \textstyle\sum_{k=1}^N\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k.
\end{aligned}\tag{10}$$

Since the rigid body dynamics (5) is of a second-order system, we consider the following Lyapunov function candidate to design the controls $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$:

$$\begin{aligned}
U_1 &= \tfrac{1}{2}\|\boldsymbol{x}_c\|_E^2+\tfrac{m_s}{2}\|k_1\boldsymbol{x}_c+\boldsymbol{u}_c\|_E^2\\
&\quad+\tfrac{1}{2}\|\boldsymbol{\eta}\|_E^2+\tfrac{1}{2}(k_2\boldsymbol{\eta}+\boldsymbol{\omega})^T\boldsymbol{J}_s(k_2\boldsymbol{\eta}+\boldsymbol{\omega}),
\end{aligned}\tag{11}$$

where k_1 and k_2 are constants to be chosen. It holds that

$$\bar{k}_{01}\|\boldsymbol{Y}_s\|_E^2\le U_1\le\bar{k}_{02}\|\boldsymbol{Y}_s\|_E^2,\tag{12}$$

where $\boldsymbol{Y}_s = \mathrm{col}(\boldsymbol{x}_c,\boldsymbol{u}_c,\boldsymbol{\eta},\boldsymbol{\omega})$, and we chose k_1 and k_2 such that

$$\begin{aligned}
\bar{k}_{01} &= \tfrac{1}{2}\min\big((1+m_sk_1^2-m_s|k_1|_E),(m_s-|k_1|_Em_s),\\
&\quad(1+k_2^2\lambda_m(\boldsymbol{J}_s)-|k_2|_E\lambda_M(\boldsymbol{J}_s)),\\
&\quad(\lambda_m(\boldsymbol{J}_s)-|k_2|_E\lambda_M(\boldsymbol{J}_s))\big)>0,\\
\bar{k}_{02} &= \tfrac{1}{2}\max\big((1+m_sk_1^2+m_s|k_1|_E),(m_s+|k_1|_Em_s),\\
&\quad(1+k_2^2\lambda_M(\boldsymbol{J}_s)+|k_2|_E\lambda_M(\boldsymbol{J}_s)),\\
&\quad(\lambda_M(\boldsymbol{J}_s)+|k_2|_E\lambda_M(\boldsymbol{J}_s))\big),
\end{aligned}\tag{13}$$

with $\lambda_m(\boldsymbol{J}_s)$ and $\lambda_M(\boldsymbol{J}_s)$ being the minimum and maximum eigenvalue of $\boldsymbol{J}_s$, respectively.

Differentiating (11) along the solutions of (5) yields

$$\begin{aligned}
\tfrac{dU_1}{dt} &= \boldsymbol{x}_c\cdot\boldsymbol{u}_c+(k_1\boldsymbol{x}_c+\boldsymbol{u}_c)\cdot\big(m_sk_1\boldsymbol{u}_c+\boldsymbol{F}_s\big)+\boldsymbol{\eta}\cdot\boldsymbol{R}\boldsymbol{\omega}\\
&\quad+(k_2\boldsymbol{\eta}+\boldsymbol{\omega})\cdot\big(k_2\boldsymbol{J}_s\boldsymbol{R}\boldsymbol{\omega}-\boldsymbol{\omega}\times(\boldsymbol{J}_s\boldsymbol{\omega})+\boldsymbol{M}_s\big)+\varpi,
\end{aligned}\tag{14}$$

where

$$\begin{aligned}
\varpi &= (k_1\boldsymbol{x}_c+\boldsymbol{u}_c)\cdot\int_{\partial\Omega_s}\boldsymbol{\sigma}_f\boldsymbol{n}d\boldsymbol{\tau}\\
&\quad+(k_2\boldsymbol{\eta}+\boldsymbol{\omega})\cdot\int_{\partial\Omega_s}\boldsymbol{r}\times(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}\\
&= k_{12}\int_{\partial\Omega_s}\boldsymbol{x}_c\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}+k_2\int_{\partial\Omega_s}\boldsymbol{x}_s\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}\\
&\quad+\int_{\partial\Omega_s}\boldsymbol{u}_s\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau},
\end{aligned}\tag{15}$$

with

$$\begin{aligned}
k_{12} &= k_1-k_2,\\
\boldsymbol{x}_s &= \boldsymbol{x}_c+\boldsymbol{\eta}\times\boldsymbol{r},
\end{aligned}\tag{16}$$

which satisfies $\mathrm{div}(\boldsymbol{x}_s)=0$.

From (14), we design the controls $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ as follows:

$$\begin{aligned}
\boldsymbol{F}_s &= -k_3\boldsymbol{x}_c-k_4(k_1\boldsymbol{x}_c+\boldsymbol{u}_c),\\
\boldsymbol{M}_s &= -k_5(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\eta}-k_6(1+\|\boldsymbol{\eta}\|_E^2)(k_2\boldsymbol{\eta}+\boldsymbol{\omega}),
\end{aligned}\tag{17}$$

where $k_i, i=3,\cdots,6$ are positive constants to be chosen. It is noted that instead of cancelling controls, we designed the controls $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ as in (17) so that they can be written in the form:

$$\begin{bmatrix}\boldsymbol{F}_s\\ \boldsymbol{M}_s\end{bmatrix} = -\boldsymbol{K}\big(\tfrac{\partial U_1}{\partial\boldsymbol{Y}_s}\big)^T,\tag{18}$$

where $\boldsymbol{K}$ is a positive definite matrix. This form means that the controls $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ are inverse pre-optimal and can be easily extended to be inverse optimal by multiplying themselves by a positive constant larger than 2, see [20,21] for extending $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ given by (17) to inverse optimal controls, and many desired properties of inverse optimal controls. However, we do not detail this issue here as we focus on the fluid loads on the rigid body in this paper.

Substituting (17) into (14) and using Young's inequality gives

$$\frac{dU_1}{dt} \leq -\bar{k}_1\|\boldsymbol{x}_c\|_E^2 - \bar{k}_2\|\boldsymbol{u}_c\|_E^2 - \bar{k}_3\|\boldsymbol{\eta}\|_E^2 - \bar{k}_4\|\boldsymbol{\omega}\|_E^2 + \varpi, \tag{19}$$

where we chose the constants $k_i, i = 1, \cdots, 6$ such that

$$\begin{aligned} &1 + k_1^2 m_s - k_3 - 2k_1k_4 = 0, \\ &\bar{k}_1 = k_1k_3 + k_1^2 k_4 > 0, \\ &\bar{k}_2 = k_4 - k_1 m_s > 0, \\ &\tfrac{1}{2} - 2k_2k_6 + k_5 = 0, \\ &\bar{k}_3 = k_5 - \tfrac{1}{8}k_2^2\lambda_M(\boldsymbol{J}_s) + k_2^2 k_6 > 0, \\ &\bar{k}_4 = k_6 - \tfrac{1}{8}k_2^2\lambda_M(\boldsymbol{J}_s) - |k_2|_E\lambda_M(\boldsymbol{J}_s) > 0. \end{aligned} \tag{20}$$

It is easy to see that there exist constants $k_i, i = 1, \cdots, 6$ such that all the conditions in (20) and (13) hold, and that $\bar{k}_i, i = 1, \cdots, 4$ are as large as required. From (17) and (10), we can solve for the controls $\boldsymbol{F}_k$, see Section 6.

Remark 3.1 *At this point, we cannot conclude any stability of the rigid body dynamics based on (11) and (19) because we do not have a bound on the fluid force* $\int_{\partial\Omega_s} \boldsymbol{\sigma}_f \boldsymbol{n} d\boldsymbol{\tau}$ *and the fluid moment* $\int_{\partial\Omega_s} \boldsymbol{r} \times (\boldsymbol{\sigma}_f \boldsymbol{n}) d\boldsymbol{\tau}$. *This means that we cannot use Young's inequality to bound the term* ϖ *defined in (15). Therefore, we will analyze stability of the rigid body dynamics and convergence of its states in Section 5 after we show existence of the weak solution of the whole system (i.e., both the fluid and the rigid body) and consider the NSEs for the fluid separately in Section 4.*

4 Existence of a weak solution of the closed-loop system

In this section, we show the closed-loop system consisting of the fluid (1) and the rigid body (5) with $\boldsymbol{F}_k$ obtained from (17) and (10) has at least one weak solution.

4.1 Formulation of a weak solution

We define the characteristic function $\chi_{\Omega_s}(t, \boldsymbol{x})$ on Ω_s, a global velocity $\boldsymbol{u}$, and a global density $\rho(t, \boldsymbol{x})$, and assign a function $h(t, \boldsymbol{x})$ to $\chi_{\Omega_s}(t, \boldsymbol{x})$ (for simplicity of presentation and for penalization purpose later) as follows:

$$\begin{aligned} &\chi_{\Omega_s}(t, \boldsymbol{x}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{x} \in \Omega_s \\ 0 & \text{otherwise}, \end{cases} \\ &\boldsymbol{u} = \begin{cases} \boldsymbol{u}_f & \text{in} \quad \Omega_f \\ \boldsymbol{u}_s & \text{in} \quad \Omega_s, \end{cases} \\ &\rho(t, \boldsymbol{x}) = \rho_s \chi_{\Omega_s}(t, \boldsymbol{x}) + \rho_f(1 - \chi_{\Omega_s}(t, \boldsymbol{x})), \\ &h(t, \boldsymbol{x}) = \chi_{\Omega_s}(t, \boldsymbol{x}). \end{aligned} \tag{21}$$

Clearly, we have

$$\Omega_s(t) = \{\boldsymbol{x} \in \Omega, h(t, \boldsymbol{x}) = 1\}. \tag{22}$$

Let us denote the initial data for those variables defined in (21) as

$$\begin{aligned} &\boldsymbol{u}_0 := \boldsymbol{u}(0, \boldsymbol{x}) = \begin{cases} \boldsymbol{u}_f(0, \boldsymbol{x}) \text{ in } \Omega_f(0) \\ \boldsymbol{u}_s(0, \boldsymbol{x}) \text{ in } \Omega_s(0). \end{cases} \\ &\rho_0 := \rho(0, \boldsymbol{x}) = \rho_s \chi_{\Omega_s}(0, \boldsymbol{x}) + \rho_f(1 - \chi_{\Omega_s}(0, \boldsymbol{x})), \\ &h_0 := h(0, \boldsymbol{x}) = \chi_{\Omega_s(0)}(0, \boldsymbol{x}). \end{aligned} \tag{23}$$

Then, $h(t, \boldsymbol{x})$ and $\rho(t, \boldsymbol{x})$ satisfy transport equations [7] in a weak form as

$$\begin{aligned} &\forall \psi \in \mathcal{C}^1(Q), \psi(T) = 0 : \\ &\int_0^T \int_\Omega \left(h\frac{\partial \psi}{\partial t} + h\boldsymbol{u}\cdot\nabla\psi\right) d\boldsymbol{x}dt + \int_\Omega h_0\psi(0) = 0, \\ &\int_0^T \int_\Omega \left(\rho\frac{\partial \psi}{\partial t} + \rho\boldsymbol{u}\cdot\nabla\psi\right) d\boldsymbol{x}dt + \int_\Omega \rho_0\psi(0) = 0. \end{aligned} \tag{24}$$

Next, we perform tedious but straightforward calculations from (1) and (5) to obtain

$$\begin{aligned} &\frac{d\boldsymbol{x}_c}{dt} = \boldsymbol{u}_c, \\ &\frac{d\boldsymbol{\eta}}{dt} = \boldsymbol{R}\boldsymbol{\omega}, \\ &\forall \boldsymbol{\xi} \in H^1(Q) \cap L^2(0, T; \mathcal{K}(t)) : \\ &\frac{d}{dt}\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{\xi} d\boldsymbol{x} = \int_\Omega \big[\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{\xi} + (\rho\boldsymbol{u}\otimes\boldsymbol{u} - 2\mu\boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{\xi}) \\ &\qquad + \textstyle\sum_{k=1}^N (\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k)\cdot\boldsymbol{\xi}\big] d\boldsymbol{x}, \end{aligned} \tag{25}$$

where $\boldsymbol{\delta}_{Fk}$ denotes the diagonal matrix of the dirac delta function of $\boldsymbol{x} - \boldsymbol{x}_{Fk}$ with $\boldsymbol{x}_{Fk}$ being the position of the force $\boldsymbol{F}_k$, i.e., $\boldsymbol{x}_{Fk} = \boldsymbol{x}_c + \boldsymbol{r}_{Fk}$. We have included the first two equations in (25) for convenience of referring later. In derivation of (25), we have multiplied the first equation in (1) by $\boldsymbol{\xi} \in H^1(Q) \cap L^2(0, T; \mathcal{K}(t))$, then calculated $\frac{d}{dt}\int_{\Omega_s} \rho\boldsymbol{u}\cdot\boldsymbol{\xi} d\boldsymbol{x}$, and noted that

$$\begin{aligned} \boldsymbol{v}_\xi\cdot\boldsymbol{F}_s + \boldsymbol{\omega}_\xi\cdot\boldsymbol{M}_s &= \textstyle\sum_{i=1}^N (\boldsymbol{v}_{\boldsymbol{\xi}}\cdot\boldsymbol{F}_k + \boldsymbol{\omega}_{\boldsymbol{\xi}}\cdot(\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k)) \\ &= \textstyle\sum_{i=1}^N \int_{\Omega_s} (\boldsymbol{v}_{\boldsymbol{\xi}}\cdot\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k + \boldsymbol{\omega}_{\boldsymbol{\xi}}\cdot(\boldsymbol{r}_{Fk}\times\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k)) d\boldsymbol{x} \\ &= \textstyle\sum_{i=1}^N \int_\Omega (\boldsymbol{v}_{\boldsymbol{\xi}}\cdot\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k + \boldsymbol{\omega}_{\boldsymbol{\xi}}\cdot(\boldsymbol{r}_{Fk}\times\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k)) d\boldsymbol{x} \\ &= \textstyle\sum_{i=1}^N \int_\Omega \boldsymbol{\xi}\cdot\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k d\boldsymbol{x}. \end{aligned} \tag{26}$$

Now, setting $\psi = h^q$ with $q \geq 1$ in the first equation of (24) and $\psi = \rho^q$ with $q \geq 1$ in the second equation of (24) yields

$$\rho, h \in \mathcal{C}(0, T; L^q(\Omega)), \quad \forall\, q \geq 1. \tag{27}$$

Moreover, setting $\boldsymbol{\xi} = \boldsymbol{u}$ in (26) and considering the "energy" $\mathcal{E}$ defined by

$$\mathcal{E} = \frac{\hat{k}_1}{2}\|\boldsymbol{x}_c\|_E^2 + 2\hat{k}_2\|\boldsymbol{\eta}\|_E^2 + \frac{1}{2}\int_\Omega \rho\|\boldsymbol{u}\|_E^2, \tag{28}$$

where $\hat{k}_1 = k_3 + k_1k_4 > 0$ and $\hat{k}_2 = k_5 + k_2k_6 > 0$ under conditions (20). Differentiating (28) along the solutions of (25), we formally obtain:

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= -2\mu\int_\Omega \|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x} - k_4 m_s\|\boldsymbol{u}_c\|_E^2 \\ &\quad - k_6(1 + \|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}. \end{aligned} \tag{29}$$

From (27), (28), and (29), we can derive that

$$\begin{aligned} &\boldsymbol{x}_c \in L^\infty(0, T; \mathbb{R}^3),\ \boldsymbol{\eta} \in L^\infty(0, T; \mathbb{R}^3), \\ &\boldsymbol{u} \in L^\infty(0, T; H) \cap L^2(0, T; V). \end{aligned} \tag{30}$$

The above derivations motivate the following weak solution definition.

Definition 4.1 *Under the initial data (9), where the initial data of global variables are given in (23),* $(\boldsymbol{x}_c(t), \boldsymbol{\eta}(t), \rho(t,\boldsymbol{x}), \boldsymbol{u}(t,\boldsymbol{x}), h(t,\boldsymbol{x}))$ *is a weak solution of the closed-loop system consisting of (1), (5), and (17) if they satisfy (24), (25), (27), and (30).*

The above definition of a weak solution is similar to a definition of a weak solution to standard nonhomogeneous NSEs except for the fact that we take the constraint of rigidity into account. On the other hand, the transport equation on ρ can be deduced from that of h and the definition of ρ_0; however h is added for passing the passage to the limit in the problem of penalization.

4.2 Penalized system

In order to find $(\boldsymbol{x}_c(t), \boldsymbol{\eta}(t), \rho(t,\boldsymbol{x}), \boldsymbol{u}(t,\boldsymbol{x}), h(t,\boldsymbol{x}))$ such that they satisfy Definition 4.1, we use a penalization approach. Due to Lemma 2.1, there are two methods to penalize the rigidity constraint: i) penalizing the difference between the fluid velocity and rigid body velocity [5]; ii) penalizing the spatial derivative of the rigid body velocity [31]. We use the first method due to its several advantages (such as simpler calculations in proof, and numerical computation) over the second method.

Let $\delta > 0$ be a penalized parameter. Given the initial data

$$\begin{aligned}&\boldsymbol{x}_{c\delta}(0) = \boldsymbol{x}_c(0),\ \boldsymbol{\eta}_\delta(0) = \boldsymbol{\eta}(0),\ \boldsymbol{u}_\delta(0) = \boldsymbol{u}_0,\\ &\rho_\delta(0) = \rho_0,\ h_\delta(0) = h_0,\end{aligned} \tag{31}$$

we wish to find $(\boldsymbol{x}_{c\delta}(t), \boldsymbol{\eta}_\delta(t), \rho_\delta(t,\boldsymbol{x}), \boldsymbol{u}_\delta(t,\boldsymbol{x}), h_\delta(t,\boldsymbol{x}), p_\delta(t,\boldsymbol{x}))$ such that they satisfy

$$\begin{aligned}&\boldsymbol{x}_{c\delta}, \boldsymbol{\eta}_\delta \in L^\infty(0,T;\mathbb{R}^3);\ \rho_\delta, h_\delta \in L^\infty(Q);\\ &\boldsymbol{u}_\delta \in L^\infty(0,T;H)\cap L^2(0,T;V);\ p_\delta \in L^2(Q),\end{aligned} \tag{32}$$

and are a solution to the penalized system:

$$\begin{cases}\rho_\delta(\partial_t\boldsymbol{u}_\delta + (\boldsymbol{u}_\delta\cdot\nabla)\boldsymbol{u}_\delta) - \operatorname{div}(\boldsymbol{\sigma}_\delta)\\ +\frac{1}{\delta}\rho_\delta h_\delta(\boldsymbol{u}_\delta - \boldsymbol{u}_{\delta,s}) - \sum_{k=1}^N(\boldsymbol{\delta}_k\boldsymbol{F}_k) = 0, \\ \operatorname{div}(\boldsymbol{u}_\delta) = 0, & \text{in } Q\\ \partial_t\rho_\delta + \boldsymbol{u}_\delta\cdot\nabla\rho_\delta = 0,\\ \partial_t h_\delta + \boldsymbol{u}_{\delta,s}\cdot\nabla h_\delta = 0,\end{cases}$$
$$\begin{cases}\frac{d\boldsymbol{x}_{c\delta}}{dt} = \boldsymbol{u}_{c\delta}, \\ \frac{d\boldsymbol{\eta}_\delta}{dt} = \boldsymbol{R}_\delta\boldsymbol{\omega}_\delta, & \text{in } \mathbb{R}^3\end{cases} \tag{33}$$

where

$$\begin{aligned}&\boldsymbol{\sigma}_\delta = 2\mu\boldsymbol{D}(\boldsymbol{u}_\delta) - p_\delta\boldsymbol{I}_3,\\ &\boldsymbol{u}_{\delta,s} = \frac{1}{m_\delta}\int_\Omega \rho_\delta\boldsymbol{u}_\delta h_\delta d\boldsymbol{x} + \big(\boldsymbol{J}_\delta^{-1}\int_\Omega \rho_\delta\boldsymbol{r}_\delta\times\boldsymbol{u}_\delta h_\delta d\boldsymbol{x}\big)\times\boldsymbol{r}_\delta,\\ &\boldsymbol{R}_\delta = \frac{1}{2}\big(\boldsymbol{I} - \boldsymbol{S}(\boldsymbol{\eta}_\delta) + \boldsymbol{\eta}_\delta\boldsymbol{\eta}_\delta^T - \frac{1+\|\boldsymbol{\eta}_\delta\|_E^2}{2}\boldsymbol{I}_3\big),\end{aligned} \tag{34}$$

with

$$\begin{aligned}&\boldsymbol{r}_\delta = \boldsymbol{x} - \boldsymbol{x}_{c\delta},\ \boldsymbol{x}_{c\delta} = \frac{1}{m_\delta}\int_\Omega \rho_\delta h_\delta\boldsymbol{x}d\boldsymbol{x},\\ &m_\delta = \int_\Omega \rho_\delta h_\delta d\boldsymbol{x},\\ &\boldsymbol{J}_\delta = \int_\Omega \rho_\delta(\|\boldsymbol{r}_\delta\|_E^2\boldsymbol{I}_3 - \boldsymbol{r}_\delta\otimes\boldsymbol{r}_\delta)h_\delta d\boldsymbol{x}\end{aligned} \tag{35}$$

In addition, we impose the homogeneous Dirichlet boundary condition:

$$\boldsymbol{u}_\delta = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{36}$$

and define $\Omega_s^\delta = \{\boldsymbol{x}\in\Omega_s, h_\delta(t,\boldsymbol{x}) = 1\}$. Several observations on the penalized system (33) are given in the following remark.

Remark 4.1

- *The penalized system (33) is based on [5], where action of the controls* $\boldsymbol{F}_s$ *and* $\boldsymbol{M}_s$ *given by (17) are taken into account.*
- *It is clear that* $|\Omega_s^\delta|_E = \int_\Omega h_\delta d\boldsymbol{x} = |\Omega_s(0)|_E$ *because* $\boldsymbol{u}_{\delta,s}$ *is divergence free and* h_δ *vanishes on* $\partial\Omega$ *as we assume there is no collision between the rigid body and* $\partial\Omega$. *We also have*
$$\begin{aligned}&m_\delta = \int_{\Omega_s^\delta}\rho_\delta d\boldsymbol{x},\\ &\boldsymbol{J}_\delta = \int_{\Omega_s^\delta}\rho_\delta(\|\boldsymbol{r}_\delta\|_E^2\boldsymbol{I}_3 - \boldsymbol{r}_\delta\otimes\boldsymbol{r}_\delta)d\boldsymbol{x}.\end{aligned} \tag{37}$$
Hence, m_δ *is positive (as* $m_\delta \geq \min(\rho_f,\rho_s)\int_{\Omega_s^\delta}d\boldsymbol{x}$*) and* $\boldsymbol{J}_\delta$ *is positive definite (as* $\boldsymbol{a}\cdot\boldsymbol{J}_\delta\boldsymbol{a} = \int_{\Omega_s^\delta}\rho_\delta\|\boldsymbol{r}_\delta\times\boldsymbol{a}\|_E^2d\boldsymbol{x} \geq \min(\rho_f,\rho_s)\int_{\Omega_s^\delta}\|\boldsymbol{r}_\delta\times\boldsymbol{a}\|_E^2d\boldsymbol{x}$ *for all* $\boldsymbol{a}\in\mathbb{R}^3\setminus\{0\}$*).*
- *In the first equation of (33), the term* $\boldsymbol{u}_{\delta,s}$ *defined in (34) is the projection of* $\boldsymbol{u}_\delta$ *onto the velocity fields which are rigid on* Ω_s^δ *because one can prove that, see [5]:*
$$\int_\Omega \rho_\delta h_\delta(\boldsymbol{u}_\delta - \boldsymbol{u}_{\delta,s})\cdot\boldsymbol{\zeta}d\boldsymbol{x} = 0, \tag{38}$$
where $\boldsymbol{\zeta}$ *is a rigid velocity field, i.e., there exist* $(\boldsymbol{v}_\zeta,\boldsymbol{\omega}_\zeta)\in\mathbb{R}^3\times\mathbb{R}^3$ *such that* $\boldsymbol{\zeta} = \boldsymbol{v}_\zeta + \boldsymbol{\omega}_\zeta\times\boldsymbol{r}(t,\boldsymbol{x})$*, and* $\boldsymbol{u}_{\delta,s}$ *is given by (34). Hence, the penalized term* $\frac{1}{\delta}\rho_\delta h_\delta(\boldsymbol{u}_\delta - \boldsymbol{u}_{\delta,s})$ *in the first equation of (33) is the difference between* $\boldsymbol{u}_\delta$ *and its projection onto rigid velocity fields in the rigid body domain, i.e.,* $\boldsymbol{u}_{\delta,s}$.
- *The density is transported with the velocity field* $\boldsymbol{u}_\delta$*. This eases calculations in estimating bounds for the penalized system.*

Existence of a weak solution $(\boldsymbol{x}_{c\delta}(t)$, $\boldsymbol{\eta}_\delta(t)$, $\rho_\delta(t,\boldsymbol{x})$, $\boldsymbol{u}_\delta(t,\boldsymbol{x})$, $h_\delta(t,\boldsymbol{x})$, $p_\delta(t,\boldsymbol{x}))$ to the penalized system (33) is stated in the following lemma.

Lemma 4.1 *There is at least one weak solution* $(\boldsymbol{x}_{c\delta}(t)$, $\boldsymbol{\eta}_\delta(t)$, $\rho_\delta(t,\boldsymbol{x})$, $\boldsymbol{u}_\delta(t,\boldsymbol{x})$, $h_\delta(t,\boldsymbol{x})$, $p_\delta(t,\boldsymbol{x}))$ *to the penalized system (33) that satisfies (32) for all* $t\in[0,T]$*, where* T *is such that* $\Omega_s^\delta(t)\Subset\Omega$ *and* $\boldsymbol{\eta}_\delta(t)\in\mathcal{D}_\eta$ *for all* $t\in[0,T]$.

Proof. Proof of this lemma principally follows the part of a priori estimates and convergence arguments in proof of Theorem 2.1 in [5]. The only main difference is that a priori estimates should use the penalized energy $\mathcal{E}_\delta$ as in (28) with $(\boldsymbol{x}_c,\boldsymbol{\eta},\boldsymbol{u})$ being substituted by $(\boldsymbol{x}_{c\delta},\boldsymbol{\eta}_\delta,\boldsymbol{u}_\delta)$. This is due to inclusion of $(\boldsymbol{x}_c,\boldsymbol{\eta})$ and controls $(\boldsymbol{F}_s,\boldsymbol{M}_s)$ in this paper.

4.3 Existence of a weak solution

Having obtained a weak solution of the penalized system (33) in Lemma 4.1, existence of a weak solution stated in Definition 4.1 is given in the following theorem.

Theorem 4.1 *Let* $(\boldsymbol{x}_{c\delta}(t)$, $\boldsymbol{\eta}_\delta(t)$, $\rho_\delta(t,\boldsymbol{x})$, $\boldsymbol{u}_\delta(t,\boldsymbol{x})$, $h_\delta(t,\boldsymbol{x})$, $p_\delta(t,\boldsymbol{x}))$ *be a weak solution to the penalized system (33). Then, under the initial data (23) and (9), there exists a subsequence of* $(\boldsymbol{x}_{c\delta}(t)$, $\boldsymbol{\eta}_\delta(t)$, $\rho_\delta(t,\boldsymbol{x})$, $\boldsymbol{u}_\delta(t,\boldsymbol{x})$,

$h_\delta(t,\boldsymbol{x})$) such that $\boldsymbol{x}_{c\delta} \to \boldsymbol{x}_c$ strongly in $L^\infty(Q)$; $\boldsymbol{\eta}_\delta \to \boldsymbol{\eta}$ strongly in $L^\infty((0,T)\times\mathcal{D}_\eta)$; $\rho_\delta \to \rho$, $h_\delta \to h$ strongly in $\mathcal{C}(0,T;L^q(\Omega))$; $\boldsymbol{u}_\delta \to \boldsymbol{u}$ strongly in $L^2(Q)$ and weakly in $L^\infty(0,T;H)\cap L^2(0,T;V)$ such that $(\boldsymbol{x}_c,\boldsymbol{\eta},\rho,h,\boldsymbol{u})$ is a weak solution of the closed-loop system consisting of (1), (5), and (17) as defined in Definition 4.1. The constant T is such that $\Omega_s(t) \Subset \Omega$ and $\boldsymbol{\eta}(t) \in \mathcal{D}_\eta$ for all $t \in [0,T]$.

Proof. Proof of this theorem can be readily obtained from that of Theorem 2.1 in [5] with a note as in the proof of Lemma 4.1.

5 Stability and convergence of the closed-loop system

This section provides stability and convergence analysis of the closed-loop system, which can be based on (11), (19), (28), and (29) once we handle the term ϖ in (15).

5.1 Detail of ϖ

Since we already showed existence of a weak solution of the closed-loop system (including both the fluid and rigid body) in Theorem 4.1, the idea to handle the term ϖ is to multiply the first equation in (1) by appropriate test functions to detail the terms:

$$\begin{aligned} A_1 &= \textstyle\int_{\partial\Omega_s} \boldsymbol{x}_c\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}, \\ A_2 &= \textstyle\int_{\partial\Omega_s} \boldsymbol{x}_s\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}, \\ A_3 &= \textstyle\int_{\partial\Omega_s} \boldsymbol{u}_s\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}, \end{aligned} \tag{39}$$

where $\boldsymbol{u}_s$ is defined in (3) and $\boldsymbol{x}_s$ is defined in (16). We refer A_1 and A_2 to as fluid work as they are products of the fluid force $(\boldsymbol{\sigma}_f\boldsymbol{n})$ with displacements $\boldsymbol{x}_c$ and $\boldsymbol{x}_s$, and A_3 to as fluid power as it is a product of the fluid force with velocity $\boldsymbol{u}_s$.

5.1.1 Detail of A_1 and A_2

We define the domain $\Omega_s^*(t)$, where the argument t of Ω_s^* is dropped for clarity henceforth, such that $\Omega_s \subset \Omega_s^* \subseteq \Omega$, and the minimum distance between $\partial\Omega_s$ and $\partial\Omega_s^*$ denoted by $\kappa = \inf_{t\geq 0}\operatorname{dist}(\partial\Omega_s,\partial\Omega_s^*)$ is strictly positive, see Fig. 1. There exists Ω_s^* such that this κ is strictly positive because we assumed $\Omega_s \Subset \Omega$. Let $\hat{\boldsymbol{X}}_s(t,\boldsymbol{x}) \in L^\infty(0,T;E)$, which represents either $\boldsymbol{x}_c$ or $\boldsymbol{x}_s$, we can extend $\hat{\boldsymbol{X}}_s(t,\boldsymbol{x})$ to $\boldsymbol{X}_s(t,\boldsymbol{x})$ in Ω_s^* such that $\boldsymbol{X}_s = 0$ on $\partial\Omega_s^*$ and $\operatorname{div}(\boldsymbol{X}_s) = 0$ in Ω_s^* using the smooth step function introduced in [11] as follows. Let $h(t,\boldsymbol{x})$ be the smooth step function extended to three dimensional space such that $\nabla\times h = 0$ on $\partial\Omega_s^*$ and $\nabla\times h = \hat{\boldsymbol{X}}_s$ on $\partial\Omega_s$. Then, $\boldsymbol{X}_s$ can be defined as $\boldsymbol{X}_s = \hat{\boldsymbol{X}}$ in Ω_s, and $\boldsymbol{X}_s = \nabla\times h$ in Ω_s^*. It is clear that $\operatorname{div}(\boldsymbol{X}_s) = 0$ because $\operatorname{div}(\hat{\boldsymbol{X}}) = 0$ and $\operatorname{div}(\nabla\times h) = 0$.

Now, multiplying the first equation in (1) by $\boldsymbol{X}_s$ and integrating over Ω_s^* yields

$$\begin{aligned} &\rho_f\textstyle\int_{\Omega_s^*}\partial_t\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} + \rho_f\int_{\Omega_s^*}(\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} \\ &-\textstyle\int_{\Omega_s^*}\operatorname{div}(\boldsymbol{\sigma}_f)\cdot\boldsymbol{X}_s d\boldsymbol{x} = 0. \end{aligned} \tag{40}$$

Using integration by parts, the boundary condition $\boldsymbol{X}_s = 0$ on $\partial\Omega_s^*$, and the interface condition given by the second equation in (6), we have

$$\begin{aligned} \textstyle\int_{\Omega_s^*}\partial_t\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \tfrac{d}{dt}\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} - \int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\boldsymbol{X}_s d\boldsymbol{x} \\ &\quad - \textstyle\int_{\partial\Omega_s}(\boldsymbol{u}_f\cdot\boldsymbol{X}_s)\boldsymbol{u}_f\cdot\boldsymbol{n}d\boldsymbol{\tau}, \\ \textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \textstyle\int_{\partial\Omega_s}(\boldsymbol{u}_f\cdot\boldsymbol{X}_s)\boldsymbol{u}_f\cdot\boldsymbol{n}d\boldsymbol{\tau} \\ &\quad - \textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\boldsymbol{X}_s d\boldsymbol{x}, \\ \textstyle\int_{\Omega_s^*}\operatorname{div}(\boldsymbol{\sigma}_f)\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \textstyle\int_{\partial\Omega_s}(\boldsymbol{\sigma}_f\boldsymbol{n})\cdot\hat{\boldsymbol{X}}_s d\boldsymbol{\tau} \\ &\quad -2\mu\textstyle\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\boldsymbol{X}_s)d\boldsymbol{x}. \end{aligned} \tag{41}$$

Substituting (41) into (40) gives

$$\begin{aligned} \textstyle\int_{\partial\Omega_s^*}(\boldsymbol{\sigma}_f\boldsymbol{n})\cdot\hat{\boldsymbol{X}}_s d\boldsymbol{\tau} &= 2\mu\textstyle\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\boldsymbol{X}_s)d\boldsymbol{x} \\ &\quad +\rho_f\tfrac{d}{dt}\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} - \rho_f\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\boldsymbol{X}_s d\boldsymbol{x} \\ &\quad -\rho_f\textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\boldsymbol{X}_s d\boldsymbol{x}. \end{aligned} \tag{42}$$

Letting $\tilde{\boldsymbol{x}}_c \equiv \boldsymbol{X}_s$ for the case $\hat{\boldsymbol{X}}_s = \boldsymbol{x}_c$ and $\tilde{\boldsymbol{x}}_s \equiv \boldsymbol{X}_s$ for the case $\hat{\boldsymbol{X}}_s = \boldsymbol{x}_s$, we can detail the terms A_1 and A_2 as

$$\begin{aligned} A_1 &= 2\mu\textstyle\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{x}}_c)d\boldsymbol{x} + \rho_f\tfrac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_c d\boldsymbol{x} \\ &\quad -\rho_f\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_c d\boldsymbol{x} - \rho_f\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{x}}_c d\boldsymbol{x}, \\ A_2 &= 2\mu\textstyle\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{x}}_s)d\boldsymbol{x} + \rho_f\tfrac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_s d\boldsymbol{x} \\ &\quad -\rho_f\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_s d\boldsymbol{x} - \rho_f\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{x}}_s d\boldsymbol{x}, \end{aligned} \tag{43}$$

where $\nabla\tilde{\boldsymbol{x}}_c = \boldsymbol{\kappa}(\nabla^2 h)\boldsymbol{x}_c$ and $\nabla\tilde{\boldsymbol{x}}_s = \boldsymbol{\kappa}(\nabla^2 h)\boldsymbol{x}_s$ with $\boldsymbol{\kappa}(\nabla^2 h)$ being a matrix depending on $\nabla^2 h$. Since $\Omega_s \subset \Omega_s^* \subseteq \Omega$, $\partial_t\boldsymbol{x}_c = \boldsymbol{u}_c$, $\partial_t\boldsymbol{x}_s = \boldsymbol{u}_c + \boldsymbol{R}\boldsymbol{\omega}\times(\boldsymbol{x}-\boldsymbol{x}_c(t))$ because $\partial_t(\boldsymbol{x}-\boldsymbol{x}_c(t)) = 0$ for $\boldsymbol{x}\in\Omega_s$, and we have proved (30), we can handle the terms $\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_c d\boldsymbol{x}$ and $\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_s d\boldsymbol{x}$ in (43).

5.1.2 Detail of A_3

We perform a similar extension as for the terms A_1 and A_2 but the difference is that we set $\hat{\boldsymbol{X}}_s = \boldsymbol{u}_s$ in Ω_s and choose $\boldsymbol{X}_s = \boldsymbol{u}_f$ on $\partial\Omega_s$. Now, the problem is that we will not be able to handle the term $\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\boldsymbol{X}_s d\boldsymbol{x}$. To fix this problem, we proceed as follows. As $\int_{\Omega_s^*}\partial_t\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} = \int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\boldsymbol{X}_s d\boldsymbol{x}$ for this extension, we can write (41) as

$$\begin{aligned} 2\textstyle\int_{\Omega_s^*}\partial_t\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \tfrac{d}{dt}\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} - \int_{\partial\Omega_s^*}|\boldsymbol{u}_f|_E^2\boldsymbol{u}_f\cdot\boldsymbol{n}d\boldsymbol{\tau}, \\ 2\textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \textstyle\int_{\partial\Omega_s^*}|\boldsymbol{u}_f|_E^2\boldsymbol{u}_f\cdot\boldsymbol{n}d\boldsymbol{\tau} \\ &\quad - \textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{u}}_s d\boldsymbol{x}, \\ \textstyle\int_{\Omega_s^*}\operatorname{div}(\boldsymbol{\sigma}_f)\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \textstyle\int_{\partial\Omega_s}(\boldsymbol{\sigma}_f\boldsymbol{n})\cdot\boldsymbol{u}_s d\boldsymbol{\tau} \\ &\quad -2\mu\textstyle\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\boldsymbol{X}_s)d\boldsymbol{x}, \end{aligned} \tag{44}$$

where $\nabla\tilde{\boldsymbol{u}}_s = (\boldsymbol{\kappa}(\nabla^2 h)\boldsymbol{u}_f)$.

Now, letting $\tilde{\boldsymbol{u}}_s \equiv \boldsymbol{X}_s$, we can detail the term A_3 as

$$\begin{aligned} A_3 &= 2\mu\textstyle\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{u}}_s)d\boldsymbol{x} + \tfrac{\rho_f}{2}\tfrac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{u}}_s d\boldsymbol{x} \\ &\quad -\tfrac{\rho_f}{2}\textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{u}}_s d\boldsymbol{x}. \end{aligned} \tag{45}$$

5.1.3 Detail of ϖ

With (39), (43), and (45), we can write ϖ defined in (15) as

$$\begin{aligned} \varpi &= k_{12}\rho_f\tfrac{d}{dt}\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_c d\boldsymbol{x} + \tfrac{\rho_f}{2}\tfrac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{u}}_s d\boldsymbol{x} \\ &\quad +k_2\rho_f\tfrac{d}{dt}\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_s d\boldsymbol{x} + \varpi^*, \end{aligned} \tag{46}$$

where

$$\begin{aligned}\varpi^* = {} & 2\mu k_{12}\textstyle\int_{\Omega_s^*} \boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{x}}_c)d\boldsymbol{x}\\ & +2\mu k_2\textstyle\int_{\Omega_s^*} \boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{x}}_s)d\boldsymbol{x}\\ & +2\mu\textstyle\int_{\Omega_s^*} \boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{u}}_s)d\boldsymbol{x}\\ & -k_{12}\rho_f\big[\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_c d\boldsymbol{x}\\ & \quad +\textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{x}}_c d\boldsymbol{x}\big]\\ & -k_2\rho_f\big[\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_s d\boldsymbol{x}\\ & \quad +\textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{x}}_s d\boldsymbol{x}\big]\\ & -\tfrac{\rho_f}{2}\textstyle\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{u}}_s d\boldsymbol{x}.\end{aligned} \tag{47}$$

We now derive the bound of ϖ^*. Due to the extensions $\tilde{\boldsymbol{x}}_c$, $\tilde{\boldsymbol{x}}_s$, and $\tilde{\boldsymbol{u}}_s$, we can use Hölder's inequality to obtain:

$$\begin{aligned}|\varpi^*|_E \le {} & 2\mu\big(|k_{12}|_E+|k_2|_E\epsilon_{11}+\vartheta(\tfrac{1}{\kappa})\big)\textstyle\int_{\Omega_s^*}\|\boldsymbol{D}(\boldsymbol{u}_f)\|_E^2 d\boldsymbol{x}\\ & +2\mu\vartheta(\tfrac{1}{\kappa})\textstyle\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^2 d\boldsymbol{x}\\ & +2\mu|k_{12}|_E\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E\|\boldsymbol{x}_c\|_E^2\\ & +4\mu|k_2|_E\vartheta(\tfrac{1}{\kappa})(\|\boldsymbol{x}_c\|_E^2+r_s^2\|\boldsymbol{\eta}\|_E^2)\\ & +\tfrac{1}{2}|k_{12}|_E\rho_f\textstyle\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^2 d\boldsymbol{x}\\ & +\tfrac{1}{2}|k_{12}|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa})\|\boldsymbol{u}_c\|_E^2\\ & +\tfrac{1}{2}|k_2|_E\rho_f\textstyle\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^2 d\boldsymbol{x}\\ & +|k_2|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa})(\|\boldsymbol{u}_c\|_E^2+r_s^2\|\boldsymbol{\omega}\|_E^2)\\ & +|k_{12}|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}\|\boldsymbol{x}_c\|_E\big(\textstyle\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^4 d\boldsymbol{x}\big)^{\frac{1}{2}}\\ & +|k_2|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}(\|\boldsymbol{x}_c\|_E\\ & +r_s\|\boldsymbol{\eta}\|_E)\big(\textstyle\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^4 d\boldsymbol{x}\big)^{\frac{1}{2}}\\ & +\tfrac{1}{2}\rho_f\vartheta(\tfrac{1}{\kappa})\big(\textstyle\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^4 d\boldsymbol{x}\big)^{\frac{1}{2}}\big(\textstyle\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^2 d\boldsymbol{x}\big)^{\frac{1}{2}},\end{aligned} \tag{48}$$

where $r_s=\sup_{\Omega_s}\|\boldsymbol{x}-\boldsymbol{x}_c(t)\|_E$ and $\vartheta(\frac{1}{\kappa})$ is an increasing function of $\frac{1}{\kappa}$, and $|\Omega_s^*|_E$ denotes the volume of Ω_s^*.

We now use the embedding $V\subset(L^6(\Omega_s^*))^3\subset(L^4(\Omega_s^*))^3$ to write (48) as

$$\begin{aligned}|\varpi^*|_E \le {} & \big(\epsilon_{11}+\epsilon_{12}|\boldsymbol{u}_f|_{\Omega_s^*}^2+\epsilon_{13}\|\boldsymbol{x}_c\|_E^2+\epsilon_{14}\|\boldsymbol{\eta}\|_E^2\big)\\ & \cdot\|\boldsymbol{u}_f\|_{\Omega_s^*}^2+\epsilon_{21}\|\boldsymbol{x}_c\|_E^2+\epsilon_{22}\|\boldsymbol{\eta}\|_E^2+\epsilon_{23}\|\boldsymbol{u}_c\|_E^2\\ & +\epsilon_{24}\|\boldsymbol{\omega}\|_E^2+\epsilon_{25}|\boldsymbol{u}_f|_{\Omega_s^*}^2,\end{aligned} \tag{49}$$

where

$$\begin{aligned}\epsilon_{11} &= c\big(2\mu\big(|k_{12}|_E+|k_2|_E+\vartheta(\tfrac{1}{\kappa})\big)+\tfrac{1}{4}\rho_f\vartheta(\tfrac{1}{\kappa})\\ &\quad +|k_2|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}+\tfrac{1}{2}|k_{12}|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}\big),\\ \epsilon_{12} &= \tfrac{1}{4}c\rho_f\vartheta(\tfrac{1}{\kappa}),\\ \epsilon_{13} &= c\big(\tfrac{1}{2}|k_2|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}+\tfrac{1}{2}|k_{12}|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}\big),\\ \epsilon_{14} &= c\tfrac{1}{2}|k_2|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}r_s^2,\\ \epsilon_{21} &= 2\mu|k_{12}|_E\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E+4\mu|k_2|_E\vartheta(\tfrac{1}{\kappa}),\\ \epsilon_{22} &= 4\mu|k_2|_E\vartheta(\tfrac{1}{\kappa})r_s^2,\\ \epsilon_{23} &= |k_2|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa})+\tfrac{1}{2}|k_{12}|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa}),\\ \epsilon_{24} &= |k_2|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa})r_s^2,\\ \epsilon_{25} &= 2\mu\big(|k_{12}|_E+\tfrac{1}{2}|k_{12}|_E\rho_f+\tfrac{1}{2}|k_2|_E\rho_f\end{aligned} \tag{50}$$

with c being the embedding constant depending on only Ω_s^*.

5.2 Convergence of the closed-loop system

With ϖ detailed by (46), we consider the following Lyapunov function candidate for the closed-loop system:

$$U=U_1+\epsilon_{01}\mathcal{E}+\tfrac{\epsilon_{02}}{2}\mathcal{E}^2+U_2, \tag{51}$$

where U_1 is given by (11), $\mathcal{E}$ is given by (28), ϵ_{01} and ϵ_{02} are positive constants to be chosen, and

$$\begin{aligned}U_2 = {} & (k_1-k_2)\rho_f\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_c d\boldsymbol{x}\\ & -\tfrac{\rho_f}{2}\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{u}}_s d\boldsymbol{x}-k_2\rho_f\textstyle\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_s d\boldsymbol{x}.\end{aligned} \tag{52}$$

Using Hölder's inequality, we can find the bound of U_2 as

$$|U_2|_E \le \epsilon_{31}\|\boldsymbol{x}_c\|_E^2+\epsilon_{32}\|\boldsymbol{\eta}\|_E^2+\epsilon_{33}\rho_f|\boldsymbol{u}_f|_{\Omega_s^*}^2. \tag{53}$$

where

$$\begin{aligned}\epsilon_{31} &= \rho_f\big(\tfrac{1}{2}|k_{12}|_E+|k_2|_E|\Omega_s^*|_E\big),\\ \epsilon_{32} &= |k_2|_E\rho_f|\Omega_s^*|_E r_s^2,\\ \epsilon_{33} &= \tfrac{1}{2}|k_{12}|_E+\tfrac{1}{4}\vartheta(\tfrac{1}{\kappa}).\end{aligned} \tag{54}$$

Since $\rho_f|\boldsymbol{u}_f|_{\Omega_s^*}^2 \le \int_\Omega\rho\|\boldsymbol{u}\|_E^2$ due to $\Omega_s^*\subset\Omega$ and definition of $\boldsymbol{u}$ in (21), we can find the bound for U as:

$$\begin{aligned}& U_1+\underline{\epsilon}_{01}\mathcal{E}+\tfrac{1}{2}\epsilon_{02}\mathcal{E}^2 \le U \le U_1+\bar{\epsilon}_{01}\mathcal{E}+\tfrac{1}{2}\epsilon_{02}\mathcal{E}^2,\\ & \alpha_0\big(\tfrac{\hat{k}_1}{2}\|\boldsymbol{x}_c\|_E^2+\tfrac{\hat{k}_2}{2}\|\boldsymbol{\eta}\|_E^2+\tfrac{1}{2}\textstyle\int_\Omega\rho\|\boldsymbol{u}\|_E^2\big)+\bar{k}_{01}\|\boldsymbol{Y}_s\|_E^2,\end{aligned} \tag{55}$$

where $\boldsymbol{Y}_s$ is defined just below (12), and we choose a sufficiently large ϵ_{01} such that

$$\begin{aligned}\underline{\epsilon}_{01} &= \min\big(\epsilon_{01}\tfrac{\hat{k}_1}{2}-\epsilon_{31},\epsilon_{01}\tfrac{\hat{k}_2}{2}-\epsilon_{32},\tfrac{1}{2}\epsilon_{01}-\epsilon_{33}\big)>0,\\ \bar{\epsilon}_{01} &= \max\big(\epsilon_{01}\tfrac{\hat{k}_1}{2}+\epsilon_{31},\epsilon_{01}\tfrac{\hat{k}_2}{2}+\epsilon_{32},\tfrac{1}{2}\epsilon_{01}+\epsilon_{33}\big)>0.\end{aligned} \tag{56}$$

Differentiating (51) along the solutions of (19), (29), using (46), and noting that $|\boldsymbol{u}_f|_{\Omega_s^*}^2\le|\boldsymbol{u}|^2$ and $\|\boldsymbol{u}_f\|_{\Omega_s^*}^2\le\|\boldsymbol{u}\|^2$ due to $\Omega_s^*\subset\Omega$ and definition of $\boldsymbol{u}$ in (21), we have

$$\begin{aligned}\tfrac{dU}{dt} &= \tfrac{dU_1}{dt}+(\epsilon_{01}+\epsilon_{02}\mathcal{E})\tfrac{d\mathcal{E}}{dt}+\tfrac{dU_2}{dt}\\ &\le -\tfrac{1}{2}\bar{k}_1\|\boldsymbol{x}_c\|_E^2-\tfrac{1}{2}\bar{k}_2\|\boldsymbol{u}_c\|_E^2-\tfrac{1}{2}\bar{k}_3\|\boldsymbol{\eta}\|_E^2-\tfrac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2\\ &\quad -\tfrac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\tfrac{\mu}{\rho_f}\textstyle\int_\Omega\rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x}+k_4 m_s\|\boldsymbol{u}_c\|_E^2\\ &\quad +k_6(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}\big)+\varpi_0,\end{aligned} \tag{57}$$

where

$$\begin{aligned}\varpi_0 = {} & -\tfrac{1}{2}\bar{k}_1\|\boldsymbol{x}_c\|_E^2-\tfrac{1}{2}\bar{k}_2\|\boldsymbol{u}_c\|_E^2-\tfrac{1}{2}\bar{k}_3\|\boldsymbol{\eta}\|_E^2-\tfrac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2\\ & -\tfrac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\tfrac{\mu}{\rho_f}\textstyle\int_\Omega\rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x}+k_4 m_s\|\boldsymbol{u}_c\|_E^2\\ & +k_6(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}\big)+\varpi^*.\end{aligned} \tag{58}$$

Substituting the bound of ϖ^* in (49) into (58), and choosing

$$\bar{k}_1\ge\epsilon_{21},\quad \bar{k}_2\ge\epsilon_{23},\quad \bar{k}_3\ge\epsilon_{22},\quad \bar{k}_4\ge\epsilon_{24}, \tag{59}$$

which is always feasible because we can choose $k_i, i=1,\cdots,6$ such that $\bar{k}_i, i=1,\cdots,4$ are as large as required, see the paragraph just under (20), and sufficiently large ϵ_{01} and ϵ_{02}, we can use the Poincaré inequality to ensure that

$$\varpi_0\le 0. \tag{60}$$

Substituting (60) in to (57) yields

$$
\begin{aligned}
\tfrac{dU}{dt} \leq & -\tfrac{1}{2}\bar{k}_1\|\boldsymbol{x}_c\|_E^2 - \tfrac{1}{2}\bar{k}_2\|\boldsymbol{u}_c\|_E^2 - \tfrac{1}{2}\bar{k}_3\|\boldsymbol{\eta}\|_E^2 - \tfrac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2 \\
& -\tfrac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\tfrac{\mu}{\rho_f}\textstyle\int_\Omega \rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x} + k_4 m_s\|\boldsymbol{u}_c\|_E^2 \\
& \quad + k_6(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}\big).
\end{aligned} \tag{61}
$$

Integrating (61) from 0 to ∞ yields

$$
\begin{aligned}
\textstyle\int_0^\infty \Big[& \tfrac{1}{2}\bar{k}_1\|\boldsymbol{x}_c\|_E^2 + \tfrac{1}{2}\bar{k}_2\|\boldsymbol{u}_c\|_E^2 + \tfrac{1}{2}\bar{k}_3\|\boldsymbol{\eta}\|_E^2 + \tfrac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2 \\
& + \tfrac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\tfrac{\mu}{\rho_f}\textstyle\int_\Omega \rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x} + k_4 m_s\|\boldsymbol{u}_c\|_E^2 \\
& + k_6(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}\big)\Big] dt \leq U(0) - U(\infty) \\
& \leq U(0).
\end{aligned} \tag{62}
$$

Since we have already proved existence of the solution of the closed-loop system consisting of (1), (5), and (17), the inequality (62) implies that

$$
\begin{aligned}
\lim_{t\to\infty} \Big[& \tfrac{1}{2}\bar{k}_1\|\boldsymbol{x}_c(t)\|_E^2 + \tfrac{1}{2}\bar{k}_2\|\boldsymbol{u}_c(t)\|_E^2 + \tfrac{1}{2}\bar{k}_3\|\boldsymbol{\eta}(t)\|_E^2 \\
& + \tfrac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2 + \tfrac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\tfrac{\mu}{\rho_f}\textstyle\int_\Omega \rho\|\boldsymbol{D}(\boldsymbol{u}(t))\|_E^2 d\boldsymbol{x} \\
& + k_4 m_s\|\boldsymbol{u}_c(t)\|_E^2 + k_6(1+\|\boldsymbol{\eta}(t)\|_E^2)\boldsymbol{\omega}(t)\cdot\boldsymbol{J}_s\boldsymbol{\omega}(t)\big)\Big] = 0,
\end{aligned} \tag{63}
$$

which shows global asymptotic stability of the closed-loop system. We now show local exponential stability of the closed-loop system, i.e. $\Upsilon(t) \leq \beta(t, \Upsilon(0))$, where $\Upsilon(t) = \|\boldsymbol{x}_c(t)\|_E^2 + \|\boldsymbol{\eta}(t)\|_E^2 + \int_\Omega \|\boldsymbol{u}(t,\boldsymbol{x})\|_E^2 d\boldsymbol{x}$, $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}_\infty$-function. When $(\boldsymbol{x}_c, \boldsymbol{u}_c, \boldsymbol{\eta}, \boldsymbol{\omega}, \int_\Omega \rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x})$ are small in magnitude (i.e., when the closed-loop system evolutes for a sufficiently long time, say $t \geq t_0$ for some $t_0 \geq 0$), we obtain from (61), 55), (11), and (28) that

$$
\tfrac{dU}{dt} \leq -c_0 U, \quad \forall\, t \geq t_0 \geq 0 \tag{64}
$$

where c_0 is a positive constant. From (64), it holds that $U(t) \leq U(t_0)e^{-c_0(t-t_0)}$, and hence local exponential stability of the closed-loop system is ensured.

We summarize the main results in the following theorem.

Theorem 5.1 *Under the initial data (9), the controls* $\boldsymbol{F}_k$*, which are obtained from (17), solves Control Objective 2.1 for all* $t \in [0,T]$*, where* T *is such that* $\Omega_s(t) \Subset \Omega$*. In particular, the closed-loop system consisting of (1), (5), and (17) has at least one weak solution, which is defined in Definition 4.1 for all* $t \in [0,T]$ *such that*

$$
\begin{aligned}
& \boldsymbol{x}_c \in \Omega,\ \boldsymbol{\eta} \in \mathcal{D}_\eta,\ \rho, h \in L^\infty(Q), \\
& \boldsymbol{u} \in L^\infty(0,T;H) \cap L^2(0,T;V),\ p \in L^2(Q),
\end{aligned} \tag{65}
$$

where $(\rho, h, \boldsymbol{u})$ *are defined in (21). Moreover, the closed-loop system is globally asymptotically and locally stable at the origin provided that there is no collision between the rigid body and the boundary of the fluid domain, i.e.,*

$$
\Upsilon(t) \leq \beta(t, \Upsilon(0)), \tag{66}
$$

where $\Upsilon(t) = \|\boldsymbol{x}_c(t)\|_E^2 + \|\boldsymbol{\eta}(t)\|_E^2 + \int_\Omega \|\boldsymbol{u}(t,\boldsymbol{x})\|_E^2 d\boldsymbol{x}$*,* $\beta(\cdot,\cdot)$ *is a class* $\mathcal{KL}_\infty$*-function, and if* $\Upsilon(t_0)$*, where* $t_0 \geq 0$*, is sufficiently small, then* $\Upsilon(t) \leq \Upsilon(t_0)e^{-c_0(t-t_0)}$*, where* c_0 *is a positive constant.*

6 Simulations

In this section, we perform a simulation to illustrate the effectiveness of the control law given by (17). We take a rectangular prism as the domain Ω with dimensions $[L_1 \times L_2 \times L_3] = [-\frac{1}{2}\pi, \frac{1}{2}\pi]\mathrm{m} \times [-\frac{1}{2}\pi, \frac{1}{2}\pi]\mathrm{m} \times [-\frac{3}{2}\pi, \frac{3}{2}\pi]\mathrm{m}$. For the fluid, we take water as the fluid with $\mu = 1.793 \times 10^{-3}$kg/ms and $\rho_f = 980$kg/m^3. For the rigid body, we take the physical shape of a rectangular prism with dimensions: $\frac{\pi}{10}$m $\times$ $\frac{\pi}{10}$m $\times$ $\frac{3\pi}{10}$m and the mass: $m_s = 10$kg, which give $\boldsymbol{J}_s = \mathrm{diag}(0.1645, 0.1645, 0.8225)$kgm^2. We approximate all the sharp corners of Ω and Ω_s by rounding them off to make $\partial\Omega$ and $\partial\Omega_f$ Lipschitz. We assume that there are six forces $\boldsymbol{F}_k, k = 1, \cdots, 6$ located at six locations $\boldsymbol{R}_k$, which are configured as

$$
\begin{aligned}
& \boldsymbol{F}_1 = \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{F}_2 = \begin{bmatrix} f_2 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{F}_3 = \begin{bmatrix} 0 \\ f_3 \\ 0 \end{bmatrix}, \boldsymbol{F}_4 = \begin{bmatrix} 0 \\ f_4 \\ 0 \end{bmatrix}, \boldsymbol{F}_5 = \begin{bmatrix} 0 \\ 0 \\ f_5 \end{bmatrix}, \boldsymbol{F}_6 = \begin{bmatrix} 0 \\ 0 \\ f_6 \end{bmatrix}, \\
& \boldsymbol{R}_1 = \begin{bmatrix} 0 \\ r_1 \\ 0 \end{bmatrix}, \boldsymbol{R}_2 = \begin{bmatrix} 0 \\ r_2 \\ 0 \end{bmatrix}, \boldsymbol{R}_3 = \begin{bmatrix} 0 \\ 0 \\ r_3 \end{bmatrix}, \boldsymbol{R}_4 = \begin{bmatrix} 0 \\ 0 \\ r_4 \end{bmatrix}, \boldsymbol{R}_5 = \begin{bmatrix} r_5 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{R}_6 = \begin{bmatrix} r_6 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned} \tag{67}
$$

Then we can write (10) as

$$
\boldsymbol{f} = \boldsymbol{Q}^{-1}\begin{bmatrix} \boldsymbol{F}_s \\ -\boldsymbol{M}_s \end{bmatrix} \tag{68}
$$

where $\boldsymbol{f} = \mathrm{col}(f_1, \cdots, f_6)$ and

$$
\boldsymbol{Q} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & r_3 & r_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_5 & r_6 \\ r_1 & r_2 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{69}
$$

The determinant $\boldsymbol{Q}$ is $\det(\boldsymbol{Q}) = r_1r_3r_5 - r_1r_3r_6 - r_1r_4r_5 - r_2r_3r_5 + r_1r_4r_6 + r_2r_3r_6 + r_2r_4r_5 - r_2r_4r_6$ and can be made nonzero to make $\boldsymbol{Q}$ invertible by a simple choice: $r_1 = -r_2 \neq 0$, $r_3 = -r_4 \neq 0$ and $r_5 = -r_6 \neq 0$. This choice yields $\det(\boldsymbol{Q}) = 8r_1r_3r_5$, which is nonzero due to $r_k \neq 0$ for all $k = 1, \cdots, 6$. In the simulations, we choose $r_1 = -r_2 = \frac{\pi}{10}$m, $r_3 = -r_4 = \frac{\pi}{10}$m, $r_5 = -r_6 = \frac{3\pi}{10}$m. The formula (68) is to calculate the individual forces $\boldsymbol{F}_k$ as $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ are given by (17). We pre-eliminate the difference between buoyancy and gravity forces before applying (17).

We will use the semi-Galerkin method to the penalized system (33) to obtain a numerical weak solution, where we approximate

$$
\boldsymbol{u}_\delta^n(t,\boldsymbol{x}) = \textstyle\sum_{l=1}^n c_l^n(t)\boldsymbol{a}_l(\boldsymbol{x}), \tag{70}
$$

where $c_l^n(t)$ are scalar functions of time, $\boldsymbol{a}_l(\boldsymbol{x})$ are eigenfunctions of the Stokes operator. We substitute (70) into the first equation of (33) and multiply it by $\boldsymbol{\xi} = \mathrm{Spann}\{\boldsymbol{a}_l(\boldsymbol{x}); l = 1, \cdots, n\}$ to obtain a system of ODEs for $c_l^n(t)$, which is numerically solvable. The transport equations (the third and fourth equations of (33)) are solved by using the characteristic method. Next, we choose the penalized parameter as $\delta = \frac{1}{n}$. We now need to derive eigenfunctions for our domain Ω. To do so, we need the following lemma [16, Theorem III.2.3].

Lemma 6.1 *If Ω is a bounded open set in $\mathbb{R}^3$ with Lipschitz boundary, then H coincides with the space of divergence free functions in $L^2(\Omega)$ such that $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$, where $\boldsymbol{n}$ is the normal unit vector to $\partial\Omega$.*

With this lemma, eigenfunctions of the Stokes problem are equivalent to those of the Laplace operator with the condition $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$ as we consider a weak solution in H. Hence, we look for $\boldsymbol{a}_l$ such that

i) $\Delta\boldsymbol{a}_l=-\lambda_l\boldsymbol{a}_l$, $\mathrm{div}(\boldsymbol{a}_l)=0$ in Ω; $\boldsymbol{a}_l\cdot\boldsymbol{n}=0$ on $\partial\Omega$,
ii) $\boldsymbol{a}_l$ is an orthornormal basis of $H(\Omega)$,
iii) $\boldsymbol{a}_l$ is an orthogonal basis of $V(\Omega)$,
iv) $0<\lambda_1\leq\lambda_2\leq\cdots$ and $\lambda_l\to\infty$ as $l\to\infty$.

A nontrivial calculation gives $\boldsymbol{a}_l$ (we neglect the round off corners), which satisfies all the above properties, as follows:

$$\boldsymbol{a}_l=\frac{\bar{a}_l}{\lambda_l}\begin{bmatrix}L_1\cos(l_1x_1)\sin(l_2x_2)\sin(l_3x_3)\\ L_2\sin(l_1x_1)\cos(l_2x_2)\sin(l_3x_3)\\ L_3\sin(l_1x_1)\sin(l_2x_{)}\cos(l_3x_3)\end{bmatrix}, \tag{71}$$

where

$$L_1=l_2^2+l_3^2+l_1(l_2-l_3),\ L_2=-(l_1^2+l_3^2+l_2(l_1+l_3)),$$
$$L_3=(l_1^2+l_2^2+l_3(l_2-l_1)),\ \bar{a}_l=\sqrt{\frac{8\lambda_l^2}{\pi^3(L_1^2+L_2^2+L_3^2)}}, \tag{72}$$
$$\lambda_l=l_1^2+l_2^2+l_3^2,$$

for $(l_1,l_2,l_3)\in\mathbb{Z}^3$ such that $l^2=l_1^2+l_2^2+l_3^2$, which are taken into account to have summing combination in calculating (70). We perform two simulations. In both simulations, we choose the control gains as follows: $k_1=0.05$, $k_4=8$, $k_2=0.1$, and $k_6=3$. This choice gives $k_3=1.825$, $k_5=0.1$, $\bar{k}_1=0.11$, $\bar{k}_2=7.5$, $\bar{k}_3=0.13$, and $\bar{k}_4=2.92$ according to (20). Clearly, the conditions in (13) and (20) hold. Moreover, we choose $n=10^8$, which gives $\delta=10^{-8}$.

In the first simulation, for the initial values of the fluid velocity we take $c_l^n(0)$ to be random number in $\frac{1}{n^2}[-1,1]$. The initial values of the rigid body are taken as $\boldsymbol{x}_c(0)=\mathrm{col}(0.2,-0.2,0.4)$m, $\boldsymbol{\eta}(0)=\mathrm{col}(1.6,0.4,2.5)$, which yields a principal axis/angle pair $\boldsymbol{e}=\mathrm{col}(0.4782,0.2050,0.8540)$ and $\gamma=4.9665$ rad. The initial values of the velocities $\boldsymbol{u}_c(0)$ and $\boldsymbol{\omega}(0)$ of the rigid body are determined via (31), (23), and the interface condition given by the second equation in (6), where $\boldsymbol{u}(0,\boldsymbol{x})$ is substituted by $\boldsymbol{u}_\delta^n(0,\boldsymbol{x})$.

The position vector $\boldsymbol{x}_c$, orientation vector $\boldsymbol{\eta}$, linear velocity vector $\boldsymbol{u}_c$, angular velocity vector $\boldsymbol{\omega}$, and H-norm of the global velocity $\int_\Omega\|\boldsymbol{u}\|_E^2d\boldsymbol{x}$ are plotted in Fig. 2. The control force vector $\boldsymbol{F}_s$, control moment vector $\boldsymbol{M}_s$, and control forces $f_k, k=1,\cdots,6$, see (68), are plotted in Fig. 3. It is seen from these figures that all the states $\boldsymbol{x}_c$, $\boldsymbol{\eta}$, $\boldsymbol{u}_c$, and $\boldsymbol{\omega}$, $|\boldsymbol{u}|=(\int_\Omega\|\boldsymbol{u}\|_E^2d\boldsymbol{x})^{\frac{1}{2}}$; and the controls $\boldsymbol{F}_s$, $\boldsymbol{M}_s$, and f_k converge to zero. It is noted that convergence of the rigid body states $\boldsymbol{x}_c$, $\boldsymbol{\eta}$, $\boldsymbol{u}_c$, and $\boldsymbol{\omega}$ to zero is affected by that of $|\boldsymbol{u}|$ due to the fluid forces and fluid moments on the rigid body.

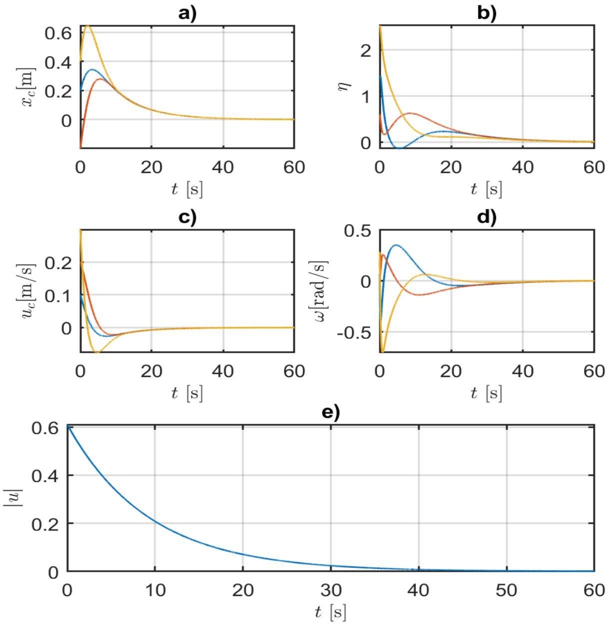

Fig. 2. First simulation - states: $\boldsymbol{x}_c$, $\boldsymbol{\eta}$, $\boldsymbol{u}_c$, $\boldsymbol{\omega}$, and $|\boldsymbol{u}|=(\int_\Omega\|\boldsymbol{u}\|_E^2d\boldsymbol{x})^{\frac{1}{2}}$.

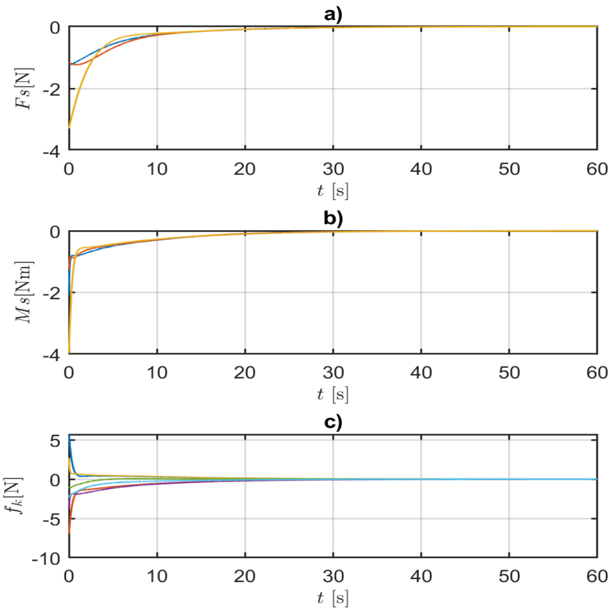

Fig. 3. First simulation - controls: $\boldsymbol{F}_{cs}$, $\boldsymbol{M}_s$, and $\boldsymbol{f}_k$.

To illustrate the robustness/performance of the proposed stabilization controller under the same control gains, we perform the second simulation with the initial values $\boldsymbol{x}_c(0)=\mathrm{col}(0.4,-0.4,0.8)$m while all other initial values and parameters are taken the same as in the first solution. Simulation results are plotted in Fig. 4 and Fig. 5. Explanation of Fig. 4 and Fig. 5 is similar to that of Fig. 2 and Fig. 3. Comparing Fig. 2 and Fig. 4; Fig. 3 and Fig. 5 shows that the proposed stabilization controller stabilizes the rigid body very well under different positions of the rigid body.

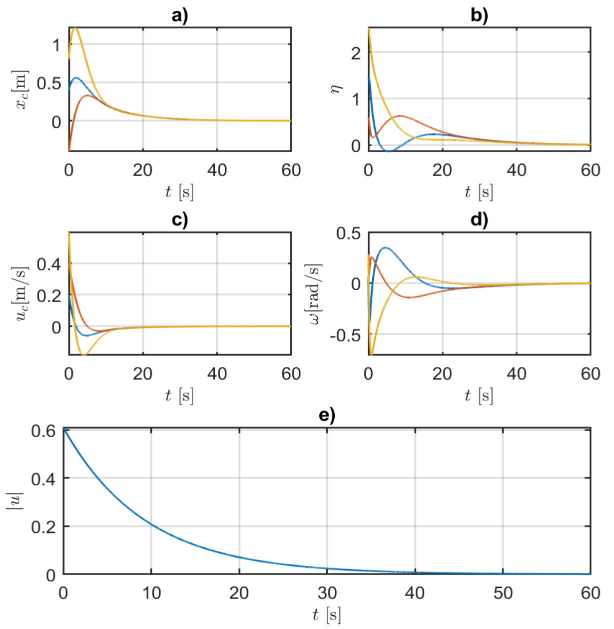

Fig. 4. Second simulation - states: $\boldsymbol{x}_c$, $\boldsymbol{\eta}$, $\boldsymbol{u}_c$, $\boldsymbol{\omega}$, and $|\boldsymbol{u}| = (\int_\Omega \|\boldsymbol{u}\|_E^2 d\boldsymbol{x})^{\frac{1}{2}}$.

7 Conclusions

Global asymptotic and local exponential stabilization of a rigid body in an incompressible viscous fluid under potential body force with the fluid velocity $\boldsymbol{u}_f(0, \boldsymbol{x}) \in H$ was solved in this paper under an assumption that there is no collision between the rigid body and the boundary of the fluid domain. Since the fluid forces and fluid moments on the rigid body are not able to bound in an Euclidean norm due to $\boldsymbol{u}_f(0, \boldsymbol{x}) \in H$, the "fluid work and fluid power" on the rigid body can be bound and should be used for stability and convergence analysis. Future work is to extend to stabilization of a rigid body in multiple fluids to cover practical cases such as floating rigid bodies.

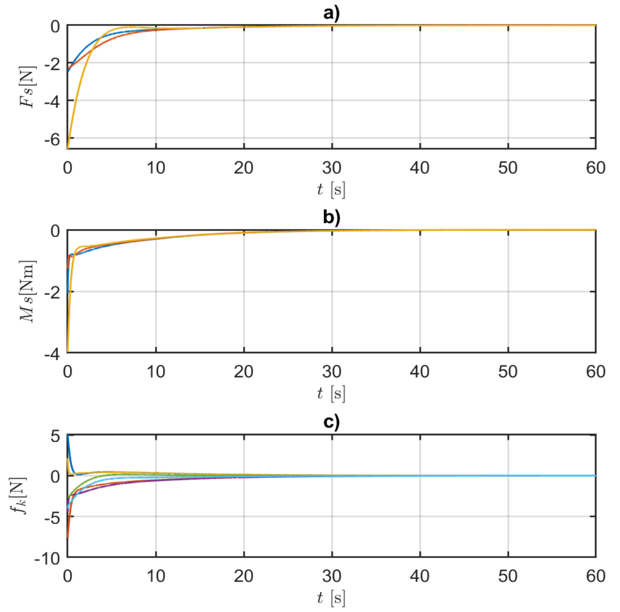

Fig. 5. First simulation - controls: $\boldsymbol{F}_{cs}$, $\boldsymbol{M}_s$, and $\boldsymbol{f}_k$.

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Stabilization of a rigid body in a viscous incompressible fluid

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Abstract

This paper addresses the problem of global asymptotic and local exponential stabilization of a rigid body inside a viscous incompressible fluid described by Navier-Stokes equations within a bounded domain in three dimensional space provided that there is no collision between the rigid body and the boundary of the fluid domain. Due to consideration of less regular initial values of the fluid velocity, the forces and moments induced by the fluid on the rigid body are not able to bound. Therefore, the paper handles "fluid work and fluid power" on the rigid body in stability and convergence analysis of the closed-loop system. The control design ensures global asymptotic and local exponential stability of the rigid body while the initial fluid velocity is not required to be small and regular but only under no collision between the rigid body and the boundary of the fluid domain.

Key words: Rigid body; Stabilization; Navier-Stokes equations; Existence; Weak solution.

1 Introduction

Stabilization of a rigid body (e.g., an ocean vehicle), in a viscous fluid has many practical applications in offshore engineering. The fluid loads on a rigid body are usually considered by an approximation approach, see [12, 14, 22, 24] and references therein. In this approach, the fluid loads on a rigid body are approximated and decoupled into two parts. The first part (related to added mass) depends on the acceleration and velocity of the rigid body. The second part depends on the fluid velocity and is considered to be bounded in the Euclidean (pointwise) norm. These approximations are overlooked from the fundamental viewpoint of the fluid-structure interaction, which can be elaborated as follows. The requirement of the first part is oversimplified because it actually depends on the fluid acceleration as shown in Section 5. The requirement of the second part to be bounded in the Euclidean norm requires a strong solution of the Navier-Stokes equations (NSEs) for a viscous incompressible fluid. Currently, existence of this strong solution is local in either time or small initial values [25] under sufficient regularity of initial values.

In comparison with the aforementioned approximation approach, a fundamental NSE approach has been recently considered. In this approach, fully coupled dynamics of both a rigid body and a fluid, where motion of the rigid body is described by nonlinear ordinary differential equations (ODEs) and motion of the fluid is described by NSEs in three dimensional space, is addressed. This approach results in much more complexities but actualities in fluid-structure interactions because we have to deal with: i) existence of an appropriate solution of the fluid and rigid body system; ii) time-varying domain of the fluid; iii) bound of the forces and moments induced by the fluid on the rigid body. There are several works related to the fundamental NSE approach. Existence of a weak solution for a system of a fluid and multiple rigid bodies was proved in [7–9, 13], where dynamics of the fluid and rigid bodies is written as a global fluid and the initial values of the fluid are assumed in $H_0^1(\Omega)$, which is the usual Sobolev space of order 1 with compact support in the domain Ω, because an estimate of the fluid acceleration in a weak form is needed for compactness argument in passing to the limit due to the nonlinear convection term. A similar result was obtained in [6, 18, 19, 27, 28] but using a coordinate transformation to handle difficulty caused by the fluid time-varying domain. In [29], a proportional and derivative control law was designed to stabilize a rigid ball in a fluid with the initial values of the fluid are also assumed in $H_0^1(\Omega)$. This allows to estimate the fluid acceleration in a weak form so that the fluid force acting on the rigid ball is bounded. Feedback stabilization of a rigid body in 1D, 2D, and 3D under similar regularity of the initial data was considered in [2–4], see also [15] for the case of stabilizing a flexible body in a fluid. Stabilization of a rigid ball in compressible fluid was considered in [26], where the global-in-time existence of strong solutions for the corresponding system under a smallness condition on the initial velocities and on the distance between the initial position of the center of the ball was proved, see also [8, 23].

In this paper, we consider the initial values of the fluid

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velocity in $H(\Omega)$, see (7) for definition of this functional space, which is less regular than $H_0^1(\Omega)$. This less regularity of the initial values of the fluid velocity will result in a global solution but will cause a major difficulty: no information on bound of the fluid loads on the rigid body because we do not have an estimate of the fluid acceleration in a weak form for both compactness argument and the bound of the fluid loads on the rigid body. To handle this difficulty, we consider the effect of the "fluid work and fluid power" (instead of the fluid forces and moments) on the rigid body, see discussion just below (39) for detail of the "fluid work and fluid power". Although uniqueness of a weak solution is neither proved nor disproved (this is also a problem for standard NSEs [30]), we show that its boundedness in appropriate norms is sufficient for ensuring global asymptotic and local exponential stability of the closed-loop system provided that there is no collision between the rigid body and the boundary of the fluid domain. Hence, in comparison with the existing works on the approximation approach [12, 14, 22, 24] our control design does not suffer from oversimplifications used in this approach, see the first paragraph of this section. In comparison with the fundamental NSE approach [6–9, 13, 18, 19, 27, 28], our work does not require existence of a strong solution because we only require the initial values of the fluid velocity in $H(\Omega)$. This results in a global solution as long as there is no collision between the rigid body and the boundary of the fluid domain.

In Section 3, a control law is designed in an appropriate form such that it can be amended to be inverse optimal [20, 21], and suitable for stability analysis of stability of the closed-loop system in Section 5. In Section 4, existence of at least one weak solution of the closed-loop system is shown via a penalization approach. In Section 5, we prove global asymptotic and local exponential stability of the closed-loop system provided that there is no collision between the rigid body and the boundary of the fluid domain. We derive the affect of the fluid on the rigid body via the "fluid work and fluid power". This enables us to consider a proper Lyapunov function for stability analysis of the closed-loop system.

Notation: Let Ω be a open bounded set in $\mathbb{R}^3$, and $T > 0$. $L^p(\Omega)$, where $1 \le p < \infty$, denotes the standard Lebesgue space of measurable p-integrable functions; $L^\infty(\Omega)$ denotes the space of essentially bounded functions; $H^1(\Omega)$ is the usual Sobolev space of order 1, see [1]; $H_0^1(\Omega)$ denotes $H^1(\Omega)$ with compact support; $L^p(0,T;X)$, where $1 \le p < \infty$ and X is a Banach space with the norm denoted by $\|\cdot\|_X$, denotes a Brochner space with the norm $\|\boldsymbol{u}\|_{L^p(0,T;X)} = (\int_0^T \|\boldsymbol{u}\|_X^p dt)^{1/p}$. We also use $\|\cdot\|_E$ to denote the Euclidean norm, i.e., $\|\boldsymbol{x}\|_E = \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ with $\boldsymbol{x}\cdot\boldsymbol{y} = \sum_i (x_i y_i)$. For a scalar, we use $|\cdot|_E$ to denote the absolute value.

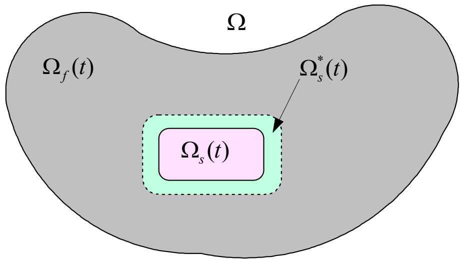

Fig. 1. Domain definition

2 Problem statement

Let $\Omega \subset \mathbb{R}^3$ be a $\mathcal{C}^1$ domain occupied by a viscous incompressible fluid surrounding a rigid body represented by $\Omega_s(t)$, which is a bounded open connected subdomain of Ω, at time t, see Fig. 1, where the domain $\Omega_s^*(t)$ is defined and used in Section 5. We assume that $\Omega_s(0) \Subset \Omega$. The fluid has density $\rho_f > 0$, dynamic viscosity $\mu > 0$, pressure p, velocity $\boldsymbol{u}_f$, and is governed by the NSEs for viscous incompressible fluids [30]:

$$\begin{array}{l}\rho_f(\partial_t\boldsymbol{u}_f+(\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f)-\mathrm{div}(\boldsymbol{\sigma}_f)=0, \text{ in } \Omega_f(t)\\ \mathrm{div}(\boldsymbol{u}_f)=0 \quad \text{in} \quad \Omega_f(t),\end{array} \tag{1}$$

where the stress tensor of the fluid $\boldsymbol{\sigma}_f$ is given by

$$\boldsymbol{\sigma}_f = 2\mu\boldsymbol{D}(\boldsymbol{u}_f)-p\boldsymbol{I}_3 \tag{2}$$

with $\boldsymbol{D}(\boldsymbol{u}_f) = \frac{1}{2}(\nabla\boldsymbol{u}_f + (\nabla\boldsymbol{u}_f)^T)$ being the rate tensor of the fluid ($\bullet^T$ denotes the transpose of $\bullet$ and it should not be confused with the time constant T), $\Omega_f(t) \subset \Omega$ being the fluid domain at time t (the boundary (interface between the fluid and the rigid body) of Ω_f depends on t), $\boldsymbol{I}_3$ being the 3×3 identity matrix, and we assume the body force is potential such as gravity, and hence is merged to pressure p.

In what follows, we briefly describe equations of motion of the rigid body in the incompressible fluid. For details of derivation, the reader is referred to [7, 13, 17].

For the rigid body, we define the mass m_s, the density $\rho_s > 0$, the vector of the center of gravity $\boldsymbol{x}_c(t)$ and its velocity vector $\boldsymbol{u}_c(t)$, the modified Rodrigues parameter vector $\boldsymbol{\eta}(t)$ representing the rigid body orientation, see [32], (this vector is related to the principal axis $\boldsymbol{e}$ and the principal angle γ through $\boldsymbol{\eta} = \boldsymbol{e}\tan(\frac{\gamma}{4})$, which is well-defined for all eigenaxis rotations in the range $[0, 2\pi)$; this range can be further relaxed by using quaternion as in [10], and should not be confused with global stability in this paper), angular velocity vector $\boldsymbol{\omega}$, the inertial matrix $\boldsymbol{J}_s$, the transformation matrix $\boldsymbol{R}$, and the velocity vector filed $\boldsymbol{u}_s$ by

$$\begin{array}{l}\rho_s = \frac{m_s}{|\Omega_s(0)|_E},\ \boldsymbol{x}_c = \frac{1}{|\Omega_s(t)|_E}\int_{\Omega_s(t)}\boldsymbol{x}d\boldsymbol{x},\\ \boldsymbol{u}_s(t,\boldsymbol{x}) = \boldsymbol{u}_c(t)+\boldsymbol{\omega}(t)\times\boldsymbol{r}(t) \text{ for } \boldsymbol{x}\in\Omega_s(t),\\ \boldsymbol{a}^T\boldsymbol{J}_s\boldsymbol{b} = \rho_s\int_{\Omega_s(0)}(\boldsymbol{a}\times\boldsymbol{r}(t))^T(\boldsymbol{b}\times\boldsymbol{r}(t))\,\forall,\ \boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3,\\ \boldsymbol{R} = \frac{1}{2}\big(\boldsymbol{I}-\boldsymbol{S}(\boldsymbol{\eta})+\boldsymbol{\eta}\boldsymbol{\eta}^T-\frac{1+\|\boldsymbol{\eta}\|_E^2}{2}\boldsymbol{I}_3\big),\end{array} \tag{3}$$

where

$$\boldsymbol{r}(t) = \boldsymbol{x}-\boldsymbol{x}_c(t), \tag{4}$$

$|\Omega_s(t)|_E$ denotes the volume of $\Omega_s(t)$, $\boldsymbol{S}(\boldsymbol{\eta})$ denotes the 3×3 skew-symmetric matrix of $\boldsymbol{\eta}$, and it holds that $\boldsymbol{\eta}\cdot\boldsymbol{R}\boldsymbol{\omega} = \frac{1}{4}(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\eta}\cdot\boldsymbol{\omega}$ for all $\boldsymbol{\eta},\boldsymbol{\omega}\in\mathbb{R}^3$. Then, equations of motion of the rigid body are given by

$$\begin{aligned}
&\tfrac{d\boldsymbol{x}_c}{dt} = \boldsymbol{u}_c,\\
&\tfrac{d\boldsymbol{\eta}}{dt} = \boldsymbol{R}\boldsymbol{\omega},\\
&m_s\tfrac{d\boldsymbol{u}_c}{dt} = \textstyle\int_{\partial\Omega_s(t)}\boldsymbol{\sigma}_f\boldsymbol{n}d\boldsymbol{\tau}+\sum_{k=1}^N\boldsymbol{F}_k,\\
&\boldsymbol{J}_s\tfrac{d\boldsymbol{\omega}}{dt} = -\boldsymbol{\omega}\times(\boldsymbol{J}_s\boldsymbol{\omega})+\textstyle\int_{\partial\Omega_s(t)}\boldsymbol{r}(t)\times(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}\\
&\qquad\qquad+\textstyle\sum_{k=1}^N\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k,
\end{aligned}\tag{5}$$

where $\boldsymbol{n}$ is the normal unit vector pointing outside of the rigid body, $\boldsymbol{F}_k$ is the control force, $\boldsymbol{r}_{F_k}$ denotes the relative position vector of $\boldsymbol{F}_k$ with respect to $\boldsymbol{x}_c(t)$ such that $\sum_{k=1}^N\boldsymbol{F}_k\in\mathbb{R}^3$ and $\sum_{k=1}^N\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k\in\mathbb{R}^3$, i.e., the rigid body is fully actuated.

We impose a homogeneous Dirichlet boundary condition at the boundary $\partial\Omega\cap\partial\Omega_f(t)$ and a continuous condition of the velocity at the interface between the rigid body and the fluid:

$$\begin{aligned}
\boldsymbol{u}_f &= 0 \text{ on } \partial\Omega\cap\partial\Omega_f(t),\\
\boldsymbol{u}_s &= \boldsymbol{u}_f \quad\text{on}\quad \partial\Omega_s(t).
\end{aligned}\tag{6}$$

In derivation of (5), we have used an interface condition that the stress is continuous in normal direction, i.e, $\boldsymbol{\sigma}_f\boldsymbol{n}=\boldsymbol{\sigma}_s\boldsymbol{n}$ on $\partial\Omega_s$ with $\boldsymbol{\sigma}_s$ being the Cauchy stress tensor, i.e., $-\boldsymbol{\sigma}_s\boldsymbol{n}$ is the force applied by the rigid body on the fluid.

From now onwards, we will drop the argument t of Ω_f, Ω_s, and $\boldsymbol{r}$ when it does not lead to a confusion. For use in the rest of the paper, we denote $Q=(0,T)\times\Omega$, and introduce the following function spaces:

$$\begin{aligned}
&V=\{\boldsymbol{v}\in H_0^1(\Omega),\operatorname{div}(\boldsymbol{v})=0\},\\
&H=\{\boldsymbol{v}\in L^2(\Omega),\operatorname{div}(\boldsymbol{v})=0,\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega}=0\},\\
&\mathcal{K}(t)=\{\boldsymbol{v}\in V,\exists(\boldsymbol{v}_v,\boldsymbol{\omega}_v)\in\mathbb{R}^3\times\mathbb{R}^3,\boldsymbol{v}|_{\Omega_s}=\boldsymbol{v}_v+\boldsymbol{\omega}_v\times\boldsymbol{r}\}.
\end{aligned}\tag{7}$$

Note that the elements of $\mathcal{K}(t)$ are given by the rigid body velocity in Ω_s. One can prove the following lemma on the space $\mathcal{K}(t)$, see [5, 31].

Lemma 2.1 *The space $\mathcal{K}(t)$ is equivalent to*

$$\mathcal{K}(t)=\{\boldsymbol{v}\in V,\ \boldsymbol{D}(\boldsymbol{v})=0 \text{ in } \Omega_s\}.\tag{8}$$

In this paper, we address the following control objective.

Control Objective 2.1 *Under the initial data:*

$$\begin{aligned}
&\Omega_s(0)\Subset\Omega,\\
&\boldsymbol{u}_f(0,\boldsymbol{x})\in H,\\
&(\boldsymbol{x}_c(0),\boldsymbol{\eta}(0),\boldsymbol{u}_c(0),\boldsymbol{\omega}(0))\in\underline{\Omega}\times\mathcal{D}_\eta\times\mathbb{R}^3\times\mathbb{R}^3,
\end{aligned}\tag{9}$$

where $\underline{\Omega}\Subset\Omega$ and $\mathcal{D}_\eta=(-\infty,\infty)^3$, design the control forces $\sum_{i=1}^k\boldsymbol{F}_i$ and control moments $\sum_{k=1}^N\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k$ to globally asymptotically and locally exponentially stabilize the rigid body at the origin provided that there is no collision between the rigid body and the boundary of the fluid domain. Moreover, the designed control forces and control moments must ensure existence of the weak solution of the closed-loop system consisting of both the rigid body and the fluid, see Definition 4.1 in Section 4 for definition of the weak solution.

Note that the initial values $\boldsymbol{x}_c(0)\in\underline{\Omega}$ is imposed to be compatible with $\Omega_s(0)\Subset\Omega$, and $\boldsymbol{\eta}(0)\in\mathcal{D}_\eta$ is made due to the domain of the modified Rodrigues parameter vector $\boldsymbol{\eta}$.

3 Control design

For convenience, we denote the control force $\boldsymbol{F}_s$ and the control moment $\boldsymbol{M}_s$ as follows

$$\begin{aligned}
\boldsymbol{F}_s &= \textstyle\sum_{k=1}^N\boldsymbol{F}_k,\\
\boldsymbol{M}_s &= \textstyle\sum_{k=1}^N\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k.
\end{aligned}\tag{10}$$

Since the rigid body dynamics (5) is of a second-order system, we consider the following Lyapunov function candidate to design the controls $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$:

$$\begin{aligned}
U_1 &= \tfrac{1}{2}\|\boldsymbol{x}_c\|_E^2+\tfrac{m_s}{2}\|k_1\boldsymbol{x}_c+\boldsymbol{u}_c\|_E^2\\
&\quad+\tfrac{1}{2}\|\boldsymbol{\eta}\|_E^2+\tfrac{1}{2}(k_2\boldsymbol{\eta}+\boldsymbol{\omega})^T\boldsymbol{J}_s(k_2\boldsymbol{\eta}+\boldsymbol{\omega}),
\end{aligned}\tag{11}$$

where k_1 and k_2 are constants to be chosen. It holds that

$$\bar{k}_{01}\|\boldsymbol{Y}_s\|_E^2\le U_1\le\bar{k}_{02}\|\boldsymbol{Y}_s\|_E^2,\tag{12}$$

where $\boldsymbol{Y}_s=\operatorname{col}(\boldsymbol{x}_c,\boldsymbol{u}_c,\boldsymbol{\eta},\boldsymbol{\omega})$, and we chose k_1 and k_2 such that

$$\begin{aligned}
\bar{k}_{01} &= \tfrac{1}{2}\min\big((1+m_sk_1^2-m_s|k_1|_E),(m_s-|k_1|_Em_s),\\
&\qquad(1+k_2^2\lambda_m(\boldsymbol{J}_s)-|k_2|_E\lambda_M(\boldsymbol{J}_s)),\\
&\qquad(\lambda_m(\boldsymbol{J}_s)-|k_2|_E\lambda_M(\boldsymbol{J}_s))\big)>0,\\
\bar{k}_{02} &= \tfrac{1}{2}\max\big((1+m_sk_1^2+m_s|k_1|_E),(m_s+|k_1|_Em_s),\\
&\qquad(1+k_2^2\lambda_M(\boldsymbol{J}_s)+|k_2|_E\lambda_M(\boldsymbol{J}_s)),\\
&\qquad(\lambda_M(\boldsymbol{J}_s)+|k_2|_E\lambda_M(\boldsymbol{J}_s))\big),
\end{aligned}\tag{13}$$

with $\lambda_m(\boldsymbol{J}_s)$ and $\lambda_M(\boldsymbol{J}_s)$ being the minimum and maximum eigenvalue of $\boldsymbol{J}_s$, respectively.

Differentiating (11) along the solutions of (5) yields

$$\begin{aligned}
\tfrac{dU_1}{dt} &= \boldsymbol{x}_c\cdot\boldsymbol{u}_c+(k_1\boldsymbol{x}_c+\boldsymbol{u}_c)\cdot\big(m_sk_1\boldsymbol{u}_c+\boldsymbol{F}_s\big)+\boldsymbol{\eta}\cdot\boldsymbol{R}\boldsymbol{\omega}\\
&\quad+(k_2\boldsymbol{\eta}+\boldsymbol{\omega})\cdot\big(k_2\boldsymbol{J}_s\boldsymbol{R}\boldsymbol{\omega}-\boldsymbol{\omega}\times(\boldsymbol{J}_s\boldsymbol{\omega})+\boldsymbol{M}_s\big)+\varpi,
\end{aligned}\tag{14}$$

where

$$\begin{aligned}
\varpi &= (k_1\boldsymbol{x}_c+\boldsymbol{u}_c)\cdot\textstyle\int_{\partial\Omega_s}\boldsymbol{\sigma}_f\boldsymbol{n}d\boldsymbol{\tau}\\
&\quad+(k_2\boldsymbol{\eta}+\boldsymbol{\omega})\cdot\textstyle\int_{\partial\Omega_s}\boldsymbol{r}\times(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}\\
&= k_{12}\textstyle\int_{\partial\Omega_s}\boldsymbol{x}_c\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}+k_2\int_{\partial\Omega_s}\boldsymbol{x}_s\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}\\
&\quad+\textstyle\int_{\partial\Omega_s}\boldsymbol{u}_s\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau},
\end{aligned}\tag{15}$$

with

$$\begin{aligned}
k_{12} &= k_1-k_2,\\
\boldsymbol{x}_s &= \boldsymbol{x}_c+\boldsymbol{\eta}\times\boldsymbol{r},
\end{aligned}\tag{16}$$

which satisfies $\operatorname{div}(\boldsymbol{x}_s)=0$.

From (14), we design the controls $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ as follows:

$$\begin{aligned}
\boldsymbol{F}_s &= -k_3\boldsymbol{x}_c-k_4(k_1\boldsymbol{x}_c+\boldsymbol{u}_c),\\
\boldsymbol{M}_s &= -k_5(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\eta}-k_6(1+\|\boldsymbol{\eta}\|_E^2)(k_2\boldsymbol{\eta}+\boldsymbol{\omega}),
\end{aligned}\tag{17}$$

where $k_i, i=3,\cdots,6$ are positive constants to be chosen. It is noted that instead of cancelling controls, we designed the controls $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ as in (17) so that they can be written in the form:

$$\begin{bmatrix}\boldsymbol{F}_s\\ \boldsymbol{M}_s\end{bmatrix} = -\boldsymbol{K}\big(\tfrac{\partial U_1}{\partial\boldsymbol{Y}_s}\big)^T,\tag{18}$$

where $\boldsymbol{K}$ is a positive definite matrix. This form means that the controls $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ are inverse pre-optimal and can be easily extended to be inverse optimal by multiplying themselves by a positive constant larger than 2, see [20,21] for extending $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ given by (17) to inverse optimal controls, and many desired properties of inverse optimal controls. However, we do not detail this issue here as we focus on the fluid loads on the rigid body in this paper.

Substituting (17) into (14) and using Young's inequality gives

$$\frac{dU_1}{dt} \leq -\bar{k}_1\|\boldsymbol{x}_c\|_E^2 - \bar{k}_2\|\boldsymbol{u}_c\|_E^2 - \bar{k}_3\|\boldsymbol{\eta}\|_E^2 - \bar{k}_4\|\boldsymbol{\omega}\|_E^2 + \varpi, \tag{19}$$

where we chose the constants $k_i, i = 1, \cdots, 6$ such that

$$\begin{aligned}
&1 + k_1^2 m_s - k_3 - 2k_1k_4 = 0,\\
&\bar{k}_1 = k_1k_3 + k_1^2 k_4 > 0,\\
&\bar{k}_2 = k_4 - k_1 m_s > 0,\\
&\tfrac{1}{2} - 2k_2k_6 + k_5 = 0,\\
&\bar{k}_3 = k_5 - \tfrac{1}{8}k_2^2\lambda_M(\boldsymbol{J}_s) + k_2^2 k_6 > 0,\\
&\bar{k}_4 = k_6 - \tfrac{1}{8}k_2^2\lambda_M(\boldsymbol{J}_s) - |k_2|_E\lambda_M(\boldsymbol{J}_s) > 0.
\end{aligned} \tag{20}$$

It is easy to see that there exist constants $k_i, i = 1, \cdots, 6$ such that all the conditions in (20) and (13) hold, and that $\bar{k}_i, i = 1, \cdots, 4$ are as large as required. From (17) and (10), we can solve for the controls $\boldsymbol{F}_k$, see Section 6.

Remark 3.1 *At this point, we cannot conclude any stability of the rigid body dynamics based on (11) and (19) because we do not have a bound on the fluid force* $\int_{\partial\Omega_s}\boldsymbol{\sigma}_f\boldsymbol{n}d\boldsymbol{\tau}$ *and the fluid moment* $\int_{\partial\Omega_s}\boldsymbol{r}\times(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}$. *This means that we cannot use Young's inequality to bound the term* ϖ *defined in (15). Therefore, we will analyze stability of the rigid body dynamics and convergence of its states in Section 5 after we show existence of the weak solution of the whole system (i.e., both the fluid and the rigid body) and consider the NSEs for the fluid separately in Section 4.*

4 Existence of a weak solution of the closed-loop system

In this section, we show the closed-loop system consisting of the fluid (1) and the rigid body (5) with $\boldsymbol{F}_k$ obtained from (17) and (10) has at least one weak solution.

4.1 Formulation of a weak solution

We define the characteristic function $\chi_{\Omega_s}(t,\boldsymbol{x})$ on Ω_s, a global velocity $\boldsymbol{u}$, and a global density $\rho(t,\boldsymbol{x})$, and assign a function $h(t,\boldsymbol{x})$ to $\chi_{\Omega_s}(t,\boldsymbol{x})$ (for simplicity of presentation and for penalization purpose later) as follows:

$$\begin{aligned}
&\chi_{\Omega_s}(t,\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \Omega_s \\ 0 & \text{otherwise,} \end{cases}\\
&\boldsymbol{u} = \begin{cases} \boldsymbol{u}_f & \text{in } \Omega_f \\ \boldsymbol{u}_s & \text{in } \Omega_s, \end{cases}\\
&\rho(t,\boldsymbol{x}) = \rho_s\chi_{\Omega_s}(t,\boldsymbol{x}) + \rho_f(1 - \chi_{\Omega_s}(t,\boldsymbol{x})),\\
&h(t,\boldsymbol{x}) = \chi_{\Omega_s}(t,\boldsymbol{x}).
\end{aligned} \tag{21}$$

Clearly, we have

$$\Omega_s(t) = \{\boldsymbol{x} \in \Omega, h(t,\boldsymbol{x}) = 1\}. \tag{22}$$

Let us denote the initial data for those variables defined in (21) as

$$\begin{aligned}
&\boldsymbol{u}_0 := \boldsymbol{u}(0,\boldsymbol{x}) = \begin{cases} \boldsymbol{u}_f(0,\boldsymbol{x}) \text{ in } \Omega_f(0) \\ \boldsymbol{u}_s(0,\boldsymbol{x}) \text{ in } \Omega_s(0). \end{cases}\\
&\rho_0 := \rho(0,\boldsymbol{x}) = \rho_s\chi_{\Omega_s}(0,\boldsymbol{x}) + \rho_f(1 - \chi_{\Omega_s}(0,\boldsymbol{x})),\\
&h_0 := h(0,\boldsymbol{x}) = \chi_{\Omega_s(0)}(0,\boldsymbol{x}).
\end{aligned} \tag{23}$$

Then, $h(t,\boldsymbol{x})$ and $\rho(t,\boldsymbol{x})$ satisfy transport equations [7] in a weak form as

$$\begin{aligned}
&\forall\psi \in \mathcal{C}^1(Q), \psi(T) = 0:\\
&\int_0^T\int_\Omega\left(h\frac{\partial\psi}{\partial t} + h\boldsymbol{u}\cdot\nabla\psi\right)d\boldsymbol{x}dt + \int_\Omega h_0\psi(0) = 0,\\
&\int_0^T\int_\Omega\left(\rho\frac{\partial\psi}{\partial t} + \rho\boldsymbol{u}\cdot\nabla\psi\right)d\boldsymbol{x}dt + \int_\Omega \rho_0\psi(0) = 0.
\end{aligned} \tag{24}$$

Next, we perform tedious but straightforward calculations from (1) and (5) to obtain

$$\begin{aligned}
&\frac{d\boldsymbol{x}_c}{dt} = \boldsymbol{u}_c,\\
&\frac{d\boldsymbol{\eta}}{dt} = \boldsymbol{R}\boldsymbol{\omega},\\
&\forall\boldsymbol{\xi} \in H^1(Q)\cap L^2(0,T;\mathcal{K}(t)):\\
&\frac{d}{dt}\int_\Omega\rho\boldsymbol{u}\cdot\boldsymbol{\xi}d\boldsymbol{x} = \int_\Omega\big[\rho\boldsymbol{u}\cdot\partial_t\boldsymbol{\xi} + (\rho\boldsymbol{u}\otimes\boldsymbol{u} - 2\mu\boldsymbol{D}(\boldsymbol{u})) : \boldsymbol{D}(\boldsymbol{\xi})\\
&\qquad + \textstyle\sum_{k=1}^N(\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k)\cdot\boldsymbol{\xi}\big]d\boldsymbol{x},
\end{aligned} \tag{25}$$

where $\boldsymbol{\delta}_{Fk}$ denotes the diagonal matrix of the dirac delta function of $\boldsymbol{x} - \boldsymbol{x}_{Fk}$ with $\boldsymbol{x}_{Fk}$ being the position of the force $\boldsymbol{F}_k$, i.e., $\boldsymbol{x}_{Fk} = \boldsymbol{x}_c + \boldsymbol{r}_{Fk}$. We have included the first two equations in (25) for convenience of referring later. In derivation of (25), we have multiplied the first equation in (1) by $\boldsymbol{\xi} \in H^1(Q)\cap L^2(0,T;\mathcal{K}(t))$, then calculated $\frac{d}{dt}\int_{\Omega_s}\rho\boldsymbol{u}\cdot\boldsymbol{\xi}d\boldsymbol{x}$, and noted that

$$\begin{aligned}
\boldsymbol{v}_\xi\cdot\boldsymbol{F}_s + \boldsymbol{\omega}_\xi\cdot\boldsymbol{M}_s &= \textstyle\sum_{i=1}^N(\boldsymbol{v}_{\boldsymbol{\xi}}\cdot\boldsymbol{F}_k + \boldsymbol{\omega}_{\boldsymbol{\xi}}\cdot(\boldsymbol{r}_{Fk}\times\boldsymbol{F}_k))\\
&= \textstyle\sum_{i=1}^N\int_{\Omega_s}(\boldsymbol{v}_{\boldsymbol{\xi}}\cdot\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k + \boldsymbol{\omega}_{\boldsymbol{\xi}}\cdot(\boldsymbol{r}_{Fk}\times\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k))d\boldsymbol{x}\\
&= \textstyle\sum_{i=1}^N\int_{\Omega}(\boldsymbol{v}_{\boldsymbol{\xi}}\cdot\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k + \boldsymbol{\omega}_{\boldsymbol{\xi}}\cdot(\boldsymbol{r}_{Fk}\times\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k))d\boldsymbol{x}\\
&= \textstyle\sum_{i=1}^N\int_\Omega\boldsymbol{\xi}\cdot\boldsymbol{\delta}_{Fk}\boldsymbol{F}_k d\boldsymbol{x}.
\end{aligned} \tag{26}$$

Now, setting $\psi = h^q$ with $q \geq 1$ in the first equation of (24) and $\psi = \rho^q$ with $q \geq 1$ in the second equation of (24) yields

$$\rho, h \in \mathcal{C}(0,T;L^q(\Omega)), \quad \forall\, q \geq 1. \tag{27}$$

Moreover, setting $\boldsymbol{\xi} = \boldsymbol{u}$ in (26) and considering the "energy" $\mathcal{E}$ defined by

$$\mathcal{E} = \frac{\hat{k}_1}{2}\|\boldsymbol{x}_c\|_E^2 + 2\hat{k}_2\|\boldsymbol{\eta}\|_E^2 + \frac{1}{2}\int_\Omega\rho\|\boldsymbol{u}\|_E^2, \tag{28}$$

where $\hat{k}_1 = k_3 + k_1k_4 > 0$ and $\hat{k}_2 = k_5 + k_2k_6 > 0$ under conditions (20). Differentiating (28) along the solutions of (25), we formally obtain:

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= -2\mu\int_\Omega\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x} - k_4 m_s\|\boldsymbol{u}_c\|_E^2\\
&\quad - k_6(1 + \|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}.
\end{aligned} \tag{29}$$

From (27), (28), and (29), we can derive that

$$\begin{aligned}
&\boldsymbol{x}_c \in L^\infty(0,T;\mathbb{R}^3),\ \boldsymbol{\eta} \in L^\infty(0,T;\mathbb{R}^3),\\
&\boldsymbol{u} \in L^\infty(0,T;H)\cap L^2(0,T;V).
\end{aligned} \tag{30}$$

The above derivations motivate the following weak solution definition.

Definition 4.1 *Under the initial data (9), where the initial data of global variables are given in (23),* $(\boldsymbol{x}_c(t), \boldsymbol{\eta}(t), \rho(t,\boldsymbol{x}), \boldsymbol{u}(t,\boldsymbol{x}), h(t,\boldsymbol{x}))$ *is a weak solution of the closed-loop system consisting of (1), (5), and (17) if they satisfy (24), (25), (27), and (30).*

The above definition of a weak solution is similar to a definition of a weak solution to standard nonhomogeneous NSEs except for the fact that we take the constraint of rigidity into account. On the other hand, the transport equation on ρ can be deduced from that of h and the definition of ρ_0; however h is added for passing the passage to the limit in the problem of penalization.

4.2 Penalized system

In order to find $(\boldsymbol{x}_c(t), \boldsymbol{\eta}(t), \rho(t,\boldsymbol{x}), \boldsymbol{u}(t,\boldsymbol{x}), h(t,\boldsymbol{x}))$ such that they satisfy Definition 4.1, we use a penalization approach. Due to Lemma 2.1, there are two methods to penalize the rigidity constraint: i) penalizing the difference between the fluid velocity and rigid body velocity [5]; ii) penalizing the spatial derivative of the rigid body velocity [31]. We use the first method due to its several advantages (such as simpler calculations in proof, and numerical computation) over the second method.

Let $\delta > 0$ be a penalized parameter. Given the initial data

$$\begin{aligned}&\boldsymbol{x}_{c\delta}(0) = \boldsymbol{x}_c(0),\ \boldsymbol{\eta}_\delta(0) = \boldsymbol{\eta}(0),\ \boldsymbol{u}_\delta(0) = \boldsymbol{u}_0,\\ &\rho_\delta(0) = \rho_0,\ h_\delta(0) = h_0,\end{aligned} \tag{31}$$

we wish to find $(\boldsymbol{x}_{c\delta}(t), \boldsymbol{\eta}_\delta(t), \rho_\delta(t,\boldsymbol{x}), \boldsymbol{u}_\delta(t,\boldsymbol{x}), h_\delta(t,\boldsymbol{x}), p_\delta(t,\boldsymbol{x}))$ such that they satisfy

$$\begin{aligned}&\boldsymbol{x}_{c\delta}, \boldsymbol{\eta}_\delta \in L^\infty(0,T;\mathbb{R}^3);\ \rho_\delta, h_\delta \in L^\infty(Q);\\ &\boldsymbol{u}_\delta \in L^\infty(0,T;H)\cap L^2(0,T;V);\ p_\delta \in L^2(Q),\end{aligned} \tag{32}$$

and are a solution to the penalized system:

$$\begin{cases}\rho_\delta(\partial_t\boldsymbol{u}_\delta + (\boldsymbol{u}_\delta\cdot\nabla)\boldsymbol{u}_\delta) - \operatorname{div}(\boldsymbol{\sigma}_\delta)\\ \quad + \frac{1}{\delta}\rho_\delta h_\delta(\boldsymbol{u}_\delta - \boldsymbol{u}_{\delta,s}) - \sum_{k=1}^N(\boldsymbol{\delta}_k\boldsymbol{F}_k) = 0,\\ \operatorname{div}(\boldsymbol{u}_\delta) = 0, & \text{in } Q\\ \partial_t\rho_\delta + \boldsymbol{u}_\delta\cdot\nabla\rho_\delta = 0,\\ \partial_t h_\delta + \boldsymbol{u}_{\delta,s}\cdot\nabla h_\delta = 0,\end{cases}$$
$$\begin{cases}\frac{d\boldsymbol{x}_{c\delta}}{dt} = \boldsymbol{u}_{c\delta},\\ \frac{d\boldsymbol{\eta}_\delta}{dt} = \boldsymbol{R}_\delta\boldsymbol{\omega}_\delta,\end{cases} \quad \text{in } \mathbb{R}^3 \tag{33}$$

where

$$\begin{aligned}&\boldsymbol{\sigma}_\delta = 2\mu\boldsymbol{D}(\boldsymbol{u}_\delta) - p_\delta\boldsymbol{I}_3,\\ &\boldsymbol{u}_{\delta,s} = \frac{1}{m_\delta}\int_\Omega \rho_\delta\boldsymbol{u}_\delta h_\delta d\boldsymbol{x} + \big(\boldsymbol{J}_\delta^{-1}\int_\Omega \rho_\delta\boldsymbol{r}_\delta\times\boldsymbol{u}_\delta h_\delta d\boldsymbol{x}\big)\times\boldsymbol{r}_\delta,\\ &\boldsymbol{R}_\delta = \frac{1}{2}\big(\boldsymbol{I} - \boldsymbol{S}(\boldsymbol{\eta}_\delta) + \boldsymbol{\eta}_\delta\boldsymbol{\eta}_\delta^T - \frac{1+\|\boldsymbol{\eta}_\delta\|_E^2}{2}\boldsymbol{I}_3\big),\end{aligned} \tag{34}$$

with

$$\begin{aligned}&\boldsymbol{r}_\delta = \boldsymbol{x} - \boldsymbol{x}_{c\delta},\ \boldsymbol{x}_{c\delta} = \frac{1}{m_\delta}\int_\Omega \rho_\delta h_\delta\boldsymbol{x}d\boldsymbol{x},\\ &m_\delta = \int_\Omega \rho_\delta h_\delta d\boldsymbol{x},\\ &\boldsymbol{J}_\delta = \int_\Omega \rho_\delta(\|\boldsymbol{r}_\delta\|_E^2\boldsymbol{I}_3 - \boldsymbol{r}_\delta\otimes\boldsymbol{r}_\delta)h_\delta d\boldsymbol{x}\end{aligned} \tag{35}$$

In addition, we impose the homogeneous Dirichlet boundary condition:

$$\boldsymbol{u}_\delta = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{36}$$

and define $\Omega_s^\delta = \{\boldsymbol{x} \in \Omega_s, h_\delta(t,\boldsymbol{x}) = 1\}$. Several observations on the penalized system (33) are given in the following remark.

Remark 4.1

- *The penalized system (33) is based on [5], where action of the controls* $\boldsymbol{F}_s$ *and* $\boldsymbol{M}_s$ *given by (17) are taken into account.*
- *It is clear that* $|\Omega_s^\delta|_E = \int_\Omega h_\delta d\boldsymbol{x} = |\Omega_s(0)|_E$ *because* $\boldsymbol{u}_{\delta,s}$ *is divergence free and* h_δ *vanishes on* $\partial\Omega$ *as we assume there is no collision between the rigid body and* $\partial\Omega$. *We also have*
$$\begin{aligned}&m_\delta = \int_{\Omega_s^\delta}\rho_\delta d\boldsymbol{x},\\ &\boldsymbol{J}_\delta = \int_{\Omega_s^\delta}\rho_\delta(\|\boldsymbol{r}_\delta\|_E^2\boldsymbol{I}_3 - \boldsymbol{r}_\delta\otimes\boldsymbol{r}_\delta)d\boldsymbol{x}.\end{aligned} \tag{37}$$
Hence, m_δ *is positive (as* $m_\delta \geq \min(\rho_f,\rho_s)\int_{\Omega_s^\delta}d\boldsymbol{x}$*) and* $\boldsymbol{J}_\delta$ *is positive definite (as* $\boldsymbol{a}\cdot\boldsymbol{J}_\delta\boldsymbol{a} = \int_{\Omega_s^\delta}\rho_\delta\|\boldsymbol{r}_\delta\times\boldsymbol{a}\|_E^2 d\boldsymbol{x} \geq \min(\rho_f,\rho_s)\int_{\Omega_s^\delta}\|\boldsymbol{r}_\delta\times\boldsymbol{a}\|_E^2 d\boldsymbol{x}$ *for all* $\boldsymbol{a} \in \mathbb{R}^3\setminus\{0\}$*).*
- *In the first equation of (33), the term* $\boldsymbol{u}_{\delta,s}$ *defined in (34) is the projection of* $\boldsymbol{u}_\delta$ *onto the velocity fields which are rigid on* Ω_s^δ *because one can prove that, see [5]:*
$$\int_\Omega \rho_\delta h_\delta(\boldsymbol{u}_\delta - \boldsymbol{u}_{\delta,s})\cdot\boldsymbol{\zeta}d\boldsymbol{x} = 0, \tag{38}$$
where $\boldsymbol{\zeta}$ *is a rigid velocity field, i.e., there exist* $(\boldsymbol{v}_\zeta, \boldsymbol{\omega}_\zeta) \in \mathbb{R}^3\times\mathbb{R}^3$ *such that* $\boldsymbol{\zeta} = \boldsymbol{v}_\zeta + \boldsymbol{\omega}_\zeta\times\boldsymbol{r}(t,\boldsymbol{x})$, *and* $\boldsymbol{u}_{\delta,s}$ *is given by (34). Hence, the penalized term* $\frac{1}{\delta}\rho_\delta h_\delta(\boldsymbol{u}_\delta - \boldsymbol{u}_{\delta,s})$ *in the first equation of (33) is the difference between* $\boldsymbol{u}_\delta$ *and its projection onto rigid velocity fields in the rigid body domain, i.e.,* $\boldsymbol{u}_{\delta,s}$.
- *The density is transported with the velocity field* $\boldsymbol{u}_\delta$. *This eases calculations in estimating bounds for the penalized system.*

Existence of a weak solution $(\boldsymbol{x}_{c\delta}(t), \boldsymbol{\eta}_\delta(t), \rho_\delta(t,\boldsymbol{x}), \boldsymbol{u}_\delta(t,\boldsymbol{x}), h_\delta(t,\boldsymbol{x}), p_\delta(t,\boldsymbol{x}))$ to the penalized system (33) is stated in the following lemma.

Lemma 4.1 *There is at least one weak solution* $(\boldsymbol{x}_{c\delta}(t), \boldsymbol{\eta}_\delta(t), \rho_\delta(t,\boldsymbol{x}), \boldsymbol{u}_\delta(t,\boldsymbol{x}), h_\delta(t,\boldsymbol{x}), p_\delta(t,\boldsymbol{x}))$ *to the penalized system (33) that satisfies (32) for all* $t \in [0,T]$, *where* T *is such that* $\Omega_s^\delta(t) \Subset \Omega$ *and* $\boldsymbol{\eta}_\delta(t) \in \mathcal{D}_\eta$ *for all* $t \in [0,T]$.

Proof. Proof of this lemma principally follows the part of a priori estimates and convergence arguments in proof of Theorem 2.1 in [5]. The only main difference is that a priori estimates should use the penalized energy $\mathcal{E}_\delta$ as in (28) with $(\boldsymbol{x}_c, \boldsymbol{\eta}, \boldsymbol{u})$ being substituted by $(\boldsymbol{x}_{c\delta}, \boldsymbol{\eta}_\delta, \boldsymbol{u}_\delta)$. This is due to inclusion of $(\boldsymbol{x}_c, \boldsymbol{\eta})$ and controls $(\boldsymbol{F}_s, \boldsymbol{M}_s)$ in this paper.

4.3 Existence of a weak solution

Having obtained a weak solution of the penalized system (33) in Lemma 4.1, existence of a weak solution stated in Definition 4.1 is given in the following theorem.

Theorem 4.1 *Let* $(\boldsymbol{x}_{c\delta}(t), \boldsymbol{\eta}_\delta(t), \rho_\delta(t,\boldsymbol{x}), \boldsymbol{u}_\delta(t,\boldsymbol{x}), h_\delta(t,\boldsymbol{x}), p_\delta(t,\boldsymbol{x}))$ *be a weak solution to the penalized system (33). Then, under the initial data (23) and (9), there exists a subsequence of* $(\boldsymbol{x}_{c\delta}(t), \boldsymbol{\eta}_\delta(t), \rho_\delta(t,\boldsymbol{x}), \boldsymbol{u}_\delta(t,\boldsymbol{x}),$

$h_\delta(t,\boldsymbol{x})$) *such that* $\boldsymbol{x}_{c\delta} \to \boldsymbol{x}_c$ *strongly in* $L^\infty(Q)$; $\boldsymbol{\eta}_\delta \to \boldsymbol{\eta}$ *strongly in* $L^\infty((0,T)\times\mathcal{D}_\eta)$; $\rho_\delta \to \rho$, $h_\delta \to h$ *strongly in* $\mathcal{C}(0,T;L^q(\Omega))$; $\boldsymbol{u}_\delta \to \boldsymbol{u}$ *strongly in* $L^2(Q)$ *and weakly in* $L^\infty(0,T;H)\cap L^2(0,T;V)$ *such that* $(\boldsymbol{x}_c,\boldsymbol{\eta},\rho,h,\boldsymbol{u})$ *is a weak solution of the closed-loop system consisting of (1), (5), and (17) as defined in Definition 4.1. The constant* T *is such that* $\Omega_s(t) \Subset \Omega$ *and* $\boldsymbol{\eta}(t)\in\mathcal{D}_\eta$ *for all* $t\in[0,T]$.

Proof. Proof of this theorem can be readily obtained from that of Theorem 2.1 in [5] with a note as in the proof of Lemma 4.1.

5 Stability and convergence of the closed-loop system

This section provides stability and convergence analysis of the closed-loop system, which can be based on (11), (19), (28), and (29) once we handle the term ϖ in (15).

5.1 Detail of ϖ

Since we already showed existence of a weak solution of the closed-loop system (including both the fluid and rigid body) in Theorem 4.1, the idea to handle the term ϖ is to multiply the first equation in (1) by appropriate test functions to detail the terms:

$$\begin{aligned} A_1 &= \int_{\partial\Omega_s} \boldsymbol{x}_c\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau},\\ A_2 &= \int_{\partial\Omega_s} \boldsymbol{x}_s\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau},\\ A_3 &= \int_{\partial\Omega_s} \boldsymbol{u}_s\cdot(\boldsymbol{\sigma}_f\boldsymbol{n})d\boldsymbol{\tau}, \end{aligned} \tag{39}$$

where $\boldsymbol{u}_s$ is defined in (3) and $\boldsymbol{x}_s$ is defined in (16). We refer A_1 and A_2 to as fluid work as they are products of the fluid force $(\boldsymbol{\sigma}_f\boldsymbol{n})$ with displacements $\boldsymbol{x}_c$ and $\boldsymbol{x}_s$, and A_3 to as fluid power as it is a product of the fluid force with velocity $\boldsymbol{u}_s$.

5.1.1 Detail of A_1 and A_2

We define the domain $\Omega_s^*(t)$, where the argument t of Ω_s^* is dropped for clarity henceforth, such that $\Omega_s \subset \Omega_s^* \subseteq \Omega$, and the minimum distance between $\partial\Omega_s$ and $\partial\Omega_s^*$ denoted by $\kappa = \inf_{t\geq 0}\operatorname{dist}(\partial\Omega_s,\partial\Omega_s^*)$ is strictly positive, see Fig. 1. There exists Ω_s^* such that this κ is strictly positive because we assumed $\Omega_s \Subset \Omega$. Let $\hat{\boldsymbol{X}}_s(t,\boldsymbol{x})\in L^\infty(0,T;E)$, which represents either $\boldsymbol{x}_c$ or $\boldsymbol{x}_s$, we can extend $\hat{\boldsymbol{X}}_s(t,\boldsymbol{x})$ to $\boldsymbol{X}_s(t,\boldsymbol{x})$ in Ω_s^* such that $\boldsymbol{X}_s = 0$ on $\partial\Omega_s^*$ and $\operatorname{div}(\boldsymbol{X}_s)=0$ in Ω_s^* using the smooth step function introduced in [11] as follows. Let $h(t,\boldsymbol{x})$ be the smooth step function extended to three dimensional space such that $\nabla\times h = 0$ on $\partial\Omega_s^*$ and $\nabla\times h = \hat{\boldsymbol{X}}_s$ on $\partial\Omega_s$. Then, $\boldsymbol{X}_s$ can be defined as $\boldsymbol{X}_s = \hat{\boldsymbol{X}}$ in Ω_s, and $\boldsymbol{X}_s = \nabla\times h$ in Ω_s^*. It is clear that $\operatorname{div}(\boldsymbol{X}_s)=0$ because $\operatorname{div}(\hat{\boldsymbol{X}})=0$ and $\operatorname{div}(\nabla\times h)=0$.

Now, multiplying the first equation in (1) by $\boldsymbol{X}_s$ and integrating over Ω_s^* yields

$$\begin{aligned} &\rho_f\int_{\Omega_s^*}\partial_t\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} + \rho_f\int_{\Omega_s^*}(\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x}\\ &-\int_{\Omega_s^*}\operatorname{div}(\boldsymbol{\sigma}_f)\cdot\boldsymbol{X}_s d\boldsymbol{x} = 0. \end{aligned} \tag{40}$$

Using integration by parts, the boundary condition $\boldsymbol{X}_s = 0$ on $\partial\Omega_s^*$, and the interface condition given by the second equation in (6), we have

$$\begin{aligned} \int_{\Omega_s^*}\partial_t\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} - \int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\boldsymbol{X}_s d\boldsymbol{x}\\ &\quad -\int_{\partial\Omega_s}(\boldsymbol{u}_f\cdot\boldsymbol{X}_s)\boldsymbol{u}_f\cdot\boldsymbol{n}d\boldsymbol{\tau},\\ \int_{\Omega_s^*}(\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \int_{\partial\Omega_s}(\boldsymbol{u}_f\cdot\boldsymbol{X}_s)\boldsymbol{u}_f\cdot\boldsymbol{n}d\boldsymbol{\tau}\\ &\quad -\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\boldsymbol{X}_s d\boldsymbol{x},\\ \int_{\Omega_s^*}\operatorname{div}(\boldsymbol{\sigma}_f)\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \int_{\partial\Omega_s}(\boldsymbol{\sigma}_f\boldsymbol{n})\cdot\hat{\boldsymbol{X}}_s d\boldsymbol{\tau}\\ &\quad -2\mu\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\boldsymbol{X}_s)d\boldsymbol{x}. \end{aligned} \tag{41}$$

Substituting (41) into (40) gives

$$\begin{aligned} \int_{\partial\Omega_s^*}(\boldsymbol{\sigma}_f\boldsymbol{n})\cdot\hat{\boldsymbol{X}}_s d\boldsymbol{\tau} &= 2\mu\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\boldsymbol{X}_s)d\boldsymbol{x}\\ &\quad +\rho_f\frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} - \rho_f\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\boldsymbol{X}_s d\boldsymbol{x}\\ &\quad -\rho_f\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\boldsymbol{X}_s d\boldsymbol{x}. \end{aligned} \tag{42}$$

Letting $\tilde{\boldsymbol{x}}_c \equiv \boldsymbol{X}_s$ for the case $\hat{\boldsymbol{X}}_s = \boldsymbol{x}_c$ and $\tilde{\boldsymbol{x}}_s \equiv \boldsymbol{X}_s$ for the case $\hat{\boldsymbol{X}}_s = \boldsymbol{x}_s$, we can detail the terms A_1 and A_2 as

$$\begin{aligned} A_1 &= 2\mu\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{x}}_c)d\boldsymbol{x} + \rho_f\frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_c d\boldsymbol{x}\\ &\quad -\rho_f\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_c d\boldsymbol{x} - \rho_f\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{x}}_c d\boldsymbol{x},\\ A_2 &= 2\mu\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{x}}_s)d\boldsymbol{x} + \rho_f\frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_s d\boldsymbol{x}\\ &\quad -\rho_f\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_s d\boldsymbol{x} - \rho_f\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{x}}_s d\boldsymbol{x}, \end{aligned} \tag{43}$$

where $\nabla\tilde{\boldsymbol{x}}_c = \boldsymbol{\kappa}(\nabla^2 h)\boldsymbol{x}_c$ and $\nabla\tilde{\boldsymbol{x}}_s = \boldsymbol{\kappa}(\nabla^2 h)\boldsymbol{x}_s$ with $\boldsymbol{\kappa}(\nabla^2 h)$ being a matrix depending on $\nabla^2 h$. Since $\Omega_s \subset \Omega_s^* \subseteq \Omega$, $\partial_t\boldsymbol{x}_c = \boldsymbol{u}_c$, $\partial_t\boldsymbol{x}_s = \boldsymbol{u}_c + \boldsymbol{R\omega}\times(\boldsymbol{x}-\boldsymbol{x}_c(t))$ because $\partial_t(\boldsymbol{x}-\boldsymbol{x}_c(t)) = 0$ for $\boldsymbol{x}\in\Omega_s$, and we have proved (30), we can handle the terms $\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_c d\boldsymbol{x}$ and $\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_s d\boldsymbol{x}$ in (43).

5.1.2 Detail of A_3

We perform a similar extension as for the terms A_1 and A_2 but the difference is that we set $\hat{\boldsymbol{X}}_s = \boldsymbol{u}_s$ in Ω_s and choose $\boldsymbol{X}_s = \boldsymbol{u}_f$ on $\partial\Omega_s$. Now, the problem is that we will not be able to handle the term $\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\boldsymbol{X}_s d\boldsymbol{x}$. To fix this problem, we proceed as follows. As $\int_{\Omega_s^*}\partial_t\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} = \int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\boldsymbol{X}_s d\boldsymbol{x}$ for this extension, we can write (41) as

$$\begin{aligned} 2\int_{\Omega_s^*}\partial_t\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} - \int_{\partial\Omega_s^*}|\boldsymbol{u}_f|_E^2\boldsymbol{u}_f\cdot\boldsymbol{n}d\boldsymbol{\tau},\\ 2\int_{\Omega_s^*}(\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \int_{\partial\Omega_s^*}|\boldsymbol{u}_f|_E^2\boldsymbol{u}_f\cdot\boldsymbol{n}d\boldsymbol{\tau}\\ &\quad -\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{u}}_s d\boldsymbol{x},\\ \int_{\Omega_s^*}\operatorname{div}(\boldsymbol{\sigma}_f)\cdot\boldsymbol{X}_s d\boldsymbol{x} &= \int_{\partial\Omega_s}(\boldsymbol{\sigma}_f\boldsymbol{n})\cdot\boldsymbol{u}_s d\boldsymbol{\tau}\\ &\quad -2\mu\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\boldsymbol{X}_s)d\boldsymbol{x}, \end{aligned} \tag{44}$$

where $\nabla\tilde{\boldsymbol{u}}_s = (\boldsymbol{\kappa}(\nabla^2 h)\boldsymbol{u}_f)$.

Now, letting $\tilde{\boldsymbol{u}}_s \equiv \boldsymbol{X}_s$, we can detail the term A_3 as

$$\begin{aligned} A_3 &= 2\mu\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{u}}_s)d\boldsymbol{x} + \frac{\rho_f}{2}\frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{u}}_s d\boldsymbol{x}\\ &\quad -\frac{\rho_f}{2}\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{u}}_s d\boldsymbol{x}. \end{aligned} \tag{45}$$

5.1.3 Detail of ϖ

With (39), (43), and (45), we can write ϖ defined in (15) as

$$\begin{aligned} \varpi &= k_{12}\rho_f\frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_c d\boldsymbol{x} + \frac{\rho_f}{2}\frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{u}}_s d\boldsymbol{x}\\ &\quad +k_2\rho_f\frac{d}{dt}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_s d\boldsymbol{x} + \varpi^*, \end{aligned} \tag{46}$$

where

$$\begin{aligned}\varpi^* = & 2\mu k_{12}\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{x}}_c)d\boldsymbol{x}\\ & +2\mu k_2\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{x}}_s)d\boldsymbol{x}\\ & +2\mu\int_{\Omega_s^*}\boldsymbol{D}(\boldsymbol{u}_f):\boldsymbol{D}(\tilde{\boldsymbol{u}}_s)d\boldsymbol{x}\\ & -k_{12}\rho_f\big[\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_c d\boldsymbol{x}\\ & \quad+\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{x}}_c d\boldsymbol{x}\big]\\ & -k_2\rho_f\big[\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\partial_t\tilde{\boldsymbol{x}}_s d\boldsymbol{x}\\ & \quad+\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{x}}_s d\boldsymbol{x}\big]\\ & -\frac{\rho_f}{2}\int_{\Omega_s^*}(\boldsymbol{u}_f\otimes\boldsymbol{u}_f):\nabla\tilde{\boldsymbol{u}}_s d\boldsymbol{x}.\end{aligned}\tag{47}$$

We now derive the bound of ϖ^*. Due to the extensions $\tilde{\boldsymbol{x}}_c$, $\tilde{\boldsymbol{x}}_s$, and $\tilde{\boldsymbol{u}}_s$, we can use Hölder's inequality to obtain:

$$\begin{aligned}|\varpi^*|_E \le & 2\mu\big(|k_{12}|_E+|k_2|_E\epsilon_{11}+\vartheta(\tfrac{1}{\kappa})\big)\int_{\Omega_s^*}\|\boldsymbol{D}(\boldsymbol{u}_f)\|_E^2 d\boldsymbol{x}\\ & +2\mu\vartheta(\tfrac{1}{\kappa})\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^2 d\boldsymbol{x}\\ & +2\mu|k_{12}|_E\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E\|\boldsymbol{x}_c\|_E^2\\ & +4\mu|k_2|_E\vartheta(\tfrac{1}{\kappa})(\|\boldsymbol{x}_c\|_E^2+r_s^2\|\boldsymbol{\eta}\|_E^2)\\ & +\tfrac{1}{2}|k_{12}|_E\rho_f\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^2 d\boldsymbol{x}\\ & +\tfrac{1}{2}|k_{12}|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa})\|\boldsymbol{u}_c\|_E^2\\ & +\tfrac{1}{2}|k_2|_E\rho_f\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^2 d\boldsymbol{x}\\ & +|k_2|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa})(\|\boldsymbol{u}_c\|_E^2+r_s^2\|\boldsymbol{\omega}\|_E^2)\\ & +|k_{12}|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}\|\boldsymbol{x}_c\|_E\big(\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^4 d\boldsymbol{x}\big)^{\frac{1}{2}}\\ & +|k_2|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}(\|\boldsymbol{x}_c\|_E\\ & +r_s\|\boldsymbol{\eta}\|_E)\big(\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^4 d\boldsymbol{x}\big)^{\frac{1}{2}}\\ & +\tfrac{1}{2}\rho_f\vartheta(\tfrac{1}{\kappa})\big(\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^4 d\boldsymbol{x}\big)^{\frac{1}{2}}\big(\int_{\Omega_s^*}\|\boldsymbol{u}_f\|_E^2 d\boldsymbol{x}\big)^{\frac{1}{2}},\end{aligned}\tag{48}$$

where $r_s=\sup_{\Omega_s}\|\boldsymbol{x}-\boldsymbol{x}_c(t)\|_E$ and $\vartheta(\frac{1}{\kappa})$ is an increasing function of $\frac{1}{\kappa}$, and $|\Omega_s^*|_E$ denotes the volume of Ω_s^*.

We now use the embedding $V\subset(L^6(\Omega_s^*))^3\subset(L^4(\Omega_s^*))^3$ to write (48) as

$$\begin{aligned}|\varpi^*|_E \le & \big(\epsilon_{11}+\epsilon_{12}|\boldsymbol{u}_f|_{\Omega_s^*}^2+\epsilon_{13}\|\boldsymbol{x}_c\|_E^2+\epsilon_{14}\|\boldsymbol{\eta}\|_E^2\big)\\ & \cdot\|\boldsymbol{u}_f\|_{\Omega_s^*}^2+\epsilon_{21}\|\boldsymbol{x}_c\|_E^2+\epsilon_{22}\|\boldsymbol{\eta}\|_E^2+\epsilon_{23}\|\boldsymbol{u}_c\|_E^2\\ & +\epsilon_{24}\|\boldsymbol{\omega}\|_E^2+\epsilon_{25}|\boldsymbol{u}_f|_{\Omega_s^*}^2,\end{aligned}\tag{49}$$

where

$$\begin{aligned}\epsilon_{11} &= c\big(2\mu\big(|k_{12}|_E+|k_2|_E+\vartheta(\tfrac{1}{\kappa})\big)+\tfrac{1}{4}\rho_f\vartheta(\tfrac{1}{\kappa})\\ &\quad+|k_2|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}+\tfrac{1}{2}|k_{12}|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}\big),\\ \epsilon_{12} &= \tfrac{1}{4}c\rho_f\vartheta(\tfrac{1}{\kappa}),\\ \epsilon_{13} &= c\big(\tfrac{1}{2}|k_2|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}+\tfrac{1}{2}|k_{12}|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}\big),\\ \epsilon_{14} &= c\tfrac{1}{2}|k_2|_E\rho_f\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E^{\frac{1}{2}}r_s^2,\\ \epsilon_{21} &= 2\mu|k_{12}|_E\vartheta(\tfrac{1}{\kappa})|\Omega_s^*|_E+4\mu|k_2|_E\vartheta(\tfrac{1}{\kappa}),\\ \epsilon_{22} &= 4\mu|k_2|_E\vartheta(\tfrac{1}{\kappa})r_s^2,\\ \epsilon_{23} &= |k_2|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa})+\tfrac{1}{2}|k_{12}|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa}),\\ \epsilon_{24} &= |k_2|_E\rho_f|\Omega_s^*|_E\vartheta(\tfrac{1}{\kappa})r_s^2,\\ \epsilon_{25} &= 2\mu\big(|k_{12}|_E+\tfrac{1}{2}|k_{12}|_E\rho_f+\tfrac{1}{2}|k_2|_E\rho_f\end{aligned}\tag{50}$$

with c being the embedding constant depending on only Ω_s^*.

5.2 *Convergence of the closed-loop system*

With ϖ detailed by (46), we consider the following Lyapunov function candidate for the closed-loop system:

$$U=U_1+\epsilon_{01}\mathcal{E}+\tfrac{\epsilon_{02}}{2}\mathcal{E}^2+U_2,\tag{51}$$

where U_1 is given by (11), $\mathcal{E}$ is given by (28), ϵ_{01} and ϵ_{02} are positive constants to be chosen, and

$$\begin{aligned}U_2 = & (k_1-k_2)\rho_f\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_c d\boldsymbol{x}\\ & -\frac{\rho_f}{2}\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{u}}_s d\boldsymbol{x}-k_2\rho_f\int_{\Omega_s^*}\boldsymbol{u}_f\cdot\tilde{\boldsymbol{x}}_s d\boldsymbol{x}.\end{aligned}\tag{52}$$

Using Hölder's inequality, we can find the bound of U_2 as

$$|U_2|_E\le\epsilon_{31}\|\boldsymbol{x}_c\|_E^2+\epsilon_{32}\|\boldsymbol{\eta}\|_E^2+\epsilon_{33}\rho_f|\boldsymbol{u}_f|_{\Omega_s^*}^2.\tag{53}$$

where

$$\begin{aligned}\epsilon_{31} &= \rho_f\big(\tfrac{1}{2}|k_{12}|_E+|k_2|_E|\Omega_s^*|_E\big),\\ \epsilon_{32} &= |k_2|_E\rho_f|\Omega_s^*|_E r_s^2,\\ \epsilon_{33} &= \tfrac{1}{2}|k_{12}|_E+\tfrac{1}{4}\vartheta(\tfrac{1}{\kappa}).\end{aligned}\tag{54}$$

Since $\rho_f|\boldsymbol{u}_f|_{\Omega_s^*}^2\le\int_\Omega\rho\|\boldsymbol{u}\|_E^2$ due to $\Omega_s^*\subset\Omega$ and definition of $\boldsymbol{u}$ in (21), we can find the bound for U as:

$$\begin{aligned}& U_1+\underline{\epsilon}_{01}\mathcal{E}+\tfrac{1}{2}\epsilon_{02}\mathcal{E}^2\le U\le U_1+\bar{\epsilon}_{01}\mathcal{E}+\tfrac{1}{2}\epsilon_{02}\mathcal{E}^2,\\ & \alpha_0\big(\tfrac{\hat{k}_1}{2}\|\boldsymbol{x}_c\|_E^2+\tfrac{\hat{k}_2}{2}\|\boldsymbol{\eta}\|_E^2+\tfrac{1}{2}\int_\Omega\rho\|\boldsymbol{u}\|_E^2\big)+\bar{k}_{01}\|\boldsymbol{Y}_s\|_E^2,\end{aligned}\tag{55}$$

where $\boldsymbol{Y}_s$ is defined just below (12), and we choose a sufficiently large ϵ_{01} such that

$$\begin{aligned}\underline{\epsilon}_{01} &= \min\big(\epsilon_{01}\tfrac{\hat{k}_1}{2}-\epsilon_{31},\epsilon_{01}\tfrac{\hat{k}_2}{2}-\epsilon_{32},\tfrac{1}{2}\epsilon_{01}-\epsilon_{33}\big)>0,\\ \bar{\epsilon}_{01} &= \max\big(\epsilon_{01}\tfrac{\hat{k}_1}{2}+\epsilon_{31},\epsilon_{01}\tfrac{\hat{k}_2}{2}+\epsilon_{32},\tfrac{1}{2}\epsilon_{01}+\epsilon_{33}\big)>0.\end{aligned}\tag{56}$$

Differentiating (51) along the solutions of (19), (29), using (46), and noting that $|\boldsymbol{u}_f|_{\Omega_s^*}^2\le|\boldsymbol{u}|^2$ and $\|\boldsymbol{u}_f\|_{\Omega_s^*}^2\le\|\boldsymbol{u}\|^2$ due to $\Omega_s^*\subset\Omega$ and definition of $\boldsymbol{u}$ in (21), we have

$$\begin{aligned}\frac{dU}{dt} = & \frac{dU_1}{dt}+(\epsilon_{01}+\epsilon_{02}\mathcal{E})\frac{d\mathcal{E}}{dt}+\frac{dU_2}{dt}\\ \le & -\tfrac{1}{2}\bar{k}_1\|\boldsymbol{x}_c\|_E^2-\tfrac{1}{2}\bar{k}_2\|\boldsymbol{u}_c\|_E^2-\tfrac{1}{2}\bar{k}_3\|\boldsymbol{\eta}\|_E^2-\tfrac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2\\ & -\tfrac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\tfrac{\mu}{\rho_f}\int_\Omega\rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x}+k_4 m_s\|\boldsymbol{u}_c\|_E^2\\ & +k_6(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}\big)+\varpi_0,\end{aligned}\tag{57}$$

where

$$\begin{aligned}\varpi_0 = & -\tfrac{1}{2}\bar{k}_1\|\boldsymbol{x}_c\|_E^2-\tfrac{1}{2}\bar{k}_2\|\boldsymbol{u}_c\|_E^2-\tfrac{1}{2}\bar{k}_3\|\boldsymbol{\eta}\|_E^2-\tfrac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2\\ & -\tfrac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\tfrac{\mu}{\rho_f}\int_\Omega\rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x}+k_4 m_s\|\boldsymbol{u}_c\|_E^2\\ & +k_6(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}\big)+\varpi^*.\end{aligned}\tag{58}$$

Substituting the bound of ϖ^* in (49) into (58), and choosing

$$\bar{k}_1\ge\epsilon_{21},\quad\bar{k}_2\ge\epsilon_{23},\quad\bar{k}_3\ge\epsilon_{22},\quad\bar{k}_4\ge\epsilon_{24},\tag{59}$$

which is always feasible because we can choose $k_i, i=1,\cdots,6$ such that $\bar{k}_i, i=1,\cdots,4$ are as large as required, see the paragraph just under (20), and sufficiently large ϵ_{01} and ϵ_{02}, we can use the Poincaré inequality to ensure that

$$\varpi_0\le 0.\tag{60}$$

Substituting (60) in to (57) yields

$$\frac{dU}{dt} \leq -\frac{1}{2}\bar{k}_1\|\boldsymbol{x}_c\|_E^2 - \frac{1}{2}\bar{k}_2\|\boldsymbol{u}_c\|_E^2 - \frac{1}{2}\bar{k}_3\|\boldsymbol{\eta}\|_E^2 - \frac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2 - \frac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\frac{\mu}{\rho_f}\int_\Omega \rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x} + k_4 m_s\|\boldsymbol{u}_c\|_E^2 + k_6(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}\big). \tag{61}$$

Integrating (61) from 0 to ∞ yields

$$\int_0^\infty \Big[\frac{1}{2}\bar{k}_1\|\boldsymbol{x}_c\|_E^2 + \frac{1}{2}\bar{k}_2\|\boldsymbol{u}_c\|_E^2 + \frac{1}{2}\bar{k}_3\|\boldsymbol{\eta}\|_E^2 + \frac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2 + \frac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\frac{\mu}{\rho_f}\int_\Omega \rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x} + k_4 m_s\|\boldsymbol{u}_c\|_E^2 + k_6(1+\|\boldsymbol{\eta}\|_E^2)\boldsymbol{\omega}\cdot\boldsymbol{J}_s\boldsymbol{\omega}\big)\Big]dt \leq U(0)-U(\infty) \leq U(0). \tag{62}$$

Since we have already proved existence of the solution of the closed-loop system consisting of (1), (5), and (17), the inequality (62) implies that

$$\lim_{t\to\infty}\Big[\frac{1}{2}\bar{k}_1\|\boldsymbol{x}_c(t)\|_E^2 + \frac{1}{2}\bar{k}_2\|\boldsymbol{u}_c(t)\|_E^2 + \frac{1}{2}\bar{k}_3\|\boldsymbol{\eta}(t)\|_E^2 + \frac{1}{2}\bar{k}_4\|\boldsymbol{\omega}\|_E^2 + \frac{1}{2}(\epsilon_{01}+\epsilon_{02}\mathcal{E})\big(\frac{\mu}{\rho_f}\int_\Omega \rho\|\boldsymbol{D}(\boldsymbol{u}(t))\|_E^2 d\boldsymbol{x} + k_4 m_s\|\boldsymbol{u}_c(t)\|_E^2 + k_6(1+\|\boldsymbol{\eta}(t)\|_E^2)\boldsymbol{\omega}(t)\cdot\boldsymbol{J}_s\boldsymbol{\omega}(t)\big)\Big] = 0, \tag{63}$$

which shows global asymptotic stability of the closed-loop system. We now show local exponential stability of the closed-loop system, i.e. $\Upsilon(t) \leq \beta(t, \Upsilon(0))$, where $\Upsilon(t) = \|\boldsymbol{x}_c(t)\|_E^2 + \|\boldsymbol{\eta}(t)\|_E^2 + \int_\Omega \|\boldsymbol{u}(t,\boldsymbol{x})\|_E^2 d\boldsymbol{x}$, $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}_\infty$-function. When $(\boldsymbol{x}_c, \boldsymbol{u}_c, \boldsymbol{\eta}, \boldsymbol{\omega}, \int_\Omega \rho\|\boldsymbol{D}(\boldsymbol{u})\|_E^2 d\boldsymbol{x})$ are small in magnitude (i.e., when the closed-loop system evolutes for a sufficiently long time, say $t \geq t_0$ for some $t_0 \geq 0$), we obtain from (61), 55), (11), and (28) that

$$\frac{dU}{dt} \leq -c_0 U, \quad \forall\, t \geq t_0 \geq 0 \tag{64}$$

where c_0 is a positive constant. From (64), it holds that $U(t) \leq U(t_0)e^{-c_0(t-t_0)}$, and hence local exponential stability of the closed-loop system is ensured.

We summarize the main results in the following theorem.

Theorem 5.1 *Under the initial data (9), the controls $\boldsymbol{F}_k$, which are obtained from (17), solves Control Objective 2.1 for all $t \in [0, T]$, where T is such that $\Omega_s(t) \Subset \Omega$. In particular, the closed-loop system consisting of (1), (5), and (17) has at least one weak solution, which is defined in Definition 4.1 for all $t \in [0, T]$ such that*

$$\begin{aligned} &\boldsymbol{x}_c \in \Omega,\ \boldsymbol{\eta} \in \mathcal{D}_\eta,\ \rho, h \in L^\infty(Q), \\ &\boldsymbol{u} \in L^\infty(0,T;H) \cap L^2(0,T;V),\ p \in L^2(Q), \end{aligned} \tag{65}$$

where $(\rho, h, \boldsymbol{u})$ are defined in (21). Moreover, the closed-loop system is globally asymptotically and locally stable at the origin provided that there is no collision between the rigid body and the boundary of the fluid domain, i.e.,

$$\Upsilon(t) \leq \beta(t, \Upsilon(0)), \tag{66}$$

where $\Upsilon(t) = \|\boldsymbol{x}_c(t)\|_E^2 + \|\boldsymbol{\eta}(t)\|_E^2 + \int_\Omega \|\boldsymbol{u}(t,\boldsymbol{x})\|_E^2 d\boldsymbol{x}$, $\beta(\cdot,\cdot)$ is a class $\mathcal{KL}_\infty$-function, and if $\Upsilon(t_0)$, where $t_0 \geq 0$, is sufficiently small, then $\Upsilon(t) \leq \Upsilon(t_0)e^{-c_0(t-t_0)}$, where c_0 is a positive constant.

6 Simulations

In this section, we perform a simulation to illustrate the effectiveness of the control law given by (17). We take a rectangular prism as the domain Ω with dimensions $[L_1 \times L_2 \times L_3] = [-\frac{1}{2}\pi, \frac{1}{2}\pi]\text{m} \times [-\frac{1}{2}\pi, \frac{1}{2}\pi]\text{m} \times [-\frac{3}{2}\pi, \frac{3}{2}\pi]\text{m}$. For the fluid, we take water as the fluid with $\mu = 1.793 \times 10^{-3}$kg/ms and $\rho_f = 980$kg/m^3. For the rigid body, we take the physical shape of a rectangular prism with dimensions: $\frac{\pi}{10}\text{m} \times \frac{\pi}{10}\text{m} \times \frac{3\pi}{10}\text{m}$ and the mass: $m_s = 10$kg, which give $\boldsymbol{J}_s = \text{diag}(0.1645, 0.1645, 0.8225)\text{kgm}^2$. We approximate all the sharp corners of Ω and Ω_s by rounding them off to make $\partial\Omega$ and $\partial\Omega_f$ Lipschitz. We assume that there are six forces $\boldsymbol{F}_k, k = 1, \cdots, 6$ located at six locations $\boldsymbol{R}_k$, which are configured as

$$\begin{aligned} &\boldsymbol{F}_1 = \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{F}_2 = \begin{bmatrix} f_2 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{F}_3 = \begin{bmatrix} 0 \\ f_3 \\ 0 \end{bmatrix}, \boldsymbol{F}_4 = \begin{bmatrix} 0 \\ f_4 \\ 0 \end{bmatrix}, \boldsymbol{F}_5 = \begin{bmatrix} 0 \\ 0 \\ f_5 \end{bmatrix}, \boldsymbol{F}_6 = \begin{bmatrix} 0 \\ 0 \\ f_6 \end{bmatrix}, \\ &\boldsymbol{R}_1 = \begin{bmatrix} 0 \\ r_1 \\ 0 \end{bmatrix}, \boldsymbol{R}_2 = \begin{bmatrix} 0 \\ r_2 \\ 0 \end{bmatrix}, \boldsymbol{R}_3 = \begin{bmatrix} 0 \\ 0 \\ r_3 \end{bmatrix}, \boldsymbol{R}_4 = \begin{bmatrix} 0 \\ 0 \\ r_4 \end{bmatrix}, \boldsymbol{R}_5 = \begin{bmatrix} r_5 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{R}_6 = \begin{bmatrix} r_6 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{67}$$

Then we can write (10) as

$$\boldsymbol{f} = \boldsymbol{Q}^{-1}\begin{bmatrix} \boldsymbol{F}_s \\ -\boldsymbol{M}_s \end{bmatrix} \tag{68}$$

where $\boldsymbol{f} = \text{col}(f_1, \cdots, f_6)$ and

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & r_3 & r_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_5 & r_6 \\ r_1 & r_2 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{69}$$

The determinant $\boldsymbol{Q}$ is $\det(\boldsymbol{Q}) = r_1r_3r_5 - r_1r_3r_6 - r_1r_4r_5 - r_2r_3r_5 + r_1r_4r_6 + r_2r_3r_6 + r_2r_4r_5 - r_2r_4r_6$ and can be made nonzero to make $\boldsymbol{Q}$ invertible by a simple choice: $r_1 = -r_2 \neq 0$, $r_3 = -r_4 \neq 0$ and $r_5 = -r_6 \neq 0$. This choice yields $\det(\boldsymbol{Q}) = 8r_1r_3r_5$, which is nonzero due to $r_k \neq 0$ for all $k = 1, \cdots, 6$. In the simulations, we choose $r_1 = -r_2 = \frac{\pi}{10}$m, $r_3 = -r_4 = \frac{\pi}{10}$m, $r_5 = -r_6 = \frac{3\pi}{10}$m. The formula (68) is to calculate the individual forces $\boldsymbol{F}_k$ as $\boldsymbol{F}_s$ and $\boldsymbol{M}_s$ are given by (17). We pre-eliminate the difference between buoyancy and gravity forces before applying (17).

We will use the semi-Galerkin method to the penalized system (33) to obtain a numerical weak solution, where we approximate

$$\boldsymbol{u}_\delta^n(t,\boldsymbol{x}) = \sum_{l=1}^n c_l^n(t)\boldsymbol{a}_l(\boldsymbol{x}), \tag{70}$$

where $c_l^n(t)$ are scalar functions of time, $\boldsymbol{a}_l(\boldsymbol{x})$ are eigenfunctions of the Stokes operator. We substitute (70) into the first equation of (33) and multiply it by $\boldsymbol{\xi} = \text{Spann}\{\boldsymbol{a}_l(\boldsymbol{x}); l = 1, \cdots, n\}$ to obtain a system of ODEs for $c_l^n(t)$, which is numerically solvable. The transport equations (the third and fourth equations of (33)) are solved by using the characteristic method. Next, we choose the penalized parameter as $\delta = \frac{1}{n}$. We now need to derive eigenfunctions for our domain Ω. To do so, we need the following lemma [16, Theorem III.2.3].

Lemma 6.1 *If Ω is a bounded open set in $\mathbb{R}^3$ with Lipschitz boundary, then H coincides with the space of divergence free functions in $L^2(\Omega)$ such that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, where $\boldsymbol{n}$ is the normal unit vector to $\partial\Omega$.*

With this lemma, eigenfunctions of the Stokes problem are equivalent to those of the Laplace operator with the condition $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ as we consider a weak solution in H. Hence, we look for $\boldsymbol{a}_l$ such that

i) $\Delta\boldsymbol{a}_l = -\lambda_l\boldsymbol{a}_l$, $\mathrm{div}(\boldsymbol{a}_l) = 0$ in Ω; $\boldsymbol{a}_l\cdot\boldsymbol{n} = 0$ on $\partial\Omega$,
ii) $\boldsymbol{a}_l$ is an orthornormal basis of $H(\Omega)$,
iii) $\boldsymbol{a}_l$ is an orthogonal basis of $V(\Omega)$,
iv) $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_l \to \infty$ as $l \to \infty$.

A nontrivial calculation gives $\boldsymbol{a}_l$ (we neglect the round off corners), which satisfies all the above properties, as follows:

$$\boldsymbol{a}_l = \frac{\bar{a}_l}{\lambda_l}\begin{bmatrix} L_1\cos(l_1x_1)\sin(l_2x_2)\sin(l_3x_3) \\ L_2\sin(l_1x_1)\cos(l_2x_2)\sin(l_3x_3) \\ L_3\sin(l_1x_1)\sin(l_2x_{)}\cos(l_3x_3)\end{bmatrix}, \tag{71}$$

where

$$\begin{aligned} &L_1 = l_2^2+l_3^2+l_1(l_2-l_3),\ L_2 = -(l_1^2+l_3^2+l_2(l_1+l_3)),\\ &L_3 = (l_1^2+l_2^2+l_3(l_2-l_1),\ \bar{a}_l = \sqrt{\frac{8\lambda_l^2}{\pi^3(L_1^2+L_2^2+L_3^2)}},\\ &\lambda_l = l_1^2+l_2^2+l_3^2, \end{aligned} \tag{72}$$

for $(l_1, l_2, l_3) \in \mathbb{Z}^3$ such that $l^2 = l_1^2 + l_2^2 + l_3^2$, which are taken into account to have summing combination in calculating (70). We perform two simulations. In both simulations, we choose the control gains as follows: $k_1 = 0.05$, $k_4 = 8$, $k_2 = 0.1$, and $k_6 = 3$. This choice gives $k_3 = 1.825$, $k_5 = 0.1$, $\bar{k}_1 = 0.11$, $\bar{k}_2 = 7.5$, $\bar{k}_3 = 0.13$, and $\bar{k}_4 = 2.92$ according to (20). Clearly, the conditions in (13) and (20) hold. Moreover, we choose $n = 10^8$, which gives $\delta = 10^{-8}$.

In the first simulation, for the initial values of the fluid velocity we take $c_l^n(0)$ to be random number in $\frac{1}{n^2}[-1, 1]$. The initial values of the rigid body are taken as $\boldsymbol{x}_c(0) = \mathrm{col}(0.2, -0.2, 0.4)$m, $\boldsymbol{\eta}(0) = \mathrm{col}(1.6, 0.4, 2.5)$, which yields a principal axis/angle pair $\boldsymbol{e} = \mathrm{col}(0.4782, 0.2050, 0.8540)$ and $\gamma = 4.9665$ rad. The initial values of the velocities $\boldsymbol{u}_c(0)$ and $\boldsymbol{\omega}(0)$ of the rigid body are determined via (31), (23), and the interface condition given by the second equation in (6), where $\boldsymbol{u}(0, \boldsymbol{x})$ is substituted by $\boldsymbol{u}_\delta^n(0, \boldsymbol{x})$.

The position vector $\boldsymbol{x}_c$, orientation vector $\boldsymbol{\eta}$, linear velocity vector $\boldsymbol{u}_c$, angular velocity vector $\boldsymbol{\omega}$, and H-norm of the global velocity $\int_\Omega \|\boldsymbol{u}\|_E^2 d\boldsymbol{x}$ are plotted in Fig. 2. The control force vector $\boldsymbol{F}_s$, control moment vector $\boldsymbol{M}_s$, and control forces $f_k, k = 1, \cdots, 6$, see (68), are plotted in Fig. 3. It is seen from these figures that all the states $\boldsymbol{x}_c$, $\boldsymbol{\eta}$, $\boldsymbol{u}_c$, and $\boldsymbol{\omega}$, $|\boldsymbol{u}| = (\int_\Omega \|\boldsymbol{u}\|_E^2 d\boldsymbol{x})^{\frac{1}{2}}$; and the controls $\boldsymbol{F}_s$, $\boldsymbol{M}_s$, and f_k converge to zero. It is noted that convergence of the rigid body states $\boldsymbol{x}_c$, $\boldsymbol{\eta}$, $\boldsymbol{u}_c$, and $\boldsymbol{\omega}$ to zero is affected by that of $|\boldsymbol{u}|$ due to the fluid forces and fluid moments on the rigid body.

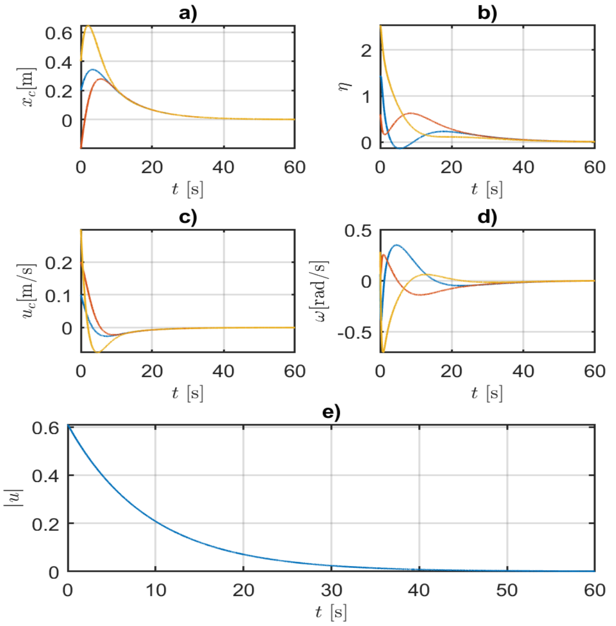

Fig. 2. First simulation - states: $\boldsymbol{x}_c$, $\boldsymbol{\eta}$, $\boldsymbol{u}_c$, $\boldsymbol{\omega}$, and $|\boldsymbol{u}| = (\int_\Omega \|\boldsymbol{u}\|_E^2 d\boldsymbol{x})^{\frac{1}{2}}$.

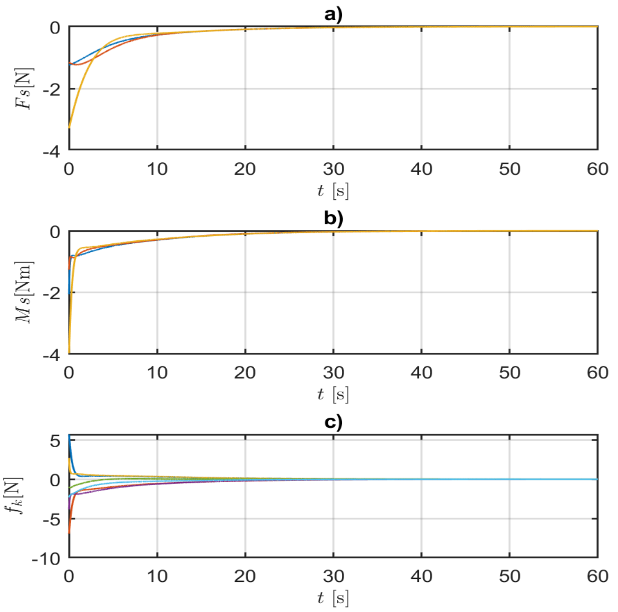

Fig. 3. First simulation - controls: $\boldsymbol{F}_{cs}$, $\boldsymbol{M}_s$, and $\boldsymbol{f}_k$.

To illustrate the robustness/performance of the proposed stabilization controller under the same control gains, we perform the second simulation with the initial values $\boldsymbol{x}_c(0) = \mathrm{col}(0.4, -0.4, 0.8)$m while all other initial values and parameters are taken the same as in the first solution. Simulation results are plotted in Fig. 4 and Fig. 5. Explanation of Fig. 4 and Fig. 5 is similar to that of Fig. 2 and Fig. 3. Comparing Fig. 2 and Fig. 4; Fig. 3 and Fig. 5 shows that the proposed stabilization controller stabilizes the rigid body very well under different positions of the rigid body.

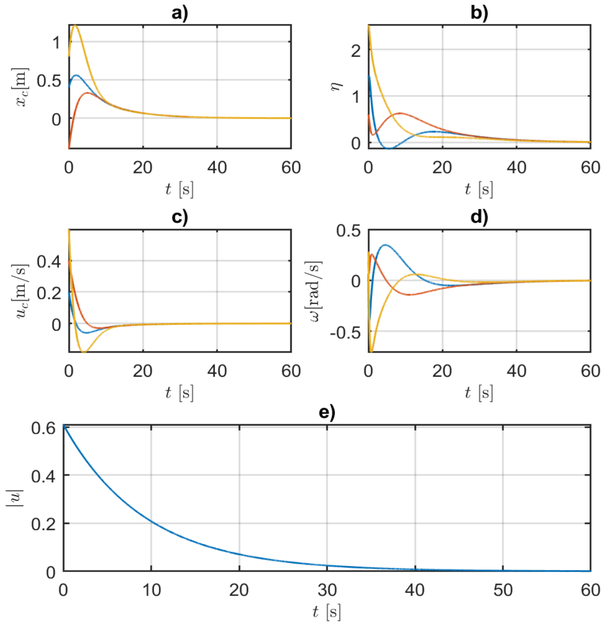

Fig. 4. Second simulation - states: $\boldsymbol{x}_c$, $\boldsymbol{\eta}$, $\boldsymbol{u}_c$, $\boldsymbol{\omega}$, and $|\boldsymbol{u}| = (\int_\Omega \|\boldsymbol{u}\|_E^2 d\boldsymbol{x})^{\frac{1}{2}}$.

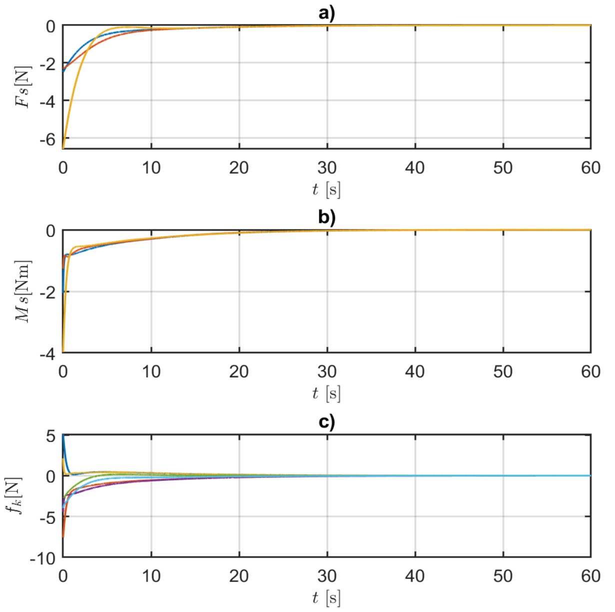

Fig. 5. First simulation - controls: $\boldsymbol{F}_{cs}$, $\boldsymbol{M}_s$, and $\boldsymbol{f}_k$.

7 Conclusions

Global asymptotic and local exponential stabilization of a rigid body in an incompressible viscous fluid under potential body force with the fluid velocity $\boldsymbol{u}_f(0, \boldsymbol{x}) \in H$ was solved in this paper under an assumption that there is no collision between the rigid body and the boundary of the fluid domain. Since the fluid forces and fluid moments on the rigid body are not able to bound in an Euclidean norm due to $\boldsymbol{u}_f(0, \boldsymbol{x}) \in H$, the "fluid work and fluid power" on the rigid body can be bound and should be used for stability and convergence analysis. Future work is to extend to stabilization of a rigid body in multiple fluids to cover practical cases such as floating rigid bodies.

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